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Analogies between Lattice QCD and the Truncated Nambu–Jona-Lasinio Model

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Abstract: A modified Nambu–Jona-Lasinio Model with lattice structure is very instructive. It shows several similar problems and their solutions as the Lattice QCD. We study the limits of the large box size, small cell size and realistic pion mass. In particular, we study the relation of the discrete (bound state) solutions to the physical scattering states, for example the pion–pion scattering.

Keywords: Nambu–Jona-Lasinio Model; pion-pion scattering; sigma meson

1. Introduction

Nambu and Jona-Lasinio formulated, in 1961, a conceptually very interesting model [1,2], designed to describe the spontaneous chiral symmetry breaking, the formation of massive constituent quarks and the behaviour of pion as a pseudo-Goldstone boson. At that time, the model was formulated in terms of nucleons interacting with a chirally invariant contact interaction. Later, the model was successfully reformulated in terms of interacting quarks. The Nambu–Jona-Lasinio Lagrangian is usually written in terms of relativistically covariant quark fields and it contains the quark kinetic energy, a "bare mass" term explicitly breaking the chiral symmetry, and a quadratic chirally invariant contact interaction:

$$\mathcal{L} = \bar{\psi}\,(\mathrm{i}\,\gamma^{\mu}\partial_{\mu} - m)\,\psi + G\,[(\bar{\psi}\psi)^2 + (\bar{\psi}\,\mathrm{i}\,\gamma_5\tau_i\,\psi)^2] \tag{1}$$

Here, τ is the isospin operator for the case of two quark flavours. For large enough coupling G, the lowest energy state, the vacuum, contains a nonzero vacuum condensate of quark–antiquark pairs $\langle\bar{\psi}\psi\rangle$, which acts as a constituent mass $M = m + \langle\bar{\psi}\psi\rangle$. Without the bare mass term, the first excited state would be a zero-frequency Goldstone mode. Therefore, the bare mass term is essential, since it changes the Goldstone mode into a nonzero pseudo-Goldstone mode and the pion acquires a small but nonzero mass. However, it does not noticeably influence the chiral condensate and the constituent mass.

For our purposes, it is more illustrative to use the first quantization and work with a finite number of quarks in the Fermi and Dirac levels, following the ideas of da Providencia (Equation (3.1), written for one flavour) [3]:

$$H = \sum_{i=1}^{N} \gamma_5(i)\vec{\sigma}(i)\cdot\nabla_i - G\sum_{i\neq j}\delta(\vec{x}_i - \vec{x}_j)\,[\beta(i)\beta(j) - \beta(i)\gamma_5(i)\,\beta(j)\gamma_5(j)] \tag{2}$$

More recently, electromagnetic and weak decays of scalar and vector mesons have been calculated in leading orders of Feynman graphs. The one loop order calculation provided a satisfactory agreement with the data for the mesonic spectrum and for radiative decays [4]. A renormalized version with the mean field expansion gave a direct link to the mesonic degrees of freedom [5]. Later, the model has been generalized to three flavours of quarks replacing the isospin operators with SU(3) operators and introducing a three-body interaction of the form of the t'Hooft determinant. This extension allowed

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the spectra and radiative decays of more numerous mesons, including strange ones [6]. The description of diquarks [7] and the competition between the chiral condensate and the diquark condensate [8] are also interesting.

At present, there is very lively activity using the Nambu–Jona-Lasinio model to study the relation of the Nambu–Jona-Lasinio model to QCD, to chiral models as well as to phenomenology, for example [9] and many more. The large number of different applications is a good sign regarding how useful the model is in achieving a qualitative understanding where the fundamental theory is too difficult.

The purpose of the present study is twofold.

(i) A solvable model is formulated which can support the validity of the popular Hartree–Fock approximation for massive constituent quarks together with the random phase approximation (RPA) for pions in the full Nambu–Jona-Lasinio model. It also supports the meaningfulness of the limit of large numbers of colours.
(ii) Lattice models as well as few-body models with a finite Hilbert space do not provide a continuum description of the two-body decay channel. Instead, the diagonalization of the Hamiltonian yields a discrete spectrum which hides a lot of information about the relevant continuum which one is trying to extract. As an example, approximate methods for $\pi - \pi$ scattering at low energy, as well as via the σ meson resonance, are studied. It is shown how the discrete eigenvalue spectrum can provide some information on scattering using the first order Born approximation, in analogy to the Luscher formula used in Lattice QCD for the same dilemma of how to extract scattering from a discrete spectrum.

A good lesson can be learned from a truncated Nambu–Jona-Lasinio model (NJL), "the quasispin model", in which quarks are enclosed in a periodic box $\mathcal{V}$ and have a momentum cut-off Λ, where $b = \sqrt[3]{6\pi^2}/\Lambda$ is a parameter analogous to the size of the lattice cell in Lattice QCD.

In Section 2, the formulation and some salient features of the quasispin model are recapitulated [10–14]. In Section 3, the lessons offered by the model are discussed. Finally (Section 4), the width of the sigma meson is estimated using an analytic extrapolation.

2. The Two-Level Quasispin Model

The model is characterized by a finite number N of quarks occupying a finite number $\mathcal{N}$ of states in the Dirac sea and the same number of states in the valence space. This allows us to use the first quantization and an explicit wavefunction.

The following simplifications are made:

(i) Periodic box of volume $\mathcal{V}$;
(ii) A sharp three-momentum cut-off Λ;
(iii) An average kinetic energy for all momentum states $|\vec{p}_i| \to P = \frac{3}{4}\Lambda$;
(iv) Restriction to one flavour of quarks $n_f = 1$;
(v) Truncation of interaction.

(while, in the NJL model, the interaction conserves the sum of the momenta of both quarks, it is assumed that each quark conserves its momentum and only switches between the Dirac level and Fermi level).

The finite number of discrete momentum states is then $\mathcal{N} = n_h n_c n_f n_p$, where n_h, n_c, n_f and $n_p = \mathcal{V}\Lambda^3/6\pi^2$ are the number of quark helicities, colours, flavours and momentum states.

The model Hamiltonian can then be written as

$$H = \sum_{k=1}^{N}\Big(\gamma_5(k)h(k)P + m_0\beta(k)\Big) + \\ - \frac{g}{2}\Big(\sum_{k=1}^{N}\beta(k)\sum_{l=1}^{N}\beta(l) + \sum_{k=1}^{N} \mathrm{i}\beta(k)\gamma_5(k)\sum_{l=1}^{N}\mathrm{i}\beta(l)\gamma_5(l)\Big). \tag{3}$$

where γ_5 and β are Dirac matrices, and $h = \vec{\sigma} \cdot \vec{p}/p$ is helicity.

There are three model parameters: $m_0 = 4.58\,\text{MeV}$ is the bare quark mass, $P = \frac{3}{4}\Lambda$ with $\Lambda = 648\,\text{MeV}$ is the average momentum and $g = 4G/\mathcal{V}$, where $G = 40.6\,\text{MeV}$ is the interaction strength in the original (continuum) NJL. These parameters have been fitted to the experimental or phenomenological values of the pion mass $m_\pi = 136\,\text{MeV}$, constituent quark mass $M = 335\,\text{MeV}$ and quark condensate $Q = 250^3\,\text{MeV}^3$ [13]. The values of the model parameters turn out to be very close to the popular values of full NJL [15,16].

It is usually overlooked that the following operators obey (quasi)spin commutation relations $j_x = \frac{1}{2}\beta, j_y = \frac{1}{2}\,\mathrm{i}\beta\gamma_5, j_z = \frac{1}{2}\gamma_5$. The (quasi)spin commutation relations are also obeyed by separate sums over quarks with right and left helicity, as well as by the total sum over all quarks ($\alpha = x,\, y,\, z$).

$$R_\alpha = \sum_{k=1}^{N} \frac{1+h(k)}{2} j_\alpha(k)\ , \quad L_\alpha = \sum_{k=1}^{N} \frac{1-h(k)}{2} j_\alpha(k)\ , \quad J_\alpha = R_\alpha + L_\alpha = \sum_{k=1}^{N} j_\alpha(k)\,.$$

The model Hamiltonian can then be rewritten as

$$H = 2P(R_z - L_z) + 2m_0 J_x - 2g(J_x^2 + J_y^2). \tag{4}$$

The Hamiltonian commutes with R^2 and L^2 but not with R_z and L_z. Nevertheless, it is convenient to work on the basis of $|\, R, L, R_z, L_z\,\rangle$. The Hamiltonian matrix elements can be easily calculated using the angular momentum algebra. For example,

$$\begin{aligned} \hat{R}_z|\, R, L, R_z, L_z\,\rangle &= R|\, R, L, R_z, L_z\,\rangle\,, \\ \hat{R}_x|\, R, L, R_z, L_z\,\rangle &= \tfrac{1}{2}\sqrt{(R-R_z)(R+R_z+1)}|\, R, L, R_z+1, L_z\,\rangle \\ &+ \tfrac{1}{2}\sqrt{(R+R_z)(R-R_z+1)}|\, R, L, R_z-1, L_z\,\rangle, \text{etc.} \end{aligned}$$

By diagonalization of the Hamiltonian matrix, one obtains the energy spectrum of the system (Table 1).

Table 1. The spectrum of the quasispin model with $N = 144$ and $N = 192$, and the ground state quantum numbers $R = L = N/4$. All energies are in MeV.

n	Parity	$E-E_0$ N = 144	ΔE N = 144	$\bar{V}$ N = 144	$E-E_0$ N = 192	ΔE N = 192	$\bar{V}$ N = 192
8	+	771	4	−11.3	861	59	−8.3
7	−	767	121	−8.8	802	93	−7.3
6	+	646	66	−11.4	709	98	−7.3
6	+	**634**		(−12.2)	**655**		(−10.9)
5	−	580	98	−10.0	611	108	−7.2
4	+	482	114	−10.5	503	115	−7.1
3	−	378	117	−10.1	388	122	−7.1
2	+	261	125	−10.3	266	129	−7.1
1	−	136	136		137	137	
0	+	0			0		

The salient features are as follows:

1. In the large N limit, the exact results of the quasispin model tend, in fact, to the Hartree–Fock and RPA values, which is a popular approximation for full NJL.
2. The spectrum of the "ground state band" (Table 1) is almost equidistant and can be interpreted as multipion states. The energy deficit can be assumed to be due to an attractive average pion–pion interaction: $E - E_0 = nm_\pi + \frac{1}{2}n(n-1)\bar{V}$.

3. This average potential is, in fact, proportional to the density of each pion, $\bar{V} \propto 1/\mathcal{V} \propto 1/N$, which supports such an interpretation.
4. The idea of an average pion–pion potential allows us to calculate the pion–pion scattering length a in the first order Born approximation (equivalent to the so-called Lüscher formula which is frequently used in the literature [17–19]) $am_\pi = \frac{m_\pi^2}{4\pi}\int V(\vec{r})\,\mathrm{d}^3r = \frac{m_\pi}{4\pi}\bar{V}\mathcal{V} = -0.077$, which is qualitatively consistent with the two-flavour experimental analysis of Lesniak, $am_\pi = -0.034$ or -0.044 [20].
5. The parity of multipion states alternates. There are, however, intruders which do not follow the alternation. In Table 1, they are written in boldface and the lowest can be interpreted as the σ meson (now called a(500)). The sigma meson is not a six-pion state but an intruder at the position around six pions; it has an overlap with a decaying two-pion state. Also, the states around $n = 7$ may be perturbed by admixtures of $\sigma + \pi$.

3. Some Lessons for Lattice-like Models

Studying the salient features of the two-level quasispin model, one can learn a few lessons.

One obtains a ground-state band of almost equidistant discrete states with alternating 0^+ and 0^-, which suggest multipion states. It is expected that, in other lattice-like models, such as the Lattice QCD, the energy deficit with respect to equidistant values is also due to an attractive average pion–pion interaction.

Even if we have only discrete states, we can mimic scattering states by using the effective pion–pion interaction to calculate scattering amplitudes. Resonances such as σ meson can be recognized from irregularities of the discrete spectrum (the "intruders").

In the simple model, the results depend on the product $\mathcal{N} = n_h n_c n_f n_p$ and not on individual factors (the number of helicities, colours, flavours and momentum states). It is equivalent to have a large number of colours and poor resolution (small n_p) or vice versa. Alternatively, one obtains the same limit $N \to \infty$ whether one takes the large N_c limit or a large box $\mathcal{V}$. This fact helps us to appreciate the meaning of the large N_c limit, which suggests a good Hartree–Fock approximation and suppression of off-diagonal terms of full NJL Hamiltonian.

For the chosen model parameters and for $N = \mathcal{N} = 192$, the "size of the box" is $B = \sqrt[3]{\mathcal{V}} = \sqrt[3]{\pi^2\mathcal{N}}/\Lambda = 3.7$ fm. The size of the "lattice cell" is $b = B/\sqrt[3]{\mathcal{N}/n_h n_c n_f} = B/32^{1/3} = 1.2$ fm. Then, B is only about three times larger than b; nevertheless, the model works well. The explanation is that, in one dimension with the same n_p, B would be 32 times larger than b, which is nice. Since the Hamiltonian is not very sensitive to the number of dimensions, the momenta $p(i)$ act only as "house numbers" and there are no spacial correlations; the quality of the three-dimensional solution is equally good. This is a general feature of Nambu–Jona-Lasinio models.

The convergence and the quality of the results in the quasispin model seem very good for $N = 194$ but not so good for $N = 144$, indicating the critical number of particles and the corresponding B/b ratio.

4. The Width of the Sigma Meson

In the spectrum in Table 1, one can clearly distinguish the presence of the sigma meson by noticing the doubling of the positive parity states at 634 and 646 MeV for $N = 144$ (655 and 709 MeV for $N = 192$). Moreover, the states at 646 MeV (655 MeV), indicated in boldface, have strong one-body transition matrix elements from the ground state.

For its width, we are trying to obtain the complex pole. For that purpose, we explore the method of analytic continuation from the bound state [21–23]. For this purpose, we vary the bare quark mass m from the region where the σ meson would be bound ($E_\sigma < E_{2\pi}$) down to the physical value of $m \to m_0$ (where $E_\sigma >> E_{2\pi}$) [11]. The method consists of the following steps:

- Determine the threshold value m_{th} and calculate $\epsilon = E_\sigma - E_{2\pi}$ as a function of m for $m > m_{\text{th}}$.
- Introduce a variable $x = \sqrt{m - m_{\text{th}}}$; calculate $k(x) = \mathrm{i}\sqrt{-\epsilon}$ in the bound state region.
- Fit $k(x)$ by a polynomial $k(x) = \mathrm{i}(c_0 + c_1 x + c_2 x^2 + \ldots + c_{2M} x^{2M})$ (Figure 1).
- Construct a Padé approximant: $k(x) = \mathrm{i}\,\frac{a_0 + a_1 x + \ldots + a_M x^M}{1 + b_1 x + \ldots + b_M x^M}$ (Figure 1).
- Analytically continue $k(x)$ to the region $m < m_{\text{th}}$ (i.e., to imaginary x) where $k(x)$ becomes complex.
- Determine the position and the width of the resonance as follows: $E_{res} = \mathrm{Re}\,[k^2(m \to m_0)]$, $\Gamma = -2\,\mathrm{Im}\,[k^2(m \to m_0)]$ (Figures 2 and 3).

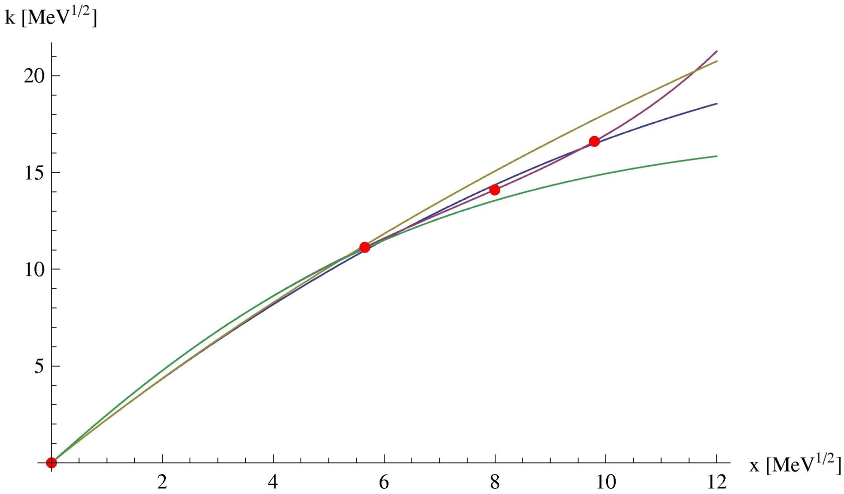

Figure 1. The fit of $k(x)$ with quadratic (lower middle) and quartic polynomial (upper middle) and with Padé approximants of order 1 (below) and 2 (above).

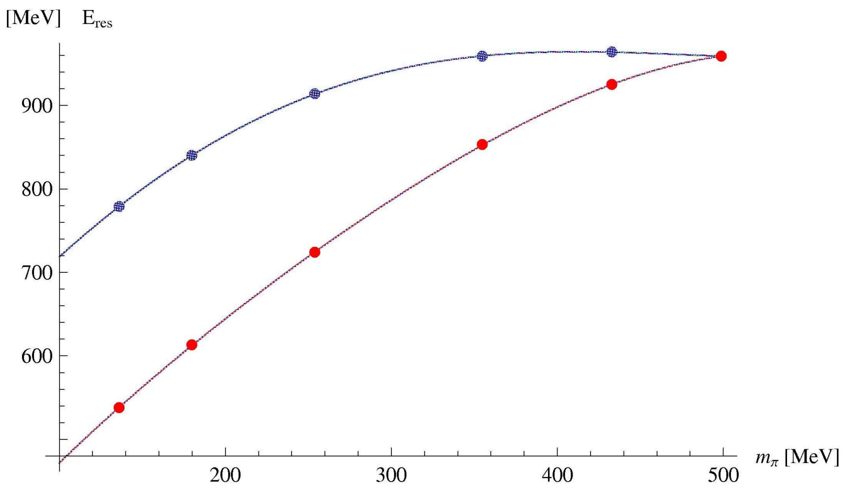

Figure 2. The resonance energy of the σ meson as a function of the pion mass–extrapolation using Padé approximants of order 1 (below) and 2 (above).

One can notice that the results for E_{res} and Γ in Table 2 deviate strongly for first- and second-order Padé approximants. This is due to the large stretch for the analytic continuation, so that convergence at higher orders cannot be expected. Nevertheless, it is rewarding that the physical values for E_{res} and Γ lie somewhere in the middle between both curves. Intentionally, the energy and width of the σ meson are plotted as a function of the corresponding pion mass rather than as a function of the model parameter m. This is reminiscent of the extrapolation of pion mass from about 500 Mev towards its physical value in typical lattice calculations.

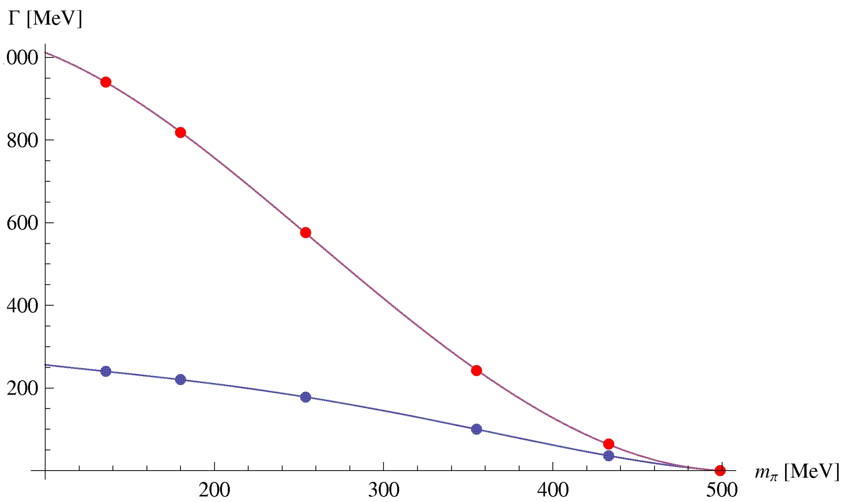

Figure 3. The width Γ of the σ meson as a function of the pion mass–extrapolation using Padé approximants of order 1 (below) and 2 (above).

Table 2. The resonance energy E_{res} and the width Γ of the σ meson as a function of the pion mass–extrapolation using Padé approximants, in [MeV].

Pion Mass	136	180	254	355	433	499
E_{res}(order 1)	779	840	914	959	964	959
E_{res}(order 2)	538	613	724	853	925	959
Γ(order 1)	240	220	178	100	36	0
Γ(order 2)	940	818	576	242	64	0

5. Conclusions

The two-level "quasispin model" has been proposed as a solvable approximation of the full Nambu–Jona-Lasinio model. It supports the Hartree–Fock approximation for massive constituent quarks, together with the random phase approximation for pions. It also gives a meaning to the limit of a large number of colours, which acts, similarly, as a large number of participating spatial states (large ratio of box size to cell size).

The discrete spectrum of excited states can be interpreted as multipion states and can serve to derive the pion scattering length using the first-order Born approximation. The intruder state between the multipion states can be interpreted as the sigma meson. Its width can be estimated using analytical continuation from bound states to resonant states (with complex energy) by varying the model parameter corresponding to the (too large) pion mass. This is analogous to Lattice QCD, where the pion mass is also extrapolated towards the physical value.

In the present article, only two applications have been shown: the low energy pion–pion scattering and the widths of the sigma resonance. The two-level "quasispin model", however, may offer many more applications in the future. Examples are mesonic spectra other than scalars and pseudoscalars, as well as baryonic spectra, and also the equation of state. Work is in progress.

I would like to encourage readers to use this simple model in preliminary studies in order to obtain a qualitative understanding of the results of any lattice-like system with discrete energy spectra.

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