symmetry

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Relativistic Formulation in Dual Minkowski Spacetime

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Abstract: The objective of this work is to derive the structure of Minkowski spacetime using a Hermitian spin basis. This Hermitian spin basis is analogous to the Pauli spin basis. The derived Minkowski metric is then employed to obtain the corresponding Lorentz factors, potential Lie algebra, effects on gamma matrices and complex representations of relativistic time dilation and length contraction. The main results, a discussion of the potential applications and future research directions are provided.

Keywords: Hermitian spin basis; Minkowski spacetime; Lorentz factor; Lie algebra; complex representations; metamaterials

Citation: Ganesan, T. Relativistic Formulation in Dual Minkowski Spacetime. *Symmetry* **2024**, *16*, 482. https://doi.org/10.3390/sym16040482

Academic Editors: Michele Arzano, Valeriy Sbitnev and Markus Scholle

Received: 8 March 2024
Revised: 8 April 2024
Accepted: 12 April 2024
Published: 16 April 2024

1. Introduction

The primary objective of this work is to use a Hermitian spin basis to derive the structure of Minkowski spacetime. This Hermitian spin basis is analogous to the Pauli spin basis. Using this metric, the corresponding Lorentz factors, the behaviour of gamma matrices, the structure of the Lie algebra and the complex representations of time dilations and length contractions are obtained.

Since the advent of the Theory of Relativity, the geometry and structure of spacetime have been studied extensively. Spacetime manifolds have been utilized to perform such explorations in relativity [1]. An exploration into the mathematical structure of Minkowski spacetimes is seen in the work of Lizzi et al. (2020) [2]. In that work, the authors studied the development of physical models on the κ-Minkowski noncommutative spacetime in curved momentum space. A unified approach was proposed in Lizzi et al. (2020) [2] to describe all momentum spaces and identify the associated quantum group and spacetime symmetries. In Cocco and Babic (2021) [3], an axiomatic approach was taken towards understanding Minkowski spacetimes. Targeting applications in the formalization of physical theories in Minkowski spacetime, the authors of Cocco and Babic (2021) [3] developed an elementary system of axioms for the geometry of Minkowski spacetime. In Foo et al. (2023) [4], the authors studied the quantum-gravitational effects generated by superpositions of periodically identified Minkowski spacetime. The authors of that work demonstrated that it is possible in principle to measure the field-theoretic effects generated by the coupling of a relativistic quantum matter to fields on such a spacetime background.

In Vilasini and Colbeck (2022) [5], the causal structure of Minkowski spacetime was investigated. In that research, the authors proposed a framework for identifying conditions when a causal model could coexist with relativistic principles. Examples of such relativistic principles include the following: (i) no superluminal signaling while allowing for cyclic and nonclassical causal influences and (ii) the possibility of causation without signaling. The authors of Vilasini and Colbeck (2022) [5] also demonstrated the mathematical possibility of embedding causal loops in Minkowski spacetime without leading to superluminal signaling. Another interesting work on the structure of Minkowski spacetime is seen in Liu and Majid (2022) [6]. In that research, the authors applied a quantum geodesic formulation to quantum Minkowski spacetime as a method to model quantum gravitational effects. That research work discovered that within quantum perturbation, because of quantum gravitational effects, a point particle cannot be modeled as an infinitely sharp Gaussian. In Gasperin (2024) [7], the research was focused on the investigation of the asymptotic behaviour of massless spin-0 fields close to spatial and null infinity in Minkowski spacetime. In that work, Friedrich's cylinder at spatial infinity was a key framework utilized in that investigation. The work of Meljanac et al. (2022) [8] investigated the symmetric ordering and Weyl realizations for non-commutative Minkowski spaces. In that research, examples of Weyl realizations were provided, and the authors also showed that for an original Snyder space, there exists a symmetric ordering with no Weyl realization.

Besides studies involving Minkowski spacetimes in a quantum context, recent research works have also been carried out on the cosmological aspects of Minkowski spacetimes. For instance, in Lombriser (2023) [9], the authors explored the formulation of cosmological Minkowski spacetimes using metric transformations uncovering the underlying mass, length and time scales across spacetimes—i.e., the evolution of fundamental constants. In this spirit, the authors reinspected the cosmological constants and potential candidates to explain the geometric origin of baryogenesis, dark matter/energy and inflation. In Volovik (2023) [10], the Minkowski metric was modified using two Planck constants in the context of the composite fermion approach to quantum gravity.

Researchers have also taken a more mathematical approach to exploring the depths of the Minkowski metric. For instance, in Chappell et al. (2023) [11], the authors presented a new derivation to the Minkowski metric. They demonstrated that the Minkowski metric could be obtained as an emergent property from physical space modeled as a Clifford algebra, $Cl(\mathbb{R}^3)$. They also showed that the formulation predicts a range of new physical effects such as superluminal light propagation. Another recent research work in this direction is seen in Li et al. (2023) [12]. In that work, the authors mathematically explored the structure of Minkowski space with respect to spacelike circular surfaces (one-parameter family of Lorentzian circles with a fixed radius regarding a non-null curve). In that work, spacelike circular surfaces were parameterized, and their geometric and singularity properties were classified. In Stone (2022) [13], the author analyzed the connection between Minkowski and Euclidean signatures with respect to Majorana fermions and gamma matrices as well as discreet/continuous symmetries in all spacetime dimensions. Minkowski spacetimes have also been explored in the context of the Yang–Mills equations. This can be seen in the work of Kumar et al. (2022) [14], where the authors obtained a two-parameter family of solutions for the Yang–Mills equations on Minkowski space (for the gauge group, $SO(1,3)$). The authors generated this result from the foliation of different parts of the Minkowski space with a non-compact coset space with $SO(1,3)$ isometry.

Metamaterials have been a focus of much research because of the unique way they interact with electromagnetic and acoustic waves—e.g., the design of materials with a negative index of refraction [15,16]. The Minkowski spacetime metric is a key component in the study of metamaterials [17,18]. For instance, in Iemma and Palma (2020) [18], the Minkowski metric was used as a baseline to perform metric corrections via spacetime transformations. These transformations were aimed at manipulating the mechanical properties of acoustic metacontinua. Similar works on the utilization of spacetime metrics for metamaterial models are seen in Caloz et al. (2019) [19], Bahrami et al. (2023) [20] and Caloz et al. (2022) [21].

The analogue Hermitian spin basis (given in Ganesan (2023) [22]) was employed in this work to derive the structure of Minkowski spacetime. The proposed dual Minkowski metrics and their associated relativistic formulation are presented in this work. This paper is organized as follows: The second section describes the construction of the dual Minkowski metric. In the third section, the derivation of the dual Lorentz factors using the dual Minkowski metric is carried out. In the fourth and fifth sections, the associated gamma matrices and the structure of the Lie algebra from the dual Minkowski metric are explored. The sixth section describes the complex forms of the length contraction, time dilation and Minkowski metric obtained using the dual Lorentz factors. This paper ends with the key findings, a discussion of the potential application of the proposed formulation on the spacetime metric for metamaterial modelling and design. Some ideas on future research directions are also included at the end of this paper.

2. Analogue Spin Basis

A set of novel Hermitian spin matrices along with its symmetry relation was introduced in Ganesan (2023) [22]. These spin matrices are as follows:

$$\sigma'_1 = \begin{bmatrix} 0 & \frac{1}{n}(1+i) \\ \frac{1}{n}(1-i) & 0 \end{bmatrix}, \ \sigma'_2 = \begin{bmatrix} 0 & \frac{1}{n}(1-i) \\ \frac{1}{n}(1+i) & 0 \end{bmatrix}, \ \sigma'_3 = \begin{bmatrix} \frac{1}{\sqrt{n}} & 0 \\ 0 & -\frac{1}{\sqrt{n}} \end{bmatrix}. \tag{1}$$

where $\sigma'_1 = \overline{\sigma'_2}$, and the spin matrices given in Equation (1) contain the following symmetry:

$$\frac{n^2}{2}\left(\sigma'_1\right)^2 = \frac{n^2}{2}(\sigma\prime_2)^2 = n(\sigma\prime_3)^2 = -\frac{i}{2}\sqrt{n}^{\,5}\sigma'_1\,\sigma'_2\sigma'_3 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2}$$

The integer parameter n is fixed to 2. It is interesting to note that if $n \neq 2$ (excluding 0), directional symmetry is loss—e.g., since the weighting between the spin matrices for each direction would be equivalent only if $n = 2$, then $2(\sigma'_1)^2 = 2(\sigma\prime_2)^2 = 2(\sigma\prime_3)^2 = I_2$. This property is key in a physical sense to ensure that the spin operators maintain directional symmetry—similar to the Pauli matrices. As stated in Ganesan (2023) [22], the analogue Hermitian spin matrices could be constructed from the Pauli spin matrices as follows:

$$\sigma'_1 = \frac{1}{n}(\sigma_1 - \sigma_2), \ \sigma\prime_2 = \frac{1}{n}(\sigma_1 + \sigma_2), \ \sigma'_3 = \frac{\sigma_3}{\sqrt{n}} \tag{3}$$

where the Pauli spin matrices as quantum operators corresponding to observables for the fermionic spin at each spatial direction are $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Similar to the Pauli group, the analogue spin matrices could also be defined within a 16-element cyclic group structure:

$$G(n) = \{\pm I_2, \ \pm iI_2, \ \pm\sigma'_1, \ \pm\sigma'_2, \ \pm\sigma'_3, \ \pm i\sigma'_1, \ \pm i\sigma'_2, \ \pm i\sigma'_3\}. \tag{4}$$

The following commutation relations hold for the analogue spin matrices:

$$[\sigma'_1, \ \sigma'_2] = \frac{4i}{\sqrt{n^3}}\sigma\prime_3, \ [\sigma'_1, \ \sigma'_3] = -\frac{2i}{\sqrt{n}}\sigma'_2, \ [\sigma'_2, \ \sigma'_3] = \frac{2i}{\sqrt{n}}\sigma'_1. \tag{5}$$

where $[\sigma'_i\,, \ \sigma'_j] = -[\sigma'_j\,, \ \sigma'_i]$ and $[\sigma'_j\,, \ \sigma'_j] = O$. For fermionic systems, $n = 2$, the commutation relations reduce to:

$$[\sigma'_1\,, \ \sigma'_2] = \sqrt{2}i\sigma'_3\,, \ [\sigma'_1\,, \ \sigma'_3] = -\sqrt{2}\,i\sigma'_2\,, \ [\sigma'_2\,, \ \sigma'_3] = \sqrt{2}\,i\sigma'_1. \tag{6}$$

The generalized commutation relation for the analogue spin matrices is then:

$$[\sigma'_i\,, \ \sigma'_j] = \sqrt{2}i\ \varepsilon_{ijk}\sigma'_k \tag{7}$$

Unlike Pauli matrices, the analogue spin matrices given in Equation (1) are not involutary. Therefore, the inverse form of the analogue spin matrices for $n = 2$ is represented as follows:

$$\begin{gathered}(\sigma'_i)^{-1} = \Sigma'_i, \\ \Sigma'_1 = \begin{bmatrix} 0 & (1-i) \\ (1+i) & 0 \end{bmatrix}, \ \Sigma'_2 = \begin{bmatrix} 0 & (1+i) \\ (1-i) & 0 \end{bmatrix}, \ \Sigma'_3 = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}.\end{gathered} \tag{8}$$

where $\Sigma'_1 = \overline{\Sigma'_2}$ and the spin matrices given in Equation (8) contain the following symmetry:

$$\frac{1}{2}\left(\Sigma'_1\right)^2 = \frac{1}{2}\left(\Sigma'_2\right)^2 = \frac{1}{2}\left(\Sigma'_3\right)^2 = \frac{i}{2\sqrt{2}}\,\Sigma'_1\Sigma'_2\Sigma'_3 = I_2. \tag{9}$$

The inverse matrices are also Hermitian, where $\left(\overline{\Sigma'_i}\right)^T = \Sigma'_i$. Similar to its counterpart (non-inverted analogue spin matrices), the symmetry property of each spin matrix is symmetrically weighted in all directions by the factor $\frac{1}{2}$. The matrix form of the dual Minkowski metrics X and Y are then constructed using the Weyl representation [23]. This is performed by defining the action of the special linear group, $SL(2,\mathbb{C})$, on the Minkowski metric. A point of spacetime, X, is then represented as a two-dimensional Hermitian matrix:

$$X = ctI_2 + x\sigma'_1 + y\sigma'_2 + z\sigma'_3 = \begin{bmatrix} ct + \frac{z}{\sqrt{2}} & \left(\frac{1+i}{2}\right)x + \left(\frac{1-i}{2}\right)y \\ \left(\frac{1-i}{2}\right)x + \left(\frac{1+i}{2}\right)y & ct - \frac{z}{\sqrt{2}} \end{bmatrix}. \tag{10}$$

where c is the speed of light and x, y and z are spatial coordinates. In the conventional Weyl representation, the form of the Minkowski spacetime is constructed using the Pauli matrices, σ_i: $\det(X') = c^2t^2 - x^2 - y^2 - z^2$; where $X' = ctI_2 + x\sigma_1 + y\sigma_2 + z\sigma_3$. On the other hand, using the analogue spin matrices shown in Equation (1), the analogue Minkowski spacetime is obtained:

$$\det\,(X) = c^2t^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 \tag{11}$$

A similar construction of the Minkowski metric, Y, could be carried out using the inverse analogue spin matrices, yielding the following results:

$$\begin{gathered} Y = ctI_2 + x\Sigma'_1 + y\Sigma'_2 + z\Sigma'_3 = \begin{bmatrix} ct + \sqrt{2}z & (1-i)x + (1+i)y \\ (1+i)x + (1-i)y & ct - \sqrt{2}z \end{bmatrix}, \\ \det\,(Y) = c^2t^2 - 2x^2 - 2y^2 - 2z^2. \end{gathered} \tag{12}$$

The investigation of the implications of the dual Minkowski metrics X and Y on the Lorentz factor in relativistic dynamics is given in the next section.

3. Dual Lorentz Factors

The matrix form of the dual Minkowski metrics X and Y are then defined using Equations (10) and (12):

$$\eta_X = \frac{1}{2}\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \eta_Y = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{13}$$

It can be shown that both η_X and η_Y are Hermitian matrices, similar to the conventional Minkowski metric. Transforming the spacetime coordinates $O(x,y,z)$ (column vector form) to the reference frame ${O_X}'$ or ${O_Y}'$ results in:

$$\begin{gathered} {O_X}' = \eta_X O = \tfrac{1}{2}\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} -2ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} \\ {O_Y}' = \eta_Y O = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -ct \\ 2x \\ 2y \\ 2z \end{bmatrix} = \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}. \end{gathered} \tag{14}$$

Using the dual Minkowski metrics, the Galilean transformation between the inertial reference frame O and $O_X{}'$ or $O_Y{}'$ is defined as follows:

$$\begin{array}{c} O \to O_X{}' : x' = a_1 \frac{x}{2} + a_2 t \,; y' = y;\; z' = z; t' = b_1 \frac{x}{2} + b_2 t. \\ O \to O_Y{}' : x' = 2a_1 x + a_2 t \,; y' = y;\; z' = z; t' = 2b_1 x + b_2 t. \end{array} \tag{15}$$

where a_1, a_2, b_1 and b_2 are the transformation parameters. For $O \to O_X{}'$, it can be readily seen that $x = \frac{-2a_2 t}{a_1} = vt$ if $x' = 0$. The relation x' is then factorized to obtain $x' = \frac{a_1}{2}\left(x + \frac{2a_2}{a_1} t\right) = \frac{a_1}{2}(x - vt)$ since $v = \frac{-2a_2}{a_1}$. The equivalence between the two reference frames is:

$$(x')^2 + (y')^2 + (z')^2 - c^2 (t')^2 = x^2 + y^2 + z^2 - c^2 t^2. \tag{16}$$

Since $y' = y$, $z' = z$ and $x' = \frac{a_1}{2}(x - vt)$, equivalence simplifies to:

$$(x')^2 - c^2 (t')^2 = x^2 - c^2 t^2 \text{ or} \left(\frac{a_1(x - vt)}{2}\right)^2 - c^2 \left(b_1 \frac{x}{2} + b_2 t\right)^2 = x^2 - c^2 t^2. \tag{17}$$

From Equation (17), the derivation to obtain the dual Lorentz factor, γ_X, is included in Appendix A:

$$\gamma_X = \frac{2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{18}$$

For $O \to O_Y{}'$, $x = \frac{-a_2 t}{2a_1} = vt$ if $x' = 0$. The relation x' is then factorized to obtain $x' = 2a_1\left(x + \frac{a_2}{2a_1} t\right) = 2a_1(x - vt)$ since $v = -\frac{a_2}{2a_1}$. The equivalence between the two reference frames is:

$$(x')^2 + (y')^2 + (z')^2 - c^2 (t')^2 = x^2 + y^2 + z^2 - c^2 t^2. \tag{19}$$

Since $y' = y$, $z' = z$ and $x' = 2a_1(x - vt)$, Equation (19) becomes:

$$\begin{array}{c} (x')^2 - c^2 (t')^2 = x^2 - c^2 t^2 \text{ or} \\ (2a_1(x - vt))^2 - c^2 (2b_1 x + b_2 t)^2 = x^2 - c^2 t^2. \end{array} \tag{20}$$

The derivation to obtain the other dual Lorentz factor, γ_Y, using the metric η_Y is given in Appendix B:

$$\gamma_Y = \frac{1}{2\sqrt{1 - \frac{v^2}{c^2}}} \tag{21}$$

The standard Lorentz factor, γ, could then be represented in terms of the dual Lorentz factors:

$$\gamma_i = k_i \gamma \text{ for } i = X, Y \text{ where the factors: } k_X = 2 \text{ and } k_Y = \frac{1}{2}.\gamma = \sqrt{\gamma_X \gamma_Y}. \tag{22}$$

The next section provides some details on the gamma matrices as a result of the obtained dual Minkowski metrics.

4. Representation of Gamma Matrices

Gamma matrices play a central role in relativistic quantum mechanics and mathematical physics. Gamma matrices within the Dirac basis are defined as follows:

$$\gamma_j^X = \begin{bmatrix} 0 & \sigma'_j \\ -\sigma'_j & 0 \end{bmatrix} \text{ and } \gamma_0^X = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \text{ for } j \in [1, 3]. \tag{23}$$

where σ'_j is the analogue Pauli spin matrices given in Equation (1). The anticommutation property $\left\{\gamma_i^X, \gamma_j^X\right\} = \left\{\gamma_j^X, \gamma_i^X\right\}$ holds. The fifth gamma matrix is given as:

$$\gamma_5^X = i\gamma_0^X\gamma_1^X\gamma_2^X\gamma_3^X = -\frac{1}{2\sqrt{2}}\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{24}$$

Similarly, for the inverse analogue spin matrices, Σ'_j:

$$\gamma_j^Y = \begin{bmatrix} 0 & \Sigma'_j \\ -\Sigma'_j & 0 \end{bmatrix} \text{ and } \gamma_0^Y = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix} \text{ for } j \in [1,3]. \tag{25}$$

The anticommutation property $\left\{\gamma_i^Y, \gamma_j^Y\right\} = \left\{\gamma_j^Y, \gamma_i^Y\right\}$ holds. The fifth gamma matrix is given as:

$$\gamma_5^Y = -i\gamma_0^Y\gamma_1^Y\gamma_2^Y\gamma_3^Y = -2\sqrt{2}\begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}. \tag{26}$$

It follows that $\gamma_5^X\gamma_5^Y = \begin{bmatrix} I_2 & 0 \\ 0 & I_2 \end{bmatrix}$. Based on the algebra of the gamma matrices defined using the analogue spin matrices, the dual Minkowski metrics are obtained using the anticommutator in the following theorems:

Theorem 1. $\left\{\gamma_i^X, \gamma_j^X\right\} = \gamma_i^X\gamma_j^X + \gamma_j^X\gamma_i^X = -2(\eta_X)_{ij}I_4$ *for* $i, j \in [0,3]$.

Proof of Theorem 1. For $(i,j) = (1,3),\ (2,3)$ and $(1,2) \implies \left\{\gamma_i^X, \gamma_j^X\right\} = 0$, and if $i = j \implies \left\{\gamma_i^X, \gamma_j^X\right\} = -I_4$. In the case where $i = 0$ or $j = 0 \implies \left\{\gamma_i^X, \gamma_j^X\right\} = 0$.

The first element: $(i,j) = (0,0) \implies \left\{\gamma_i^X, \gamma_j^X\right\} = 2I_4$.

$$\left\{\gamma_i^X, \gamma_j^X\right\} = -2(\eta_X)_{ij}I_4$$

□

Theorem 2. $\left\{\gamma_i^Y, \gamma_j^Y\right\} = \gamma_i^Y\gamma_j^Y + \gamma_j^Y\gamma_i^Y = -2(\eta_Y)_{ij}I_4$ *for* $i, j \in [0,3]$.

Proof of Theorem 2. For $(i,j) = (1,3),\ (2,3)$ and $(1,2) \implies \left\{\gamma_i^X, \gamma_j^X\right\} = 0$, and if $i = j \implies \left\{\gamma_i^Y, \gamma_j^Y\right\} = -4I_4$. In the case where $i = 0$ or $j = 0 \implies \left\{\gamma_i^Y, \gamma_j^Y\right\} = 0$.

The first element: $(i,j) = (0,0) \implies \left\{\gamma_i^Y, \gamma_j^Y\right\} = 2I_4$.

$$\left\{\gamma_i^Y, \gamma_j^Y\right\} = -2(\eta_Y)_{ij}I_4\,.$$

□

Theorem 3. $\left\{\gamma_i^X, \gamma_j^Y\right\} = \left\{\gamma_i^Y, \gamma_j^X\right\} = \gamma_i^X\gamma_j^Y + \gamma_j^Y\gamma_i^X = 2I_4$ *for* $i, j \in [0,3]$.

Proof of Theorem 3. For $(i,j) = (1,3),\ (2,3)$ and $(1,2) \implies \left\{\gamma_i^X, \gamma_j^Y\right\} = 0$, and if $i = j \implies \left\{\gamma_i^X, \gamma_j^Y\right\} = \left\{\gamma_i^Y, \gamma_j^X\right\} = 2I_4$. In the case where $i = 0$ or $j = 0 \implies \left\{\gamma_i^X, \gamma_j^Y\right\} = \left\{\gamma_i^Y, \gamma_j^X\right\} = 0$.

The first element: $(i,j) = (0,0) \Longrightarrow \left\{\gamma_i^X, \gamma_j^Y\right\} = \left\{\gamma_i^Y, \gamma_j^X\right\} = 2I_4$.

$$\left\{\gamma_i^X, \gamma_j^Y\right\} = \left\{\gamma_i^Y, \gamma_j^X\right\} = 2I_4 .$$

□

The following identities for the anticommutators are then obtained:

$$\begin{gathered} \eta_X \eta_Y = I_4, \\ \left\{\gamma_i^X, \gamma_j^X\right\}\left\{\gamma_i^Y, \gamma_j^Y\right\} = 4(\eta_X)_{ij}(\eta_Y)_{ij} I_4 = 4I_4 \text{ for } i,j \in [0,3]. \end{gathered} \tag{27}$$

5. Lie Algebra Structure

The dual Minkowski metrics in this work are not subjected to the conventional boost and rotation generators of the Lorentz group. However, it is possible to develop generators using the basis for the Lie algebra for the dual Minkowski metrics, η_X and η_Y. In this section, two example generators for the associated Lie algebra are provided. One example in the four-vector representation is as follows:

$$J_1 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -1 & 0 \end{bmatrix}, J_2 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, J_3 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{28}$$

The metric η_i is Hermitian where $\eta_j = \left(\overline{\eta_j}\right)^T$ for $j = X, Y$. There exists a transformation between the metrics η_X and η_Y using Lie generators since $\eta_X \eta_Y = I_4$:

$$\eta_X J_i = \frac{1}{2} J_i \text{ or } J_i = \frac{1}{2} J_i \eta_Y , \eta_X J_i \eta_X = \frac{J_i}{4} \text{ or } J_i = \eta_Y \frac{J_i}{4} \eta_Y \text{ for } i = 1,2,3. \tag{29}$$

Another example for the four-vector representation the generators for the Lie algebra is as follows:

$$K_1 = i\begin{bmatrix} 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, K_2 = i\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, K_3 = i\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}. \tag{30}$$

The remarks to prove the vanishing Jacobi identity using the commutator properties of J and K to establish it as a Lie algebra is given in Appendix C.

As with the previous example for J, the transformation between the metrics η_X and η_Y using the Lie generators for K are:

$$\begin{gathered} \eta_X K_i \eta_X = -\tfrac{K_i}{2} \text{ or } K_i = -\eta_Y \tfrac{K_i}{2} \eta_Y \text{ for } i = 1,2 \text{ and} \\ \eta_X K_3 \eta_X = \tfrac{K_3}{4} \text{ or } \eta_Y K_3 \eta_Y = \tfrac{K_3}{4}. \end{gathered} \tag{31}$$

The transformations are two examples of a set of potential transformations of the dual Minkowski metrics η_X and η_Y.

6. Complex Representations of Time Dilation and Length Contraction

In contrast to the conventional Lorentz factor, the dual Lorentz factors, γ_X and γ_Y, introduce a factor in the length contraction and time dilation in relativistic dynamics. This then directly influences the physical and mathematical utilization of the dual metric framework introduced in this work. The central idea is to implement the dual Lorentz factors such that the laws of special relativity are respected. In special relativity, time dilation is represented as $\Delta t = \Delta t_0\, \gamma$, where Δt_0 is the time interval at a rest reference frame and γ is the

Lorentz factor. This relation could then be represented in terms of the dual Lorentz factors as $\Delta t = \Delta t_0\sqrt{\gamma_X\gamma_Y}$. Then, $(\Delta t)^2 = (\Delta t_0)^2\gamma_X\gamma_Y = (\gamma_X\Delta t_0)(\gamma_Y\Delta t_0)$. The following definition aims to construct a complex representation for the time dilation using the dual Lorentz factors, γ_X and γ_Y. The expressions for complex times $T = \tau + i\Delta t$ and $\overline{T} = \tau - i\Delta t$ are considered, where $T\overline{T} = \tau^2 + (\Delta t)^2$.

Definition 1. *The complex representation of the time dilation at the initial time $\tau = 0$ is:*

$$i\Delta t = \frac{1}{\sqrt{2}}\gamma_X \Delta t_0(i+1) \text{ and } -i\Delta t = \frac{1}{\sqrt{2}}\gamma_Y\Delta t_0(i-1).$$

Based on Definition 1, there exists two relations for Δt in terms of each dual Lorentz factor:

$$\Delta t = \frac{1}{\sqrt{2}}\gamma_X\Delta t_0(1-i) \text{ and } \Delta t = -\frac{1}{\sqrt{2}}\gamma_Y \Delta t_0(1+i). \tag{32}$$

Recovering the square form of the time dilation relation, the complex form of the relativistic time dilation relation is obtained:

$$\begin{gathered}(\Delta t)^2 = \left[\tfrac{1}{\sqrt{2}}\gamma_X\Delta t_0(1-i)\right]\left[-\tfrac{1}{\sqrt{2}}\gamma_Y\Delta t_0(1+i)\right] = -\gamma_X\gamma_Y(\Delta t_0)^2\,, \\ \therefore \Delta t = -\sqrt{\gamma_X\gamma_Y}\Delta t_0 = i\gamma\Delta t_0.\end{gathered} \tag{33}$$

The expressions for complex length $L = l + i\Delta L$ and $\overline{L} = l - i\Delta L$ are considered, where $L\overline{L} = l^2 + (\Delta L)^2$. The complex representation for length contraction using the dual Lorentz factors, γ_X and γ_Y, is given in Definition 2:

Definition 2. *The complex representation of the length contraction dilation at the initial length, $l = 0$, is:*

$$i\Delta L = \frac{1}{\sqrt{2}}\gamma_X^{-1} \Delta L_0(i+1) \text{ and } -i\Delta L = \frac{1}{\sqrt{2}}\gamma_Y^{-1} \Delta L_0(i-1).$$

In this form, the complex relativistic length contractions in terms of the dual Lorentz factors are as follows:

$$\begin{gathered}\Delta L = \tfrac{1}{\sqrt{2}}\gamma_X^{-1}\Delta L_0(1-i) \text{ and } \Delta L = -\tfrac{1}{\sqrt{2}}\gamma_Y^{-1}\Delta L_0(1+i), \\ (\Delta L)^2 = \left[\tfrac{1}{\sqrt{2}}\gamma_X^{-1}\Delta L_0(1-i)\right]\left[-\tfrac{1}{\sqrt{2}}\gamma_Y^{-1}\Delta L_0(1+i)\right] = -\tfrac{(\Delta L_0)^2}{\gamma_X\gamma_Y}\,, \\ \therefore \Delta L = i\tfrac{\Delta L_0}{\sqrt{\gamma_X\gamma_Y}} = i\gamma^{-1}\Delta L_0\,.\end{gathered} \tag{34}$$

In view of Definitions 1 and 2, the dual matrix representation of the length contraction and the time dilation complex transformation is represented as follows:

$$\begin{bmatrix}\Delta L^x\\ \Delta L^y\\ \Delta L^z\\ \Delta t\end{bmatrix} = \overbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&i\end{bmatrix}}^{\eta_1}\begin{bmatrix}\gamma^{-1}\Delta L_0^x\\ \gamma^{-1}\Delta L_0^y\\ \gamma^{-1}\Delta L_0^z\\ \gamma\Delta t_0\end{bmatrix} \text{ and } \begin{bmatrix}\Delta L^x\\ \Delta L^y\\ \Delta L^z\\ \Delta t\end{bmatrix} = \overbrace{\begin{bmatrix}i&0&0&0\\0&i&0&0\\0&0&i&0\\0&0&0&1\end{bmatrix}}^{\eta_2}\begin{bmatrix}\gamma^{-1}\Delta L_0^x\\ \gamma^{-1}\Delta L_0^y\\ \gamma^{-1}\Delta L_0^z\\ \gamma\Delta t_0\end{bmatrix}. \tag{35}$$

where timelike and spacelike skew-Hermitian Minkowski metrics are η_1 and η_2. The standard timelike and spacelike Minkowski metrics are then recovered by taking the square of the skew-Hermitian Minkowski metrics $\eta = \eta_1^2$ and $\eta = \overline{\eta}_2^2$, respectively.

7. Discussion

The mathematical framework presented in this work may provide useful tools to approach certain classes of problems in metamaterial modelling. The related Lie algebra, the structure of the gamma matrices and the complex representation of time dilation and length contraction provide techniques to effectively explore the relativistic quantum dynamics in such metamaterials (or other forms of exotic material structures). The dual Lorentz factors (γ_X and γ_Y) obtained using the dual Minkowski metrics introduces an additional factor of 2 or $\frac{1}{2}$ to the conventional Lorentz factor. Since the Lorentz factors are obtained from the Minkowski metric, these factors directly result from the structure of the spacetime metric. Therefore, the resulting dual Lorentz factors and their respective Minkowski spacetimes metrics may provide deeper insights into the potential structure of the spacetime metric and its corresponding dynamics in metamaterial models. In negative-index metamaterial models, the permittivity and permeability coefficients are usually complex-valued. A complex representation described in the previous section may be useful for incorporating the permittivity and permeability coefficients into the metric structure. This could provide an alternative approach for describing relativistic electrodynamics phenomena in such metamaterials.

The key concepts of the mathematical formulation proposed in this work are as follows:

i. A dual structure of the conventional Minkowski metric, η, is obtained from the analogue Hermitian spin matrices ${\sigma'}_i$ and ${\Sigma 6'}_i$, where the Minkowski metric $\eta = \eta_X \eta \eta_Y = \eta_Y \eta \eta_X$.
ii. A gamma matrix formulation and examples of Lie algebra generators are obtained using the proposed framework.
iii. Using the dual Lorentz factors (derived from the dual Minkowski metrics), a complex representation of the length contraction and time dilations is revealed:
    - Lorentz factor: $\gamma = \sqrt{\gamma_X \gamma_Y}$.
    - Complex length contraction: $\Delta L = i\gamma^{-1} \Delta L_0$.
    - Complex time dilation: $\Delta t = i\gamma \Delta t_0$.

In the future, deeper investigations into other potential Lie algebra structures of the dual Minkowski metric could be performed. Quaternions are isomorphic to the algebra of Pauli spin matrices [24]. Since the analogue spin matrices are also constructed from Pauli spin matrices, the analogue spin matrices are similarly isomorphic to quaternions. Therefore, future research works could be directed towards understanding the mathematical and physical implications of a real and complex quaternionic formulation of the Lorentz transform using the dual analogue spin matrices. This could also be extended to applications involving hyperbolic spinors [25–27].

A potential research question is if it is possible to develop an electromagnetism representation (and the corresponding spinor formulation) using the dual analogue spin matrices as well as the dual Minkowski metrics presented in this work. If such a representation is possible, then it would be interesting to investigate and ascertain its physical ramifications—e.g., in the field of metamaterial design. Another potential area of research is to study the mathematical and physical properties of the proposed formulation for anisotropic spacetimes. This can be performed by setting the integer parameter in Equation (2) to $n \neq 2$. In addition to metamaterials, future research may also be focused on broader connections to analogue gravity models as well as phenomena related to nonlinear optics [28–30].

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The author thanks the Department of Physics and Astronomy (PHAS) at the University of Calgary for their support throughout this research work.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Derivation for the Dual Lorentz Factor (γ_X)

Expanding Equation (17) results in:

$$\frac{1}{4}a_1^2\left(x^2 - 2vtx + v^2t^2\right) - \frac{1}{4}c^2b_1^2x^2 - c^2b_1b_2xt - c^2b_2^2t^2 = x^2 - c^2t^2. \tag{A1}$$

Grouping the terms will then give:

$$\left(\frac{a_1^2}{4} - \frac{c^2b_1^2}{4}\right)x^2 + \left(-\frac{1}{2}a_1^2vtx - c^2b_1b_2xt\right) + \left(\frac{1}{4}a_1^2v^2 - c^2b_2^2\right)t^2 = x^2 - c^2t^2,$$

where

$$a_1^2 - c^2b_1^2 = 4 \;\rightarrow\; b_1^2c^2 = a_1^2 - 4. \tag{A2}$$

$$\frac{1}{4}a_1^2v^2 - c^2b_2^2 = -c^2 \rightarrow\; b_2^2c^2 = \frac{1}{4}a_1^2v^2 + c^2\,. \tag{A3}$$

$$-\frac{1}{2}a_1^2vxt - c^2b_1b_2xt = 0 \;\rightarrow\; a_1^2 = \frac{-2c^2b_1b_2}{v} \text{ or } a_1^4 = \frac{4c^4b_1^2b_2^2}{v^2} \tag{A4}$$

Substituting relations (A2) and (A3) into (A4) gives the following expression for a_1:

$$\begin{gathered} a_1^4 = \tfrac{4}{v^2}\left(\tfrac{1}{4}a_1^2v^2 + c^2\right)\left(a_1^2 - 4\right) = \left(a_1^2 + \tfrac{4c^2}{v^2}\right)\left(a_1^2 - 4\right) \\ = a_1^4 - 4a_1^2 + \tfrac{4c^2}{v^2}a_1^2 - \tfrac{16c^2}{v^2}. \\ \left(1 - \tfrac{c^2}{v^2}\right)a_1^2 + \tfrac{4c^2}{v^2} = 0 \;\rightarrow\; a_1^2 = -\tfrac{4c^2}{v^2}\left(\tfrac{v^2}{v^2-c^2}\right) = \tfrac{4}{1-\frac{v^2}{c^2}}. \\ \therefore a_1 = \gamma_X = \tfrac{2}{\sqrt{1-\frac{v^2}{c^2}}}. \end{gathered} \tag{A5}$$

Considering a_1 as the analogue Lorentz factor, γ_X, the parameters a_2, b_1 and b_2 could be represented as follows:

$$a_2 = -\frac{v}{2}\gamma_0\,,\; b_1 = \frac{1}{c}\sqrt{\gamma_0^2 - 4}\,,\; b_2 = \frac{1}{c}\sqrt{\frac{1}{4}\gamma_0^2v^2 + c^2}. \tag{A6}$$

Appendix B. Derivation for the Dual Lorentz Factor (γ_Y)

The expansion of Equation (20) results in:

$$4a_1^2(x^2 - 2vxt + v^2t^2\big) - 4c^2b_1^2x^2 - 4c^2b_1b_2xt - c^2b_2^2t^2 = x^2 - c^2t^2 \tag{A7}$$

Regrouping the terms will then give:

$$\left(4a_1^2 - 4c^2b_1^2\right)x^2 + \left(-8a_1^2v - 4c^2b_1b_2\right)xt + \left(4a_1^2v^2 - c^2b_2^2\right)t^2 = x^2 - c^2t^2$$

where

$$a_1^2 - c^2b_1^2 = \frac{1}{4} \;\rightarrow\; b_1^2c^2 = a_1^2 - \frac{1}{4}. \tag{A8}$$

$$4a_1^2v^2 - c^2b_2^2 = -c^2 \rightarrow\; b_2^2c^2 = 4a_1^2v^2 + c^2\,. \tag{A9}$$

$$-8a_1^2vxt - 4c^2b_1b_2xt = 0 \;\rightarrow\; a_1^2 = \frac{-c^2b_1b_2}{2v} \text{ or } a_1^4 = \frac{c^4b_1^2b_2^2}{4v^2}. \tag{A10}$$

Substituting relations (A8) and (A9) into (A10) gives the following expression for a_1:

$$\begin{gathered} a_1^4 = \frac{1}{4v^2}\left(4a_1^2v^2 + c^2\right)\left(a_1^2 - \frac{1}{4}\right) = \left(a_1^2 + \frac{c^2}{4v^2}\right)\left(a_1^2 - \frac{1}{4}\right). \\ = a_1^4 - \frac{a_1^2}{4} + \frac{a_1^2c^2}{4v^2} - \frac{c^2}{16v^2}. \\ \left(1 - \frac{c^2}{v^2}\right)a_1^2 + \frac{c^2}{4v^2} = 0 \rightarrow a_1^2 = -\frac{c^2}{4v^2}\left(\frac{v^2}{v^2 - c^2}\right) = \frac{1}{4\left(1 - \frac{v^2}{c^2}\right)}, \\ \therefore a_1 = \gamma_Y = \frac{1}{2\sqrt{1 - \frac{v^2}{c^2}}}. \end{gathered} \tag{A11}$$

Considering a_1 as the analogue Lorentz factor, γ_Y, the parameters a_2, b_1 and b_2 could be represented as follows:

$$a_2 = -2v\gamma_Y\ ,\ b_1 = \frac{1}{c}\sqrt{\gamma_Y^2 - \frac{1}{4}}\ ,\ b_2 = \frac{1}{c}\sqrt{4\gamma_0^2v^2 + c^2}. \tag{A12}$$

Appendix C. Proofs for the Jacobi Identities

Remark A1. *Jacobi Identity:* $[J_1,\ [J_2, J_3]] + [J_2,\ [J_1, J_3]] + [J_3,\ [J_1, J_2]] = 0.$

Proof. The commutators for the basis in Equation (28) are:

$$\begin{gathered} [J_1,\ J_2] = -\tfrac{1}{2}iJ_3\ ,\ [J_2,\ J_3] = -\tfrac{1}{2}iJ_1 \text{ and } [J_1,\ J_3] = iJ_2 \\ [J_1,\ [J_2, J_3]] + [J_2,\ [J_1, J_3]] + [J_3,\ [J_1, J_2]] = \left[J_1, -\tfrac{1}{2}iJ_1\right] + [J_2, iJ_2] + \left[J_3, -\tfrac{1}{2}iJ_3\right]. \end{gathered}$$

Since each commutator: $\left[J_1, -\tfrac{1}{2}iJ_1\right] = 0$, $[J_2, iJ_2] = 0$ and $\left[J_3, -\tfrac{1}{2}iJ_3\right] = 0$.
$\therefore [J_1,\ [J_2, J_3]] + [J_2,\ [J_1, J_3]] + [J_3,\ [J_1, J_2]] = 0$. □

Remark A2. *Jacobi Identity:* $[K_1,\ [K_2, K_3]] + [K_2,\ [K_1, K_3]] + [K_3,\ [K_1, K_2]] = 0.$

Proof. The commutators for the basis in Equation (30) are:

$$\begin{gathered} [K_1,\ K_2] = -iK_3\ ,\ [K_2,\ K_3] = -\tfrac{1}{2}iK_1 \text{ and } [K_1,\ K_3] = \tfrac{1}{2}iK_2. \\ [K_1,\ [K_2, K_3]] + [K_2,\ [K_1, K_3]] + [K_3,\ [K_1, K_2]] = \left[K_1, -\tfrac{1}{2}iK_1\right] + \left[K_2, \tfrac{1}{2}iK_2\right] + [K_3,\ -iK_3]. \end{gathered}$$

Since each commutator: $\left[K_1, -\tfrac{1}{2}iK_1\right] = 0$, $\left[K_2, \tfrac{1}{2}iK_2\right] = 0$ and $[K_3, -iK_3] = 0$,
$\therefore [K_1,\ [K_2, K_3]] + [K_2,\ [K_1, K_3]] + [K_3,\ [K_1, K_2]] = 0$. □

References

1. Misner, C.W.; Thorne, K.S.; Wheeler, J.A. *Gravitation*; Macmillan: New York, NY, USA, 1973.
2. Lizzi, F.; Manfredonia, M.; Mercati, F. The momentum spaces of κ-Minkowski noncommutative spacetime. *Nucl. Phys. B* **2020**, *958*, 115117. [CrossRef]
3. Cocco, L.; Babic, J. A system of axioms for Minkowski spacetime. *J. Philos. Log.* **2021**, *50*, 149–185. [CrossRef]
4. Foo, J.; Arabaci, C.S.; Zych, M.; Mann, R.B. Quantum superpositions of Minkowski spacetime. *Phys. Rev. D* **2023**, *107*, 045014. [CrossRef]
5. Vilasini, V.; Colbeck, R. Impossibility of superluminal signaling in Minkowski spacetime does not rule out causal loops. *Phys. Rev. Lett.* **2022**, *129*, 110401. [CrossRef]
6. Liu, C.; Majid, S. Quantum geodesics on quantum Minkowski spacetime. *J. Phys. A Math. Theor.* **2022**, *55*, 424003. [CrossRef]
7. Gasperin, E. Polyhomogeneous spin-0 fields in Minkowski space–time. *Philos. Trans. R. Soc. A* **2024**, *382*, 20230045. [CrossRef] [PubMed]
8. Meljanac, S.; Škoda, Z.; Krešić–Jurić, S. Symmetric ordering and Weyl realizations for quantum Minkowski spaces. *J. Math. Phys.* **2022**, *63*, 123508. [CrossRef]
9. Lombriser, L. Cosmology in Minkowski space. *Class. Quantum Gravity* **2023**, *40*, 155005. [CrossRef]
10. Volovik, G.E. Planck Constants in the Symmetry Breaking Quantum Gravity. *Symmetry* **2023**, *15*, 991. [CrossRef]

11. Chappell, J.M.; Hartnett, J.G.; Iannella, N.; Iqbal, A.; Berkahn, D.L.; Abbott, D. A new derivation of the Minkowski metric. *J. Phys. Commun.* **2023**, *7*, 065001. [CrossRef]
12. Li, Y.; Aldossary, M.T.; Abdel-Baky, R.A. Spacelike circular surfaces in Minkowski 3-space. *Symmetry* **2023**, *15*, 173. [CrossRef]
13. Stone, M. Gamma matrices, Majorana fermions, and discrete symmetries in Minkowski and Euclidean signature. *J. Phys. A Math. Theor.* **2022**, *55*, 205401. [CrossRef]
14. Kumar, K.; Lechtenfeld, O.; Costa, G.P.; Röhrig, J. Yang–Mills solutions on Minkowski space via non-compact coset spaces. *Phys. Lett. B* **2022**, *835*, 137564. [CrossRef]
15. Liu, Y.; Wang, G.P.; Pendry, J.B.; Zhang, S. All-angle reflectionless negative refraction with ideal photonic Weyl metamaterials. *Light Sci. Appl.* **2022**, *11*, 276. [CrossRef] [PubMed]
16. Chang, Q.; Liu, X.; Wang, Z.; Saito, N.; Fan, T. Design of three-dimensional isotropic negative-refractive-index metamaterials with wideband response based on an effective-medium approach. *Appl. Phys. A* **2022**, *128*, 440. [CrossRef]
17. Smolyaninov, I.I. Modeling of causality with metamaterials. *J. Opt.* **2013**, *15*, 025101. [CrossRef]
18. Iemma, U.; Palma, G. Design of metacontinua in the aeroacoustic spacetime. *Sci. Rep.* **2020**, *10*, 18192. [CrossRef]
19. Caloz, C.; Deck-Léger, Z.L. Spacetime metamaterials—Part I: General concepts. *IEEE Trans. Antennas Propag.* **2019**, *68*, 1569–1582. [CrossRef]
20. Bahrami, A.; Deck-Léger, Z.L.; Caloz, C. Electrodynamics of accelerated-modulation space-time metamaterials. *Phys. Rev. Appl.* **2023**, *19*, 054044. [CrossRef]
21. Caloz, C.; Deck-Léger, Z.L.; Bahrami, A.; Vicente, O.C.; Li, Z. Generalized Space-Time Engineered Modulation (GSTEM) Metamaterials: A global and extended perspective. *IEEE Antennas Propag. Mag.* **2022**, *65*, 50–60. [CrossRef]
22. Ganesan, T. Exotic Particle Dynamics Using Novel Hermitian Spin Matrices. *Axioms* **2023**, *12*, 1052. [CrossRef]
23. Zhelnorovich, V.A. *Theory of Spinors and Its Application in Physics and Mechanics*; Springer International Publishing: Berlin/Heidelberg, Germany, 2019.
24. Hong, I.K.; Kim, C.S. Quaternion electromagnetism and the relation with two-spinor formalism. *Universe* **2019**, *5*, 135. [CrossRef]
25. Erişir, T.; Ali Güngör, M.; Tosun, M. Geometry of the hyperbolic spinors corresponding to alternative frame. *Adv. Appl. Clifford Algebras* **2015**, *25*, 799–810. [CrossRef]
26. Ketenci, Z.; Erişir, T.; Güngör, M.A. A construction of hyperbolic spinors according to Frenet frame in Minkowski space. *J. Dyn. Syst. Geom. Theor.* **2015**, *13*, 179–193. [CrossRef]
27. Tarakçioğlu, M.; Erişir, T.; Güngör, M.A.; Tosun, M. The hyperbolic spinor representation of transformations in R 1 3 by means of split quaternions. *Adv. Appl. Clifford Algebras* **2018**, *28*, 26. [CrossRef]
28. Bekenstein, R.; Segev, M. Self-accelerating optical beams in highly nonlocal nonlinear media. *Opt. Express* **2011**, *19*, 23706–23715. [CrossRef]
29. Steinhauer, J. Observation of quantum Hawking radiation and its entanglement in an analogue black hole. *Nat. Phys.* **2016**, *12*, 959–965. [CrossRef]
30. Almeida, C.R.; Jacquet, M.J. Analogue gravity and the Hawking effect: Historical perspective and literature review. *Eur. Phys. J. H* **2023**, *48*, 15. [CrossRef]