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New Model for Hill's Problem in the Framework of Continuation Fractional Potential

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Abstract: In this work, we derived a new type model for spatial Hill's system considering the created perturbation by the parameter effect of the continuation fractional potential. The new model is considered a reduced system from the restricted three-body problem under the same effect for describing Hill's problem. We identified the associated Lagrangian and Hamiltonian functions of the new system, and used them to verify the existence of the new equations of motion. We also proved that the new model has different six valid solutions under different six symmetries transformations as well as the original solution, where the new model is an invariant under these transformations. The several symmetries of Hill's model can extremely simplify the calculation and analysis of preparatory studies for the dynamical behavior of the system. Finally, we confirm that these symmetries also authorize us to explore the similarities and differences among many classes of paths that otherwise differ from the obtained trajectories by restricted three-body problem.

Keywords: RTBP; Hill's problem; CFP; symmetry

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1. Introduction

The diversity of activities in space, especially those related to the motion of celestial objects or spacecrafts, leads also to the diversity in applying of mathematical models that describe the motion of such bodies, such as the motion of two, three, or more bodies, or even modified ones, to simplify their analysis while maintaining the accuracy of obtaining results. Some considerable works, which represent a large range in in both celestial mechanics and astrodynamics, were studied in [1–8].

In general, the three-body problem describes the motion of three particles that move under the mutual attraction forces among them. This problem can be applied in outer space to the motion of stars. But scientists realized that there is no general solution to such a problem, and this prompted them to make restrictions and simplifications about the motion through scientific thinking in accordance with the mechanics laws to find some solutions or study the dynamic behavior of motion or its characteristics. For instance, the mass of the Sun or Jupiter is clearly much greater than a comet's mass, so the effect of a comet's gravity on the motion of Jupiter or the Sun is so small that it does not affect the motion of either of them. This leads immediately to the restricted models of the N-body problem [9–16].

In addition to the aforementioned restrictions, variables changing can lead to special models for a restricted three- or four-bodies problem to the so-called Hill's systems [17–23]. A precise mathematical technique for specifying these limiting processes through the Hamiltonian formula and symmetric scaling methods is presented. Hill's system approximations in the framework of restricted three-body problem (RTBP) provide the first term in its own Hamiltonian's expansion when canonical transformations are performed. After Hill's system was constructed, the early researchers devoted their efforts to identify the families of periodic orbits in this dynamical system. Perturbation approaches can be applied to demonstrate the existence of periodic orbits and KAM tori, which reveal regions of limited motion [24–29].

Although the RTBP has been considered as the underlying framework in many studied, to analyze the dynamical properties of celestial objects motion or design space missions, there are numerous motivating transfers that can be developed under utilizing Hill's system, which was constructed by Hill in 1978 to study the Moon's motion. This system, depicted as "luminous" by Henrard [30], offers the non-negligible feature of unpretentiousness while precisely investigating the nonlinear dynamics of interest. Thus, this little transfer in modeling admits us to explore a range of physical cases while greatly simplifying equations of motions. A wide range of researchers are attracted to Hill's system because it represents a simple formulation of a nonintegrable model. Furthermore, it can be employed in both dynamical astronomy and astrodynamics. It is used to investigate dynamical properties of star clusters, the analysis of motion in the Earth–Moon–Sun system, as comets and asteroids [31–33].

The periodic orbits (POs) play a significant role in the analysis of dynamical system, particularly in the analysis of astronomical dynamical system. In this sense, a considerable work on the dynamics of Hill's system was investigated by using the amended "Grid Search Method" for the global calculation the family of periodic trajectories orbits [34]. In fact, this method is very effective in finding POs, and aiming to obtain precise trajectories for these orbits, the old version of this method was improved through implementation of two steps. The first improvement is related to the effectiveness speed of the method for its main proceedings of global sampling in the phase space for initial conditions. While the second improvement is accomplished by complement it with a second phase which is a data processing proceeding for classifying the families of the POs which are identified in the first phase. The improved method was examined through running it to retrieval some famous POs in the Hill's problem. Furthermore, it has been successfully employed to identify some new results in Hill's system. The method was used to calculate the whole tree branches of the main family f of retrograde satellites in the classical Hill's system, and the mechanism of the bifurcation of these orbits from f was verified based on the relevant self-resonance conditions. This approach has a great advantage in the calculations of irregular families of POs regard to the complete search in its first phase. In fact, it was applied to calculate many families of POs not only in the Hill's system but also in the RTBP considering several perturbation effects [35,36]. This method is recommended for computing the family of POs in a perturbed Hill's system, which their calculations are very difficult by using continuation methods.

The analysis of an unperturbed dynamical system is more simple than the perturbed, but it does not often achieve what is needed, because it is not accurate in depicting the real features of the problem, and does not gather into consideration the perturbing forces such as the asphericity effect, which has an efficient impact on the motion of bodies. In this paper, we develop a new type of model for spatial Hill's problem depending on the potential of a massive body which is different in its own structure from that of the spherical body, but its potential is similar to the continuation fractional potential (CFP), which gives the same feature of the oblate body without involving the singularity when the separation distance between two bodies equals zero. This is the substantial motivation for fulfilling this study.

2. Model Description

Let us consider the motion of the infinitesimal body in the framework of a circular restricted three-body problem (CRTBP). In this sense, we assume that m_1 and m_2 are the masses of two bodies such as the Earth and Moon, respectively. According to the fact that the Earth is not spherical body, its gravitational field can be described by a CFP, but the Moon generates a potential similar to the created by a spherical body [37]. Let m be the infinitesimal body mass, and it is moving under mutual gravitational forces of m_1 and m_2, which are called the primaries bodies, where this mass is negligible with respect to the primaries masses and does not affect their motions. We also impose that the primaries bodies are moving in circular orbits around their center of mass.

Now we admit some terminology to normalize the physical quantities such as distances and masses. Thus, we consider that the sum of primaries' masses equals one, and denote the mass of smaller primary as μ; hence, $m_1 = 1-\mu$ and $m_2 = \mu$. The separation distance between the primaries and the gravitational constant are also taken as unity. Thus, the equations of motion of the infinitesimal body in rotating reference frame can be written as in [38] by

$$\begin{aligned} \ddot{x}_1 - 2\omega\dot{y}_1 &= \Theta_{x_1}, \\ \ddot{y}_1 + 2\omega\dot{x}_1 &= \Theta_{y_1}, \\ \ddot{z}_1 &= \Theta_{z_1}, \end{aligned} \tag{1}$$

where $\Theta(x_1, y_1, z_1)$ is called the effective potential and written as

$$\Theta(x_1, y_1, z_1) = \frac{1}{2}\omega^2(x_1^2 + y_1^2) + \frac{(1-\mu)\, r_{11}}{r_{11}^2 + \varepsilon} + \frac{\mu}{r_{12}}, \tag{2}$$

$$\omega^2 = \frac{1-\varepsilon}{(1+\varepsilon)^2}, \tag{3}$$

and $r_{11} = |\mathbf{r}_{11}|$ and $r_{12} = |\mathbf{r}_{12}|$ are the separation distances among the infinitesimal body, the massive, and smaller bodies, respectively, where these distances are identified by

$$\begin{aligned} r_{11}^2 &= (x_1+\mu)^2 + y_1^2 + z_1^2, \\ r_{12}^2 &= (x_1+\mu-1)^2 + y_1^2 + z_1^2. \end{aligned} \tag{4}$$

In Equations (2) and (3), the parameter ε represents the effect of perturbation force, which comes from the CFP, while ω indicates the perturbed mean motion. We would like to remark that the parameter ε plays the same role of oblateness or zonal harmonic coefficients effects; see [37] for more details. Hence, the order of ε effect belongs to the interval $[\mathcal{O}(10^{-3}), \mathcal{O}(10^{-6})]$ from the major force. Furthermore, the functions Θ_{x_1}, Θ_{y_1}, and Θ_{z_1} denote the first-order partial derivatives of $\Theta(x_1, y_1, z_1)$ with respect to x_1, y_1, and z_1, respectively; thus, formulae of these derivatives can be written:

$$\begin{aligned} \Theta_{x_1} &= (x_1+\mu)[\lambda_1(r_{11}) + \lambda_2(r_{12})] - \lambda_2(r_{12}), \\ \Theta_{y_1} &= y_1[\lambda_1(r_{11}) + \lambda_2(r_{12})], \\ \Theta_{z_1} &= z_1\left[\lambda_1(r_{11}) + \lambda_2(r_{12}) - \omega^2\right], \end{aligned} \tag{5}$$

where

$$\begin{aligned} \lambda_1(r_{11}) &= (1-\mu)\left(\omega^2 - \frac{r_{11}^2 - \varepsilon}{r_{11}(r_{11}^2+\varepsilon)^2}\right) \\ \lambda_2(r_{12}) &= \mu\left(\omega^2 - \frac{1}{r_{12}^3}\right). \end{aligned} \tag{6}$$

After integrating Equation (1), we obtain

$$2\Theta(x_1, y_1, z_1) - (\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) = C, \tag{7}$$

Relation (7) represents the first integral of motion, where C is the constant of integration and is well known as the Jacobian constant. The importance of this constant is behind its own properties, which can be employed to calculate the curves of zero velocity and recognize the regions of admissible and prohibited motion.

3. Perturbed Hill's System

Now we assume that the infinitesimal body (planetesimal) orbits in the proximity of the earth (protoplanet) which has lower mass than the sun (protosun). This means that the motion can be formulated by Hill's approximation to the circular restricted three-body problem (CRTBP). To employ this approximation, we will follow Szebehely's transformation [17]. Firstly, we have to subordinate the equations of motion of the perturbed CRTBP in System (1) and its related formulae to a translation along the X_1-axis. In this manner, we admit that the origin of coordinates moves to the center of the smaller primary body. Thus, the old variables (x_1, y_1, z_1) and new (x_2, y_2, z_2) are linked by the following transformation:

$$\begin{aligned} x_1 &= x_2 - \mu + 1, \\ y_1 &= y_2, \\ z_1 &= z_2. \end{aligned} \tag{8}$$

Utilizing Equation (8) with Equations (1) and (2), we obtain

$$\begin{aligned} \ddot{x}_2 - 2\omega \dot{y}_2 &= \frac{\partial}{\partial x_2} \Upsilon(x_2, y_2, z_2), \\ \ddot{y}_2 + 2\omega \dot{x}_2 &= \frac{\partial}{\partial y_2} \Upsilon(x_2, y_2, z_2), \\ \ddot{z}_2 &= \frac{\partial}{\partial z_2} \Upsilon(x_2, y_2, z_2), \end{aligned} \tag{9}$$

where the new derivatives, which related to Equation (5), can be rewritten as

$$\begin{aligned} \Upsilon_{x_2} &= (1 + x_2)[\lambda_1(\rho_{11}) + \lambda_2(\rho_{12})] - \lambda_2(\rho_{12}), \\ \Upsilon_{y_2} &= y_2[\lambda_1(\rho_{11}) + \lambda_2(\rho_{12})], \\ \Upsilon_{z_2} &= z_2\left[\lambda_1(\rho_{11}) + \lambda_2(\rho_{12}) - \omega^2\right], \end{aligned} \tag{10}$$

Here, $\lambda_1(\rho_{11})$ and $\lambda_2(\rho_{12})$ are identified by Equation (6), while $\Upsilon \equiv \Upsilon(x_2, y_2, z_2)$ and is given by

$$\Upsilon = \frac{1}{2}\omega^2\left(x_2 - \mu + 1)^2 + y_2^2\right) + \frac{(1-\mu)\,\rho_{11}}{\rho_{11}^2 + \varepsilon} + \frac{\mu}{\rho_{12}}, \tag{11}$$

and substituting Equation (8) into Equation (4), we obtain

$$\begin{aligned} \rho_{11}^2 &= (1 + x_2)^2 + y_2^2 + z_2^2, \\ \rho_{12}^2 &= x_2^2 + +y_2^2 + z_2^2. \end{aligned} \tag{12}$$

Secondly, we again follow Szebehely in [17] to scale the variables by introducing

$$\begin{aligned} x_2 &= \mu^\beta x, \\ y_2 &= \mu^\beta y, \\ z_2 &= \mu^\beta z. \end{aligned} \tag{13}$$

The aforementioned scale will preserve the magnitude of Coriolis and centrifugal terms in the same order of the previous equations.

Substituting Equation (13) into in Equations (9)–(12), we obtain

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} &= \mu^{-2\beta}\frac{\partial}{\partial x}U(x,y,z) \\ \ddot{y} + 2\omega\dot{x} &= \mu^{-2\beta}\frac{\partial}{\partial y}U(x,y,z) \\ \ddot{z} &= \mu^{-2\beta}\frac{\partial}{\partial z}U(x,y,z), \end{aligned} \tag{14}$$

where $U \equiv U(x,y,z)$ is given by

$$U = \frac{1}{2}\omega^2\left[(\mu^\beta x - \mu + 1)^2 + \mu^{2\beta}y^2\right] + \frac{(1-\mu)\,\rho_1}{\rho_1^2 + \varepsilon} + \frac{\mu}{\rho_2}, \tag{15}$$

and

$$\begin{aligned} &\rho_1^2 = 1 + 2\mu^\beta x + \mu^{2\beta}\rho^2, \\ &\rho_2^2 = \mu^{2\beta}\rho^2, \\ &\rho^2 = x^2 + y^2 + z^2, \end{aligned} \tag{16}$$

Utilizing Equations (15) and (16), the derivatives U_x, U_y, and U_z are identified by the following formulae:

$$\begin{aligned} U_x &= \mu^\beta(1-\mu)(1+\mu^\beta x)\left[\omega^2 - \Gamma(\rho_1)\right] + \mu^{2\beta}x\left[\mu^{2\beta}\omega^2 - \frac{\mu^{1-3\beta}}{\rho^3}\right], \\ U_y &= \mu^{2\beta}y\left[\mu^{2\beta}\omega^2 - \frac{\mu^{1-3\beta}}{\rho^3} - (1-\mu)\Gamma(\rho_1)\right], \\ U_z &= -\mu^{2\beta}z\left[\frac{\mu^{1-3\beta}}{\rho^3} + (1-\mu)\Gamma(\rho_1)\right]. \end{aligned} \tag{17}$$

where $\Gamma(\rho_1)$ is defined by

$$\Gamma(\rho_1) = \frac{\rho_1^2 - \varepsilon}{\rho_1(\rho_1^2 + \varepsilon)^2} \tag{18}$$

Thirdly, after utilizing Equations (14) and (17), and letting the mass parameter μ tend to zero, the existing of such limit leads to Hill's equations. Thereby, Equation (14) can be rewritten as

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} &= L_{x1} + L_{x2} \\ \ddot{y} + 2\omega\dot{x} &= L_{y1} \\ \ddot{z} &= L_{z1}, \end{aligned} \tag{19}$$

where the limits L_{x1}, L_{x2}, L_{y1}, and L_{z1} are identified by

$$\begin{aligned} L_{x1} &= \lim_{\mu\to 0}(1-\mu)(1+\mu^\beta x)\frac{1}{\mu^\beta}\left[\omega^2 - \Gamma(\rho_1)\right], \\ L_{x2} &= \lim_{\mu\to 0}\left[\mu^{2\beta}\omega^2 - \frac{\mu^{1-3\beta}}{\rho^3}\right]x, \\ L_{y1} &= \lim_{\mu\to 0}\left[\omega^2 - \frac{\mu^{1-3\beta}}{\rho^3} - (1-\mu)\Gamma(\rho_1)\right]y, \\ L_{z1} &= -\lim_{\mu\to 0}\left[\frac{\mu^{1-3\beta}}{\rho^3} + (1-\mu)\Gamma(\rho_1)\right]z. \end{aligned} \tag{20}$$

From Equations (16) and (18), $\rho_1 \to 0$ and $\Gamma(\rho_1) \to \omega^2$ when $\mu \to 0$ if $\beta > 0$. Hence, it is clear that from Equation (20), the last three limits can be calculated by directly substituting $\mu = 0$, while the first limit represents an indeterminate quantity. Furthermore, β represents an arbitrary positive value but the proper selection is $1 - 3\beta = 0$ ($\beta = 1/3$), because this choice provides the associated terms to both of centrifugal and mutual gravitational forces for second and third equations of motion (19). In this context, ρ_1, ρ_2, and the values of limits L_{x1}, L_{x2}, L_y, and L_z are given by

$$\begin{aligned} \rho_1^2 &= 1 + 2\mu^{1/3}x + \mu^{2/3}\rho^2, \\ \rho_2^2 &= \mu^{2/3}\rho^2 \end{aligned} \tag{21}$$

and

$$\begin{aligned} L_{x1} &= \frac{0}{0}, \\ L_{x2} &= -\frac{x}{\rho^3}, \\ L_{y1} &= -\frac{y}{\rho^3}, \\ L_{z1} &= -\left(\omega^2 + \frac{1}{\rho^3}\right)z. \end{aligned} \tag{22}$$

The indeterminate quantity of L_{x1} means that we have to use some analysis methods and limits rules alongside proper approximation to estimate its value instead of using direct substituting. Thus, we will expand the values of $\omega(\varepsilon)$ and $\Gamma(\rho_1)$ to first order term of ε; thus, we obtain

$$\omega(\varepsilon) = 1 - \frac{3}{2}\varepsilon + \mathcal{O}(\varepsilon^2), \tag{23}$$

and

$$\Gamma(\rho_1) = \frac{1}{\rho_1^3} - \frac{3\varepsilon}{\rho_1^5} + \mathcal{O}(\varepsilon^2), \tag{24}$$

Utilizing Equations (21) and (24) with expanding in terms of μ, we obtain

$$\Gamma(\rho_1) = 1 - 3\,\varepsilon - 3\mu^{1/3}(1 - 5\,\varepsilon)x + \mathcal{O}(\varepsilon^2, \mu^{2/3}), \tag{25}$$

Now utilizing Equations (22)–(25), we obtain

$$\begin{aligned} L_{x1} &= 3(1 - 5\varepsilon)x, \\ L_{x2} &= -\frac{x}{\rho^3}, \\ L_{y1} &= -\frac{y}{\rho^3}, \\ L_{z1} &= -\left(1 - 3\varepsilon + \frac{1}{\rho^3}\right)z. \end{aligned} \tag{26}$$

Substituting Equations (23) and (26) into Equation (19), we obtain

$$\begin{aligned} \ddot{x} - 2(1 - \frac{3}{2}\varepsilon)\dot{y} &= 3(1 - 5\varepsilon)x - \frac{x}{\rho^3}, \\ \ddot{y} + 2(1 - \frac{3}{2}\varepsilon)\dot{x} &= -\frac{y}{\rho^3}, \\ \ddot{z} &= -(1 - 3\varepsilon)z - \frac{z}{\rho^3}. \end{aligned} \tag{27}$$

For simple form, Equation (27) can be rewritten as

$$\begin{aligned} \ddot{x} - 2\,\omega\,\dot{y} &= V_x, \\ \ddot{y} + 2\,\omega\,\dot{x} &= V_y, \\ \ddot{z} &= V_z. \end{aligned} \tag{28}$$

where

$$V = \frac{1}{2}\Big[(5\,\omega^2 - 2)\,x^2 - \omega^2 z^2\Big] + \frac{1}{\rho}, \tag{29}$$

We would like to advise that from now on, the values of ω and ω^2 are approximated by $(1 - 3\,\varepsilon/2)$ and $(1 - 3\,\varepsilon)$. Furthermore, we remark that Equations (28) and (29) represent a new perturbed dynamical system for the spacial Hill's problem in the framework of the CFP effect. We emphasize that this system is generalized for the classical Hill's system and it can be reduced to it when the perturbation parameter equals zero ($\varepsilon = 0$) [17,19].

4. Lagrangian and Hamiltonian Approaches

The Lagrangian and Hamiltonian techniques give the basics for more profound results in classical mechanics, and propose equivalent expressions in quantum mechanics such as the path integral formula and the Schrödinger equation. The Lagrangian mechanics is preferable to the Hamiltonian mechanics when the numerical solutions are needed for typical undergraduate problems in classical mechanics, whilst Hamiltonian mechanics has an obvious feature for addressing more deeper and philosophical questions in physical sciences.

In classical mechanics, the Hamiltonian approach is a reformulation of the Lagrangian approach, where in the first, the generalized velocities are replaced by generalized momentum. However, both of them give an explanation for classical mechanics and each of them can be used to describe the same physical system. But one of the most important properties of Hamiltonian mechanics is that it has a close relationship with geometry (notably, symplectic geometry and Poisson structures), and this serves as a connection between classical and quantum mechanics.

4.1. *Lagrangian Equations for Perturbed Hill's Problem*

Lagrangian mechanics is based on the principle of least action; it can be used to describe a dynamical system in both physical sciences and classical mechanics. Since the differentiable function is stationary at its local extreme, the Lagrange equations have some advantageous-to-solve optimization systems that provide us with some functions that minimize or maximize some required functionals. Lagrange equations will provide the same equations as Newton's laws by using the kinetic and potential energy instead of forces, where Lagrangian functions will characterize the dynamics of the entire systems. In general, the Lagrangian has energy unit but there is no one expression for all dynamical systems. The function that can be used to find the proper equations of motion in the conformity of mechanics laws can be considered as a Lagrangian function.

To find Lagrangian equations, we impose that $(\mathcal{Q}, \mathcal{L})$ defines a real dynamical system of N degree of freedom, where $\mathcal{Q}$ is the configuration space and $\mathcal{L} \equiv \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ is a Lagrangian function, $\mathbf{q} = (q_1, q_2, \ldots, q_N) \in \mathcal{Q}$, and $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_N)$ is the corresponding vector velocity to the position vector $\mathbf{q}$. Thus, Lagrange's equations can be written as [39]

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \tag{30}$$

where $i = 1, 2, \ldots, N$, $\mathcal{L} = \mathcal{T} - \mathcal{U}$, $\mathcal{T}, \mathcal{U}$ are the kinetic and potential energy of the dynamical system, respectively.

Equations (28) and (29) represent Hill's system in a rotating reference frame. To find the Lagrangian function for such system, we have to evaluate the kinetic and potential energy in this frame. For convenient of applying Lagrangian equations, we impose that $(x, y, z) = (q_1, q_2, q_3) = (q_x, q_y, q_z) = \mathbf{q}$; since the angular velocity of reference frame about the Z-axis is ω, then the velocity vector in rotating reference frame $\dot{\mathbf{q}} = (\dot{q}_x - \omega\, q_y, \dot{q}_y + \omega\, q_x, \dot{q}_z)$. Thereby, the kinetic and potential energy of the dynamical system are identified by

$$\begin{aligned} \mathcal{T} =& \frac{1}{2}\Big[\left(\dot{q}_x - \omega\, q_y\right)^2 + \left(\dot{q}_y + \omega\, q_x\right)^2 + \dot{q}_z^2\Big], \\ \mathcal{U} =& \frac{1}{2}\,\omega^2\left[q_x^2 + q_y^2 + q_z^2\right] - \frac{1}{2}\,(5\,\omega^2 - 2)\,q_x^2 - \frac{1}{q}, \end{aligned} \tag{31}$$

where q is a function in the generalized variable q_x, q_y, q_z, and is evaluated by $q\,(q_x, q_y, q_z) = \sqrt{q_x^2 + q_y^2 + q_z^2}$.

Since the Lagrangian function is defined by $\mathcal{L} = \mathcal{T} - \mathcal{U}$, then by using Equation (31), we obtain

$$\begin{aligned} \mathcal{L} = \frac{1}{2}&\Big[\left(\dot{q}_x - \omega\, q_y\right)^2 + \left(\dot{q}_y + \omega\, q_x\right)^2 + \dot{q}_z^2\Big] + \frac{1}{q} \\ &- \frac{1}{2}\,\omega^2\left[q_x^2 + q_y^2 + q_z^2\right] + \frac{1}{2}\,(5\,\omega^2 - 2)\,q_x^2. \end{aligned} \tag{32}$$

For finding the equations of motion, Equation (30) can be separated to

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_x}\right) - \frac{\partial \mathcal{L}}{\partial q_x} = 0, \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_y}\right) - \frac{\partial \mathcal{L}}{\partial q_y} = 0, \\ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_z}\right) - \frac{\partial \mathcal{L}}{\partial q_z} = 0. \end{aligned} \tag{33}$$

We can obtain the perturbed dynamical System (28) by substituting Equation (32) into Equation (33), which can be rewritten in the following form:

$$\begin{aligned} \ddot{q}_x - 2\,\omega \dot{q}_y =& (5\,\omega^2 - 2)q_x - \frac{q_x}{q^3}, \\ \ddot{q}_y + 2\,\omega \dot{q}_x =& -\frac{q_y}{q^3}, \\ \ddot{q}_z =& -\omega^2\, z - \frac{q_z}{q^3}. \end{aligned} \tag{34}$$

One of the most paramount features of the Lagrangian function is that we can read off from it easily conserved physical quantities. For example, the generalized momentum "canonically conjugate" to the coordinate q_i is identified by $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. By applying this property to Lagrangian relation, we obtain $\dot{q}_x = p_x + q_y$, $\dot{q}_y = p_y - q_x$, and $\dot{q}_z = p_z$.

4.2. Hamiltonian Equations for Perturbed Hill's Problem

Hamiltonian mechanics can be employed to characterize some simple dynamical systems such as a bouncing ball or a pendulum problem, in which the total energy is constant, because a change in increase or decrease in kinetic energy leads to a decrease or increase of the same amount in the potential energy, and vice versa. But the power of Hamiltonian mechanics is that it can be used to describe more complex dynamical systems, which express the motions of natural bodies such as planetary and stellar motion in celestial mechanics [40,41], where an increase in the degrees of freedom of the system leads to an increase in the complexity of the system's dynamics and its time evolution.

In classical mechanics, the state of a dynamical system can be described by identifying its Lagrangian as a function in both the generalized coordinates and their associated velocities, but it is proper, in some cases, to describe the state of system by defining its Hamiltonian. In this context, the Hamiltonian satisfies the following relation [42,43]:

$$\mathcal{H}(\mathbf{p},\mathbf{q},t) = \mathbf{p}(t).\mathbf{q}(t) - \mathcal{L}(\mathbf{q},\dot{\mathbf{q}},t), \tag{35}$$

where the position vector $\mathbf{q}$ and its conjugate momenta $\mathbf{p}$ are

$$\begin{aligned} \mathbf{q} &= (q_x, q_y, q_z), \\ \dot{\mathbf{p}} &= (p_x, p_y, p_z). \end{aligned} \tag{36}$$

Utilizing Equations (32), (35), and (36), the Hamiltonian of the perturbed Hill's system by the continued fractional potential is

$$\mathcal{H} = \frac{1}{2}\Big[\left(p_x + \omega\, q_y\right)^2 + \left(p_y - \omega\, q_x\right)^2 + p_z^2\Big] - \frac{1}{2}\Big[\Big(5\,\omega^2 - 2\Big)\, q_x^2 - \omega^2 q_z^2\Big] - \frac{1}{q}. \tag{37}$$

where the time evolution of Hamilton's equations is given by

$$\begin{aligned} \dot{q}_x &= \frac{\partial\mathcal{H}}{\partial p_x}, \quad \dot{p}_x = -\frac{\partial\mathcal{H}}{\partial q_x}, \\ \dot{q}_y &= \frac{\partial\mathcal{H}}{\partial p_y}, \quad \dot{p}_y = -\frac{\partial\mathcal{H}}{\partial q_y}, \\ \dot{q}_z &= \frac{\partial\mathcal{H}}{\partial p_z}, \quad \dot{p}_z = -\frac{\partial\mathcal{H}}{\partial q_z}, \end{aligned} \tag{38}$$

Substituting Equation (37) into (38), the spatial perturbed Hill's equations are given by

$$\begin{aligned} \dot{q}_x &= p_x + \omega\, q_y, \\ \dot{q}_y &= p_y - \omega\, q_x, \\ \dot{q}_z &= p_z, \\ \dot{p}_x &= \omega\, \dot{q}_y + \Big(5\,\omega^2 - 2\Big)\, q_x - \frac{q_x}{q^3}, \\ \dot{p}_y &= -\,\omega\, \dot{q}_x - \frac{q_y}{q^3}, \\ \dot{p}_z &= -\,\omega^2\, \dot{q}_z - \frac{q_z}{q^3}, \end{aligned} \tag{39}$$

For an effective formula from practical point of view, we will write the Hamiltonian in the following form:

$$\mathcal{H} = \mathcal{H}_0 + \varepsilon\mathcal{H}_1, \tag{40}$$

where

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2}\Big[\left(p_x + q_y\right)^2 + \left(p_y - q_x\right)^2 + p_z^2\Big] - \frac{1}{2}\Big[3q_x^2 - q_z^2\Big] - \frac{1}{q}, \\ \mathcal{H}_1 &= \frac{3}{2}\Big[4\, q_x^2 - q_y^2 - q_z^2 - p_x q_y + p_y q_x\Big]. \end{aligned} \tag{41}$$

The first expression $\mathcal{H}_0$ in Equation (41) represents the classical motion of Hill's system, while the second $\mathcal{H}_1$ gives the perturbation in such motion.

Thus, the perturbed Hamiltonian (40) can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{O}(\varepsilon). \tag{42}$$

The formula (40) for Hamiltonian can be used to develop the dynamical properties of the classical Hill's system to prove that such properties can be extended to the perturbed problem under the effect of the continuation fraction potential parameter.

5. Symmetries

Symmetry is a substantial subject in different areas in both physical and mathematical sciences [44–46]. There are many systems that describe different physical phenomena in nature that possess some symmetry features that somehow affect their functionality. Considering such properties may significantly facilitate the analysis of the proposed systems. The symmetry of a dynamical system is preserved in mathematical approaches used to formulate it. We can explore this concept in Lagrangian and Hamiltonian systems, where the symmetry of Lagrangian and Hamiltonian functions can be shown in their own equations of motion, which are invariant under the same transfer symmetry [47–49]. Briefly, if the transformations take systems solutions to other solutions where all solutions satisfy equations that describe the dynamical system, then this system is called an invariant under these transformations.

Let $(x(t), y(t), z(t))$ be solutions of System (28); then, this system is invariant and the obtained trajectories under the following transformations are solutions too.

1. S_1: $(x(t), y(t), z(t)) \rightarrow (-x(-t), y(-t), z(-t))$,
2. S_2: $(x(t), y(t), z(t)) \rightarrow (x(-t), -y(-t), z(-t))$,
3. S_3: $(x(t), y(t), z(t)) \rightarrow (x(t), y(t), -z(t))$,
4. S_4: $(x(t), y(t), z(t)) \rightarrow (-x(t), -y(t), z(t))$,
5. S_5: $(x(t), y(t), z(t)) \rightarrow (-x(t), -y(t), -z(t))$,
6. S_6: $(x(t), y(t), z(t)) \rightarrow (x(-t), -y(-t), -z(-t))$.

Since the Lagrangian function $\mathcal{L} \equiv \mathcal{L}(q_x, q_y, q_z, \dot{q}_x, \dot{q}_y, \dot{q}_z)$ and the Hamiltonian $\mathcal{H} \equiv \mathcal{H}(q_x, q_y, q_z, p_x, p_y, p_z)$, it is clear that both functions are invariant over the previous transformation S_1, S_2, S_3, S_4, S_5, and S_6. Thus the perturbed Hill's system (34) or (39) is an invariant under the same transformations and the obtained trajectories over these symmetries are also valid solutions. Furthermore, this property can be verified directly by substituting components of the velocity and acceleration which are associated with coordinates in each transformation into Hill's system (34) or (39), where these components are

1. $S_1: \rightarrow \begin{cases} \text{Velocity: } (\dot{x}(-t), -\dot{y}(-t), -\dot{z}(-t)), \\ \text{Acceleration: } (-\ddot{x}(-t), \ddot{y}(-t), \ddot{z}(-t)), \end{cases}$
2. $S_2: \rightarrow \begin{cases} \text{Velocity: } (-\dot{x}(-t), \dot{y}(-t), -\dot{z}(-t)), \\ \text{Acceleration: } (\ddot{x}(-t), -\ddot{y}(-t), \ddot{z}(-t)), \end{cases}$
3. $S_3: \rightarrow \begin{cases} \text{Velocity: } (\dot{x}(t), \dot{y}(t), -\dot{z}(t)), \\ \text{Acceleration: } (\ddot{x}(t), \ddot{y}(t), -\ddot{z}(t)), \end{cases}$
4. $S_4: \rightarrow \begin{cases} \text{Velocity: } (-\dot{x}(t), -\dot{y}(t), \dot{z}(t)), \\ \text{Acceleration: } (-\ddot{x}(t), -\ddot{y}(t), \ddot{z}(t)), \end{cases}$
5. $S_5: \rightarrow \begin{cases} \text{Velocity: } (-\dot{x}(t), -\dot{y}(t), -\dot{z}(t)), \\ \text{Acceleration: } (-\ddot{x}(t), -\ddot{y}(t), -\ddot{z}(t)), \end{cases}$
6. $S_6: \rightarrow \begin{cases} \text{Velocity: } (-\dot{x}(-t), \dot{y}(-t), \dot{z}(-t)), \\ \text{Acceleration: } (\ddot{x}(-t), -\ddot{y}(-t), -\ddot{z}(-t)). \end{cases}$

The transformations S_1, S_2, S_3, S_4, S_5, and S_6 show that the perturbed Hill's system (27) or (28) has different valid six solutions as well as the original one under the effect of continuation fraction potential parameter, where the system is an invariant under these transformations.

The symmetry S_1 comprises a reflection about the YZ-plane and S_2 comprises a reflection about the XZ-plane, and both of them implicate a reflection in time. The symmetry S_3 includes reflection about the XY-plane, which admits us to perform the trajectory calculations with respect to positive Z-coordinate only. This will be applied to simplify the evaluate of the Poincaré map. The symmetries S_4, S_5, and S_6 directly result from compositions of S_1, S_2, and S_3, where the compositions of S_1 = and S_2 give the symmetry S_4, and S_1, S_2, and S_3 yield S_5, while the compositions of S_2 and S_3 provide the symmetry S_6. The symmetry S_4 is a result for two successive reflections about the YZ and XZ planes. But the symmetry S_5 is a result for three successive reflections about the YZ, XZ, and XY planes, which are equivalently purely symmetry about the origin point. The last transformation S_6 includes the reflection in time as well as two successive reflections about XZ and XY.

We note that the several symmetries found in the Hill's system can greatly simplify the calculation and analysis of primary studies, and authorize us to explore similarities and differences among many classes of trajectories that otherwise differ in the framework of the RTBP. Several extended studies on the periodic orbits and symmetries of the classical Hill's lunar problem are developed with emphasis on the stability evolution of these orbits [50,51]. Furthermore, we demonstrate that if the trajectory of a dynamical system has a property $p(\phi)$, where this system is an invariant under the symmetry S, then $S(p(\phi))$ will be transformed to the symmetric trajectories.

6. Conclusions

Although Hill's problem was defined as a limiting case for a restricted three-body problem, it can be obtained from the general three-body problem as in the motions of Earth and the Moon around the Sun, where the two masses are very small with respect to the Sun's mass. In a general case, there are three zero assumptions: the solar parallax, the eccentricity, and the lunar inclinations all of them are assumed to equal zero value. The Hill's problem construction can be established from a general three-body problem by expanding its Hamiltonian using the assumptions of smallness parameters which are related to the smallness of two masses and the separation distance between them, where we should keep only the first-order terms in small parameters.

Hill's approximation can be applied when two masses are nearby and small compared to a third mass. Thus, it is a complementary approach to the restricted case, while the Hill's approximation is a reduction to the restricted problem. In this case, the mass of the third body is considered to be very small in comparison to the other two bodies, which is considered the main reason for assuming that the two bodies are moving in Kepler circular orbits. Thus, we used the modified CRTBP under the effect of continuation fraction potential to construct a novel type model for Hill's problem.

In this paper, an analytical development for deriving a new model type was introduced to describe Hill's problem. First, an extensive review of the problem was stated. Hence, the model description of the RTBP was proclaimed, in which case the bigger primary creates a potential that is similar to the continuation fraction potential, while the smaller primary creates a potential as a spherical body. After that, the obtained perturbed system was subjected to the transformation along X-axes to allow the origin of coordinates to move to the center of the smaller primary. Furthermore, the variables were scaled by using the parameter of mass ratio, which yields Hill's problem if limiting process is considered when the mass ratio tends to zero. The associated Lagrangian and Hamiltonian functions of the new system were also calculated and used to verify their validation for obtaining the new Hill's system of motion.

We studied the symmetry analysis of the new system and proved that such a system has different six valid solutions under different six symmetries' transformations, as well as the original solution, where the system is an invariant under these transformations. We remark that the several symmetries of the Hill's system can greatly simplify the calculation and analysis of preliminary investigations for the dynamical behavior of the system, and this authorizes us to explore the similarities and differences among many classes of trajectories that otherwise differ in the framework of the RTBP. Finally, we emphasize that Hill's system has many applications in celestial mechanics, particularly in the dynamics of exploring planetary rings and binary encounters.

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References

1. Hahn, S.G.; Lindquist, R.W. The two-body problem in geometrodynamics. *Ann. Phys.* **1964**, *29*, 304–331. [CrossRef]
2. Diacu, F.N.; Mingarelli, A.; Mioc, V.; Stoica, C. The Manev two-body problem: Quantitative and qualitative theory. In *Dynamical Systems and Applications*; World Scientific: Singapore, 1995; pp. 213–227.
3. Abouelmagd, E.I.; Mostafa, A.; Guirao, J.L. A first order automated Lie transform. *Int. J. Bifurc. Chaos* **2015**, *25*, 1540026. [CrossRef]
4. Ershkov, S.; Abouelmagd, E.I.; Rachinskaya, A. A novel type of ER3BP introduced for hierarchical configuration with variable angular momentum of secondary planet. *Arch. Appl. Mech.* **2021**, *91*, 4599–4607. [CrossRef]
5. Ershkov, S.; Leshchenko, D.; Abouelmagd, E.I. About influence of differential rotation in convection zone of gaseous or fluid giant planet (Uranus) onto the parameters of orbits of satellites. *Eur. Phys. J. Plus* **2021**, *136*, 387. [CrossRef]
6. Alrebdi, H.; Alsaif, N.A.; Suraj, M.S.; Zotos, E.E. Investigating the properties of equilibrium points of the collinear restricted 4-body problem. *Planet. Space Sci.* **2023**, *237*, 105767. [CrossRef]
7. Ershkov, S.; Mohamdien, G.F.; Idrisi, M.J.; Abouelmagd, E.I. Revisiting the Dynamics of Two-Body Problem in the Framework of the Continued Fraction Potential. *Mathematics* **2024**, *12*, 590. [CrossRef]
8. Abouelmagd, E.I. New dynamical system for circular satellites relative motion. *Chaos Solitons Fractals* **2024**, *182*, 114879. [CrossRef]
9. Kalantonis, V.S.; Perdiou, A.E.; Perdios, E.A. On the stability of the triangular equilibrium points in the elliptic restricted three-body problem with radiation and oblateness. In *Mathematical Analysis and Applications*; Springer: Cham, Switzerland, 2019; pp. 273–286.
10. Kalantonis, V.S.; Vincent, A.E.; Gyegwe, J.M.; Perdios, E.A. Periodic solutions around the out-of-plane equilibrium points in the restricted three-body problem with radiation and angular velocity variation. In *Nonlinear Analysis and Global Optimization*; Springer: Cham, Switzerland, 2021; pp. 251–275.
11. Alrebdi, H.; Smii, B.; Zotos, E.E. Equilibrium dynamics of the restricted three-body problem with radiating prolate bodies. *Results Phys.* **2022**, *34*, 105240. [CrossRef]
12. Alrebdi, H.I.; Dubeibe, F.L.; Zotos, E.E. Equilibrium points and their stability in a new generalized planar version of the collinear restricted four-body problem. *Commun. Nonlinear Sci. Numer. Simul.* **2023**, *120*, 107196. [CrossRef]
13. Sachan, P.; Suraj, M.S.; Aggarwal, R.; Mittal, A.; Asique, M.C. A Study of the Axisymmetric Restricted Five-Body Problem within the Frame of Variable Mass: The Concave Case. *Astron. Rep.* **2023**, *67*, 404–423. [CrossRef]
14. Sachan, P.; Suraj, M.S.; Aggarwal, R.; Asique, M.C.; Mittal, A. On the axisymmetric restricted five-body problem within the frame of variable mass: The convex case. *New Astron.* **2022**, *92*, 101697. [CrossRef]
15. Verma, R.K.; Kushvah, B.S.; Pal, A.K. Dynamics of the perturbed restricted three-body problem with quantum correction and modified gravitational potential. *Arch. Appl. Mech.* **2024**, *94*, 651–665. [CrossRef]
16. Kumar, M.; Pal, A.K.; Verma, R.K.; Kushvah, B.S. Analysis of albedo and disc effects in the generalized restricted four-body problem. *Adv. Space Res.* **2024**, *73*, 4284–4295. [CrossRef]
17. Szebehely, V. *Theory of Orbit: The Restricted Problem of Three Bodies*; Academic Press Inc.: New York, NY, USA; London, UK, 1967.
18. Hénon, M. Numerical exploration of the restricted problem. VI. Hill's case: Non-periodic orbits. *Astron. Astrophys.* **1970**, *9*, 24–36.
19. Gómez, G.; Marcote, M.; Mondelo, J. The invariant manifold structure of the spatial Hill's problem. *Dyn. Syst.* **2005**, *20*, 115–147. [CrossRef]
20. Burgos-García, J.; Gidea, M. Hill's approximation in a restricted four-body problem. *Celest. Mech. Dyn. Astron.* **2015**, *122*, 117–141. [CrossRef]
21. Burgos-García, J. Families of periodic orbits in the planar Hill's four-body problem. *Astrophys. Space Sci.* **2016**, *361*, 1–21. [CrossRef]
22. Bouaziz, F.; Ansari, A.A. Perturbed Hill's problem with variable mass. *Astron. Nachrichten* **2021**, *342*, 666–674. [CrossRef]
23. Moneer, E.M.; Alanazi, M.; Elaissi, S.; Allawi, Y.; Dubeibe, F.L.; Zotos, E.E. Orbital dynamics in the Hill problem with oblateness. *Results Phys.* **2023**, *53*, 106936. [CrossRef]
24. Michalodimitrakis, M. Hill's problem: Families of three-dimensional periodic orbits (part I). *Astrophys. Space Sci.* **1980**, *68*, 253–268. [CrossRef]
25. Henon, M. New families of periodic orbits in Hill's problem of three bodies. *Celest. Mech. Dyn. Astron.* **2003**, *85*, 223–246. [CrossRef]
26. Henon, M. Families of asymmetric periodic orbits in Hill's problem of three bodies. *Celest. Mech. Dyn. Astron.* **2005**, *93*, 87–100. [CrossRef]

27. Chenciner, A.; Llibre, J. A note on the existence of invariant punctured tori in the planar circular restricted three-body problem. *Ergod. Theory Dyn. Syst.* **1988**, *8*, 63–72.
28. Simó, C.; Stuchi, T.d.J. Central stable/unstable manifolds and the destruction of KAM tori in the planar Hill problem. *Phys. D Nonlinear Phenom.* **2000**, *140*, 1–32. [CrossRef]
29. Meyer, K.R.; Palacián, J.F.; Yanguas, P. Geometric averaging of Hamiltonian systems: Periodic solutions, stability, and KAM tori. *SIAM J. Appl. Dyn. Syst.* **2011**, *10*, 817–856. [CrossRef]
30. Henrard, J.; Navarro, J.F. Spiral structures and chaotic scattering of coorbital satellites. *Celest. Mech. Dyn. Astron.* **2001**, *79*, 297–314. [CrossRef]
31. Steves, B. *The Restless Universe Applications of Gravitational N-Body Dynamics to Planetary Stellar and Galactic Systems*; CRC Press: Boca Raton, FL, USA, 2019.
32. Scheeres, D.J. The restricted Hill four-body problem with applications to the Earth–Moon–Sun system. *Celest. Mech. Dyn. Astron.* **1998**, *70*, 75–98. [CrossRef]
33. Scheeres, D.; Marzari, F. Spacecraft Dynamics in the Vicinity of a Comet. *J. Astronaut. Sci.* **2002**, *50*, 35–52. [CrossRef]
34. Tsirogiannis, G.A.; Perdios, E.A.; Markellos, V.V. Improved grid search method: An efficient tool for global computation of periodic orbits: Application to Hill's problem. *Celest. Mech. Dyn. Astron.* **2009**, *103*, 49–78. [CrossRef]
35. Perdiou, A.E.; Perdios, E.A.; Kalantonis, V.S. Periodic orbits of the Hill problem with radiation and oblateness. *Astrophys. Space Sci.* **2012**, *342*, 19–30. [CrossRef]
36. Kalantonis, V.S.; Douskos, C.N.; Perdios, E.A. Numerical determination of homoclinic and heteroclinic orbits at collinear equilibria in the restricted three-body problem with oblateness. *Celest. Mech. Dyn. Astron.* **2006**, *94*, 135–153. [CrossRef]
37. Abouelmagd, E.I. Periodic Solution of the Two–Body Problem by KB Averaging Method Within Frame of the Modified Newtonian Potential. *J. Astronaut. Sci.* **2018**, *65*, 291–306. [CrossRef]
38. Bairwa, L.K.; Pal, A.K.; Kumari, R.; Alhowaity, S.; Abouelmagd, E.I. Study of Lagrange Points in the Earth–Moon System with Continuation Fractional Potential. *Fractal Fract.* **2022**, *6*, 321. [CrossRef]
39. Hand, L.N.; Finch, J.D. *Analytical Mechanics*; Cambridge University Press: Cambridge, UK, 1998.
40. Dantas, M.P.; Llibre, J. The global phase space for the 2-and 3-dimensional Kepler problems. *Qual. Theory Dyn. Syst.* **2009**, *8*, 45. [CrossRef]
41. Moeckel, R. *Central Configurations, Periodic Orbits, and Hamiltonian Systems*; Springer: Berlin/Heidelberg, Germany, 2015.
42. Calkin, M.G.; Weinstock, R. Lagrangian and Hamiltonian Mechanics. *Am. J. Phys.* **1998**, *66*, 261–262. [CrossRef]
43. Goldstein, H. *Classical Mechanics*; Pearson Education India: Bengaluru, India, 2011.
44. Oliver, P. *Applications of Lie Groups to Mechanics and Symmetry*; Springer: Berlin/Heidelberg, Germany, 1986.
45. Marsden, J.E.; Ratiu, T.S.; Hermann, R. Introduction to mechanics and symmetry. *SIAM Rev.* **1997**, *39*, 152.
46. Golubitsky, M.; Stewart, I. *The Symmetry Perspective: From Equilibrium to Chaos in Phase Space and Physical Space*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2003; Volume 200.
47. Cicogna, G.; Gaeta, G. *Symmetry and Perturbation Theory in Nonlinear Dynamics*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2003; Volume 57.
48. Spong, M.W.; Bullo, F. Controlled symmetries and passive walking. *IEEE Trans. Autom. Control.* **2005**, *50*, 1025–1031. [CrossRef]
49. Field, M. *Dynamics and Symmetry*; World Scientific: Singapore, 2007; Volume 3.
50. Kalantonis, V.S. Numerical investigation for periodic orbits in the Hill three-body problem. *Universe* **2020**, *6*, 72. [CrossRef]
51. Xu, X. Determination of the doubly symmetric periodic orbits in the restricted three-body problem and Hill's lunar problem. *Celest. Mech. Dyn. Astron.* **2023**, *135*, 8. [CrossRef]