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Differential Subordination and Superordination Using an Integral Operator for Certain Subclasses of p-Valent Functions

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Abstract: This work presents a novel investigation that utilizes the integral operator $I_{p,\lambda}^{n}$ in the field of geometric function theory, with a specific focus on sandwich theorems. We obtained findings about the differential subordination and superordination of a novel formula for a generalized integral operator. Additionally, certain sandwich theorems were discovered.

Keywords: analytic function; integral operator; Hadamard product; differential subordination and superordination

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1. Introduction and Definitions

In the past, people have used complex numbers to solve real cubic equations, which has facilitated the development of a fascinating theory known as the theory of functions of a complex variable (complex analysis). This field has a historical origin dating back to the 17th century. Noteworthy figures in the field include Riemann, Gauss, Euler, Cauchy, Mittag-Leffler, and several more scientists. Riemann introduced the Riemann mapping theorem in 1851 during the 19th century, giving rise to geometric function theory (GFT), a notable and captivating theoretical framework [1]. It has seen significant development and has been applied in several scientific domains, including operator theory, differential inequality theory, and other related topics. To enhance the Riemann mapping theorem, Koebe [1] utilized a univalent function defined on an open unit disk in 1907. In 1909, Lindeöf introduced the subordinate idea. The Schwarz function is employed to examine two complex functions. Diverse subordination theory on a complex domain may be understood as an extension of differential inequality theory on a real domain. This topic was extensively explored by Miller and Mocanu in their seminal works published in 1978 [2], 1981 [3], and 2000 [4]. Miller and Mocanu [5] (2003) introduced the concept of differential subordination theory, specifically referred to as differential superordination. Differential subordination and superordination are crucial techniques in GFT that are employed in studies to obtain sandwich results. This theory has great importance, and several proficient analysts have made exceptional contributions to studying the related issues, including Srivastava et al. [6], Ghanim et al. [7], Lupas and Oros [8], Attiya et al. [9], and others. In 2015, Ibrahim et al. [10] introduced a novel operator that combines a fractional integral operator with the Carlson–Shaffer operator. This operator was employed to investigate the characteristics of subordination and superordination. The fractional derivative operator for higher-order derivatives of certain analytic multivalent functions was expanded by Morais and Zayed [11] in the year 2021. The subordination and superordination features were investigated by Lupas and Oros [8] in 2021 by the utilization of the fractional integral of the confluent hypergeometric function. In the year 2022, other authors conducted investigations pertaining to subordination and its associated qualities [12–14].

The fractional integral operator is a fundamental mathematical operation employed across several domains within the realms of science and engineering. It possesses applicability in several fields. Recent decades have witnessed the successful use of fractional calculus in physical models. The generalized Mittag-Leffler function has been utilized in several mathematical and physical domains due to its inherent ability to express solutions to fractional integral and differential equations. The utilization of fractional-order calculus is prevalent in several practical applications, such as [15–19]. By employing fractional operators in the resolution of differential equations, this study contributes to the field of mathematical applications. Furthermore, it emphasizes the importance of these operators in the fields of physics and engineering, particularly for the advancement of geometric function theory, a specialist field within complex analysis.

The application of the subordination technique is employed in relation to pertinent categories of permissible functions. According to Antonino and Miller [20], the acceptable functions are defined as follows:

Let $\mathcal{H}(\mathbb{D})$ denote the class of functions analytic in the open unit disk

$$\mathbb{D} := \{\zeta : \zeta \in \mathbb{C} \quad and \quad |\zeta| < 1\}$$

and $\mathcal{H}[a,n]$ denote the subclass $f \in \mathcal{H}(\mathbb{D})$ consisting of the functions of the form $f(\zeta) = a + a_n\zeta^n + a_{n+1}\zeta^{n+1} + \ldots$, with $\mathcal{H}_0 = \mathcal{H}[0,1]$ and $\mathcal{H} = \mathcal{H}[1,1]$

Also, let $\mathcal{H}_p$ be the subclass of $f \in \mathcal{H}(\mathbb{D})$ of the form

$$f(\zeta) = \zeta^p + \sum_{l=p+1}^{\infty} a_l \zeta^l \qquad (p \in \mathbb{N}), \tag{1}$$

and set $\mathcal{H} \equiv \mathcal{H}_1$. For functions $f(\zeta) \in \mathcal{H}_p$, given by (1) and $g(\zeta)$ given by

$$g(\zeta) = \zeta^p + \sum_{l=p+1}^{\infty} b_l \zeta^l \qquad (p \in \mathbb{N}), \tag{2}$$

the Hadamard product (or convolution) of $f(\zeta)$ and $g(\zeta)$ is defined by

$$(f * g)(\zeta) = \zeta^p + \sum_{l=p+1}^{\infty} a_l b_l \zeta^l = (g * f)(\zeta) \qquad (\zeta \in \mathbb{D};\ p \in \mathbb{N}). \tag{3}$$

For that, $f(\zeta)$ and $g(\zeta)$ are in $\mathcal{H}(\mathbb{D})$. We say that $f(\zeta)$ is subordinate to $g(\zeta)$ (or $g(\zeta)$ is superordinate to $f(\zeta)$), written as

$$f \prec g \quad in \quad \mathbb{D} \quad or \quad f(\zeta) \prec g(\zeta) \qquad (\zeta \in \mathbb{D}),$$

if there exists a function $\omega \in \mathcal{H}$ satisfying the conditions of the Schwarz lemma (i.e., $\omega(0) = 0$ and $|\omega(\zeta)| < 1$) such that

$$f(\zeta) = g(\omega(\zeta)) \qquad (\zeta \in \mathbb{D}),$$

it follows that

$f(\zeta) \prec g(\zeta)$ $(\zeta \in \mathbb{D})$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ (see [4,21,22]).

Definition 1 ([5])**.** *Supposing that $p(\zeta)$ and $h(\zeta)$ are two analytic functions in $\mathbb{D}$, let*

$$\nu(r,s,t;\zeta) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}.$$

If $p(\zeta)$ and $\nu(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta)$ are univalent functions in $\mathbb{D}$. If h satisfies the second-order superordination

$$h(\zeta) \prec \nu(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta), \tag{4}$$

then $p(\zeta)$ is a solution of the differential superordination (4). A function $\kappa \in \mathcal{H}(\mathbb{D})$ is called a subordinant of (4) if $\kappa(\zeta) \prec p(\zeta)$ for all the functions h satisfies (4). A univalent subordinant $\tilde{\kappa}$ that satisfies $\kappa(\zeta) \prec \tilde{\kappa}(\zeta)$ for all of the subordinants κ of (4) is the best subordinant.

Definition 2 ([4])**.** *Supposing that $p(\zeta)$ and $h(\zeta)$ are two analytic functions in $\mathbb{D}$, let*

$$\nu(r,s,t;\zeta):\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}.$$

If $p(\zeta)$ is analytic in $\mathbb{D}$ and satisfies the second-order differential subordination

$$\nu(p(\zeta),\zeta p'(\zeta),\zeta^2 p''(\zeta);\zeta)\prec h(\zeta), \tag{5}$$

then $p(\zeta)$ is called a solution of the differential subordination (5). The univalent function $\kappa(\zeta)$ is called a dominant of the solution of the differential subordination (5), or more simply dominant, if $p(\zeta)\prec\kappa(\zeta)$ for all $p(\zeta)$, satisfying (5). A dominant $\tilde{\kappa}(\zeta)$ that satisfies $\tilde{\kappa}(\zeta)\prec\kappa(\zeta)$ for all dominant $\kappa(\zeta)$ of (5) is called the best dominant of (5).

The following inference holds for the functions h,κ, and ν according to sufficient conditions, as obtained by many authors (see [5,23–28]).

$$h(\zeta)\prec\nu(p(\zeta),\zeta p'(\zeta),\zeta^2 p''(\zeta);\zeta)\Rightarrow\kappa_2(\zeta)\prec p(\zeta)\qquad(\zeta\in\mathbb{D}).$$

Bulboaca [21] investigated first-order differential superordinations and superordination-preserving integral operators [29]. Ali et al. [23] used the results of [21] to develop adequate requirements for certain normalized analytic functions to satisfy

$$\kappa_1(\zeta)\prec\frac{\zeta f'(\zeta)}{f(\zeta)}\prec\kappa_2(\zeta),$$

where κ_1 and κ_2 represent univalent normalized functions in $\mathbb{D}$. Shanmugam et al. [24,30–32] recently reported sandwich results for specific analytic function classes. Further subordination results are available in [33–39].

For $p\in\mathbb{N},\ n\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \mu>0$ and $f\in\mathcal{H}_p$, we consider the integral operator defined as follows [40]:

$$\mathcal{I}^0_{p,\mu}f(\zeta)=f(\zeta)$$

$$\mathcal{I}^1_{p,\mu}f(\zeta)=\frac{p}{\mu}\zeta^{p-\frac{p}{\mu}}\int_0^\zeta t^{\frac{p}{\mu}-p-1}f(t)dt=\zeta^p+\sum_{l=p+1}^{\infty}\left[\frac{p}{p+\mu(l-p)}\right]a_l\zeta^l$$

$$\mathcal{I}^2_{p,\mu}f(\zeta)=\frac{p}{\mu}\zeta^{p-\frac{p}{\mu}}\int_0^\zeta t^{\frac{p}{\mu}-p-1}\mathcal{I}^1_{p,\mu}f(t)dt=\zeta^p+\sum_{l=p+1}^{\infty}\left[\frac{p}{p+\mu(l-p)}\right]^2 a_l\zeta^l$$

and (in general)

$$\begin{aligned}\mathcal{I}^n_{p,\mu}f(\zeta)&=\frac{p}{\mu}\zeta^{p-\frac{p}{\mu}}\int_0^\zeta t^{\frac{p}{\mu}-p-1}\mathcal{I}^{n-1}_{p,\mu}f(t)dt=\zeta^p+\sum_{l=p+1}^{\infty}\left[\frac{p}{p+\mu(l-p)}\right]^n a_l\zeta^l\\&=\underbrace{\mathcal{I}^1_{p,\mu}\left(\frac{\zeta^p}{1-\zeta}\right)*\mathcal{I}^1_{p,\mu}\left(\frac{\zeta^p}{1-\zeta}\right)*\ldots*\mathcal{I}^1_{p,\mu}\left(\frac{\zeta^p}{1-\zeta}\right)}_{n-times}*f(\zeta),\end{aligned} \tag{6}$$

then from (6), we can easily deduce that

$$\frac{\mu}{p}\zeta(\mathcal{I}^n_{p,\mu}f(\zeta))'=\mathcal{I}^{n-1}_{p,\mu}f(\zeta)-(1-\mu)\mathcal{I}^n_{p,\mu}f(\zeta)\quad(p,n\in\mathbb{N};\mu>0). \tag{7}$$

We note that:

$$(i) \quad \mathcal{I}^n_{1,\mu}f(\zeta) = \mathcal{I}^{-n}_{\mu}f(\zeta) \text{ (see [41])}$$
$$= \left\{ f(\zeta) \in A : \mathcal{I}^{-n}_{\mu}f(\zeta) = \zeta^p + \sum_{l=2}^{\infty}[1+\mu(l-1)]^{-n}a_l\zeta^l (n \in \mathbb{N}_0) \right\},$$

$$(ii) \quad \mathcal{I}^n_{1,1}f(\zeta) = \mathcal{I}^n f(\zeta) \text{ (see [42])}$$
$$= \left\{ f(\zeta) \in A : \mathcal{I}^n f(\zeta) = \zeta^p + \sum_{l=2}^{\infty} l^{-n}a_l\zeta^l (n \in \mathbb{N}_0) \right\}.$$

(iii) $\mathcal{I}^n_{p,1}f(\zeta) = \mathcal{I}^n_p f(\zeta)$, where $\mathcal{I}^n_p$ is a p-valent Salagean integral operator [40]

$$\mathcal{I}^n_p f(\zeta) = \left\{ f(\zeta) \in A(p) : \mathcal{I}^n_p f(\zeta) = \zeta^p + \sum_{l=p+1}^{\infty} \left(\frac{p}{l}\right)^n a_l\zeta^l (p \in \mathbb{N} \in \mathbb{N}_0) \right\}.$$

To prove our results, we need the following definitions and lemmas.

Denote by $\mathcal{F}$ the set of all functions $\kappa(\zeta)$ that are analytic and injective on $\bar{\mathbb{D}}\backslash E(\kappa)$, where

$$E(\kappa) = \{\eta \in \partial\mathbb{D} : \lim_{\zeta \to \eta} \kappa(\zeta) = \infty\},$$

and are such that $\kappa'(\eta) \neq 0$ for $\eta \in \partial\mathbb{D}\backslash E(\kappa)$. Further, let the subclass of $\mathcal{F}$ for which $\kappa(0) = a$ be denoted by $\mathcal{F}(a)$, $\mathcal{F}(0) \equiv \mathcal{F}_0$, and $\mathcal{F}(1) \equiv \mathcal{F}_1$.

Definition 3 ([4] , Definition 2.3a, p. 27)**.** *Let ϑ be a set in $\mathbb{C}, \kappa \in \mathcal{F}$, and n be a positive integer. The class of admissible functions $\Upsilon_n[\vartheta, \kappa]$, consists of those functions $\gamma : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\gamma(r, s, t; \zeta) \notin \vartheta$$

whenever

$$r = \kappa(\eta), \quad s = l\eta\kappa'(\eta), \ \Re\left\{\frac{t}{s} + 1\right\} \geq l\Re\left\{1 + \frac{\eta\kappa''(\eta)}{\kappa'(\eta)}\right\},$$

where $\zeta \in \mathbb{D}$, $\eta \in \partial\mathbb{D}\backslash E(\kappa)$, and $l \geq n$. We write $\Upsilon_1[\vartheta, \kappa]$ as $\Upsilon[\vartheta, \kappa]$.

In particular, when

$$\kappa(\zeta) = Q\frac{Qz + a}{Q + \bar{a}\zeta} \quad (Q > 0, \ |a| < Q),$$

then $\kappa(\mathbb{D}) = \mathbb{D}_Q = \{w : |w| < Q\}$, $\kappa(0) = a$, $E(\kappa) = \nu$, and $\kappa \in \mathcal{F}$. In this case, we set $\Upsilon_n[\vartheta, Q, a] = \Upsilon_n[\vartheta, \kappa]$, and in the special case when the set $\vartheta = \mathbb{D}_Q$, the class is simply denoted by $\Upsilon_n[Q, a]$.

Definition 4 ([5], Definition 3, p. 817)**.** *Let ϑ be a set in $\mathbb{C}$, $\kappa(\zeta) \in \mathcal{H}[a, n]$ with $\kappa'(\zeta) \neq 0$. The class of admissible functions $\Upsilon'_n[\vartheta, \kappa]$ consists of those functions $\gamma : \mathbb{C}^3 \times \bar{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\gamma(r, s, t; \eta) \in \vartheta$$

whenever

$$r = \kappa(\zeta), \ s = \frac{\zeta\kappa'(\zeta)}{q}, \ \Re\left\{\frac{t}{s} + 1\right\} \geq \frac{1}{q}\Re\left\{1 + \frac{\zeta\kappa''(\zeta)}{\kappa'(\zeta)}\right\},$$

where $\zeta \in \mathbb{D}, \eta \in \partial\mathbb{D}$, and $q \geq n \geq 1$. In particular, we write $\Upsilon'_1[\vartheta, \kappa]$ as $\Upsilon'[\vartheta, \kappa]$.

Lemma 1 ([4], Theorem 2.3b, p. 28)**.** *Let* $\gamma \in \Upsilon_n[\vartheta, \kappa]$ *with* $\kappa(0) = a$*. If the analytic function* $g(\zeta) = a + a_n\zeta^n + a_{n+1}\zeta^{n+1} + \ldots$ *satisfies*

$$\gamma(g(\zeta), zg'(\zeta), \zeta^2 g''(\zeta); \zeta) \in \vartheta,$$

then $g(\zeta) \prec \kappa(\zeta)$*.*

Lemma 2 ([5], Theorem 1, p. 818)**.** *Let* $\gamma \in \Upsilon'_n[\vartheta, \kappa]$ *with* $\kappa(0) = a$*. If* $g(\zeta) \in \mathcal{F}(a)$ *and*

$$\gamma(g(\zeta), zg'(\zeta), \zeta^2 g''(\zeta); \zeta)$$

is univalent in $\mathbb{D}$*, then*

$$\vartheta \subset \{\gamma(g(\zeta), zg'(\zeta), \zeta^2 g''(\zeta); \zeta) : \zeta \in \mathbb{D}\},$$

implies $\kappa(\zeta) \prec g(\zeta)$*.*

In this paper, we extend Miller and Mocanu's differential subordination result ([4], Theorem 2.3b, p. 28) to include functions related to the integral operator, and we also obtain some other related results. Aghalary et al. [43], Ali et al. [44], Aouf [45], Aouf et al. [46], Kim and Srivastava [47], and Seoudy [48] all investigated a comparable issue for analytical functions. Furthermore, they conducted investigations on the relevant differential superordination problem, yielding numerous sandwich-type results.

2. Subordination Results Involving $\mathcal{I}^n_{p,\mu}$

In this study, we assume that $n > 3$, $p \in \mathbb{N}$, $\zeta \in \mathbb{D}$, and all powers are principal ones, unless otherwise specified.

Definition 5. *Let* ϑ *be a set in* $\mathbb{C}$ *and* $\kappa(\zeta) \in \mathcal{F}_0 \cap \mathcal{H}[0, p]$*. The class of admissible functions* $\varphi_j[\vartheta, \kappa]$ *consists of those functions* $\nu : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ *that satisfy the admissibility condition*

$$\nu(u, v, w; \zeta) \notin \vartheta$$

whenever

$$u = \kappa(\eta), \quad v = \frac{l\mu\eta\kappa'(\eta) + p(1-\mu)\kappa(\eta)}{p},$$

$$\Re\left\{\frac{wp^2 - (1-\mu)^2 p^2 u}{vp - p(1-\mu)u} - 2p(1-\mu)\right\} \geq l\Re\left\{1 + \frac{\eta\kappa''(\eta)}{\kappa'(\eta)}\right\},$$

where $\zeta \in \mathbb{D}$, $\eta \in \partial\mathbb{D}\backslash E(\kappa)$, $\mu > 0; n, p \in \mathbb{N}$*, and* $l \geq p$*.*

Theorem 1. *Let* $\varphi_j[\vartheta, \kappa]$*. If* $f(\zeta) \in \mathcal{H}_p$ *satisfies*

$$\left\{\nu\left(\mathcal{I}^n_{p,\mu}f(\zeta), \mathcal{I}^{n-1}_{p,\mu}f(\zeta), \mathcal{I}^{n-2}_{p,\mu}f(\zeta); \zeta\right) : \zeta \in \mathbb{D}\right\} \subset \vartheta \quad (p, n \in \mathbb{N}; \mu > 0), \tag{8}$$

then

$$\mathcal{I}^n_{p,\mu}f(\zeta) \prec \kappa(\zeta) \quad (\zeta \in \mathbb{D}).$$

Proof. Define the analytic function $g(\zeta)$ in $\mathbb{D}$ by

$$g(\zeta) = \mathcal{I}^n_{p,\mu}f(\zeta) \quad (p, n \in \mathbb{N}; \mu > 0). \tag{9}$$

In view of the relation (7) from (9), we obtain

$$\mathcal{I}_{p,\mu}^{n-1}f(\zeta)=\frac{\zeta\mu g'(\zeta)+p(1-\mu)g(\zeta)}{p}. \tag{10}$$

Further computations show that

$$\mathcal{I}_{p,\mu}^{n-2}f(\zeta)=\frac{\zeta^2\mu^2 g''(\zeta)+\zeta\mu(2p(1-\mu)+\mu)g'(\zeta)+p^2(1-\mu)^2g(\zeta)}{p^2}. \tag{11}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,\ v=\frac{s+p(1-\mu)r}{p},\quad w=\frac{\mu^2 t+\mu(2p(1-\mu)+\mu)s+p^2(1-\mu)^2 r}{p^2}. \tag{12}$$

Let

$$\begin{aligned}\gamma(r,s,t;\zeta)=&\nu(u,v,w;\zeta)\\ =&\nu\left(r,\frac{s+p(1-\mu)r}{p},\frac{\mu^2 t+\mu(2p(1-\mu)+\mu)s+p^2(1-\mu)^2 r}{p^2};\zeta\right).\end{aligned} \tag{13}$$

The proof shall make use of Lemma 1. Using Equations (9)–(11), from (13), we obtain

$$\begin{gathered}\gamma(p(\zeta),\zeta p'(\zeta),\zeta^2 p''(\zeta);\zeta)=\nu\left(\mathcal{I}_{p,\mu}^{n}f(\zeta),\mathcal{I}_{p,\mu}^{n-1}f(\zeta),\mathcal{I}_{p,\mu}^{n-2}f(\zeta);\zeta\right)\\ (p,n\in\mathbb{N};\mu>0).\end{gathered} \tag{14}$$

Hence, (8) becomes

$$\gamma(p(\zeta),\zeta p'(\zeta),\zeta^2 p''(\zeta);\zeta)\in\vartheta.$$

If it can be demonstrated that the $\nu\in\varphi_j[\vartheta,\kappa]$ admissibility condition is equal to the γ admissibility requirement stated in Definition 3, the proof is considered successful. Observe that

$$\frac{t}{s}+1=\frac{wp^2-(1-\mu)^2p^2u}{vp-p(1-\mu)u}-2p(1-\mu),$$

and hence, $\gamma\in\Upsilon_j[\vartheta,\kappa]$. By Lemma 1,

$$g(\zeta)\prec\kappa(\zeta)\quad\text{or}\quad\mathcal{I}_{p,\mu}^{n}f(\zeta)\prec\kappa(\zeta)\quad(\zeta\in\mathbb{D}).$$

□

If the domain $\vartheta\neq\mathbb{C}$ is simply linked, then $\vartheta=h(\mathbb{D})$ for some conformal mapping $h(\zeta)$ of $\mathbb{D}$ onto ϑ. The class $\varphi_j'[h(\mathbb{D}),\kappa]$ can be represented as $\varphi_j'[h.\kappa]$.

Continuing as in the preceding section, Theorem 1 immediately leads to the following result.

Theorem 2. *Let $\varphi_j[h,\kappa]$. If $f(\zeta)\in\mathcal{H}_p$ satisfies*

$$\nu\left(\mathcal{I}_{p,\mu}^{n}f(\zeta),\mathcal{I}_{p,\mu}^{n-1}f(\zeta),\mathcal{I}_{p,\mu}^{n-2}f(\zeta);\zeta\right)\prec h(\zeta)\quad(p,n\in\mathbb{N};\zeta\in\mathbb{D};\mu>0), \tag{15}$$

then

$$\mathcal{I}_{p,\mu}^{n}f(\zeta)\prec\kappa(\zeta)\quad(\zeta\in\mathbb{D}).$$

In the case where $\kappa(\zeta)$ on $\partial\mathbb{D}$ has an uncertain behavior, our next result extends Theorem 1.

Corollary 1. *Let $\vartheta \subset \mathbb{C}$ and let $\kappa(\zeta)$ be univalent in $\mathbb{D}$, $\kappa(0) = 0$. Let $\nu \in \varphi_j[\vartheta, \kappa_\rho]$ for some $\rho \in (0,1)$, where $\kappa_\rho(\zeta) = \kappa(\rho\zeta)$. If $f(\zeta) \in \mathcal{H}_p$ and*

$$\nu\left(\mathcal{I}^{n}_{p,\mu}f(\zeta), \mathcal{I}^{n-1}_{p,\mu}f(\zeta), \mathcal{I}^{n-2}_{p,\mu}f(\zeta); \zeta\right) \in \vartheta \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

then

$$\mathcal{I}^{n}_{p,\mu}f(\zeta) \prec \kappa(\zeta) \quad (\zeta \in \mathbb{D}).$$

Proof. Theorem 1 yields $\mathcal{I}^{n}_{p,\mu}f(\zeta) \prec \kappa_\rho(\zeta)$. The result is now deduced from $\kappa_\rho(\zeta) \prec \kappa(\zeta)$. □

Theorem 3. *Let $h(\zeta)$ and $\kappa(\zeta)$ be univalent in $\mathbb{D}$, with $\kappa(0) = 0$, and set $\kappa_\rho(\zeta) = \kappa(\rho\zeta)$ and $h_\rho(\zeta) = h(\rho\zeta)$. Let $\nu : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfy one of the following conditions:*

1. $\nu \in \varphi_j[h, \kappa_\rho]$, $\rho \in (0,1)$;
2. *there exists $\rho_0 \in (0,1)$ such that $\nu \in \varphi_j[h_\rho, \kappa_\rho]$, for all $\rho \in (\rho_0, 1)$.*

If $f(\zeta) \in \mathcal{H}_p$ satisfies (15), then

$$\mathcal{I}^{n}_{p,\mu}f(\zeta) \prec \kappa(\zeta) \quad (\zeta \in \mathbb{D}).$$

Proof. The proof is omitted because it is comparable to the proof of ([4], Theorem 2.3d, p. 30).

The best dominant of the differential subordination is obtained by the following theorem (15). □

Theorem 4. *Let $h(\zeta)$ be univalent in $\mathbb{D}$. Let $\nu : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$. Suppose that the differential equation*

$$\nu(\kappa(\zeta), \zeta\kappa'(\zeta), \zeta^2\kappa''(\zeta); \zeta) = h(\zeta) \tag{16}$$

has a solution $\kappa(\zeta)$ with $\kappa(0) = 0$ and satisfies one of the following conditions:

1. $\kappa(\zeta) \in \mathcal{F}_0$ *and* $\nu \in \varphi_j[h, \kappa]$;
2. $\kappa(\zeta)$ *is univalent in $\mathbb{D}$ and $\nu \in \varphi_j[h, \kappa_\rho]$, for some $\rho \in (0,1)$;*
3. $\kappa(\zeta)$ *is univalent in $\mathbb{D}$ and there exists $\rho_0 \in (0,1)$ such that $\nu \in \varphi_j[h_\rho, \kappa_\rho]$ for all $\rho \in (\rho_0, 1)$.*

   If $f(\zeta) \in \mathcal{H}_p$ satisfies (15), then

$$\mathcal{I}^{n}_{p,\mu}f(\zeta) \prec \kappa(\zeta)(\zeta \in \mathbb{D}),$$

and $\kappa(\zeta)$ is the best dominant.

Proof. By using the same reasoning as in ([4], Theorem 2.3e, p. 31), we may infer from Theorems 2 and 3 that $\kappa(\zeta)$ is a dominant. Since $\kappa(\zeta)$ is a solution of (15) and fulfills (16), all dominants will dominate $\kappa(\zeta)$. The optimal dominant is, therefore, $\kappa(\zeta)$. □

In the particular case $\kappa(\zeta) = Qz, \quad Q > 0$, and in view of Definition 3, the class of admissible functions $\varphi_j[\vartheta, \kappa]$, denoted by $\varphi_j[\vartheta, Q]$, is described below.

Definition 6. *Let ϑ be a set in $\mathbb{C}$ and $Q > 0$. The class of admissible functions $\varphi_j[\vartheta, Q]$ consists of those functions $\nu : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\nu\left(Qe^{i\theta}, \frac{l + p(1-\mu)}{p}Qe^{i\theta}, \frac{\mu^2 L + [\mu(2p(1-\mu) + \mu)l + p^2(1-\mu)^2]Qe^{i\theta}}{p^2}; \zeta\right) \notin \vartheta \tag{17}$$

whenever $\zeta \in \mathbb{D}, \theta \in \mathbb{R}$, $\Re\left(Le^{-i\theta}\right) \geq (l-1)lQ$ for all real θ, $p, n \in \mathbb{N}$, $\mu > 0$, and $l \geq p$.

Corollary 2. *Let $\nu \in \varphi_j[\vartheta, Q]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\nu\left(\mathcal{I}_{p,\mu}^{n} f(\zeta), \mathcal{I}_{p,\mu}^{n-1} f(\zeta), \mathcal{I}_{p,\mu}^{n-2} f(\zeta); \zeta\right) \in \vartheta \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

then

$$\left|\mathcal{I}_{p,\mu}^{n} f(\zeta)\right| < Q \quad (\zeta \in \mathbb{D}).$$

The class $\varphi_j[\vartheta, Q]$ is easily denoted by $\varphi_j[Q]$ in the particular case $\vartheta = \kappa(\mathbb{D}) = \{\omega : |\omega| < Q\}$.

Corollary 3. *Let $\nu \in \varphi_j[Q]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\left|\nu\left(\mathcal{I}_{p,\mu}^{n} f(\zeta), \mathcal{I}_{p,\mu}^{n-1} f(\zeta), \mathcal{I}_{p,\mu}^{n-2} f(\zeta); \zeta\right)\right| < Q \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

then

$$\left|\mathcal{I}_{p,\mu}^{n} f(\zeta)\right| < Q \quad (\zeta \in \mathbb{D}).$$

Corollary 4. *If $l \geq p$ and $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\left|\mathcal{I}_{p,\mu}^{n-1} f(\zeta)\right| < Q \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

then

$$\left|\mathcal{I}_{p,\mu}^{n} f(\zeta)\right| < Q \quad (\zeta \in \mathbb{D}).$$

Proof. Corollary 3 dictates that this is performed by taking

$$\nu(u, v, w; \zeta) = v = \frac{l + p(1-\mu)r}{p} Q e^{i\theta}.$$

□

Definition 7. *Let ϑ be a set in $\mathbb{C}$ and $\kappa(\zeta) \in \mathcal{F}_0 \cap \mathcal{H}_0$. The class of admissible functions $\varphi_{j,1}[\vartheta, \kappa]$ consists of those functions $\nu : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\nu(u, v, w; \zeta) \notin \vartheta$$

whenever

$$u = \kappa(\eta), \quad v = \frac{l\eta\mu\kappa'(\eta) + (p-\mu)\kappa(\eta)}{p},$$

$$\Re\left\{\frac{wp^2 + 2vp(\mu - p) + (p-\mu)^2 u}{\mu(vp - (p-\mu)u)}\right\} \geq l\Re\left\{1 + \frac{\eta\kappa''(\eta)}{\kappa'(\eta)}\right\},$$

where $\zeta \in \mathbb{D}$, $\eta \in \partial\mathbb{D}\backslash E(\kappa)$, $\mu > 0; n, p \in \mathbb{N}$, and $l \geq 1$.

Theorem 5. *Let $\nu \in \varphi_{j,1}[\vartheta, \kappa]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\left\{\nu\left(\frac{\mathcal{I}_{p,\mu}^{n} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}_{p,\mu}^{n-1} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}_{p,\mu}^{n-2} f(\zeta)}{\zeta^{p-1}}; \zeta\right)\right\} \subset \vartheta \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0), \tag{18}$$

then

$$\frac{\mathcal{I}_{p,\mu}^{n} f(\zeta)}{\zeta^{p-1}} \prec \kappa(\zeta) \quad (\zeta \in \mathbb{D}).$$

Proof. Define an analytic function $g(\zeta)$ in $\mathbb{D}$ by

$$g(\zeta) = \frac{\mathcal{I}^{n}_{p,\mu} f(\zeta)}{\zeta^{p-1}} \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0). \tag{19}$$

By making use of (7) and (19), we obtain

$$\frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\zeta^{p-1}} = \frac{\zeta \mu g'(\zeta) + (p-\mu) g(\zeta)}{p}. \tag{20}$$

Further computations show that

$$\frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\zeta^{p-1}} = \frac{\zeta^2 \mu^2 g''(\zeta) + \mu(2p-\mu) z g'(\zeta) + (p-\mu)^2 g(\zeta)}{p^2}. \tag{21}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{\mu s + (p-\mu) r}{p}, \quad w = \frac{\mu^2 t + \mu(2p-\mu) s + (p-\mu)^2 r}{p^2}. \tag{22}$$

Let

$$\begin{aligned} \gamma(r, s, t; \zeta) &= \nu(u, v, w; \zeta) \\ &= \nu\left(r, \frac{\mu s + (p-\mu) r}{p}, \frac{\mu^2 t + \mu(2p-\mu) s + (p-\mu)^2 r}{p^2}; \zeta\right). \end{aligned} \tag{23}$$

The proof shall make use of Lemma 1. Using Equations (19)–(21), and from (23), we obtain

$$\gamma\Big(g(\zeta), z g'(\zeta), \zeta^2 g''(\zeta); \zeta\Big) = \nu\left(\frac{\mathcal{I}^{n}_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\zeta^{p-1}}; \zeta\right). \tag{24}$$

Hence, (18) becomes

$$\gamma\Big(g(\zeta), z g'(\zeta), \zeta^2 g''(\zeta); \zeta\Big) \in \vartheta.$$

If it can be demonstrated that the $\nu \in \varphi_{j,1}[\vartheta, \kappa]$ admissibility condition is equal to the γ admissibility requirement stated in Definition 3, the proof is considered successful. Observe that

$$\frac{t}{s} + 1 = \frac{w p^2 + 2 v p(\mu - p) + (p-\mu)^2 u}{\mu(v p - (p-\mu) u)},$$

and hence, $\gamma \in \Upsilon[\vartheta, \kappa]$. By Lemma 1,

$$g(\zeta) \prec \kappa(\zeta) \quad \text{or} \quad \frac{\mathcal{I}^{n}_{p,\mu} f(\zeta)}{\zeta^{p-1}} \prec \kappa(\zeta) \quad (\zeta \in \mathbb{D}).$$

□

If $\vartheta \neq \mathbb{C}$ is a simply connected domain, then $\vartheta = h(\mathbb{D})$ for some conformal mapping $h(\zeta)$ of $\mathbb{D}$ onto ϑ. In this case, the class $\varphi_{j,1}[h(\mathbb{D}), \kappa]$ is written as $\varphi_{j,1}[h, \kappa]$. In the particular case $\kappa(\zeta) = Q\zeta,\ Q > 0$, the class of admissible functions $\varphi_{j,1}[\vartheta, \kappa]$ is denoted by $\varphi_{j,1}[\vartheta, Q]$. The following outcome is a direct conclusion of Theorem 5, employing the same procedure as in the preceding section.

Theorem 6. *Let $\nu \in \varphi_{j,1}[h,\kappa]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\nu\left(\frac{\mathcal{I}_{p,\mu}^{n} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}_{p,\mu}^{n-1} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}_{p,\mu}^{n-2} f(\zeta)}{\zeta^{p-1}}; \zeta\right) \prec h(\zeta) \quad (p,n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0), \tag{25}$$

then

$$\frac{\mathcal{I}_{p,\mu}^{n-1} f(\zeta)}{\zeta^{p-1}} \prec \kappa(\zeta) \ \ (\zeta \in \mathbb{D}).$$

Definition 8. *Let ϑ be a set in $\mathbb{C}$ and $Q > 0$. The class of admissible functions $\varphi_{j,1}[\vartheta, Q]$ consists of those functions $\nu : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\nu\left(Qe^{i\theta}, \frac{\mu l + (p-\mu)}{p} Qe^{i\theta}, \frac{\mu^2 L + [\mu l(2p-\mu) + (p-\mu)^2] Q^{i\theta}}{p^2}; \zeta\right) \notin \vartheta \tag{26}$$

whenever $\zeta \in \mathbb{D}, \theta \in \mathbb{R},\ R\left(Le^{-i\theta}\right) \geq (l-1)lQ$ for all real $\theta,\ p \in \mathbb{N}$, and $l \geq 1$.

Corollary 5. *Let $\nu \in \varphi_{j,1}[\vartheta, Q]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\nu\left(\frac{\mathcal{I}_{p,\mu}^{n} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}_{p,\mu}^{n-1} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}_{p,\mu}^{n-2} f(\zeta)}{\zeta^{p-1}}; \zeta\right) \in \vartheta \quad (p,n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

then

$$\left|\frac{\mathcal{I}_{p,\mu}^{n} f(\zeta)}{\zeta^{p-1}}\right| < Q \ \ (\zeta \in \mathbb{D}).$$

The class $\varphi_{j,1}[\vartheta, Q]$ is easily denoted by $\varphi_{j,1}[Q]$ in the particular case $\vartheta = \kappa(\mathbb{D}) = \{\omega : |\omega| < Q\}$.

Corollary 6. *Let $\nu \in \varphi_{j,1}[Q]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\left|\nu\left(\frac{\mathcal{I}_{p,\mu}^{n} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}_{p,\mu}^{n-1} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}_{p,\mu}^{n-2} f(\zeta)}{\zeta^{p-1}}; \zeta\right)\right| < Q \quad (p,n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

then

$$\left|\frac{\mathcal{I}_{p,\mu}^{n} f(\zeta)}{\zeta^{p-1}}\right| < Q \ \ (\zeta \in \mathbb{D}).$$

Corollary 7. *If $l \geq 1$ and $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\left|\frac{\mathcal{I}_{p,\mu}^{n-1} f(\zeta)}{\zeta^{p-1}}\right| < Q \quad (p,n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

then

$$\left|\frac{\mathcal{I}_{p,\mu}^{n} f(\zeta)}{\zeta^{p-1}}\right| < Q \ \ (\zeta \in \mathbb{D}).$$

Proof. Corollary 6 dictates that this is performed by taking

$$\nu(u,v,w;\zeta) = v = \frac{\mu l + (p-\mu)}{p} Qe^{i\theta}.$$

□

Definition 9. *Let ϑ be a set in $\mathbb{C}$ and $\kappa(\zeta) \in \mathcal{F}_1 \cap \mathcal{H}$. The class of admissible functions $\varphi_{j,2}[\vartheta, \kappa]$ consists of those functions $\nu : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\nu(u, v, w; \zeta) \notin \vartheta$$

whenever

$$u = \kappa(\eta), v = \frac{1}{p}\left\{\frac{l\mu\eta\kappa'(\eta)}{\kappa(\eta)} + p\kappa(\eta)\right\},$$

$$\Re\left\{\frac{(wp - vp)(vp - pu + p)}{\mu(vp - pu)} + \frac{vp - 2pu - \mu}{\mu r} + 1\right\} \geq l\Re\left\{1 + \frac{\eta\kappa''(\eta)}{\kappa'(\eta)}\right\},$$

where $\zeta \in \mathbb{D}$, $\eta \in \partial\mathbb{D} \backslash E(\kappa)$, $p \in \mathbb{N}$, and $l \geq 1$.

Theorem 7. *Let $\nu \in \varphi_{j,2}[\vartheta, \kappa]$ and $\mathcal{I}^n_{p,\mu} f(\zeta) \neq 0$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\left\{\left(\frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\mathcal{I}^n_{p,\mu} f(\zeta)}, \frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}, \frac{\mathcal{I}^{n-3}_{p,\mu} f(\zeta)}{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}; \zeta\right) : \zeta \in \mathbb{D}\right\} \subset \vartheta \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0), \tag{27}$$

then

$$\frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\mathcal{I}^n_{p,\mu} f(\zeta)} \prec \kappa(\zeta) \ (\zeta \in \mathbb{D}).$$

Proof. Define an analytic function $g(\zeta)$ in $\mathbb{D}$ by

$$g(\zeta) = \frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\mathcal{I}^n_{p,\mu} f(\zeta)} \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0). \tag{28}$$

Using (28), we obtain

$$\frac{\zeta g'(\zeta)}{g(\zeta)} = \frac{\zeta\left(\mathcal{I}^{n-1}_{p,\mu} f(\zeta)\right)'}{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)} - \frac{\zeta\left(\mathcal{I}^n_{p,\mu} f(\zeta)\right)'}{\mathcal{I}^n_{p,\mu} f(\zeta)}. \tag{29}$$

By making use of (7) in (29), we obtain

$$\frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)} = \frac{1}{p}\left\{\frac{\mu\zeta g'(\zeta)}{g(\zeta)} + pg(\zeta)\right\}. \tag{30}$$

Further computations show that

$$\frac{\mathcal{I}^{n-3}_{p,\mu} f(\zeta)}{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)} = \frac{1}{p}\left\{\frac{\mu\zeta g'(\zeta)}{g(\zeta)} + pg(\zeta) + \frac{\frac{\mu^2\zeta^2 g''(\zeta)}{g(\zeta)} + \frac{\mu^2\zeta g'(\zeta)}{g(\zeta)} - \left(\frac{\mu z g'(\zeta)}{g(\zeta)}\right)^2 + p\mu z g'(\zeta)}{\mu\frac{\zeta g'(\zeta)}{g(\zeta)} + pg(\zeta)}\right\}. \tag{31}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{aligned} u &= r, \\ v &= \frac{1}{p}\left\{\frac{\mu s}{r} + pr\right\}, \\ w &= \frac{1}{p}\left\{\frac{\mu s}{r} + pr + \frac{\frac{\mu^2 t}{r} + \frac{\mu^2 s}{r} - \left(\frac{\mu s}{r}\right)^2 + p\mu s}{\frac{\mu s}{r} + pr}\right\}. \end{aligned} \tag{32}$$

Let

$$\gamma(r,s,t;\zeta) = \nu(u,v,w;\zeta)$$
$$= \nu\left(r, \frac{1}{p}\left\{\frac{\mu s}{r} + pr\right\}, \frac{1}{p}\left\{\frac{\mu s}{r} + pr + \frac{\frac{\mu^2 t}{r} + \frac{\mu^2 s}{r} - \left(\frac{\mu s}{r}\right)^2 + p\mu s}{\frac{\mu s}{r} + pr}\right\};\zeta\right). \tag{33}$$

The proof shall make use of Lemma 1. Using Equations (28), (30), and (31), from (33), we obtain

$$\gamma(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta);\zeta) = \nu\left(\frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-3}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)};\zeta\right). \tag{34}$$

Hence, (27) becomes

$$\gamma(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta);\zeta) \in \vartheta.$$

If it can be demonstrated that the $\nu \in \varphi_{j,2}[\vartheta,\kappa]$ admissibility condition is equal to the γ admissibility requirement stated in Definition 3, the proof is considered successful. Observe that

$$\frac{t}{s} + 1 = \frac{(wp - vp)vp - p\mu u}{\mu(vp - pu)} + \frac{1}{\mu},$$

and hence, $\gamma \in \Upsilon[\vartheta,\kappa]$. By Lemma 1,

$$g(\zeta) \prec \kappa(\zeta) \;\text{ or }\; \frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)} \prec \kappa(\zeta) \quad (\zeta \in \mathbb{D}).$$

□

There exists a conformal mapping $h(\zeta)$ of D onto ϑ such that $\vartheta \neq \mathbb{C}$ is a simply connected domain and $\vartheta = h(\mathbb{D})$. Here, $\varphi_{j,2}[h(\mathbb{D}),\kappa]$ is expressed as $\nu_{j,2}[h,\kappa]$. The class $\nu_{j,2}[\vartheta,\kappa]$ of admissible functions becomes $\nu_{j,2}[\vartheta,Q]$ in the specific case $\kappa(\zeta) = Qz,\ Q > 0$. Proceeding as in the previous section, the subsequent result gives a direct verification of Theorem 7.

Theorem 8. *Let $\nu \in \nu_{j,2}[h,\kappa]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\left\{\nu\left(\frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-3}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)};\zeta\right) : \zeta \in \mathbb{D}\right\} \prec h(\zeta) \quad (p,n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0), \tag{35}$$

then

$$\frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)} \prec \kappa(\zeta) \quad (\zeta \in \mathbb{D}).$$

Definition 10. *Let ϑ be a set in $\mathbb{C}$ and $Q > 0$. The class of admissible functions $\varphi_{j,2}[\vartheta,Q]$ consists of those functions $\nu : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\nu\left(Qe^{i\theta}, \frac{1}{p}\left\{l\mu + pQe^{i\theta}\right\}, \frac{1}{p}\left\{l\mu + pQe^{i\theta} + \frac{\mu^2 Le^{-i\theta} + (l\mu^2 - (l\mu)^2)Qe^{-i\theta} + lp\mu Q^2}{l\mu + pQe^{i\theta}}\right\};\zeta\right) \notin \vartheta, \tag{36}$$

whenever $\zeta \in \mathbb{D}, \theta \in \mathbb{R},\ \Re\left(Le^{-i\theta}\right) \geq (l-1)lQ$ for all real θ, $p \in \mathbb{N}$, and $l \geq 1$.

Corollary 8. *Let $\nu \in \varphi_{j,2}[\vartheta, Q]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\nu\left(\frac{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}{\mathcal{I}_{p,\mu}^{n}f(\zeta)}, \frac{\mathcal{I}_{p,\mu}^{n-2}f(\zeta)}{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}, \frac{\mathcal{I}_{p,\mu}^{n-3}f(\zeta)}{\mathcal{I}_{p,\mu}^{n-2}f(\zeta)}; \zeta\right) \in \vartheta,$$

then

$$\left|\frac{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}{\mathcal{I}_{p,\mu}^{n}f(\zeta)}\right| < Q \quad (\zeta \in \mathbb{D}).$$

The class $\nu_{j,2}[\vartheta, Q]$ is easily denoted by $\nu_{j,2}[Q]$ in the particular case $\vartheta = \kappa(\mathbb{D}) = \{\omega : |\omega| < Q\}$.

Corollary 9. *Let $\nu \in \varphi_{j,2}[Q]$. If $f(\zeta) \in \mathcal{H}_p$ satisfies*

$$\left|\nu\left(\frac{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}{\mathcal{I}_{p,\mu}^{n}f(\zeta)}, \frac{\mathcal{I}_{p,\mu}^{n-2}f(\zeta)}{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}, \frac{\mathcal{I}_{p,\mu}^{n-3}f(\zeta)}{\mathcal{I}_{p,\mu}^{n-2}f(\zeta)}; \zeta\right)\right| < Q \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

then

$$\left|\frac{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}{\mathcal{I}_{p,\mu}^{n}f(\zeta)}\right| < Q \quad (\zeta \in \mathbb{D}).$$

3. Superordination and Sandwich Results Involving $\mathcal{I}_{p,\mu}^{n}$

This section focuses on the investigation of the dual problem of differential subordination, specifically the differential superordination of the integral operator $\mathcal{I}_{p,\mu}$. The class of acceptable functions is defined as follows for this purpose.

Definition 11. *Let ϑ be a set in $\mathbb{C}$ and $\kappa(\zeta) \in \mathcal{H}[0, p]$ with $\zeta\kappa'(\zeta) \neq 0$. The class of admissible functions $\varphi_j'[\vartheta, \kappa]$ consists of those functions $\nu : \mathbb{C}^3 \times \bar{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\nu(u, v, w; \eta) \in \vartheta$$

whenever

$$u = \kappa(\zeta), \quad v = \frac{\mu\zeta\kappa'(\zeta) + qp(1-\mu)\kappa(\zeta)}{qp},$$

$$\Re\left\{\frac{wp^2 - (1-\mu)^2p^2u}{vp - p(1-\mu)u} - 2p(1-\mu)\right\} \leq \frac{1}{q}\Re\left\{1 + \frac{\zeta\kappa'(\zeta)}{\kappa'(\zeta)}\right\},$$

where $\zeta \in \mathbb{D}, \eta \in \partial\mathbb{D}$, and $q \geq p$.

Theorem 9. *Let $\nu \in \varphi_j'[\vartheta, \kappa]$. If $f(\zeta) \in \mathcal{H}_p$, $\mathcal{I}_{p,\mu}^{n}f(\zeta) \in \mathcal{F}_0$ and*

$$\nu\left(\mathcal{I}_{p,\mu}^{n}f(\zeta), \mathcal{I}_{p,\mu}^{n-1}f(\zeta), \mathcal{I}_{p,\mu}^{n-2}f(\zeta); \zeta\right)$$

is univalent in $\mathbb{D}$, then

$$\vartheta \subset \left\{\nu\left(\mathcal{I}_{p,\mu}^{n}f(\zeta), \mathcal{I}_{p,\mu}^{n-1}f(\zeta), \mathcal{I}_{p,\mu}^{n-2}f(\zeta); \zeta\right) : (\zeta \in \mathbb{D})\right\} \quad (p, n \in \mathbb{N}; \mu > 0), \tag{37}$$

implies

$$\kappa(\zeta) \prec \mathcal{I}_{p,\mu}^{n}f(\zeta) \quad (\zeta \in \mathbb{D}).$$

Proof. From (14) and (37), we have

$$\vartheta \subset \left\{ \gamma(g(\zeta), \zeta g'(\zeta), \zeta^2 g''(\zeta); \zeta) : \zeta \in \mathbb{D} \right\}.$$

The admissibility requirement for $\nu \in \varphi_j'[\vartheta, \kappa]$ may be observed from (12). It is the same as the γ admissibility criterion stated in Definition 4. Thus, by Lemma 2 and $\gamma \in \Upsilon_p'[\vartheta, \kappa]$

$$\kappa(\zeta) \prec g(\zeta) \text{ or } \kappa(\zeta) \prec \mathcal{I}_{p,\mu}^n f(\zeta) \quad (\zeta \in \mathbb{D}).$$

□

If the domain $\vartheta \neq \mathbb{C}$ is simply linked, then $\vartheta = h(\mathbb{D})$ for some conformal mapping $h(\zeta)$ of $\mathbb{D}$ onto ϑ. The class $\varphi_j'[h(\mathbb{D}), \kappa]$ can be represented as $\varphi_j'[h.\kappa]$.

Continuing as in the preceding section, Theorem 9 immediately leads to the following result.

Theorem 10. *Let $h(\zeta)$ be analytic functions in $\mathbb{D}$ and $\nu \in \varphi_j'[h, \kappa]$. If $f(\zeta) \in \mathcal{H}_p$, $\mathcal{I}_{p,\mu}^n f(\zeta) \in \mathcal{F}_0$, and*

$$\nu\left(\mathcal{I}_{p,\mu}^n f(\zeta), \mathcal{I}_{p,\mu}^{n-1} f(\zeta), \mathcal{I}_{p,\mu}^{n-2} f(\zeta); \zeta\right) \quad (\zeta \in \mathbb{D})$$

is univalent in $\mathbb{D}$, then

$$h(\zeta) \prec \nu\left(\mathcal{I}_{p,\mu}^n f(\zeta), \mathcal{I}_{p,\mu}^{n-1} f(\zeta), \mathcal{I}_{p,\mu}^{n-2} f(\zeta); \zeta\right) \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0), \tag{38}$$

implies

$$\kappa(\zeta) \prec \mathcal{I}_{p,\mu}^n f(\zeta) \quad (\zeta \in \mathbb{D}).$$

Subordinants of differential superordination of the forms (37) or (38) can only be obtained using Theorems 9 and 10. The subsequent theorem establishes the existence of the optimal subordinant of Equation (38) for a given value of ν.

Theorem 11. *Let $h(\zeta)$ be analytic in $\mathbb{D}$ and $\nu : \mathbb{C}^3 \times \bar{\mathbb{D}} \to \mathbb{C}$. Suppose that the differential equation*

$$\nu(\kappa(\zeta), \zeta\kappa'(\zeta), \zeta^2\kappa''(\zeta); \zeta) = h(\zeta)$$

has a solution $\kappa(\zeta) \in \mathcal{F}_0$. If $\nu \in \varphi_j'[h, \kappa]$, $f(\zeta) \in \mathcal{H}_p$, $\mathcal{I}_{p,\mu}^n f(\zeta) \in \mathcal{F}_0$, and

$$\nu\left(\mathcal{I}_{p,\mu}^n f(\zeta), \mathcal{I}_{p,\mu}^{n-1} f(\zeta), \mathcal{I}_{p,\mu}^{n-2} f(\zeta); \zeta\right) \quad (\zeta \in \mathbb{D})$$

is univalent in $\mathbb{D}$, then

$$h(\zeta) \prec \nu\left(\mathcal{I}_{p,\mu}^n f(\zeta), \mathcal{I}_{p,\mu}^{n-1} f(\zeta), \mathcal{I}_{p,\mu}^{n-2} f(\zeta); \zeta\right) \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

implies

$$\kappa(\zeta) \prec \mathcal{I}_{p,\mu}^n f(\zeta) \quad (\zeta \in \mathbb{D}),$$

and $\kappa(\zeta)$ is the best subordinant.

Proof. The proof is omitted since it is similar to the proof of Theorem 4. By merging Theorems 2 and 10, we obtain the subsequent sandwich-type theorem. □

Corollary 10. *Let $h_1(\zeta)$ and $\kappa_1(\zeta)$ be analytic functions in $\mathbb{D}$, $h_2(\zeta)$ be a univalent function in $\mathbb{D}$, $\kappa_2(\zeta) \in \mathcal{F}_0$ with $\kappa_1(0) = \kappa_2(0) = 0$, and $\nu \in \varphi_j[h_2, \kappa_2] \cap \varphi_j'[h_1, \kappa_1]$. If $f(\zeta) \in \mathcal{H}_p$, $\mathcal{I}^n_{p,\mu} f(\zeta) \in \mathcal{H}[0, p] \cap \mathcal{F}_0$, and*

$$\nu\Big(\mathcal{I}^n_{p,\mu} f(\zeta), \mathcal{I}^{n-1}_{p,\mu} f(\zeta), \mathcal{I}^{n-2}_{p,\mu} f(\zeta); \zeta\Big) \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

is univalent in $\mathbb{D}$, then

$$h_1(\zeta) \prec \nu\Big(\mathcal{I}^n_{p,\mu} f(\zeta), \mathcal{I}^{n-1}_{p,\mu} f(\zeta), \mathcal{I}^{n-2}_{p,\mu} f(\zeta); \zeta\Big) \prec h_2(\zeta) \quad (\zeta \in \mathbb{D}),$$

implies

$$\kappa_1(\zeta) \prec \mathcal{I}^n_{p,\mu} f(\zeta) \prec \kappa_2(\zeta) \quad (\zeta \in \mathbb{D}).$$

Definition 12. *Let ϑ be a set in $\mathbb{C}$ and $\kappa(\zeta) \in \mathcal{H}_0$ with $\zeta\kappa'(\zeta) \neq 0$. The class of admissible functions $\varphi'_{j,1}[\vartheta, \kappa]$ consists of those functions $\nu : \mathbb{C}^3 \times \bar{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\nu(u, v, w; \eta) \in \vartheta \tag{39}$$

whenever

$$u = \kappa(\zeta), \quad v = \frac{\zeta\mu\kappa'(\zeta) + q(p-\mu)\kappa(\zeta)}{qp},$$

$$\Re\left\{\frac{wp^2 + 2vp(\mu - p) + (p-\mu)^2 u}{\mu(vp - (p-\mu)u)}\right\} \leq \frac{1}{q}\Re\left\{1 + \frac{\zeta\kappa'(\zeta)}{\kappa'(\zeta)}\right\},$$

where $\zeta \in \mathbb{D}, \eta \in \partial\mathbb{D}$, and $q \geq 1$.

We shall now present the differential superordination dual conclusion of Theorem 5.

Theorem 12. *Let $\nu \in \varphi'_{j,1}[\vartheta, \kappa]$. If $f(\zeta) \in \mathcal{H}_p$, $\dfrac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}} \in \mathcal{F}_0$, and*

$$\nu\left(\frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\zeta^{p-1}}; \zeta\right)$$

is univalent in $\mathbb{D}$, then

$$\vartheta \subset \left\{\nu\left(\frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\zeta^{p-1}}; \zeta\right) : \zeta \in \mathbb{D}\right\} \quad (p, n \in \mathbb{N}; \mu > 0), \tag{40}$$

implies

$$\kappa(\zeta) \prec \frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}} \quad (\zeta \in \mathbb{D}).$$

Proof. From (24) and (40), we have

$$\vartheta \subset \left\{\gamma\Big(g(\zeta), zg'(\zeta), \zeta^2 g''(\zeta); \zeta\Big) : \zeta \in \mathbb{D}\right\} \quad (p, n \in \mathbb{N}; \mu > 0).$$

According to Equation (22), the requirement for $\nu \in \varphi'_{j,1}[\vartheta, \kappa]$ is the same as the requirement for γ as stated in Definition 4. Therefore, the value of $\gamma \in \Upsilon'[\vartheta, \kappa]$ is determined by Lemma 2.

$$\kappa(\zeta) \prec g(\zeta) \;\; \text{or} \;\; \kappa(\zeta) \prec \frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}} \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0).$$

□

If the domain $\vartheta \neq \mathbb{C}$ is simply linked, then $\vartheta = h(\mathbb{D})$ for some conformal mapping $h(\zeta)$ of $\mathbb{D}$ onto ϑ. The class $\varphi'_{j,1}[h(\mathbb{D}), \kappa]$ can be represented as $\varphi'_{j,1}[h.\kappa]$.

Continuing as in the preceding section, Theorem 12 immediately leads to the following result.

Theorem 13. *Let $\kappa(\zeta) \in \mathcal{H}_0$, $h(\zeta)$ be analytic on $\mathbb{D}$, and $\nu \in \varphi'_{j,1}[h, \kappa]$. If $f(\zeta) \in \mathcal{H}_p$, $\dfrac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}} \in \mathcal{F}_0$, and*

$$\nu\left(\frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\zeta^{p-1}}; \zeta\right),$$

is univalent in $\mathbb{D}$, then

$$h(\zeta) \prec \nu\left(\frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\zeta^{p-1}}; \zeta\right) \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0), \tag{41}$$

implies

$$\kappa(\zeta) \prec \frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}} \quad (\zeta \in \mathbb{D}).$$

The sandwich-type theorem is derived by combining Theorems 6 and 13.

Corollary 11. *Let $h_1(\zeta)$ and $\kappa_1(\zeta)$ be analytic functions in $\mathbb{D}$, $h_2(\zeta)$ be a univalent function in $\mathbb{D}$, $\kappa_2(\zeta) \in \mathcal{F}_0$ with $\kappa_1(0) = \kappa_2(0) = 0$, and $\nu \in \varphi_{j,1}[h_2, \kappa_2] \cap \varphi'_{j,1}[h_1, \kappa_1]$. If $f(\zeta) \in \mathcal{H}_p$, $\dfrac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}} \in \mathcal{H}_0 \cap \mathcal{F}_0$, and*

$$\nu\left(\frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\zeta^{p-1}}; \zeta\right),$$

is univalent in $\mathbb{D}$, then

$$h_1(\zeta) \prec \nu\left(\frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-1}_{p,\mu} f(\zeta)}{\zeta^{p-1}}, \frac{\mathcal{I}^{n-2}_{p,\mu} f(\zeta)}{\zeta^{p-1}}; \zeta\right) \prec h_2(\zeta) \quad (p, n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0),$$

implies

$$\kappa_1(\zeta) \prec \frac{\mathcal{I}^n_{p,\mu} f(\zeta)}{\zeta^{p-1}} \prec \kappa_2(\zeta) \quad (\zeta \in \mathbb{D}).$$

Definition 13. *Let ϑ be a set in $\mathbb{C}$, $\kappa(\zeta) \neq 0, \zeta\kappa'(\zeta) \neq 0$, and $\kappa(\zeta) \in \mathcal{H}$. The class of admissible functions $\nu \in \varphi'_{j,2}[\vartheta, \kappa]$ consists of those functions $\nu : \mathbb{C}^3 \times \bar{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\nu(u, v, w; \eta) \in \vartheta$$

whenever

$$u = \kappa(\zeta), \;\; v = \frac{1}{p}\left\{\frac{\mu\zeta\kappa'(\zeta)}{\kappa(\zeta)} + p\kappa(\zeta)\right\},$$

$$\Re\left\{\frac{(wp - vp)(vp - pu + p)}{\mu(vp - pu)} + \frac{vp - 2pu - \mu}{\mu r} + 1\right\} \leq \frac{1}{q}\Re\left\{1 + \frac{\zeta\kappa''(\zeta)}{\kappa'(\zeta)}\right\},$$

where $\zeta \in \mathbb{D}$, $\eta \in \partial\mathbb{D}$, $p \in \mathbb{N}$, and $q \geq 1$.

We shall now present the differential superordination dual conclusion of Theorem 7.

Theorem 14. *Let $\nu \in \varphi'_{j,2}[\vartheta,\kappa]$. If $f(\zeta) \in \mathcal{H}_p$, $\dfrac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)} \in \mathcal{F}_1$, and*

$$\nu\left(\frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-3}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)};\zeta\right)$$

is univalent in $\mathbb{D}$, then

$$\vartheta \subset \left\{\nu\left(\frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-3}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)};\zeta\right) : \zeta \in \mathbb{D}\right\} \quad (p,n \in \mathbb{N}; \mu > 0), \tag{42}$$

implies

$$\kappa(\zeta) \prec \frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)} \quad (\zeta \in \mathbb{D}).$$

Proof. From (34) and (42), we have

$$\vartheta \subset \left\{\gamma\Big(g(\zeta), zg'(\zeta), \zeta^2 g''(\zeta);\zeta\Big) : \zeta \in \mathbb{D}\right\}.$$

According to (32), the admissibility condition for $\nu \in \varphi'_{j,2}[\vartheta,\kappa]$ is the same as the admissibility condition for γ in Definition 4. Hence, $\gamma \in \Upsilon'[\vartheta,\kappa]$, and by Lemma 2.

$$\kappa(\zeta) \prec g(\zeta) \ \text{ or } \ \kappa(\zeta) \prec \frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)} \quad (\zeta \in \mathbb{D}).$$

□

If the domain $\vartheta \neq \mathbb{C}$ is simply linked, then $\vartheta = h(\mathbb{D})$ for some conformal mapping $h(\zeta)$ of $\mathbb{D}$ onto ϑ. The class $\varphi'_{j,2}[h(\mathbb{D}),\kappa]$ can be represented as $\varphi'_{j,2}[h.\kappa]$.

Continuing as in the preceding section, Theorem 14 immediately leads to the following result.

Theorem 15. *Let $\kappa(\zeta) \in \mathcal{H}$, $h(\zeta)$ be analytic in $\mathbb{D}$ and $\nu \in \varphi'_{j,2}[h.\kappa]$. If $f(\zeta) \in \mathcal{H}_p$, $\dfrac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)} \in \mathcal{F}_1$, and*

$$\nu\left(\frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-3}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)};\zeta\right)$$

is univalent in $\mathbb{D}$, then

$$h(\zeta) \prec \nu\left(\frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}, \frac{\mathcal{I}^{n-3}_{p,\mu}f(\zeta)}{\mathcal{I}^{n-2}_{p,\mu}f(\zeta)};\zeta\right) \quad (p,n \in \mathbb{N}; \zeta \in \mathbb{D}; \mu > 0), \tag{43}$$

implies

$$\kappa(\zeta) \prec \frac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)} \quad (\zeta \in \mathbb{D}).$$

The sandwich-type theorem is derived by combining Theorems 8 and 15.

Corollary 12. *Let $h_1(\zeta)$ and $\kappa_1(\zeta)$ be analytic functions in $\mathbb{D}$, $h_2(\zeta)$ be a univalent function in $\mathbb{D}$, $\kappa_2(\zeta) \in \mathcal{F}_1$, with $\kappa_1(0) = \kappa_2(0) = 1$ and $\nu \in \varphi_{j,2}[h_2,\kappa_2] \cap \varphi'_{j,2}[h_1,\kappa_1]$. If $f(\zeta) \in \mathcal{H}_p$, $\dfrac{\mathcal{I}^{n-1}_{p,\mu}f(\zeta)}{\mathcal{I}^{n}_{p,\mu}f(\zeta)} \in \mathcal{H} \cap \mathcal{F}_1$, and*

$$\nu\left(\frac{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}{\mathcal{I}_{p,\mu}^{n}f(\zeta)},\frac{\mathcal{I}_{p,\mu}^{n-2}f(\zeta)}{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)},\frac{\mathcal{I}_{p,\mu}^{n-3}f(\zeta)}{\mathcal{I}_{p,\mu}^{n-2}f(\zeta)};\zeta\right)$$

is univalent in $\mathbb{D}$, then

$$h_1(\zeta)\prec\nu\left(\frac{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}{\mathcal{I}_{p,\mu}^{n}f(\zeta)},\frac{\mathcal{I}_{p,\mu}^{n-2}f(\zeta)}{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)},\frac{\mathcal{I}_{p,\mu}^{n-3}f(\zeta)}{\mathcal{I}_{p,\mu}^{n-2}f(\zeta)};\zeta\right)\prec h_2(\zeta)\quad(p,n\in\mathbb{N};\zeta\in\mathbb{D};\mu>0),$$

implies

$$\kappa_1(\zeta)\prec\frac{\mathcal{I}_{p,\mu}^{n-1}f(\zeta)}{\mathcal{I}_{p,\mu}^{n}f(\zeta)}\prec\kappa_2(\zeta)\quad(\zeta\in\mathbb{D}).$$

Remark 1. *Putting $\mu=1$ in the above results, we obtain the corresponding results for the p-valent Salagen integral operator $\mathcal{I}_p^n$ in [45].*

In this paper, we used the same technique as in [40].

4. Conclusions

In this study, we aimed to present original findings about an integral operator $I_{p,\lambda}^n f(\zeta)$ for a certain category of analytic functions on the open unit disk $\mathbb{U}=\{\zeta\in\mathbb{C}:|\zeta|<1\}$. Our approach involved the utilization of differential subordination and superordination. The derivation of the theorems and corollaries involved an analysis of relevant lemmas pertaining to differential subordination. The paper revealed unique findings on differential subordination and superordination through the utilization of sandwich theorems. Furthermore, the study identified a multitude of specific situations. The symmetry between the properties and outcomes of differential subordination and differential superordination gives rise to the sandwich theorems. The results presented in this current publication provide novel recommendations for further investigation, and we have created opportunities for researchers to extrapolate the findings to establish novel outcomes in geometric function theory and its applications.

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