



## Article

## K-Essence Lagrangians of Polytropic and Logotropic Unified Dark Matter and Dark Energy Models

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**Abstract:** We determine the k-essence Lagrangian of a relativistic barotropic fluid. The equation of state of the fluid can be specified in different manners depending on whether the pressure is expressed in terms of the energy density (model I), the rest-mass density (model II), or the pseudo rest-mass density for a complex scalar field in the Thomas-Fermi approximation (model III). In the nonrelativistic limit, these three formulations coincide. In the relativistic regime, they lead to different models that we study exhaustively. We provide general results valid for an arbitrary equation of state and show how the different models are connected to each other. For illustration, we specifically consider polytropic and logotropic dark fluids that have been proposed as unified dark matter and dark energy models. We recover the Born-Infeld action of the Chaplygin gas in models I and III and obtain the explicit expression of the reduced action of the logotropic dark fluid in models II and III. We also derive the two-fluid representation of the Chaplygin and logotropic models. Our general formalism can be applied to many other situations such as Bose-Einstein condensates with a  $|\varphi|^4$  (or more general) self-interaction, dark matter superfluids, and mixed models.

**Keywords:** dark matter; dark energy; scalar fields; Bose-Einstein condensates; equation of state



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## 1. Introduction

Baryonic matter constitutes only 5% of the content of the universe today. The rest of the universe is made of approximately 25% dark matter (DM) and 70% dark energy (DE) [1,2]. DM can explain the flat rotation curves of the spiral galaxies. It is also necessary to form the large-scale structures of the universe. DE does not cluster but is responsible for the late time acceleration of the universe revealed by the observations of type Ia supernovae, the cosmic microwave background (CMB) anisotropies, and galaxy clustering. Although there have been many theoretical attempts to explain DM and DE, we still do not have a robust model for these dark components that can pass all the theoretical and observational tests.

The most natural and simplest model is the  $\Lambda$ CDM model which treats DM as a non-relativistic cold pressureless gas and DE as a cosmological constant  $\Lambda$  possibly representing vacuum energy [3,4]. The effect of the cosmological constant is equivalent to that of a fluid with a constant energy density  $\epsilon_\Lambda = \Lambda c^2 / 8\pi G$  and a negative pressure  $P_\Lambda = -\epsilon_\Lambda$ . Therefore, the  $\Lambda$ CDM model is a two-fluid model comprising DM with an equation of state  $P_{\text{dm}} = 0$  and DE with an equation of state  $P_{\text{de}} = -\epsilon_{\text{de}}$ . When combined with the energy conservation equation, the equation of state  $P_{\text{dm}} = 0$  implies that the DM density decreases with the scale factor as  $\epsilon_{\text{dm}} \propto a^{-3}$  and the equation of state  $P_{\text{de}} = -\epsilon_{\text{de}}$  implies that the DE density is constant:  $\epsilon_{\text{de}} = \epsilon_\Lambda$ . Therefore, the total energy density of the universe (DM + DE) evolves as

$$\epsilon = \frac{\epsilon_{\text{dm},0}}{a^3} + \epsilon_\Lambda, \quad (1)$$

where  $\epsilon_{\text{dm},0}$  is the present density of DM (when  $a = 1$ )<sup>1</sup>. DM dominates at early times when the density is high and DE dominates at late times when the density is low. The scale factor increases algebraically as  $a \propto t^{2/3}$  during the DM era (Einstein-de Sitter regime) and

exponentially as  $a \propto e^{\sqrt{\Lambda/3}t}$  during the DE era (de Sitter regime). At the present epoch, both components are important in the energy budget of the universe.

Although the  $\Lambda$ CDM model is perfectly consistent with current cosmological observations, it faces two main problems. The first problem is to explain the tiny value of the cosmological constant  $\Lambda = 1.00 \times 10^{-35} \text{ s}^{-2}$ . Indeed, if DE can be attributed to vacuum fluctuations, quantum field theory predicts that  $\Lambda$  should correspond to the Planck scale which lies 123 orders of magnitude above the observed value. This is called the cosmological constant problem [6,7]. The second problem is to explain why DM and DE are of similar magnitudes today although they scale differently with the universe's expansion. This is the cosmic coincidence problem [8,9], frequently triggering anthropic explanations. The CDM model also faces important problems at the scale of DM halos such as the core-cusp problem [10], the missing satellite problem [11–13], and the “too big to fail” problem [14]. This leads to the so-called small-scale crisis of CDM [15].

For these reasons, other types of matter with negative pressure that can behave like a cosmological constant at late time have been considered as candidates of DE: fluids of topological defects (domain walls, cosmic strings) [16–20],  $x$ -fluids with a linear equation of state [21–23], quintessence—an evolving self-interacting scalar field (SF) minimally coupled to gravity—[24] (see earlier works in [25–33])<sup>2</sup>,  $k$ -essence fields—SFs with a noncanonical kinetic term [37–39] that were initially introduced to describe inflation ( $k$ -inflation) [40,41]—and even phantom or ghost fields [42,43] which predict that the energy density of the universe may ultimately increase with time. Quintessence can be viewed as a dynamical vacuum energy following the old idea that the cosmological term could evolve [44–47]. However, these models still face the cosmic coincidence problem<sup>3</sup> because they treat DM and DE as distinct entities. Accounting for similar magnitude of DM and DE today requires very particular (fine-tuned) initial conditions. For some kind of potential terms, which have their justification in supergravity [48], this problem can be solved by the so-called tracking solution [9,49]. The self-interacting SF evolves in such a way that it approaches a cosmological constant behaviour exactly today [48]. However, this is achieved at the expense of fine-tuning the potential parameters. This unsatisfactory state of affairs motivated a search for further alternatives.

In the standard  $\Lambda$ CDM model and in quintessence CDM models, DM and DE are two distinct entities introduced to explain the clustering of matter and the cosmic acceleration, respectively. However, DM and DE could be two different manifestations of a common structure, a dark fluid. In this respect, Kamenshchik et al. [50] have proposed a simple unification of DM and DE in the form of a perfect fluid with an exotic equation of state known as the Chaplygin gas, for which

$$P = -\frac{A}{\epsilon/c^2}, \quad (2)$$

where  $A$  is a positive constant. This gas exhibits a negative pressure, as required to explain the acceleration of the universe today, but the squared speed of sound is positive ( $c_s^2 = P'(\epsilon)c^2 = Ac^4/\epsilon^2 > 0$ ). This is a very important property because many fluids with negative pressure obeying a barotropic equation of state suffer from instabilities at small scales due to an imaginary speed of sound [18,19]<sup>4</sup>. This is not the case for the Chaplygin gas.

Integrating the energy conservation equation with the Chaplygin equation of state (2) leads to

$$\epsilon/c^2 = \sqrt{\frac{A}{c^2} + \frac{B}{a^6}}, \quad (3)$$

where  $B$  is an integration constant. Therefore, the Chaplygin gas smoothly interpolates between pressureless DM ( $P \simeq 0$ ,  $\epsilon \sim a^{-3}$ ,  $c_s \simeq 0$ ) at high redshift and a cosmological constant ( $P = -\epsilon$ ,  $\epsilon \rightarrow \sqrt{Ac^2}$ ,  $c_s \rightarrow c$ ) as  $a$  tends to infinity. There is also an intermediate phase which can be described by a cosmological constant mixed with a stiff matter fluid

( $P \sim \epsilon$ ,  $\epsilon \sim a^{-6}$ ,  $c_s \sim c$ ) [50]. In the Chaplygin gas model, DM and DE are different manifestations of a single underlying substance (dark fluid) that is called “quartessence” [52]. These models where the fluid behaves as DM at early times and as DE at late times are called unified models for DM and DE (UDM models) [52]. This dual behavior avoids fine-tuning problems since the Chaplygin gas model can be interpreted as an entangled mixture of DM and DE. In this cosmological context, Kamenshchik et al. [50] introduced a real SF representation of the Chaplygin gas and determined its potential  $V(\varphi)$  explicitly.

The Chaplygin gas model has an interesting history that we briefly retrace below. Chaplygin [53] introduced his equation of state  $P = -A/\rho$  in 1904 as a convenient soluble model to study the lifting force on a plane wing in aerodynamics. The same model was rediscovered later by Tsien [54] and von Kármán [55]. It was also realized that certain deformable solids can be described by the Chaplygin equation of state [56]. The integrability of the corresponding Euler equations resides in the fact that they have a large symmetry group (see [57–60] for a modern description). Indeed, the Chaplygin gas model possesses further space-time symmetries beyond those of the Galileo group [57]. In addition, the Chaplygin gas is the only fluid which admits a supersymmetric generalization [61–65]. The Chaplygin equation of state involves a negative pressure which is required to account for the accelerated expansion of the universe<sup>5</sup>. It is possible to develop a Lagrangian description of the nonrelativistic Chaplygin gas [57–59,66–68] leading to an action of the form

$$\mathcal{L}_{\text{Chap}} = -(2A)^{1/2} \sqrt{\dot{\theta} + \frac{1}{2}(\nabla\theta)^2}, \quad (4)$$

where  $\theta$  is the potential of the flow. The relativistic generalization of the Chaplygin gas model leads to a Born-Infeld-type [69] theory for a real SF [58,59,68,70,71]. The Born-Infeld action

$$\mathcal{L}_{\text{BI}} = -(Ac^2)^{1/2} \sqrt{1 - \frac{1}{c^2} \partial_\mu \theta \partial^\mu \theta} \quad (5)$$

possesses additional symmetries beyond the Lorentz and Poincaré invariance and has an interesting connection with string/M theory [72]. The Chaplygin gas model can be motivated by a brane-world interpretation (see [73] for a review on brane world models). Indeed, the “hidden” symmetries and the associated transformation laws for the Chaplygin and Born-Infeld models may be given a coherent setting [59] by considering the Nambu-Goto action [74] for a  $d$ -brane in  $(d+1)$  spatial dimensions moving in a  $(d+1, 1)$ -dimensional spacetime. The Galileo-invariant (nonrelativistic) Chaplygin gas action (4) is obtained in the light-cone parametrization and the Poincaré-invariant (relativistic) Born-Infeld action (5) is obtained in the Cartesian parametrization [58,59]<sup>6</sup>. A fluid with a Chaplygin equation of state is also necessary to stabilize the branes [90] in black hole bulks [91,92]. This is how Kamenshchik et al. [93] came across this fluid and had the idea to consider its cosmological implications [50]. Bilic et al. [71] generalized the Chaplygin gas model in the inhomogeneous case and showed that the real SF that occurs in the Born-Infeld action can be interpreted as the phase of a complex self-interacting SF described by the Klein-Gordon (KG) equation. This SF may be given a hydrodynamic representation in terms of an irrotational barotropic flow with the Chaplygin equation of state. This explains the connection of the Born-Infeld action with fluid mechanics in the Thomas-Fermi (TF) approximation. Bilic et al. [71] determined the potential  $V(|\varphi|^2)$  of the complex SF associated with the Chaplygin gas. This potential is different from the potential  $V(\varphi)$  of the real SF introduced by Kamenshchik et al. [50] which is valid for an homogeneous SF in a cosmological context.

A generalized Chaplygin gas model (GCG) has been introduced. It has an equation of state<sup>7</sup>

$$P = -\frac{A}{(\epsilon/c^2)^\alpha} \quad (7)$$

with  $A > 0$  and a generic  $\alpha$  constant in the range  $0 \leq \alpha \leq 1$  in order to ensure the condition of stability  $c_s^2 \geq 0$  and the condition of causality  $c_s \leq c$  (the quantity  $c_s^2/c^2$  goes from 0 to  $\alpha$  when  $a$  goes from 0 to  $+\infty$ ). Combined with the energy conservation equation, we obtain

$$\epsilon/c^2 = \left[ \frac{A}{c^2} + \frac{B}{a^{3(\alpha+1)}} \right]^{\frac{1}{\alpha+1}}. \quad (8)$$

This model interpolates between a universe dominated by dust and de Sitter eras via an intermediate phase described by a linear equation of state  $P \sim \alpha\epsilon$  [50,95]. The original Chaplygin gas model is recovered for  $\alpha = 1$ . Bento et al. [95] argued that the GCG model corresponds to a generalized Nambu-Goto action which can be interpreted as a perturbed  $d$ -brane in a  $(d+1, 1)$  spacetime. Bilic et al. [100] mentioned that the generalized Nambu-Goto action lacks any geometrical interpretation, but that the generalized Chaplygin equation of state can be obtained from a moving brane in Schwarzschild-anti-de-Sitter bulk [101].

For  $\alpha = 0$ , the generalized Chaplygin equation of state (7) reduces to a constant negative pressure

$$P = -A. \quad (9)$$

In that case, the speed of sound vanishes identically ( $c_s^2 = 0$ ) and  $\epsilon/c^2 = A/c^2 + B/a^3$ . It can be shown [102,103] that this model is equivalent to the  $\Lambda$ CDM model not only to 0th order in perturbation theory (background) but to all orders, even in the nonlinear clustering regime (contrary to the initial claim made in Ref. [104]). Therefore, the  $\Lambda$ CDM model can either be considered as a two-fluid model involving a DM fluid with  $P_{\text{dm}} = 0$  and a DE fluid with  $P_{\text{de}} = -\epsilon_{\text{de}}$ , or as a single dark fluid with a constant negative pressure  $P = -\epsilon_{\Lambda}$  [98]. In this sense, it may be regarded as the simplest UDM model one can possibly conceive in which DM and DE appear as different manifestations of a single dark fluid. As a result, the GCG model includes the original Chaplygin gas model ( $\alpha = 1$ ) and the  $\Lambda$ CDM model ( $\alpha = 0$ ) as particular cases.

The GCG model has been successfully confronted with various phenomenological tests such as high precision Cosmic Microwave Background Radiation data [105–109], type Ia supernova (SNIa) data [52,94,110–113], age estimates of high- $z$  objects [112] and gravitational lensing [114]. Although the GCG model is consistent with observations related to the background cosmology (the Hubble law is almost insensitive to  $\alpha$ ) [52,110,113], Sandvik et al. [103] showed that it produces unphysical oscillations or even an exponential blow-up which are not seen in the observed matter power spectrum calculated at the present time. This is caused by the behaviour of the sound speed through the GCG fluid. At early times, the GCG behaves as DM and its sound speed vanishes. In that case, the GCG clusters like pressureless dust. At late times, when the GCG behaves as DE, its sound speed becomes relatively large yielding unphysical features in the matter power spectrum. To avoid such unphysical features, the value of  $\alpha$  must be extremely close to zero ( $|\alpha| < 10^{-5}$ ), so that the GCG model becomes indistinguishable from the  $\Lambda$ CDM model. Similar conclusions were reached by Bean and Doré [115], Carturan and Finelli [108] and Amendola et al. [109] who studied the effect of the GCG on density perturbations and on CMB anisotropies and found that the GCG strongly increases the amount of integrated Sachs-Wolfe effect. Therefore, CMB data are more selective than SN Ia data to constrain  $\alpha$ . The GCG is essentially ruled out except for a tiny region of parameter space very close to the  $\Lambda$ CDM limit. This conclusion is not restricted to the GCG model but is actually valid for all UDM models<sup>8</sup>. Some solutions to this problem have been suggested (see a short review in Section XVI of [116]) but there is no definite consensus at the present time. However, in a recent paper, Abdullah et al. [117] argued that a cosmological scenario based on the Chaplygin gas may not be ruled out from the viewpoint of structure formation as usually claimed. Indeed, a nonlinear analysis may predict collapse rather than a re-expansion of small-scale perturbations so that nonlinear clustering may occur in the Chaplygin gas. This is because pressure forces in UDM fluids decrease with increasing density so that systems that are stable against self-gravitating collapse in the linear regime may become

unstable in the nonlinear regime. As a result, the problem of acoustic oscillations in the linear power spectrum of UDM models may not be as serious as usually assumed provided the hierarchical structure formation process is adequately taken into account.

The previous results suggest that a viable UDM model should be as close as possible to the  $\Lambda$ CDM model, but sufficiently different from it in order to solve its problems. This is the basic idea that led us to introduce the logotropic model in Ref. [118] (see also [116,119–122]). The logotropic dark fluid has an equation of state

$$P = A \ln\left(\frac{\rho_m}{\rho_*}\right), \quad (10)$$

where  $\rho_m$  is the rest-mass density. This equation of state can be obtained from the polytropic (GCG) equation of state  $P = K\rho_m^\gamma$  by considering the limit  $\gamma \rightarrow 0$  and  $K \rightarrow \infty$  with  $A = K\gamma$  constant (see Section 3 of [118] and Appendix A of [116]). This yields

$$P = Ke^{\gamma \ln \rho_m} \simeq K(1 + \gamma \ln \rho_m + \dots) \simeq K + A \ln \rho, \quad (11)$$

which is equivalent to Equation (10) up to a constant term<sup>9</sup>. Since the  $\Lambda$ CDM model (viewed as a UDM model) is equivalent to a polytropic gas of index  $\gamma = 0$  (constant pressure), one can say that the logotropic model which has  $\gamma \rightarrow 0$  is the simplest extension of the  $\Lambda$ CDM model. It is argued in Ref. [118] that  $\rho_* = \rho_P = c^5/(\hbar G^2) = 5.16 \times 10^{99} \text{ g/m}^3$  is the Planck density. It is also argued that  $A/c^2 = B\rho_\Lambda = 2.10 \times 10^{-26} \text{ g m}^{-3}$  (where  $B = 3.53 \times 10^{-3}$  and  $\rho_\Lambda = \Lambda/8\pi G = 5.96 \times 10^{-24} \text{ g m}^{-3}$ ) is a fundamental constant of physics that supersedes the cosmological constant  $\Lambda$ . The logotropic model is able to account for the transition between a DM era and a DE era and is indistinguishable from the  $\Lambda$ CDM model, for what concerns the evolution of the cosmological background, up to 25 billion years in the future when it becomes phantom [118–121]. Very interestingly, the logotropic model implies that DM halos should have a constant surface density and it predicts its universal value  $\Sigma_0^{\text{th}} = 0.01955c\sqrt{\Lambda}/G = 133 M_\odot/\text{pc}^2$  [116,118–122] without adjustable parameter. This theoretical value is in good agreement with the value  $\Sigma_0^{\text{obs}} = 141_{-52}^{+83} M_\odot/\text{pc}^2$  obtained from the observations [123]. The logotropic model also predicts that the present ratio of DE and DM is equal to the Euler number  $\Omega_{\text{de},0}^{\text{th}}/\Omega_{\text{dm},0}^{\text{th}} = e = 2.71828\dots$  [116,121,122] in very good agreement with the observations giving  $\Omega_{\text{de},0}^{\text{obs}}/\Omega_{\text{dm},0}^{\text{obs}} = 2.669 \pm 0.08$ <sup>10</sup>. Using the measured present proportion of baryonic matter  $\Omega_{\text{b},0}^{\text{obs}} = 0.0486 \pm 0.0010$ , we find that the values of the present proportions of DM and DE are  $\Omega_{\text{dm},0}^{\text{th}} = \frac{1}{1+e}(1 - \Omega_{\text{b},0}) = 0.2559$  and  $\Omega_{\text{de},0}^{\text{th}} = \frac{e}{1+e}(1 - \Omega_{\text{b},0}) = 0.6955$  in very good agreement with the observational values  $\Omega_{\text{dm},0}^{\text{obs}} = 0.2589 \pm 0.0057$  and  $\Omega_{\text{de},0}^{\text{obs}} = 0.6911 \pm 0.0062$  within the error bars. This result is striking because the proportions of DE and DM change with time so it is only at the present epoch that their ratio is equal to  $e$  [122]. Unfortunately, the logotropic model suffers from the same problems as the GCG model regarding the presence of unphysical oscillations in the matter power spectrum [116,124]. It is not clear how these problems can be circumvented (see the discussion in [116]). However, as discussed above, this problem may not be as insurmountable as previously thought provided that an adequate nonlinear analysis of structure formation is developed [117]. In any case, the logotropic dark fluid (LDF) remains an interesting UDM model, especially because of its connection with the polytropic (GCG) model.

The aim of the present paper is to develop the Lagrangian formulation of the polytropic (GCG) and logotropic models. We point out that the equation of state can be specified in different manners, yielding three sorts of models. In model I, the pressure is a function  $P(\epsilon)$  of the energy density; in model II, the pressure is a function  $P(\rho_m)$  of the rest-mass density; in model III, the pressure is a function  $P(\rho)$  of the pseudo rest-mass density associated with a complex SF (in the sense given below). In the nonrelativistic regime, these three formulations coincide. However, in the relativistic regime, they lead to different models. In this paper, we describe these models in detail and show their interrelations.



For example, given  $P(\epsilon)$ , we show how one can obtain  $P(\rho_m)$  and  $P(\rho)$ , and reciprocally. We also explain how one can obtain the k-essence Lagrangian (action) for each model. We first provide general results that can be applied to an arbitrary barotropic equation of state. Then, for illustration, we obtain explicit analytical results for a polytropic and a logotropic equation of state. We recover the Born-Infeld action of the Chaplygin gas and determine the expression of the action of the GCG of type I, II and III. We also explicitly obtain the logotropic action in models II and III. We show that it can be recovered from the polytropic (GCG) action in the limit  $\gamma \rightarrow 0$  and  $K \rightarrow \infty$  with  $A = K\gamma$  constant.

The paper is organized as follows. In Section 2, we consider a nonrelativistic complex self-interacting SF which may represent the wavefunction of a Bose-Einstein condensate (BEC) described by the Gross-Pitaevskii (GP) equation. We determine its Madelung hydrodynamic representation and show that it is equivalent to an irrotational quantum fluid with a quantum potential and a barotropic equation of state  $P(\rho)$  determined by the self-interaction potential. In the TF limit, it reduces to an irrotational classical barotropic fluid. We determine its Lagrangian and “reduced” Lagrangian for an arbitrary equation of state. The reduced Lagrangian has the form of a k-essence Lagrangian  $\mathcal{L}(x)$  with  $x = \dot{\theta} + (1/2)(\nabla\theta)^2$ , where  $\theta$  is the potential of the velocity field (the phase of the wavefunction). In Section 3, we consider a relativistic complex self-interacting SF described by the KG equation which may represent the wavefunction of a relativistic BEC. We determine its de Broglie hydrodynamic representation and show that it is equivalent to an irrotational quantum fluid with a covariant quantum potential and a barotropic equation of state  $P(\rho)$  determined by the self-interaction potential. In the TF limit, it reduces to an irrotational classical barotropic fluid. We determine its Lagrangian and reduced Lagrangian for an arbitrary equation of state. Its reduced Lagrangian has the form of a k-essence Lagrangian  $\mathcal{L}(X)$  with  $X = (1/2)\partial_\mu\theta\partial^\mu\theta$ , where the real SF  $\theta$  is played by the phase of the complex SF. In Section 4, we introduce three types of equations of state (models I, II and III) and explain their physical meanings. We provide general equations allowing us to connect one model to the other and to determine the reduced Lagrangian  $\mathcal{L}(X)$  for an arbitrary equation of state. In Sections 5 and 6, we illustrate our general results by applying them to a polytropic (GCG) equation of state and a logotropic equation of state. We recover the Born-Infeld action of the Chaplygin gas and determine the reduced action of the GCG and of the logotropic gas. In Appendix A, we establish general identities valid for a nonrelativistic cold gas. In Appendix B, we consider a general k-essence Lagrangian and specifically discuss the case of a canonical and tachyonic SF. In Appendix C, we define the equation of state of model I and detail how one can obtain the potential  $V(\varphi)$  of a homogeneous real SF in an expanding universe. In Appendix D, we define the equation of state of model II and detail how one can obtain the corresponding internal energy. In Appendix E, we define the equation of state of model III (see also Section 3) and detail how the basic equations of the problem can be simplified in the case of a homogeneous SF in a cosmological context. In Appendix F, we discuss the analogies and the differences between the internal energy and the potential of a complex SF in the TF limit. In Appendix H, we list some studies devoted to polytropic and logotropic equations of state of type I, II and III. In Appendix I, we detail the Lagrangian structure and the conservation laws of a nonrelativistic and relativistic SF. Applications and generalizations of the results of this paper will be presented in future works [125].

## 2. Nonrelativistic Theory

### 2.1. The Gross-Pitaevskii Equation

We consider a complex SF  $\psi(\mathbf{r}, t)$  whose evolution is governed by the GP equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\hbar(|\psi|^2)\psi. \quad (12)$$

This equation describes, for example, the wave function of a BEC at  $T = 0$  [126]. For the sake of generality, we have introduced an arbitrary nonlinearity determined by the effective

potential  $h(|\psi|^2)$  instead of the quadratic potential  $h(|\psi|^2) = g|\psi|^2 = (4\pi a_s \hbar^2 / m^3)|\psi|^2$ , where  $a_s$  is the s-scattering length of the bosons, arising from pair contact interactions in the usual GP equation [127–130] (see, e.g., the discussion in Ref. [126]). In this manner, we can describe a larger class of systems<sup>11</sup>. The GP Equation (12) can also be derived from the Klein-Gordon (KG) equation

$$\square\varphi + \frac{m^2 c^2}{\hbar^2}\varphi + 2\frac{dV}{d|\varphi|^2}\varphi = 0, \quad (13)$$

which governs the evolution of a complex SF  $\varphi(\mathbf{r}, t)$  with a self-interaction potential  $V(|\varphi|^2)$ . The GP Equation (12) is obtained from the KG Equation (13) in the nonrelativistic limit  $c \rightarrow +\infty$  after making the Klein transformation

$$\varphi(\mathbf{r}, t) = \frac{\hbar}{m} e^{-imc^2 t / \hbar} \psi(\mathbf{r}, t). \quad (14)$$

In that case, the effective potential  $h(|\psi|^2)$  that appears in the GP equation is related to the self-interaction potential  $V(|\varphi|^2)$  present in the KG equation by (see Refs. [131,132] and Appendix C of Ref. [118])

$$h(|\psi|^2) = \frac{dV}{d|\psi|^2} \quad \text{with} \quad |\psi|^2 = \frac{m^2}{\hbar^2} |\varphi|^2. \quad (15)$$

As a result, the GP Equation (12) can be rewritten as

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \frac{dV}{d|\psi|^2} \psi. \quad (16)$$

*Remark:* The GP Equation (16) can also be derived from the KG equation governing the evolution of a real SF but, in that case, the potential  $V(|\psi|^2)$  that appears in the GP equation does not coincide with the potential  $V(\varphi)$  present in the KG equation. Indeed,  $V(|\psi|^2)$  is an effective potential obtained after averaging  $V(\varphi)$  over the fast oscillations of the SF (see Sections II and III of [133] and Appendix A of [134] for details).

## 2.2. The Madelung Transformation

We can write the GP Equation (12) under the form of hydrodynamic equations by using the Madelung [135] transformation. To that purpose, we write the wave function as

$$\psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)/\hbar}, \quad (17)$$

where  $\rho(\mathbf{r}, t)$  is the density and  $S(\mathbf{r}, t)$  is the action. They are given by

$$\rho = |\psi|^2 \quad \text{and} \quad S = -i\frac{\hbar}{2} \ln\left(\frac{\psi}{\psi^*}\right). \quad (18)$$

Following Madelung, we introduce the velocity field

$$\mathbf{u} = \frac{\nabla S}{m}. \quad (19)$$

Since the velocity derives from a potential, the flow is irrotational:  $\nabla \times \mathbf{u} = \mathbf{0}$ . Substituting Equation (17) into Equation (12) and separating the real and the imaginary parts, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (20)$$

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + m h(\rho) + Q = 0, \quad (21)$$

where

$$Q = -\frac{\hbar^2}{2m} \frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{4m} \left[ \frac{\Delta\rho}{\rho} - \frac{1}{2} \frac{(\nabla\rho)^2}{\rho^2} \right] \quad (22)$$

is the quantum potential. It takes into account the Heisenberg uncertainty principle. Equation (20) is similar to the equation of continuity in hydrodynamics. It accounts for the local conservation of mass  $M = \int \rho \, d\mathbf{r}$ . Equation (21) has a form similar to the classical Hamilton-Jacobi equation with an additional quantum potential. It can also be interpreted as a quantum Bernoulli equation for a potential flow. Taking the gradient of Equation (21), and using the well-known identity of vector analysis  $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\mathbf{u}^2/2) - \mathbf{u} \times (\nabla \times \mathbf{u})$  which reduces to  $(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla(\mathbf{u}^2/2)$  for an irrotational flow, we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla h - \frac{1}{m} \nabla Q. \quad (23)$$

Since  $h = h(\rho)$  we can introduce a function  $P = P(\rho)$  satisfying  $\nabla h = (1/\rho)\nabla P$ . Equation (23) can then be rewritten as

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho} \nabla P - \frac{1}{m} \nabla Q. \quad (24)$$

This equation is similar to the Euler equation with a pressure force  $-(1/\rho)\nabla P$  and a quantum force  $-\frac{1}{m}\nabla Q$ . Since  $P(\mathbf{r}, t) = P[\rho(\mathbf{r}, t)]$  is a function of the density, the flow is barotropic. The equation of state  $P(\rho)$  is determined by the potential  $h(\rho)$  through the relation

$$h'(\rho) = \frac{P'(\rho)}{\rho}. \quad (25)$$

Equation (25) can be integrated into

$$P(\rho) = \rho h(\rho) - V(\rho) = \rho V'(\rho) - V(\rho) = \rho^2 \left[ \frac{V(\rho)}{\rho} \right]', \quad (26)$$

where  $V$  is a primitive of  $h$ . This notation is consistent with Equation (15) which can be rewritten as

$$h(\rho) = V'(\rho), \quad (27)$$

where  $V(\rho)$  is the potential in the KG Equation (13) or in the GP Equation (16). Equation (26) determines the pressure  $P(\rho)$  as a function of the potential  $V(\rho)$ . Inversely, the potential is determined as a function of the pressure by<sup>12</sup>

$$V(\rho) = \rho \int \frac{P(\rho)}{\rho^2} d\rho. \quad (28)$$

The speed of sound is  $c_s^2 = P'(\rho) = \rho V''(\rho)$ . The GP Equation (12) is equivalent to the hydrodynamic Equations (20), (21) and (24). We shall call them the quantum Euler equations. Since there is no viscosity, they describe a superfluid. In the TF approximation  $\hbar \rightarrow 0$ , they reduce to the classical Euler equations.

*Remark:* We show in Appendix A that the effective potential  $h$  appearing in the GP equation can be interpreted, in the hydrodynamic equations, as the enthalpy (or as the chemical potential by unit of mass  $h = \mu/m$ ) and that its primitive  $V(\rho)$ , which is equal to the potential in the KG equation, can be interpreted as the internal energy density  $u$ . Thus, we have

$$u(\rho) = V(\rho), \quad h(\rho) = \frac{P(\rho) + V(\rho)}{\rho}. \quad (29)$$



On the other hand, if we define the energy by

$$E(\mathbf{r}, t) = -\frac{\partial S}{\partial t}, \quad (30)$$

the Hamilton-Jacobi Equation (21) can be rewritten as

$$E(\mathbf{r}, t) = \frac{1}{2}m\mathbf{u}^2 + mh(\rho) + Q. \quad (31)$$

### 2.3. Lagrangian of a Quantum Barotropic Gas

The action of the complex SF associated with the GP Equation (16) is given by

$$S = \int L dt \quad (32)$$

with the Lagrangian

$$L = \int \left\{ \frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \frac{\hbar^2}{2m^2} |\nabla \psi|^2 - V(|\psi|^2) \right\} d\mathbf{r}. \quad (33)$$

We can view the Lagrangian (33) as a functional of  $\psi$ ,  $\dot{\psi}$  and  $\nabla \psi$ . The least action principle  $\delta S = 0$ , which is equivalent to the Euler-Lagrange equation

$$\frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \dot{\psi}} \right) + \nabla \cdot \left( \frac{\delta L}{\delta \nabla \psi} \right) - \frac{\delta L}{\delta \psi} = 0, \quad (34)$$

returns the GP Equation (16). The Hamiltonian (energy) is obtained from the transformation

$$H = \int \frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) d\mathbf{r} - L \quad (35)$$

yielding

$$H = \frac{\hbar^2}{2m^2} \int |\nabla \psi|^2 d\mathbf{r} + \int V(|\psi|^2) d\mathbf{r}. \quad (36)$$

The first term is the kinetic energy  $\Theta = -\frac{\hbar^2}{2m^2} \int \psi^* \Delta \psi d\mathbf{r}$  and the second term is the self-interaction (internal) energy  $U$ . Since the Lagrangian does not explicitly depend on time, the Hamiltonian (energy) is conserved<sup>13</sup>. The GP Equation (16) can be written as

$$i\hbar \frac{\partial \psi}{\partial t} = m \frac{\delta H}{\delta \psi^*}, \quad i\hbar \frac{\partial \psi^*}{\partial t} = -m \frac{\delta H}{\delta \psi}, \quad (37)$$

which can be interpreted as the Hamilton equations (see Appendix I.3).

Using the Madelung transformation, we can rewrite the Lagrangian in terms of hydrodynamic variables. According to Equations (17) and (18) we have

$$\frac{\partial S}{\partial t} = -i\frac{\hbar}{2} \frac{1}{|\psi|^2} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) \quad (38)$$

and

$$|\nabla \psi|^2 = \frac{1}{4\rho} (\nabla \rho)^2 + \frac{\rho}{\hbar^2} (\nabla S)^2. \quad (39)$$

Substituting these identities into Equation (33), we get

$$L = - \int \left\{ \frac{\rho}{m} \frac{\partial S}{\partial t} + \frac{\rho}{2m^2} (\nabla S)^2 + \frac{\hbar^2}{8m^2} \frac{(\nabla \rho)^2}{\rho} + V(\rho) \right\} d\mathbf{r}. \quad (40)$$

We can view the Lagrangian (40) as a functional of  $S$ ,  $\dot{S}$ ,  $\nabla S$ ,  $\rho$ ,  $\dot{\rho}$ , and  $\nabla \rho$ . The Euler-Lagrange equation for the action

$$\frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \dot{S}} \right) + \nabla \cdot \left( \frac{\delta L}{\delta \nabla S} \right) - \frac{\delta L}{\delta S} = 0 \quad (41)$$

returns the equation of continuity (20). The Euler-Lagrange equation for the density

$$\frac{\partial}{\partial t} \left( \frac{\delta L}{\delta \dot{\rho}} \right) + \nabla \cdot \left( \frac{\delta L}{\delta \nabla \rho} \right) - \frac{\delta L}{\delta \rho} = 0 \quad (42)$$

returns the quantum Hamilton-Jacobi (or Bernoulli) Equation (21) leading to the quantum Euler Equation (24). The Hamiltonian (energy) is obtained from the transformation

$$H = - \int \frac{\rho}{m} \frac{\partial S}{\partial t} d\mathbf{r} - L \quad (43)$$

yielding

$$H = \int \frac{1}{2} \rho \mathbf{u}^2 d\mathbf{r} + \int \frac{\hbar^2}{8m^2} \frac{(\nabla \rho)^2}{\rho} d\mathbf{r} + \int V(\rho) d\mathbf{r}. \quad (44)$$

This expression is equivalent to Equation (36) as can be seen by a direct calculation using the Madelung transformation [see Equation (39)]. The first term is the classical kinetic energy  $\Theta_c$ , the second term is the quantum kinetic energy  $\Theta_Q$  (we have  $\Theta = \Theta_c + \Theta_Q$ ), and the third term is the self-interaction (internal) energy  $U$ . The quantum kinetic energy can also be written as  $\Theta_Q = \int \rho \frac{Q}{m} d\mathbf{r}$  [136]. The continuity Equation (20) and the Hamilton-Jacobi (or Bernoulli) Equation (21) can be written as

$$\frac{\partial \rho}{\partial t} = m \frac{\delta H}{\delta S}, \quad \frac{\partial S}{\partial t} = -m \frac{\delta H}{\delta \rho}, \quad (45)$$

which can be interpreted as the Hamilton equations (see Appendix I.4).

If we substitute the quantum Hamilton-Jacobi (or Bernoulli) Equation (21) into the Lagrangian (40) and use Equations (22) and (26) we find that

$$L = \int P d\mathbf{r}. \quad (46)$$

This shows that the Lagrangian density is equal to the pressure:  $\mathcal{L} = P$ . Actually, the Lagrangian density is equal to  $\mathcal{L} = P(\rho) - \frac{\hbar^2}{4m^2} \Delta \rho$ . There is an additional term  $-\frac{\hbar^2}{4m^2} \Delta \rho$  which disappears by integration. The same result is obtained by substituting the GP Equation (16) into the Lagrangian (33) and using Equation (26).

#### 2.4. Lagrangian of a Classical Barotropic Gas

Introducing the notation  $\theta = S/m$ , so that  $\psi = \sqrt{\rho} e^{im\theta/\hbar}$ ,<sup>14</sup> and taking the limit  $\hbar \rightarrow 0$  in Equation (40), we obtain the classical Lagrangian<sup>15</sup>

$$L = - \int \left[ \rho \dot{\theta} + \frac{1}{2} \rho (\nabla \theta)^2 + V(\rho) \right] d\mathbf{r}. \quad (47)$$

We can view the Lagrangian (47) as a functional of  $\theta, \dot{\theta}, \nabla\theta, \rho, \dot{\rho}$ , and  $\nabla\rho$ . The Euler-Lagrange equation for the action leads to the equation of continuity

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{u}) = 0, \quad (48)$$

with the velocity field

$$\mathbf{u} = \nabla\theta. \quad (49)$$

The Euler-Lagrange equation for the density leads to the Bernoulli (or Hamilton-Jacobi) equation

$$\dot{\theta} + \frac{1}{2}(\nabla\theta)^2 + V'(\rho) = 0. \quad (50)$$

Taking the gradient of Equation (50) and using Equation (26), we obtain the Euler equation

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P. \quad (51)$$

The Hamiltonian (energy) is obtained from the transformation

$$H = - \int \rho \dot{\theta} d\mathbf{r} - L \quad (52)$$

yielding

$$H = \int \frac{1}{2}\rho\mathbf{u}^2 d\mathbf{r} + \int V(\rho) d\mathbf{r}. \quad (53)$$

The first term is the classical kinetic energy  $\Theta_c$  and the second term is the self-interaction (internal) energy  $U$ .

*Remark:* In our presentation, we started from a quantum fluid (or from the hydrodynamic representation of the GP equation) and finally considered the classical limit  $\hbar \rightarrow 0$ . Alternatively, we can obtain the equations of this section directly from the classical Euler equations by assuming that the fluid is barotropic (so that  $P = P(\rho)$ ) and that the flow is irrotational (so that the velocity derives from a potential:  $\mathbf{u} = \nabla\theta$ ) [59]. The Lagrangians (40) and (47) were first obtained by Eckart [137] for a classical fluid and from the hydrodynamic representation of the Schrödinger equation (see also [57,138]). The Lagrangian (47) with the potential  $V = A/(2\rho)$  corresponding to the Chaplygin equation of state  $P = -A/\rho$  (see below) also appeared in the theory of membranes ( $d = 2$ ) [57,66]. It was later generalized to a  $d$ -brane moving in a  $(d + 1, 1)$  space-time [58,59].

## 2.5. Reduced Lagrangian $\mathcal{L}(x)$

We introduce the Lagrangian density

$$\mathcal{L} = - \left[ \rho \dot{\theta} + \frac{1}{2}\rho(\nabla\theta)^2 + V(\rho) \right], \quad (54)$$

so that  $L = \int \mathcal{L} d\mathbf{r}$  and  $S = \int \mathcal{L} d\mathbf{r} dt$ . Using the Bernoulli Equation (50) and the identity (26), we can eliminate  $\theta$  from the Lagrangian and obtain

$$\mathcal{L} = \rho V'(\rho) - V(\rho) = P(\rho). \quad (55)$$

Therefore, the Lagrangian density is equal to the pressure:

$$\mathcal{L} = P. \quad (56)$$

We now eliminate  $\rho$  from the Lagrangian. Introducing the notation

$$x = \dot{\theta} + \frac{1}{2}(\nabla\theta)^2, \quad (57)$$

the Bernoulli Equation (50) can be written as

$$x = -V'(\rho). \quad (58)$$

Assuming  $V'' > 0$ , this equation can be reversed to give

$$\rho = F(x) \quad (59)$$

with  $F(x) = (V')^{-1}(-x)$ <sup>16</sup>. As a result, the equation of continuity (48) can be written in terms of  $\theta$  alone as

$$\frac{\partial}{\partial t} \left\{ F \left[ \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right] \right\} + \nabla \cdot \left\{ F \left[ \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right] \nabla\theta \right\} = 0. \quad (60)$$

On the other hand, according to Equations (26) and (59), we have  $P = P(x)$ . Therefore, recalling Equation (56), we get

$$\mathcal{L} = P(x). \quad (61)$$

In this manner, we have eliminated the density  $\rho$  from the Lagrangian (54) and we have obtained a reduced Lagrangian of the form  $\mathcal{L}(x)$  that depends only on  $x$ . This kind of Lagrangian, called k-essence Lagrangian, is specifically discussed in Appendix B. We show below that

$$\rho = F(x) = -\mathcal{L}'(x) = -P'(x). \quad (62)$$

Using Equation (62), we can write Equation (48) in terms of  $\mathcal{L}'(x)$  as

$$\frac{\partial}{\partial t} [\mathcal{L}'(x)] + \nabla \cdot [\mathcal{L}'(x) \nabla\theta] = 0. \quad (63)$$

*Proof of Equation (62):* From Equations (26) and (58) we have

$$P'(\rho) = \rho V''(\rho) \quad (64)$$

and

$$\frac{dx}{d\rho} = -V''(\rho). \quad (65)$$

Starting from Equation (61) and using Equations (64) and (65) we obtain

$$\mathcal{L}'(x) = P'(x) = P'(\rho) \frac{d\rho}{dx} = \rho V''(\rho) \frac{d\rho}{dx} = -\rho \frac{dx}{d\rho} \frac{d\rho}{dx} = -\rho, \quad (66)$$

which establishes Equation (62).

The preceding results are general. In the following sections, we consider particular equations of state.

## 2.6. Polytopic Gas

We first consider the polytropic equation of state [139]

$$P = K\rho^\gamma. \quad (67)$$

It can be obtained from the potential [136]

$$V(\rho) = \frac{K}{\gamma-1} \rho^\gamma \quad i.e. \quad V(|\psi|^2) = \frac{K}{\gamma-1} |\psi|^{2\gamma}. \quad (68)$$

As discussed in [136] this potential is similar to the Tsallis free energy density  $-Ks_\gamma$ , where the polytropic constant  $K$  plays the role of a generalized temperature and  $s_\gamma = -\frac{1}{\gamma-1}(\rho^\gamma - \rho)$  is the Tsallis entropy density. The Lagrangian and the Hamiltonian are given by Equations (47) and (53) with Equation (68). The Bernoulli Equation (50) takes the form

$$\dot{\theta} + \frac{1}{2}(\nabla\theta)^2 + \frac{K\gamma}{\gamma-1}\rho^{\gamma-1} = 0. \quad (69)$$

From that equation, we obtain

$$\rho = \left[ -\frac{\gamma-1}{K\gamma} \left( \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right) \right]^{\frac{1}{\gamma-1}}, \quad (70)$$

which is similar to the Tsallis distribution. The reduced Lagrangian  $\mathcal{L}(x)$  corresponding to the polytropic gas is

$$\mathcal{L} = P = K \left[ -\frac{\gamma-1}{K\gamma} \left( \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right) \right]^{\frac{\gamma}{\gamma-1}}. \quad (71)$$

The equation of motion is

$$\frac{\partial}{\partial t} \left( \left| \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right|^{\frac{1}{\gamma-1}} \right) + \nabla \cdot \left( \left| \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right|^{\frac{1}{\gamma-1}} \nabla\theta \right) = 0. \quad (72)$$

We note that the polytropic constant  $K$  does not appear in this equation.

## 2.7. Isothermal Gas

The case  $\gamma = 1$ , corresponding to the isothermal equation of state [139]

$$P = K\rho, \quad (73)$$

must be treated specifically (here  $K$  plays the role of the temperature  $k_B T/m$ ). It can be obtained from the potential [136]

$$V(\rho) = K\rho[\ln(\rho/\rho_*) - 1] \quad \text{i.e.,} \quad V(|\psi|^2) = K|\psi|^2[\ln(|\psi|^2/\rho_*) - 1]. \quad (74)$$

As discussed in [136] this potential is similar to the Boltzmann free energy density  $-Ks_B$ , where  $K$  plays the role of the temperature and  $s_B = -\rho[\ln(\rho/\rho_*) - 1]$  is the Boltzmann entropy density. The Lagrangian and the Hamiltonian are given by Equations (47) and (53) with Equation (74). The Bernoulli Equation (50) takes the form

$$\dot{\theta} + \frac{1}{2}(\nabla\theta)^2 + K\ln(\rho/\rho_*) = 0. \quad (75)$$

From this equation, we obtain

$$\rho = \rho_* e^{-\frac{1}{K}[\dot{\theta} + \frac{1}{2}(\nabla\theta)^2]}, \quad (76)$$

which is similar to the Boltzmann distribution. The reduced Lagrangian  $\mathcal{L}(x)$  corresponding to the isothermal gas is

$$\mathcal{L} = P = K\rho_* e^{-\frac{1}{K}[\dot{\theta} + \frac{1}{2}(\nabla\theta)^2]}. \quad (77)$$

The equation of motion is

$$\frac{\partial}{\partial t} \left( e^{-\frac{1}{K}[\dot{\theta} + \frac{1}{2}(\nabla\theta)^2]} \right) + \nabla \cdot \left( e^{-\frac{1}{K}[\dot{\theta} + \frac{1}{2}(\nabla\theta)^2]} \nabla\theta \right) = 0. \quad (78)$$



We note that the constant  $K$  (temperature) cannot be eliminated from this equation contrary to the case of the polytropic gas.

## 2.8. Chaplygin Gas

The Chaplygin equation of state reads [53]

$$P = \frac{K}{\rho}. \quad (79)$$

The ordinary Chaplygin gas corresponds to  $K < 0$ . The case  $K > 0$  is called the anti-Chaplygin gas. Equation (79) is a particular polytropic equation of state (67) corresponding to  $\gamma = -1$ <sup>17</sup>. It can be obtained from the potential

$$V(\rho) = -\frac{K}{2\rho} \quad i.e. \quad V(|\psi|^2) = -\frac{K}{2|\psi|^2}. \quad (80)$$

The Lagrangian and the Hamiltonian are given by Equations (47) and (53) with Equation (80). The Bernoulli Equation (50) takes the form

$$\dot{\theta} + \frac{1}{2}(\nabla\theta)^2 + \frac{K}{2\rho^2} = 0, \quad (81)$$

yielding

$$\rho = \sqrt{\frac{-K}{2\left[\dot{\theta} + \frac{1}{2}(\nabla\theta)^2\right]}}. \quad (82)$$

The reduced Lagrangian  $\mathcal{L}(x)$  corresponding to the Chaplygin gas is

$$\mathcal{L} = P = K\sqrt{\frac{2}{-K}\left[\dot{\theta} + \frac{1}{2}(\nabla\theta)^2\right]}. \quad (83)$$

The equation of motion is

$$\frac{\partial}{\partial t} \left[ \frac{1}{\sqrt{|\dot{\theta} + \frac{1}{2}(\nabla\theta)^2|}} \right] + \nabla \cdot \left[ \frac{\nabla\theta}{\sqrt{|\dot{\theta} + \frac{1}{2}(\nabla\theta)^2|}} \right] = 0. \quad (84)$$

The Chaplygin constant  $K$  does not appear in this equation. If we consider time-independent solutions, this equation reduces to

$$\nabla \cdot \left( \frac{\nabla\theta}{\sqrt{(\nabla\theta)^2}} \right) = 0. \quad (85)$$

The same equation is obtained by taking the massless limit (recalling that  $\theta = S/m$ ). In the theory of  $d$ -branes, this equation means that the surface  $\theta(x_1, x_2, \dots, x_d) = \text{const}$  has zero extrinsic mean curvature [68]. This solution exists only when  $K < 0$ .

*Remark:* The Lagrangian (83) was obtained by [58,59] in two different manners: (i) starting from the Lagrangian (47) with Equation (80) and using the Bernoulli Equation (81) to eliminate  $\rho$  as we have done here; (ii) for a  $d$ -brane moving in a  $(d+1, 1)$  space-time. In that second case, it can be obtained from the Nambu-Goto action in the light-cone parametrization. This explains the connection between  $d$ -branes and the hydrodynamics of the Chaplygin gas.

### 2.9. Standard BEC

The potential of a standard BEC described by the ordinary GP equation is

$$V(|\psi|^2) = \frac{2\pi a_s \hbar^2}{m^3} |\psi|^4 \quad \text{i.e.} \quad V(\rho) = \frac{2\pi a_s \hbar^2}{m^3} \rho^2, \quad (86)$$

where  $a_s$  is the scattering length of the bosons (the interaction is repulsive when  $a_s > 0$  and attractive when  $a_s < 0$ ). This quartic potential accounts for two-body interactions in a weakly interacting microscopic theory of superfluidity [140]. The corresponding equation of state is

$$P = \frac{2\pi a_s \hbar^2}{m^3} \rho^2. \quad (87)$$

This is the equation of state of a polytrope of index  $\gamma = 2$  and polytropic constant  $K = 2\pi a_s \hbar^2 / m^3$ . The Lagrangian and the Hamiltonian are given by Equations (47) and (53) with Equation (86). The Bernoulli Equation (50) takes the form

$$\dot{\theta} + \frac{1}{2}(\nabla\theta)^2 + 2K\rho = 0, \quad (88)$$

yielding

$$\rho = -\frac{1}{2K} \left[ \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right]. \quad (89)$$

The reduced Lagrangian  $\mathcal{L}(x)$  corresponding to the standard BEC is

$$\mathcal{L} = P = \frac{1}{4K} \left[ \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right]^2. \quad (90)$$

The equation of motion is

$$\frac{\partial}{\partial t} \left[ \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right] + \nabla \cdot \left( \left[ \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right] \nabla\theta \right) = 0. \quad (91)$$

The BEC constant  $K$  does not appear in this equation.

### 2.10. DM Superfluid

The potential of a superfluid (BEC) with a sextic self-interaction is

$$V(|\psi|^2) = \frac{1}{2}K|\psi|^6 \quad \text{i.e.} \quad V(\rho) = \frac{1}{2}K\rho^3. \quad (92)$$

This potential accounts for three-body interactions in a weakly interacting microscopic theory of superfluidity [133]. The potential (92) may also describe a more exotic DM superfluid [141]. In that case, it has a completely different interpretation. The corresponding equation of state is

$$P = K\rho^3. \quad (93)$$

This is the equation of state of a polytrope of index  $\gamma = 3$ . The Lagrangian and the Hamiltonian are given by Equations (47) and (53) with Equation (92). The Bernoulli Equation (50) takes the form

$$\dot{\theta} + \frac{1}{2}(\nabla\theta)^2 + \frac{3K}{2}\rho^2 = 0, \quad (94)$$

yielding

$$\rho = \sqrt{-\frac{2}{3K} \left[ \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right]}. \quad (95)$$

The reduced Lagrangian  $\mathcal{L}(x)$  corresponding to the superfluid is

$$\mathcal{L} = P = K \left\{ -\frac{2}{3K} \left[ \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right] \right\}^{3/2}. \quad (96)$$

The equation of motion is

$$\frac{\partial}{\partial t} \left[ \sqrt{\left| \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right|} \right] + \nabla \cdot \left( \sqrt{\left| \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right|} \nabla\theta \right) = 0. \quad (97)$$

The superfluid constant  $K$  does not appear in this equation. If we consider time-independent solutions, this equation reduces to

$$\nabla \cdot (|\nabla\theta| \nabla\theta) = 0. \quad (98)$$

This solution exists only when  $K < 0$ . Interestingly, there is a connection between a superfluid described by Equation (98) and the modified Newtonian dynamics (MOND) theory (see, e.g., [141,142] for more details).

### 2.11. Logotropic Gas

Finally, we consider the logotropic equation of state [118]

$$P = A \ln \left( \frac{\rho}{\rho_*} \right), \quad (99)$$

which can be obtained from the potential [136]

$$V(\rho) = -A \ln \left( \frac{\rho}{\rho_*} \right) - A \quad i.e. \quad V(|\psi|^2) = -A \ln \left( \frac{|\psi|^2}{\rho_*} \right) - A. \quad (100)$$

The Lagrangian and the Hamiltonian are given by Equations (47) and (53) with Equation (100). The Bernoulli Equation (50) takes the form

$$\dot{\theta} + \frac{1}{2}(\nabla\theta)^2 - \frac{A}{\rho} = 0, \quad (101)$$

yielding

$$\rho = \frac{A}{\dot{\theta} + \frac{1}{2}(\nabla\theta)^2}. \quad (102)$$

The reduced Lagrangian  $\mathcal{L}(x)$  corresponding to the logotropic gas is

$$\mathcal{L} = P = -A \ln \left[ \frac{\rho_*}{A} \left( \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right) \right]. \quad (103)$$

The equation of motion is

$$\frac{\partial}{\partial t} \left( \frac{1}{\left| \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right|} \right) + \nabla \cdot \left( \frac{\nabla\theta}{\left| \dot{\theta} + \frac{1}{2}(\nabla\theta)^2 \right|} \right) = 0. \quad (104)$$

The logotropic constant  $A$  does not appear in this equation.

*Remark:* We can recover these results from the polytropic equation of state of Section 2.6 by considering the limit  $\gamma \rightarrow 0$ ,  $K \rightarrow \infty$  with  $A = K\gamma$  constant [118,143]. Starting from Equation (71), we get

$$\mathcal{L} = K \left[ -\frac{\gamma-1}{K\gamma} x \right]^{\frac{\gamma}{\gamma-1}} \simeq K e^{-\gamma \ln\left(\frac{x}{K\gamma}\right)} \simeq K \left[ 1 - \gamma \ln\left(\frac{x}{K\gamma}\right) + \dots \right] \simeq K - A \ln\left(\frac{x}{A}\right), \quad (105)$$

which is equivalent to Equation (103) up to a constant term (see notes 9 and 12).

### 2.12. Summary

For a polytropic equation of state  $P = K\rho^\gamma$  with  $\gamma \neq 1$ , the reduced Lagrangian is

$$\mathcal{L}(x) = K \left( -\frac{\gamma-1}{K\gamma} x \right)^{\frac{\gamma}{\gamma-1}}. \quad (106)$$

It is a pure power-law  $\mathcal{L} \propto x^{\frac{\gamma}{\gamma-1}}$ . In particular, for the Chaplygin gas ( $\gamma = -1$ ), for the standard BEC ( $\gamma = 2$ ) and for the DM superfluid ( $\gamma = 3$ ) we have  $\mathcal{L} \propto x^{1/2}$ ,  $\mathcal{L} \propto x^2$  and  $\mathcal{L} \propto x^{3/2}$  respectively. For the unitary Fermi gas ( $\gamma = 5/3$ ) we have  $\mathcal{L} \propto x^{5/2}$ . For an isothermal equation of state  $P = K\rho$ , the reduced Lagrangian is

$$\mathcal{L}(x) = K\rho_* e^{-x/K}. \quad (107)$$

For a logotropic equation of state  $P = A \ln(\rho/\rho_*)$ , it reads

$$\mathcal{L}(x) = -A \ln\left(\frac{\rho_*}{A} x\right). \quad (108)$$

## 3. Relativistic Theory

### 3.1. Klein-Gordon Equation

We consider a relativistic complex SF  $\varphi(x^\mu) = \varphi(x, y, z, t)$  which is a continuous function of space and time. It can represent the wavefunction of a relativistic BEC. The action of the SF can be written as

$$S = \int \mathcal{L} \sqrt{-g} d^4x, \quad (109)$$

where  $\mathcal{L} = \mathcal{L}(\varphi, \varphi^*, \partial_\mu \varphi, \partial_\mu \varphi^*)$  is the Lagrangian density and  $g = \det(g_{\mu\nu})$  is the determinant of the metric tensor. We consider a canonical Lagrangian density of the form

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^* \partial_\nu \varphi - V_{\text{tot}}(|\varphi|^2), \quad (110)$$

where the first term is the kinetic energy and the second term is minus the potential energy. The potential energy can be decomposed into a rest-mass energy term and a self-interaction energy term:

$$V_{\text{tot}}(|\varphi|^2) = \frac{1}{2} \frac{m^2 c^2}{\hbar^2} |\varphi|^2 + V(|\varphi|^2). \quad (111)$$

The least action principle  $\delta S = 0$  with respect to variations of the SF  $\delta\varphi$  (or  $\delta\varphi^*$ ), which is equivalent to the Euler-Lagrange equation

$$D_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)^*} \right] - \frac{\partial \mathcal{L}}{\partial \varphi^*} = 0, \quad (112)$$

yields the KG equation

$$\square \varphi + 2 \frac{dV_{\text{tot}}}{d|\varphi|^2} \varphi = 0, \quad (113)$$

where  $\square = D_\mu \partial^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$  is the d'Alembertian operator in a curved space-time. It can be written explicitly as

$$\square \varphi = D_\mu \partial^\mu \varphi = g^{\mu\nu} D_\mu \partial_\nu \varphi = g^{\mu\nu} (\partial_\mu \partial_\nu \varphi - \Gamma_{\mu\nu}^\sigma \partial_\sigma \varphi) = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \varphi). \quad (114)$$

For a free massless SF ( $V_{\text{tot}} = 0$ ), the KG equation reduces to  $\square \varphi = 0$ .

The energy-momentum (stress) tensor is given by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{-g}} \frac{\partial(\sqrt{-g} \mathcal{L})}{\partial g^{\mu\nu}} = 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu} \mathcal{L}. \quad (115)$$

For a complex SF, we have

$$T_\mu^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi^*)} \partial_\mu \varphi^* - g_\mu^\nu \mathcal{L}. \quad (116)$$

For the Lagrangian (110), we obtain

$$T_{\mu\nu} = \frac{1}{2} (\partial_\mu \varphi^* \partial_\nu \varphi + \partial_\nu \varphi^* \partial_\mu \varphi) - g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\sigma} \partial_\rho \varphi^* \partial_\sigma \varphi - V_{\text{tot}}(|\varphi|^2) \right]. \quad (117)$$

The conservation of the energy-momentum tensor, which results from the invariance of the Lagrangian density under continuous translations in space and time (Noether theorem [144]), reads

$$D_\nu T^{\mu\nu} = 0. \quad (118)$$

The energy-momentum four vector is  $P^\mu = \int T^{\mu 0} \sqrt{-g} d^3x$ . Its time component  $P^0$  is the energy while  $\mathbf{P}$  is the impulse. Each component of  $P^\mu$  is conserved in time, i.e., it is a constant of motion. Indeed, we have

$$\dot{P}^\mu = \frac{d}{dt} \int T^{\mu 0} \sqrt{-g} d^3x = c \int \partial_0 (T^{\mu 0} \sqrt{-g}) d^3x = -c \int \partial_i (T^{\mu i} \sqrt{-g}) d^3x = 0, \quad (119)$$

where we have used Equation (118) with  $D_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} V^\mu)$  to get the third equality.

The current of charge of a complex SF is given by

$$J^\mu = \frac{m}{i\hbar} \left[ \varphi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \varphi^* \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \right]. \quad (120)$$

For the Lagrangian (110), we obtain

$$J_\mu = -\frac{m}{2i\hbar} (\varphi^* \partial_\mu \varphi - \varphi \partial_\mu \varphi^*). \quad (121)$$

Using the KG Equation (113), one can show that

$$D_\mu J^\mu = 0. \quad (122)$$

This equation expresses the local conservation of the charge. The total charge of the SF is

$$Q = \frac{e}{mc} \int J^0 \sqrt{-g} d^3x, \quad (123)$$

and we easily find from Equation (122) that  $\dot{Q} = 0$ . The charge  $Q$  is proportional to the number  $N$  of bosons provided that antibosons are counted negatively [145]. Therefore, Equation (122) also expresses the local conservation of the boson number ( $Q = Ne$ ). This



conservation law results via the Noether theorem from the global  $U(1)$  symmetry of the Lagrangian, i.e., from the invariance of the Lagrangian density under a global phase transformation  $\varphi \rightarrow \varphi e^{-i\theta}$  (rotation) of the complex SF. Note that  $J_\mu$  vanishes for a real SF so the charge and the particle number are not conserved in that case.

The Einstein-Hilbert action of general relativity is

$$S_g = \frac{c^4}{16\pi G} \int R \sqrt{-g} d^4x, \quad (124)$$

where  $R$  is the Ricci scalar curvature. The least action principle  $\delta S_g = 0$  with respect to variations of the metric  $\delta g^{\mu\nu}$  yields the Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \quad (125)$$

The contracted Bianchi identity  $D_\nu G^{\mu\nu} = 0$  implies the conservation of the energy momentum tensor ( $D_\nu T^{\mu\nu} = 0$ ).

### 3.2. The de Broglie Transformation

We can write the KG Equation (113) under the form of hydrodynamic equations by using the de Broglie [146–148] transformation. To that purpose, we write the SF as

$$\varphi = \frac{\hbar}{m} \sqrt{\rho} e^{iS_{\text{tot}}/\hbar}, \quad (126)$$

where  $\rho$  is the pseudo rest-mass density<sup>18</sup> and  $S_{\text{tot}}$  is the action. They are given by

$$\rho = \frac{m^2}{\hbar^2} |\varphi|^2 \quad \text{and} \quad S_{\text{tot}} = \frac{\hbar}{2i} \ln \left( \frac{\varphi}{\varphi^*} \right). \quad (127)$$

For convenience, we define  $\theta = S_{\text{tot}}/m$  so that Equation (126) can be rewritten as<sup>19</sup>

$$\varphi = \frac{\hbar}{m} \sqrt{\rho} e^{im\theta/\hbar}. \quad (128)$$

Substituting this expression into the Lagrangian density (110), we obtain

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \rho \partial_\mu \theta \partial_\nu \theta + \frac{\hbar^2}{8m^2 \rho} g^{\mu\nu} \partial_\mu \rho \partial_\nu \rho - V_{\text{tot}}(\rho) \quad (129)$$

with

$$V_{\text{tot}}(\rho) = \frac{1}{2} \rho c^2 + V(\rho). \quad (130)$$

The Euler-Lagrange equations for  $\theta$  and  $\rho$ , expressing the least action principle, are

$$D_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \theta)} \right] - \frac{\partial \mathcal{L}}{\partial \theta} = 0, \quad (131)$$

$$D_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \rho)} \right] - \frac{\partial \mathcal{L}}{\partial \rho} = 0. \quad (132)$$

They yield

$$D_\mu (\rho \partial^\mu \theta) = 0, \quad (133)$$

$$\frac{1}{2}\partial_\mu\theta\partial^\mu\theta - \frac{\hbar^2}{2m^2}\frac{\square\sqrt{\rho}}{\sqrt{\rho}} - V'_{\text{tot}}(\rho) = 0. \quad (134)$$

The same equations are obtained by substituting the de Broglie transformation from Equation (128) into the KG Equation (113), and by separating the real and the imaginary parts. Equation (133) can be interpreted as a continuity equation and Equation (134) can be interpreted as a quantum relativistic Hamilton-Jacobi (or Bernoulli) equation with a relativistic covariant quantum potential

$$Q = \frac{\hbar^2}{2m}\frac{\square\sqrt{\rho}}{\sqrt{\rho}}. \quad (135)$$

Introducing the pseudo quadrivelocity<sup>20</sup>

$$v_\mu = -\frac{\partial_\mu S_{\text{tot}}}{m} = -\partial_\mu\theta, \quad (136)$$

we can rewrite Equations (133) and (134) as

$$D_\mu(\rho v^\mu) = 0, \quad (137)$$

$$\frac{1}{2}mv_\mu v^\mu - Q - mV'_{\text{tot}}(\rho) = 0. \quad (138)$$

Taking the gradient of the quantum Hamilton-Jacobi Equation (138) we obtain [132]

$$\frac{dv_\nu}{dt} \equiv v^\mu D_\mu v_\nu = \frac{1}{m}\partial_\nu Q + \partial_\nu V'(\rho), \quad (139)$$

which can be interpreted as a relativistic quantum Euler equation (with the limitations of note 20). The first term on the right hand side can be interpreted as a quantum force and the second term as a pressure force  $(1/\rho)\partial_\nu P$  such that  $(1/\rho)P'(\rho) = h'(\rho) = V''(\rho)$ , where  $h$  is the pseudo enthalpy. We note that the pressure is determined by Equations (25)–(28) as in the nonrelativistic case.

The energy-momentum tensor is given by Equation (115) or, in the hydrodynamic representation, by

$$T_\mu^\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\theta)}\partial_\mu\theta + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\rho)}\partial_\mu\rho - g_\mu^\nu\mathcal{L}. \quad (140)$$

For the Lagrangian (129) we obtain

$$T_{\mu\nu} = \rho\partial_\mu\theta\partial_\nu\theta + \frac{\hbar^2}{4m^2\rho}\partial_\mu\rho\partial_\nu\rho - g_{\mu\nu}\mathcal{L}. \quad (141)$$

The current of charge of a complex SF is given by

$$J^\mu = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\theta)} \quad (142)$$

For the Lagrangian (110), we obtain

$$J_\mu = -\frac{\rho}{m}\partial_\mu S_{\text{tot}} = -\rho\partial_\mu\theta = \rho v_\mu. \quad (143)$$

This result can also be obtained from Equation (121) by using Equation (126) coming from the de Broglie transformation. We then see that the continuity Equation (133) or (137)

is equivalent to Equation (122). It expresses the conservation of the charge  $Q$  of the SF (or the conservation of the boson number  $N$ )

$$Q = Ne = -\frac{e}{mc} \int \rho \partial^0 \theta \sqrt{-g} d^3x. \quad (144)$$

Assuming  $\partial_\mu \theta \partial^\mu \theta > 0$ , we can introduce the fluid quadrivelocity<sup>21</sup>

$$u_\mu = -\frac{\partial_\mu \theta}{\sqrt{\partial_\mu \theta \partial^\mu \theta}} c, \quad (145)$$

which satisfies the identity

$$u_\mu u^\mu = c^2. \quad (146)$$

Using Equations (143) and (145), we can write the current as

$$J_\mu = \frac{\rho}{c} \sqrt{\partial_\mu \theta \partial^\mu \theta} u_\mu \quad (147)$$

and the continuity equation as

$$D_\mu \left[ \rho \sqrt{\partial_\mu \theta \partial^\mu \theta} u^\mu \right] = 0. \quad (148)$$

The rest-mass density  $\rho_m = nm$  (which is proportional to the charge density  $\rho_e$ ) is defined by

$$J_\mu = \rho_m u_\mu. \quad (149)$$

Using Equation (146), we see that  $\rho_m c^2 = u_\mu J^\mu$ . The continuity Equation (122) can be written as

$$D_\mu (\rho_m u^\mu) = 0. \quad (150)$$

Comparing Equation (147) with Equation (149), we find that the rest-mass density  $\rho_m = nm$  of the SF is given by

$$\rho_m = \frac{\rho}{c} \sqrt{\partial_\mu \theta \partial^\mu \theta}. \quad (151)$$

Using the Bernoulli Equation (134), we get

$$\rho_m = \frac{\rho}{c} \sqrt{\frac{\hbar^2}{m^2} \frac{\square \sqrt{\rho}}{\sqrt{\rho}} + 2V'_{\text{tot}}(\rho)}. \quad (152)$$

*Remark:* More generally, we can define the quadrivelocity by

$$u^\mu = \frac{J^\mu}{\sqrt{J_\mu J^\mu}} c, \quad (153)$$

which satisfies the identity (146). Using Equation (149) we find that the rest-mass (or charge) density is given by

$$\rho_m = \frac{1}{c} \sqrt{J_\mu J^\mu}. \quad (154)$$

We note that  $J^0$  is not equal to the rest-mass density in general ( $\rho_m \neq J^0/c$ ), except if the SF is static in which case  $u^\mu = c \delta_0^\mu$  and  $J^0 = \rho_m c$ .

### 3.3. TF Approximation

In the classical or TF limit ( $\hbar \rightarrow 0$ ), the Lagrangian from Equation (129) reduces to

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \rho \partial_\mu \theta \partial_\nu \theta - V_{\text{tot}}(\rho). \quad (155)$$

The Euler-Lagrange Equations (131) and (132) yield the equations of motion

$$D_\mu (\rho \partial^\mu \theta) = 0, \quad (156)$$

$$\frac{1}{2} \partial_\mu \theta \partial^\mu \theta - V'_{\text{tot}}(\rho) = 0. \quad (157)$$

The same equations are obtained by making the TF approximation in Equation (134), i.e., by neglecting the quantum potential. Equation (156) can be interpreted as a continuity equation and Equation (157) can be interpreted as a classical relativistic Hamilton-Jacobi (or Bernoulli) equation. In order to determine the rest mass density, we can proceed as before. Assuming  $V'_{\text{tot}} > 0$ , and using Equation (157), we introduce the fluid quadrivelocity

$$u_\mu = - \frac{\partial_\mu \theta}{\sqrt{2V'_{\text{tot}}(\rho)}} c, \quad (158)$$

which satisfies the identity (146). Using Equations (143) and (158), we can write the current as

$$J_\mu = \frac{\rho}{c} \sqrt{2V'_{\text{tot}}(\rho)} u_\mu \quad (159)$$

and the continuity Equation (156) as

$$D_\mu \left[ \rho \sqrt{2V'_{\text{tot}}(\rho)} u^\mu \right] = 0. \quad (160)$$

Comparing Equation (159) with Equation (149), we find that the rest-mass density  $\rho_m = nm$  is given, in the TF approximation, by

$$\rho_m = \frac{\rho}{c} \sqrt{2V'_{\text{tot}}(\rho)}. \quad (161)$$

In general,  $\rho_m \neq \rho$  except (i) for a noninteracting SF ( $V = 0$ ), (ii) when  $V$  is constant, corresponding to the  $\Lambda$ CDM model (see below), and (iii) in the nonrelativistic limit  $c \rightarrow +\infty$ .

The energy-momentum tensor is given by Equation (115) or, in the hydrodynamic representation, by Equation (140). For the Lagrangian (155) we obtain

$$T_{\mu\nu} = \rho \partial_\mu \theta \partial_\nu \theta - g_{\mu\nu} \mathcal{L} \quad (162)$$

or, using Equation (158),

$$T_{\mu\nu} = 2\rho V'_{\text{tot}}(\rho) \frac{u_\mu u_\nu}{c^2} - g_{\mu\nu} \mathcal{L}. \quad (163)$$

The energy-momentum tensor (163) can be written under the perfect fluid form<sup>22</sup>

$$T_{\mu\nu} = (\epsilon + P) \frac{u_\mu u_\nu}{c^2} - P g_{\mu\nu}, \quad (165)$$

where  $\epsilon$  is the energy density and  $P$  is the pressure, provided that we make the identifications

$$P = \mathcal{L}, \quad \epsilon + P = 2\rho V'_{\text{tot}}(\rho). \quad (166)$$

Therefore, the Lagrangian plays the role of the pressure of the fluid. Combining Equation (155) with the Bernoulli Equation (157), we get

$$\mathcal{L} = \rho V'_{\text{tot}}(\rho) - V_{\text{tot}}(\rho). \quad (167)$$

Therefore, according to Equations (166) and (167), the energy density and the pressure derived from the Lagrangian (155) are given by

$$\epsilon = \rho V'_{\text{tot}}(\rho) + V_{\text{tot}}(\rho) = \rho c^2 + \rho V'(\rho) + V(\rho), \quad (168)$$

$$P = \rho V'_{\text{tot}}(\rho) - V_{\text{tot}}(\rho) = \rho V'(\rho) - V(\rho), \quad (169)$$

where we have used Equation (130) to get the second equalities. Eliminating  $\rho$  between these equations, we obtain the equation of state  $P(\epsilon)$ . Equation (169) for the pressure is exactly the same as Equation (26) obtained in the nonrelativistic limit. Therefore, knowing  $P(\rho)$ , we can obtain the SF potential  $V(\rho)$  by the formula [see Equation (28)]<sup>23</sup>

$$V(\rho) = \rho \int \frac{P(\rho)}{\rho^2} d\rho. \quad (170)$$

The squared speed of sound is

$$c_s^2 = P'(\epsilon)c^2 = \frac{\rho V''(\rho)c^2}{c^2 + \rho V'''(\rho) + 2V'(\rho)}. \quad (171)$$

*Remark:* In [149] we have considered a spatially homogeneous complex SF in an expanding universe described by the Klein-Gordon-Friedmann (KGF) equations. In the fast oscillation regime  $\omega \gg H$ , where  $\omega$  is the pulsation of the SF and  $H$  the Hubble constant, we can average the KG equation over the oscillations of the SF (see Appendix A of [149] and references therein) and obtain a virial relation leading to Equations (168) and (169). These equations can also be obtained by transforming the KG equation into hydrodynamic equations, taking the limit  $\hbar \rightarrow 0$ , and using the Bernoulli equation (see Section II of [149])<sup>24</sup>. This is similar to the derivation given here. However, the present derivation is more general since it applies to a possibly inhomogeneous SF [71]. An interest of the results of [149] is to show that the fast oscillation approximation in cosmology is equivalent to the TF approximation.

### 3.4. Reduced Lagrangian $\mathcal{L}(X)$

In the previous section, we have used the Bernoulli Equation (157) to eliminate  $\theta$  from the Lagrangian, leading to Equation (167). Here, we eliminate  $\rho$  from the Lagrangian. Introducing the notation

$$X = \frac{1}{2} \partial_\mu \theta \partial^\mu \theta, \quad (172)$$

the Bernoulli Equation (157) can be written as

$$X = V'_{\text{tot}}(\rho). \quad (173)$$



Assuming  $V''_{\text{tot}} > 0$ , this equation can be reversed to give

$$\rho = G(X) \quad (174)$$

with  $G(X) = (V'_{\text{tot}})^{-1}(X)$ . As a result, the equation of continuity (156) can be written as

$$D_\mu [G(X) \partial^\mu \theta] = 0. \quad (175)$$

On the other hand, according to Equations (168), (169) and (174), we have  $\epsilon = \epsilon(X)$  and  $P = P(X)$ . Therefore,

$$\mathcal{L} = P(X). \quad (176)$$

In this manner, we have eliminated the pseudo rest-mass density  $\rho$  from the Lagrangian (155) and we have obtained a reduced Lagrangian of the form  $\mathcal{L}(X)$  that depends only on  $X$ . This kind of Lagrangian, called  $k$ -essence Lagrangian, is specifically discussed in Appendix B. We show below that

$$\rho = G(X) = P'(X) = \mathcal{L}'(X). \quad (177)$$

Using Equation (177), we can rewrite Equation (175) in terms of  $\mathcal{L}'(X)$  as

$$D_\mu [\mathcal{L}'(X) \partial^\mu \theta] = 0. \quad (178)$$

We also show below that

$$\epsilon = 2XP'(X) - P. \quad (179)$$

If we know  $\epsilon = \epsilon(P)$  we can solve this differential equation to obtain  $P(X)$ , hence  $\mathcal{L}(X)$ .

*Proof of Equations (177) and (179):* According to Equations (169) and (173), we have

$$P'(\rho) = \rho V''_{\text{tot}}(\rho) \quad (180)$$

and

$$\frac{dX}{d\rho} = V'_{\text{tot}}(\rho). \quad (181)$$

Starting from Equation (176) and using Equations (180) and (181) we obtain

$$\mathcal{L}'(X) = P'(X) = P'(\rho) \frac{d\rho}{dX} = \rho V''_{\text{tot}}(\rho) \frac{d\rho}{dX} = \rho \frac{dX}{d\rho} \frac{d\rho}{dX} = \rho, \quad (182)$$

which establishes Equation (177). On the other hand, according to Equations (168) and (169), we have

$$\epsilon + P = 2\rho V'_{\text{tot}}(\rho). \quad (183)$$

Using Equations (173) and (182), we obtain

$$\epsilon + P = 2\rho X = 2XP'(X), \quad (184)$$

which establishes Equation (179).

*Remark:* We can obtain the preceding results in a more direct and more general manner from a  $k$ -essence Lagrangian  $\mathcal{L}(X)$  by using the results of Appendix B. The present calculations show how a  $k$ -essential Lagrangian arises from the canonical Lagrangian of a complex SF  $\varphi$  in the TF limit. In that case, the real SF  $\theta$  represents the phase of the complex SF  $\varphi$ .

### 3.5. Nonrelativistic Limit

To obtain the nonrelativistic limit of the foregoing equations, we first have to make the Klein transformation (14) then take the limit  $c \rightarrow +\infty$ . In this manner, the KG Equation (113) reduces to the GP Equation (12) and the relativistic hydrodynamic Equations (137)–(139) reduce to the nonrelativistic Equations (20)–(24). These transformations are discussed in detail in [131,132,150] for self-gravitating BECs. Here, we consider the nongravitational case and we focus on the nonrelativistic limit of the Lagrangian  $\mathcal{L}(X)$  from Section 3.4 leading to the Lagrangian  $\mathcal{L}(x)$  from Section 2.5.

Since  $\mathcal{L} = P$  in the two cases, we just have to find the relation between  $X$  and  $x$  when  $c \rightarrow +\infty$ . Making the Klein transformation<sup>25</sup>

$$\theta = \theta_{\text{NR}} - c^2 t \quad (185)$$

in Equation (172), we obtain

$$\begin{aligned} X &= \frac{1}{2} \partial_\mu \theta \partial^\mu \theta \\ &\simeq \frac{1}{2c^2} \left( \frac{\partial \theta}{\partial t} \right)^2 - \frac{1}{2} (\nabla \theta)^2 \\ &\simeq \frac{1}{2c^2} \left( \frac{\partial \theta_{\text{NR}}}{\partial t} \right)^2 - \frac{\partial \theta_{\text{NR}}}{\partial t} + \frac{c^2}{2} - \frac{1}{2} (\nabla \theta_{\text{NR}})^2. \end{aligned} \quad (186)$$

When  $c \rightarrow +\infty$ , we find that

$$X \sim \frac{c^2}{2}. \quad (187)$$

The nonrelativistic limit is then given by

$$\frac{c^2}{2} - X \rightarrow \dot{\theta}_{\text{NR}} + \frac{1}{2} (\nabla \theta_{\text{NR}})^2. \quad (188)$$

Therefore, when  $c \rightarrow +\infty$ , we can write

$$X \simeq \frac{c^2}{2} - x, \quad (189)$$

where  $x$  is defined by Equation (57).

Using Equation (189) we can easily check that the equations of Section 3.4 return the equations of Section 2.5 in the nonrelativistic limit. For example, using Equation (130), the relation  $X = V'_{\text{tot}}(\rho)$  reduces to  $x = -V'(\rho)$ . On the other hand, using  $\epsilon \sim \rho c^2$ , Equation (179) reduces to

$$\rho \sim P'(x) \frac{dx}{dX} \sim -P'(x) \sim -\mathcal{L}'(x), \quad (190)$$

which, together with Equation (177), returns Equation (62).

### 3.6. Enthalpy

Using Equations (A163), (168) and (169) we find that the enthalpy is given by

$$h = 2 \frac{\rho}{\rho_m} V'_{\text{tot}}(\rho). \quad (191)$$

Using Equation (161), we obtain

$$h = \sqrt{2V'_{\text{tot}}(\rho)} c. \quad (192)$$

According to Equation (157), the enthalpy can be written as

$$h = c\sqrt{\partial_\mu\theta\partial^\mu\theta} = c\sqrt{2X}. \quad (193)$$

Substituting Equation (130) into Equation (192), subtracting  $c^2$ , and taking the nonrelativistic limit  $c \rightarrow +\infty$ , we recover Equation (27).

#### 4. General Equation of State

In this section, we provide general results valid for an arbitrary equation of state. We consider three different manners to specify the equation of state depending on whether the pressure  $P$  is expressed as a function of (i) the energy density  $\epsilon$ ; (ii) the rest-mass density  $\rho_m$ ; (iii) the pseudo rest-mass density  $\rho$ . In each case, we determine the pressure  $P$ , the energy density  $\epsilon$ , the rest-mass density  $\rho_m$ , the internal energy  $u$ , the pseudo rest-mass density  $\rho$ , the SF potential  $V_{\text{tot}}(\rho)$ , and the k-essence Lagrangian  $\mathcal{L}(X)$ .

##### 4.1. Equation of State of Type I

We first consider an equation of state of type I (see Appendix C) where the pressure is given as a function of the energy density:  $P = P(\epsilon)$ .

##### 4.1.1. Determination of $\rho_m$ , $P(\rho_m)$ and $u(\rho_m)$

Using the results of Appendix D, we can obtain the rest-mass density  $\rho_m = nm$  and the internal energy  $u$  as follows. According to Equation (A162), we have

$$\ln \rho_m = \int \frac{d\epsilon}{P(\epsilon) + \epsilon}, \quad (194)$$

which determines  $\rho_m(\epsilon)$ . Eliminating  $\epsilon$  between  $P(\epsilon)$  and  $\rho_m(\epsilon)$  we obtain  $P(\rho_m)$ . On the other hand, according to Equation (A160), we have

$$u = \epsilon - \rho_m(\epsilon)c^2. \quad (195)$$

Eliminating  $\epsilon$  between Equations (194) and (195), we obtain  $u(\rho_m)$ . We can also obtain  $u(\rho_m)$  from  $P(\rho_m)$ , or the converse, by using Equations (A165) and (A166).

##### 4.1.2. Determination of $\rho$ , $P(\rho)$ and $V_{\text{tot}}(\rho)$

Using the results of Section 3.3, we can obtain the pseudo rest-mass density  $\rho$  and the SF potential  $V_{\text{tot}}$  as follows. According to Equations (168) and (169), we have

$$\epsilon - P = 2V_{\text{tot}}(\rho), \quad (196)$$

$$\epsilon + P = 2\rho V'_{\text{tot}}(\rho). \quad (197)$$

Differentiating Equation (196) and using Equation (197), we get

$$d(\epsilon - P) = 2V'_{\text{tot}}(\rho)d\rho = \frac{\epsilon + P}{\rho}d\rho. \quad (198)$$

This yields

$$\ln \rho = \int \frac{1 - P'(\epsilon)}{\epsilon + P(\epsilon)} d\epsilon, \quad (199)$$

which determines  $\rho(\epsilon)$ . Eliminating  $\epsilon$  between  $P(\epsilon)$  and  $\rho(\epsilon)$  we obtain  $P(\rho)$ . On the other hand, according to Equation (196), we have

$$V_{\text{tot}} = \frac{1}{2}[\epsilon - P(\epsilon)]. \quad (200)$$

Eliminating  $\epsilon$  between Equations (199) and (200), we obtain  $V_{\text{tot}}(\rho)$ . We can also obtain  $V(\rho)$  from  $P(\rho)$ , or the converse, by using Equations (169) and (170).

*Remark:* If the relation  $\epsilon(P)$  is more explicit than  $P(\epsilon)$ , we can use

$$\ln \rho = \int \frac{\epsilon'(P) - 1}{\epsilon(P) + P} dP \quad (201)$$

and

$$V_{\text{tot}} = \frac{1}{2}[\epsilon(P) - P], \quad (202)$$

instead of Equations (199) and (200). The first equation gives  $\rho(P)$ . Eliminating  $P$  between Equations (201) and (202), we obtain  $V_{\text{tot}}(\rho)$ .

#### 4.1.3. Lagrangian $\mathcal{L}(X)$

If we know  $\epsilon = \epsilon(P)$  then, according to Equation (179), we have

$$\ln X = 2 \int \frac{dP}{\epsilon(P) + P}, \quad (203)$$

which determines  $X(P)$ . If this function can be inverted we get  $P(X)$ , hence  $\mathcal{L}(X)$ . If we know  $P = P(\epsilon)$ , we can rewrite Equation (203) as

$$\ln X = 2 \int \frac{P'(\epsilon)}{\epsilon + P(\epsilon)} d\epsilon, \quad (204)$$

which determines  $X(\epsilon)$ . If this function can be inverted we get  $\epsilon(X)$ , then  $P(X) = P[\epsilon(X)]$ , hence  $\mathcal{L}(X)$ .

#### 4.2. Equation of State of Type II

We now consider an equation of state of type II (see Appendix D) where the pressure is given as a function of the rest-mass density:  $P = P(\rho_m)$ . We can then determine the internal energy  $u(\rho_m)$  from Equation (A165). Inversely, we can specify the internal energy  $u(\rho_m)$  as a function of the rest-mass density and obtain the equation of state  $P(\rho_m)$  from Equation (A166).

##### 4.2.1. Determination of $\epsilon$ and $P(\epsilon)$

According to Equations (A160) and (A166), the energy density and the pressure are given by

$$\epsilon = \rho_m c^2 + u(\rho_m), \quad (205)$$

$$P = \rho_m u'(\rho_m) - u(\rho_m). \quad (206)$$

Eliminating  $\rho_m$  between Equations (205) and (206), we obtain  $P(\epsilon)$ .

#### 4.2.2. Determination of $\rho$ , $P(\rho)$ and $V_{\text{tot}}(\rho)$

According to Equation (199), we have

$$\ln \rho = \int \frac{\epsilon'(\rho_m) - P'(\rho_m)}{\epsilon(\rho_m) + P(\rho_m)} d\rho_m. \quad (207)$$

Then, using Equations (205) and (206), we obtain

$$\ln \rho = \int \frac{c^2 + u'(\rho_m) - \rho_m u''(\rho_m)}{\rho_m c^2 + \rho_m u'(\rho_m)} d\rho_m, \quad (208)$$

which determines  $\rho(\rho_m)$ . This equation can be integrated into

$$\rho = \frac{\rho_m}{1 + \frac{1}{c^2} u'(\rho_m)}, \quad (209)$$

where the constant of integration has been determined in order to obtain  $\rho = \rho_m$  in the nonrelativistic limit. Identifying  $\epsilon + P = 2\rho V'_{\text{tot}}(\rho)$  from Equations (168) and (169) with  $\epsilon + P = \rho_m(c^2 + u'(\rho_m))$  from Equations (205) and (206) we see that Equation (209) is equivalent to Equation (161). Eliminating  $\rho_m$  between  $P(\rho_m)$  and  $\rho(\rho_m)$ , we obtain  $P(\rho)$ . On the other hand, according to Equations (200), (205) and (206), we get

$$V_{\text{tot}} = \frac{1}{2} \left[ \rho_m c^2 + 2u(\rho_m) - \rho_m u'(\rho_m) \right]. \quad (210)$$

Eliminating  $\rho_m$  between Equations (208) and (210) we obtain  $V_{\text{tot}}(\rho)$ . We can also obtain  $V(\rho)$  from  $P(\rho)$ , or the converse, by using Equations (169) and (170).

#### 4.2.3. Lagrangian $\mathcal{L}(X)$

According to Equation (203), we have

$$\ln X = 2 \int \frac{P'(\rho_m)}{\epsilon(\rho_m) + P(\rho_m)} d\rho_m. \quad (211)$$

Using Equations (205) and (206), we obtain

$$\ln X = 2 \int \frac{u''(\rho_m)}{c^2 + u'(\rho_m)} d\rho_m, \quad (212)$$

which can be integrated into

$$X = \frac{1}{2c^2} \left[ c^2 + u'(\rho_m) \right]^2. \quad (213)$$

We have determined the constant of integration so that, in the nonrelativistic limit,  $X \sim c^2/2$  (see Section 3.5). From Equation (213) we obtain  $X(\rho_m)$ . If this function can be inverted we get  $\rho_m(X)$ , then  $P(X) = P[\rho_m(X)]$ , hence  $\mathcal{L}(X)$ .

#### 4.3. Equation of State of Type III

Finally, we consider an equation of state of type III (see Section 3 and Appendix E) where the pressure is given as a function of the pseudo rest-mass density:  $P = P(\rho)$ . We can then determine the SF potential  $V(\rho)$  from Equation (170)<sup>26</sup>. Inversely, we can specify the SF potential  $V(\rho)$  and obtain the equation of state  $P(\rho)$  from Equation (169).



#### 4.3.1. Determination of $\epsilon$ and $P(\epsilon)$

According to Equations (168) and (169), the energy density and the pressure are given by

$$\epsilon = \rho V'_{\text{tot}}(\rho) + V_{\text{tot}}(\rho), \quad (214)$$

$$P = \rho V'_{\text{tot}}(\rho) - V_{\text{tot}}(\rho). \quad (215)$$

Eliminating  $\rho$  between Equations (214) and (215), we obtain  $P(\epsilon)$ .

#### 4.3.2. Determination of $\rho_m$ , $P(\rho_m)$ and $u(\rho_m)$

According to Equation (A162), we have

$$\ln \rho_m = \int \frac{\epsilon'(\rho)}{\epsilon + P(\rho)} d\rho. \quad (216)$$

Using Equations (214) and (215), we obtain

$$\ln \rho_m = \int \frac{\rho V''_{\text{tot}}(\rho) + 2V'_{\text{tot}}(\rho)}{2\rho V'_{\text{tot}}(\rho)} d\rho, \quad (217)$$

which determines  $\rho_m(\rho)$ . This equation can be integrated into

$$\rho_m = \frac{\rho}{c} \sqrt{2V'_{\text{tot}}(\rho)}, \quad (218)$$

where the constant of integration has been determined in order to obtain  $\rho_m = \rho$  in the nonrelativistic limit. This relation is equivalent to Equation (161). Eliminating  $\rho$  between  $P(\rho)$  and  $\rho_m(\rho)$  we obtain  $P(\rho_m)$ . On the other hand, according to Equation (A160), we have

$$u = \epsilon - \rho_m c^2. \quad (219)$$

Using Equations (214) and (218), we obtain

$$\begin{aligned} u &= \rho V'_{\text{tot}}(\rho) + V_{\text{tot}}(\rho) - \rho_m(\rho) c^2 \\ &= \rho V'_{\text{tot}}(\rho) + V_{\text{tot}}(\rho) - \rho c \sqrt{2V'_{\text{tot}}(\rho)}. \end{aligned} \quad (220)$$

Eliminating  $\rho$  between Equations (217) and (220) we obtain  $u(\rho_m)$ . We can also obtain  $u(\rho_m)$  from  $P(\rho_m)$ , or the converse, by using Equations (A165) and (A166).

#### 4.3.3. Lagrangian $\mathcal{L}(X)$

According to Equation (173) we have

$$X = V'_{\text{tot}}(\rho), \quad (221)$$

which determines  $X(\rho)$ . If this function can be inverted we get  $\rho(X)$ , then  $P(X) = P[\rho(X)]$ , hence  $\mathcal{L}(X)$ .

### 5. Polytropes

In this section, we apply the general results of Section 4 to the case of a polytropic equation of state.

### 5.1. Polytropic Equation of State of Type I

The polytropic equation of state of type I reads [151]

$$P = K \left( \frac{\epsilon}{c^2} \right)^\gamma, \quad (222)$$

where  $K$  is the polytropic constant and  $\gamma = 1 + 1/n$  is the polytropic index. This is the equation of state of the GCG [95]. In the nonrelativistic regime, using  $\epsilon \sim \rho c^2$ , we recover Equation (67).

(i) For  $\gamma = -1$ , we obtain

$$P = \frac{Kc^2}{\epsilon}. \quad (223)$$

This is the equation of state of the Chaplygin ( $K < 0$ ) or anti-Chaplygin ( $K > 0$ ) gas [50,71,86,98].

(ii) For  $\gamma = 2$ , we obtain

$$P = K \left( \frac{\epsilon}{c^2} \right)^2. \quad (224)$$

This is the equation of state of the standard BEC with repulsive ( $K > 0$ ) or attractive ( $K < 0$ ) self-interaction<sup>27</sup>. In that case,  $K = 2\pi a_s \hbar^2 / m^3$  (see Section 2.9).

(iii) For  $\gamma = 0$ , we obtain

$$P = K. \quad (225)$$

This is the equation of state of the  $\Lambda$ CDM ( $K < 0$ ) or anti- $\Lambda$ CDM ( $K > 0$ ) model [86,98,102,103]. In that case  $K = -\rho_\Lambda c^2$ , where  $\rho_\Lambda = \Lambda / (8\pi G)$  is the cosmological density.

(iv) For  $\gamma = 3$ , we obtain

$$P = K \left( \frac{\epsilon}{c^2} \right)^3. \quad (226)$$

This is the equation of state of a superfluid with repulsive ( $K > 0$ ) or attractive ( $K < 0$ ) self-interaction (see Section 2.10).

(v) The case  $\gamma = 1$  must be treated specifically. In that case, we have a linear equation of state [157–161]

$$P = \alpha \epsilon, \quad (227)$$

where we have defined

$$\alpha = \frac{K}{c^2}. \quad (228)$$

This linear equation of state describes pressureless matter ( $\alpha = 0$ ), radiation ( $\alpha = 1/3$ ) and stiff matter ( $\alpha = 1$ ). The nonrelativistic limit corresponds to  $\alpha \rightarrow 0$ . Using  $\epsilon \sim \rho c^2$ , we recover the isothermal equation of state (73).

#### 5.1.1. Determination of $\rho_m$ , $P(\rho_m)$ and $u(\rho_m)$

The rest-mass density is determined by Equation (194) with the equation of state (222). We have

$$\ln \rho_m = \int \frac{d\epsilon}{K \left( \frac{\epsilon}{c^2} \right)^\gamma + \epsilon}. \quad (229)$$

The integral can be calculated analytically yielding

$$\rho_m c^2 = \frac{\epsilon}{\left[1 + \frac{K}{c^2} \left(\frac{\epsilon}{c^2}\right)^{\gamma-1}\right]^{1/(\gamma-1)}}. \quad (230)$$

We have determined the constant of integration so that  $\epsilon \sim \rho_m c^2$  in the nonrelativistic limit. Equation (230) can be inverted to give

$$\epsilon = \frac{\rho_m c^2}{\left(1 - \frac{K \rho_m^{\gamma-1}}{c^2}\right)^{\frac{1}{\gamma-1}}}. \quad (231)$$

Substituting this result into Equation (222), we obtain

$$P = \frac{K \rho_m^\gamma}{\left(1 - \frac{K \rho_m^{\gamma-1}}{c^2}\right)^{\frac{\gamma}{\gamma-1}}}. \quad (232)$$

The internal energy is given by Equations (195) and (231) giving

$$u = \frac{\rho_m c^2}{\left(1 - \frac{K \rho_m^{\gamma-1}}{c^2}\right)^{\frac{1}{\gamma-1}}} - \rho_m c^2. \quad (233)$$

These results are consistent with those obtained in Appendix B.3 of [154]. In the nonrelativistic limit, using  $\epsilon \sim \rho_m c^2 [1 + \frac{K}{(\gamma-1)c^2} \rho_m^{\gamma-1}]$ , we recover Equations (67) and (68) [recalling Equation (29)].

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$\epsilon = \sqrt{(\rho_m c^2)^2 - K c^2}, \quad (234)$$

$$P = \frac{K c^2}{\sqrt{(\rho_m c^2)^2 - K c^2}}, \quad (235)$$

$$u = \sqrt{(\rho_m c^2)^2 - K c^2} - \rho_m c^2. \quad (236)$$

(ii) For  $\gamma = 2$  (BEC), we obtain

$$\epsilon = \frac{\rho_m c^2}{1 - \frac{K \rho_m}{c^2}}, \quad (237)$$

$$P = \frac{K \rho_m^2}{\left(1 - \frac{K \rho_m}{c^2}\right)^2}, \quad (238)$$

$$u = \frac{K \rho_m^2}{1 - \frac{K \rho_m}{c^2}}. \quad (239)$$

For  $K > 0$  there is a maximum density  $(\rho_m)_{\max} = c^2/K$ . The equation of state (238) was first obtained in [154].

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we obtain

$$\epsilon = \rho_m c^2 - K, \quad (240)$$

$$P = K, \quad (241)$$

$$u = -K. \quad (242)$$

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$\epsilon = \frac{\rho_m c^2}{\left(1 - \frac{K\rho_m^2}{c^2}\right)^{1/2}}, \quad (243)$$

$$P = \frac{K\rho_m^3}{\left(1 - \frac{K\rho_m^2}{c^2}\right)^{3/2}}, \quad (244)$$

$$u = \frac{\rho_m c^2}{\left(1 - \frac{K\rho_m^2}{c^2}\right)^{1/2}} - \rho_m c^2. \quad (245)$$

For  $K > 0$  there is a maximum density  $(\rho_m)_{\max} = c/\sqrt{K}$ .

(v) For  $\gamma = 1$ , Equation (194) can be integrated into

$$\rho_m = \left[ \frac{\alpha \epsilon}{\mathcal{K}(\alpha)} \right]^{\frac{1}{1+\alpha}}, \quad (246)$$

where  $\mathcal{K}(\alpha)$  is a constant that depends on  $\alpha$ . In the nonrelativistic limit  $\alpha \rightarrow 0$ , the condition  $\epsilon \sim \rho_m c^2$  implies  $\mathcal{K}(\alpha) \rightarrow \alpha c^2 = K$ . Combining Equation (246) with Equation (227), we obtain

$$P = \mathcal{K}(\alpha) \rho_m^{1+\alpha}. \quad (247)$$

This is the equation of state of a polytrope of type II (see Section 5.2) with a polytropic index  $\Gamma = 1 + \alpha$  (i.e.  $n = 1/\alpha$ ) and a polytropic constant  $\mathcal{K}(\alpha)$ <sup>28</sup>. In the nonrelativistic limit  $\alpha \rightarrow 0$ , we obtain an isothermal equation of state  $P = K\rho_m$  with a “temperature”  $K$ . The internal energy (195) is given by

$$u = \rho_m c^2 \left[ \frac{\mathcal{K}(\alpha)}{\alpha c^2} \rho_m^\alpha - 1 \right]. \quad (248)$$

It is similar to the Tsallis free energy density  $-\mathcal{K}s_q$  (where  $s_q = -\frac{1}{q-1}\rho_m^q$  is the Tsallis entropy density) of index  $q = 1 + \alpha$  with a “polytropic” temperature  $\mathcal{K}(\alpha)$ . In the nonrelativistic limit  $\alpha \rightarrow 0$  (i.e.  $q \rightarrow 1$ ), Equation (248) reduces to  $u = K\rho_m \ln \rho_m$  (up to an additive constant) and we recover Equation (74) [recalling Equation (29)]. It is similar to the Boltzmann free energy density  $-Ks_B$  (where  $s_B = -\rho_m \ln \rho_m$  is the Boltzmann entropy density) with the temperature  $K$ . In the present context, the Tsallis entropy arises from relativistic effects ( $\alpha \neq 0 \Rightarrow q \neq 1$ ).

### 5.1.2. Determination of $\rho$ , $P(\rho)$ and $V_{\text{tot}}(\rho)$

The pseudo rest-mass density and the SF potential are determined by Equations (199) and (200) with the equation of state (222). We have

$$\ln \rho = \int \frac{1 - \frac{K\gamma}{c^2} \left(\frac{\epsilon}{c^2}\right)^{\gamma-1}}{\epsilon + K \left(\frac{\epsilon}{c^2}\right)^{\gamma}} d\epsilon. \quad (249)$$

The integral can be calculated analytically yielding

$$\rho c^2 = \epsilon \left[ 1 + \frac{K}{c^2} \left(\frac{\epsilon}{c^2}\right)^{\gamma-1} \right]^{(1+\gamma)/(1-\gamma)}. \quad (250)$$

We have determined the constant of integration so that  $\epsilon \sim \rho c^2$  in the nonrelativistic limit. The SF potential is given by

$$V_{\text{tot}} = \frac{1}{2} \left[ \epsilon - K \left(\frac{\epsilon}{c^2}\right)^{\gamma} \right]. \quad (251)$$

Equations (222), (250) and (251) define  $P(\rho)$  and  $V_{\text{tot}}(\rho)$  in parametric form with parameter  $\epsilon$ . In the nonrelativistic limit, using  $\epsilon \sim \rho c^2 [1 - \frac{K(1+\gamma)}{(1-\gamma)c^2} \rho^{\gamma-1}]$  and Equation (130), we recover Equations (67) and (68).

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$\epsilon = \rho c^2, \quad (252)$$

$$P = \frac{K}{\rho}, \quad (253)$$

$$V_{\text{tot}}(\rho) = \frac{1}{2} \left( \rho c^2 - \frac{K}{\rho} \right). \quad (254)$$

Expression (254) of the SF potential was first given in [71]. We note that the energy density  $\epsilon$  coincides with the pseudo rest-mass energy density  $\rho c^2$  [see Equation (252)]. As a result,  $P(\rho)$  is a Chaplygin equation of state of type III (see Section 5.3). Therefore, the models I and III coincide in that case.

(ii) For  $\gamma = 2$  (BEC), the energy density is determined by a cubic equation

$$\rho c^2 = \frac{\epsilon}{\left(1 + \frac{K\epsilon}{c^4}\right)^3}. \quad (255)$$

The solution  $\epsilon(\rho)$  can be obtained by standard means. The total potential  $V_{\text{tot}}(\rho)$  is then given by

$$V_{\text{tot}} = \frac{1}{2} \left[ \epsilon - K \left(\frac{\epsilon}{c^2}\right)^2 \right] \quad (256)$$

with  $\epsilon$  replaced by  $\epsilon(\rho)$ . Equations (255) and (256) also determine  $V_{\text{tot}}(\rho)$  in parametric form.

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we obtain

$$\epsilon = \rho c^2 - K, \quad (257)$$

$$P = K, \quad (258)$$

$$V_{\text{tot}}(\rho) = \frac{1}{2} \rho c^2 - K. \quad (259)$$

Comparing these results with Equations (240)–(242), we note that  $\rho = \rho_m$  and  $V = u = -K$ . The potential  $V$  is constant.

(iv) For  $\gamma = 3$  (superfluid), the total potential  $V_{\text{tot}}(\rho)$  is given in parametric form by

$$\rho c^2 = \frac{\epsilon}{\left[1 + \frac{K}{c^2} \left(\frac{\epsilon}{c^2}\right)^2\right]^2}, \quad (260)$$

$$V_{\text{tot}} = \frac{1}{2} \left[ \epsilon - K \left(\frac{\epsilon}{c^2}\right)^3 \right]. \quad (261)$$

It is not possible to obtain more explicit expressions.

(v) For  $\gamma = 1$ , Equation (199) can be integrated into

$$\rho = \left[ \frac{\alpha \epsilon}{\mathcal{K}(\alpha)} \right]^{(1-\alpha)/(1+\alpha)}, \quad (262)$$

where  $\mathcal{K}(\alpha)$  is a constant that depends on  $\alpha$ . In the nonrelativistic limit  $\alpha \rightarrow 0$ , the condition  $\epsilon \sim \rho c^2$  implies  $\mathcal{K}(\alpha) \rightarrow \alpha c^2 = K$ . Combining Equation (262) with Equation (227), we obtain

$$P = \mathcal{K}(\alpha) \rho^{(1+\alpha)/(1-\alpha)}. \quad (263)$$

This is the equation of state of a polytrope of type III (see Section 5.3) with a polytropic index  $\Gamma = (1 + \alpha)/(1 - \alpha)$  (i.e.  $n = (1 - \alpha)/(2\alpha)$ ) and a polytropic constant  $\mathcal{K}(\alpha)$ <sup>29</sup>. In the nonrelativistic limit  $\alpha \rightarrow 0$ , we obtain an isothermal equation of state  $P = K\rho$  with a “temperature”  $K$ . The SF potential (200) is given by

$$V_{\text{tot}} = \frac{1 - \alpha}{2\alpha} \mathcal{K}(\alpha) \rho^{(1+\alpha)/(1-\alpha)}. \quad (264)$$

This is a power-law potential<sup>30</sup>. In the nonrelativistic limit  $\alpha \rightarrow 0$ , Equation (264) reduces to  $V_{\text{tot}} = K\rho \ln \rho$  (up to an additive constant) and we recover Equation (74). For  $\alpha = 1$  (stiff matter) we obtain  $V_{\text{tot}} = 0$ , corresponding to a free massless SF satisfying  $\square\varphi = 0$ .

### 5.1.3. Lagrangian $\mathcal{L}(X)$

The Lagrangian  $\mathcal{L}(X)$  is determined by Equation (203) or Equation (204) with the equation of state (222). We have

$$\ln X = 2 \int \frac{dP}{\left(\frac{P}{K}\right)^{1/\gamma} c^2 + P} \quad (265)$$

or

$$\ln X = 2 \int \frac{\frac{K\gamma}{c^2} \left(\frac{\epsilon}{c^2}\right)^{\gamma-1}}{\epsilon + K \left(\frac{\epsilon}{c^2}\right)^{\gamma}} d\epsilon. \quad (266)$$

The integrals can be calculated analytically yielding

$$\mathcal{L}(X) = P = K \left\{ \frac{c^2}{-K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{\frac{\gamma-1}{2\gamma}} \right] \right\}^{\frac{\gamma}{\gamma-1}}. \quad (267)$$

We have determined the constant of integration so that, in the nonrelativistic limit,  $X \sim c^2/2$  (see Section 3.5). In the nonrelativistic limit, using Equation (189), we recover

Equation (71). The Lagrangian (267) was first obtained in [95] in relation to the GCG by using the procedure of [71] that we have followed.

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$\mathcal{L}(X) = P = K \sqrt{\frac{c^2}{-K} \left(1 - \frac{2X}{c^2}\right)}. \quad (268)$$

The Lagrangian of the Chaplygin gas ( $K < 0$ ) is of the Born-Infeld type. Indeed, setting  $K = -A$  so that  $P = -Ac^2/\epsilon$  we get the Born-Infeld Lagrangian

$$\mathcal{L}_{\text{BI}} = -(Ac^2)^{1/2} \sqrt{1 - \frac{1}{c^2} \partial_\mu \theta \partial^\mu \theta}. \quad (269)$$

In the nonrelativistic limit, using Equation (189), it reduces to

$$\mathcal{L}_{\text{NR}} = -(2A)^{1/2} \sqrt{\dot{\theta} + \frac{1}{2}(\nabla\theta)^2}, \quad (270)$$

corresponding to Equation (83). Using Equation (175) or Equation (178), we obtain the equation of motion

$$D_\mu \left[ \frac{\partial^\mu \theta}{\sqrt{|1 - \frac{1}{c^2} \partial_\nu \theta \partial^\nu \theta|}} \right] = 0. \quad (271)$$

In the nonrelativistic limit, it reduces to Equation (84). The Born-Infeld Lagrangian (269) was obtained by [58,59] in two different manners: (i) starting from a heuristic relativistic Lagrangian

$$L = - \int \left[ \rho \dot{\theta} + \rho c^2 \sqrt{1 + \frac{A}{\rho^2 c^2}} \sqrt{1 + \frac{(\nabla\theta)^2}{c^2}} \right] d\mathbf{r} \quad (272)$$

which generalizes the nonrelativistic Lagrangian from Equations (47) and (80), writing the equations of motion, and eliminating  $\rho$  with the aid of the Bernoulli equation; (ii) for a  $d$ -brane moving in a  $(d+1, 1)$  space-time. In the second case, it can be obtained from the Nambu-Goto action in the Cartesian parametrization. The Born-Infeld Lagrangian (269) was also obtained in [71] for a complex SF in the TF regime by developing the procedure that we have followed. It can also be directly obtained from the k-essence formalism applied to the Chaplygin equation of state (see Appendix B).

(ii) For  $\gamma = 2$  (BEC), we obtain

$$\mathcal{L}(X) = P = K \left\{ \frac{c^2}{-K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/4} \right] \right\}^2. \quad (273)$$

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model) the k-essence Lagrangian is constant.

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$\mathcal{L}(X) = P = K \left\{ \frac{c^2}{-K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/3} \right] \right\}^{3/2}. \quad (274)$$

(v) For  $\gamma = 1$ , Equation (203) or Equation (204) can be easily integrated yielding

$$\mathcal{L}(X) = P = \mathcal{A}(\alpha) \left( \frac{2X}{c^2} \right)^{\frac{\alpha+1}{2\alpha}}, \quad (275)$$

where  $\mathcal{A}(\alpha)$  is a constant that depends on  $\alpha$ . The Lagrangian is a pure power-law. It was first given in [41]. Using Equation (175) or Equation (178), we obtain the equation of motion

$$D_\mu \left[ \left( \frac{1}{c^2} \partial_\nu \theta \partial^\nu \theta \right)^{(1-\alpha)/2\alpha} \partial^\mu \theta \right] = 0. \quad (276)$$

In the nonrelativistic limit corresponding to  $\alpha \rightarrow 0$ , using Equation (189), we recover Equations (77) and (78) with  $\mathcal{A}(\alpha) \rightarrow K\rho_*$ . In the case  $\alpha = 1$ , we obtain  $\mathcal{L} \propto X$  and  $\square\theta = 0$  (free massless SF).

### 5.2. Polytropic Equation of State of Type II

The polytropic equation of state of type II reads [162]

$$P = K\rho_m^\gamma. \quad (277)$$

Using Equation (A165), the internal energy is

$$u = \frac{K}{\gamma - 1} \rho_m^\gamma. \quad (278)$$

It is similar to the Tsallis free energy. In the nonrelativistic limit, using  $\rho_m \sim \rho$ , we recover Equations (67) and (68) [recalling Equation (29)]. Actually, Equations (277) and (278) coincide with Equations (67) and (68) with  $\rho_m$  in place of  $\rho$ . We note that  $P = (\gamma - 1)u$ .

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$P = \frac{K}{\rho_m}, \quad u = -\frac{K}{2\rho_m}. \quad (279)$$

(ii) For  $\gamma = 2$  (BEC), we obtain

$$P = K\rho_m^2, \quad u = K\rho_m^2. \quad (280)$$

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we obtain

$$P = K, \quad u = -K. \quad (281)$$

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$P = K\rho_m^3, \quad u = \frac{1}{2}K\rho_m^3. \quad (282)$$

(v) The case  $\gamma = 1$ , corresponding to an isothermal equation of state

$$P = K\rho_m, \quad (283)$$

must be treated specifically. Using Equation (A165), the internal energy is

$$u = K\rho_m \left[ \ln \left( \frac{\rho_m}{\rho_*} \right) - 1 \right]. \quad (284)$$

It is similar to the Boltzmann free energy. In the nonrelativistic limit, using  $\rho_m \sim \rho$ , we recover Equations (73) and (74) [recalling Equation (29)]. Actually, Equations (283) and (284) coincide with Equations (73) and (74) with  $\rho_m$  in place of  $\rho$ .



### 5.2.1. Determination of $\epsilon$ and $P(\epsilon)$

The energy density is determined by Equation (205) with Equation (278). We obtain

$$\epsilon = \rho_m c^2 + \frac{K}{\gamma - 1} \rho_m^\gamma. \quad (285)$$

The pressure is determined by Equation (206) with Equation (278). This returns Equation (277). Eliminating  $\rho_m$  between Equation (277) and Equation (285) we obtain  $P(\epsilon)$  under the inverse form  $\epsilon(P)$  as

$$\epsilon = \left( \frac{P}{K} \right)^{1/\gamma} c^2 + \frac{P}{\gamma - 1}. \quad (286)$$

In the nonrelativistic limit, using  $\epsilon \sim \rho c^2$ , we recover Equation (67).

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$\epsilon = \rho_m c^2 - \frac{K}{2\rho_m}, \quad (287)$$

$$\epsilon = \frac{Kc^2}{P} - \frac{P}{2}, \quad (288)$$

$$\rho_m c^2 = \frac{\epsilon \pm \sqrt{\epsilon^2 + 2Kc^2}}{2}, \quad (289)$$

$$P = -\epsilon \pm \sqrt{\epsilon^2 + 2Kc^2}. \quad (290)$$

We note that the Chaplygin gas of type II is different from the Chaplygin gas of type I.

(ii) For  $\gamma = 2$  (BEC), we obtain

$$\epsilon = \rho_m c^2 + K\rho_m^2, \quad (291)$$

$$\epsilon = \sqrt{\frac{P}{K}} c^2 + P, \quad (292)$$

$$\rho_m = \frac{-c^2 \pm \sqrt{c^4 + 4K\epsilon}}{2K}, \quad (293)$$

$$P = \frac{1}{4K} \left[ -c^2 \pm \sqrt{c^4 + 4K\epsilon} \right]^2. \quad (294)$$

The equation of state (294) was first obtained in [154,163].

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we obtain

$$\epsilon = \rho_m c^2 - K, \quad (295)$$

$$P = K. \quad (296)$$

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$\epsilon = \rho_m c^2 + \frac{1}{2} K \rho_m^3, \quad (297)$$

$$\epsilon = \left(\frac{P}{K}\right)^{1/3} c^2 + \frac{1}{2}P. \quad (298)$$

This is a third degree equation for  $P^{1/3}$  which can be solved by standard means to obtain  $P(\epsilon)$ .

(v) For  $\gamma = 1$ , the energy density is determined by Equation (205) with Equation (284). We obtain

$$\epsilon = \rho_m c^2 + K \rho_m \left[ \ln\left(\frac{\rho_m}{\rho_*}\right) - 1 \right]. \quad (299)$$

The pressure is determined by Equation (206) with Equation (284). This returns Equation (283). Eliminating  $\rho_m$  between Equation (283) and Equation (299) we obtain  $P(\epsilon)$  under the inverse form  $\epsilon(P)$  as

$$\epsilon = \frac{P}{K} c^2 + P \left[ \ln\left(\frac{P}{K \rho_*}\right) - 1 \right]. \quad (300)$$

*Remark:* For the index  $\gamma = 1/2$ , we can inverse Equation (286) to obtain

$$P = \frac{K^2}{c^2} \pm K \sqrt{\frac{K^2}{c^4} + \frac{\epsilon}{c^2}}. \quad (301)$$

For the index  $\gamma = 3/2$ , Equation (285) becomes

$$\epsilon = \rho_m c^2 + 2K \rho_m^{3/2}. \quad (302)$$

This is a third degree equation for  $\sqrt{\rho_m}$  which can be solved by standard means. One can then obtain  $P(\epsilon)$  explicitly.

### 5.2.2. Determination of $\rho$ , $P(\rho)$ and $V_{\text{tot}}(\rho)$

The pseudo rest-mass density and the SF potential are determined by Equations (209) and (210) with Equation (278). We get

$$\rho = \frac{\rho_m}{1 + \frac{\gamma}{\gamma-1} \frac{K}{c^2} \rho_m^{\gamma-1}}, \quad (303)$$

$$V_{\text{tot}} = \frac{1}{2} \left[ \rho_m c^2 - K \frac{\gamma-2}{\gamma-1} \rho_m^\gamma \right]. \quad (304)$$

Equations (277), (303) and (304) determine  $P(\rho)$  and  $V_{\text{tot}}(\rho)$  in parametric form with parameter  $\rho_m$ . In the nonrelativistic limit, using  $\rho_m \sim \rho [1 + \frac{\gamma}{\gamma-1} \frac{K}{c^2} \rho^{\gamma-1}]$  and Equation (130), we recover Equations (67) and (68).

(i) For  $\gamma = -1$  (Chaplygin gas),  $\rho_m$  is determined by a cubic equation

$$2\rho_m^3 - 2\rho\rho_m^2 - \frac{K\rho}{c^2} = 0, \quad (305)$$

which can be solved by standard means. The SF potential is given by

$$V_{\text{tot}} = \frac{1}{2} \left( \rho_m c^2 + \frac{3K}{2\rho_m} \right). \quad (306)$$

One can then obtain  $P(\rho)$  and  $V_{\text{tot}}(\rho)$  explicitly.

(ii) For  $\gamma = 2$  (BEC), we obtain

$$\rho = \frac{\rho_m}{1 + \frac{2K}{c^2} \rho_m}, \quad (307)$$

$$\rho_m = \frac{\rho}{1 - \frac{2K}{c^2}\rho}, \quad (308)$$

$$P = \frac{K\rho^2}{\left(1 - \frac{2K}{c^2}\rho\right)^2}, \quad (309)$$

$$V_{\text{tot}} = \frac{1}{2}\rho c^2 + \frac{K\rho^2}{1 - \frac{2K\rho}{c^2}}. \quad (310)$$

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we obtain

$$\rho = \rho_m, \quad P = K, \quad (311)$$

$$V_{\text{tot}} = \frac{1}{2}\rho c^2 - K. \quad (312)$$

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$\rho = \frac{\rho_m}{1 + \frac{3K}{2c^2}\rho_m^2}, \quad (313)$$

$$\rho_m = \frac{c^2}{3K\rho} \left(1 \pm \sqrt{1 - \frac{6K}{c^2}\rho^2}\right), \quad (314)$$

$$P = K \left(\frac{c^2}{3K\rho}\right)^3 \left(1 \pm \sqrt{1 - \frac{6K}{c^2}\rho^2}\right)^3, \quad (315)$$

$$V_{\text{tot}} = \frac{c^4}{6K\rho} \left(1 \pm \sqrt{1 - \frac{6K}{c^2}\rho^2}\right) \left[\frac{4}{3} - \frac{c^2}{9K\rho^2} \left(1 \pm \sqrt{1 - \frac{6K}{c^2}\rho^2}\right)\right]. \quad (316)$$

(v) For  $\gamma = 1$ , the pseudo rest-mass density and the SF potential are determined by Equations (209) and (210) with Equation (284). We get

$$\rho = \frac{\rho_m}{1 + \frac{K}{c^2} \ln\left(\frac{\rho_m}{\rho_*}\right)}, \quad (317)$$

$$V_{\text{tot}} = \frac{1}{2} \left[ \rho_m c^2 + K \rho_m \ln\left(\frac{\rho_m}{\rho_*}\right) - 2K\rho_m \right]. \quad (318)$$

Equations (283), (317) and (318) determine  $P(\rho)$  and  $V_{\text{tot}}(\rho)$  in parametric form with parameter  $\rho_m$ . In the nonrelativistic limit, using  $\rho_m \simeq \rho[1 + (K/c^2) \ln(\rho/\rho_*)]$  and Equation (130), we recover Equation (74).

*Remark:* For the index  $\gamma = 1/2$ , Equation (303) can be written as

$$\rho_m^{3/2} - \rho \rho_m^{1/2} + \frac{K\rho}{c^2} = 0. \quad (319)$$

This is a third degree equation for  $\sqrt{\rho_m}$  which can be solved by standard means. One can then obtain  $P(\rho)$  and  $V_{\text{tot}}(\rho)$  explicitly. For the index  $\gamma = 3/2$  we find that

$$P = K \left( \frac{3K}{2c^2}\rho \pm \sqrt{\frac{9K^2}{4c^4}\rho^2 + \rho} \right)^3, \quad (320)$$

$$V_{\text{tot}} = \frac{1}{2} \left( \frac{3K}{2c^2} \rho \pm \sqrt{\frac{9K^2}{4c^4} \rho^2 + \rho} \right)^2 c^2 \left[ 1 + \frac{K}{2c^2} \rho \left( \frac{3K}{c^2} \pm \sqrt{\frac{9K^2}{c^4} + \frac{4}{\rho}} \right) \right]. \quad (321)$$

### 5.2.3. Lagrangian $\mathcal{L}(X)$

The Lagrangian  $\mathcal{L}(X)$  is determined by Equations (213), (277) and (278). We get

$$X = \frac{1}{2c^2} \left[ c^2 + \frac{K\gamma}{\gamma-1} \rho_m^{\gamma-1} \right]^2. \quad (322)$$

This equation can be inverted to give

$$\rho_m^{\gamma-1} = -\frac{\gamma-1}{\gamma} \frac{c^2}{K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/2} \right]. \quad (323)$$

We then obtain

$$\mathcal{L}(X) = P = K \left\{ -\frac{\gamma-1}{\gamma} \frac{c^2}{K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/2} \right] \right\}^{\frac{\gamma}{\gamma-1}}. \quad (324)$$

In the nonrelativistic limit, using Equation (189), we recover Equation (71).

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$\mathcal{L}(X) = P = K \sqrt{\frac{2c^2}{-K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/2} \right]}. \quad (325)$$

(ii) For  $\gamma = 2$  (BEC), we obtain

$$\mathcal{L}(X) = P = K \left\{ \frac{c^2}{-2K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/2} \right] \right\}^2. \quad (326)$$

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), the k-essence Lagrangian is constant.

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$\mathcal{L}(X) = P = K \left\{ \frac{2c^2}{-3K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/2} \right] \right\}^{3/2}. \quad (327)$$

(v) For  $\gamma = 1$ , the Lagrangian  $\mathcal{L}(X)$  is determined by Equations (213), (283) and (284). This yields

$$X = \frac{1}{2c^2} \left[ c^2 + K \ln \left( \frac{\rho_m}{\rho_*} \right) \right]^2. \quad (328)$$

This equation can be inverted to give

$$\rho_m = \rho_* e^{-\frac{c^2}{K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/2} \right]}. \quad (329)$$

We then obtain

$$\mathcal{L}(X) = P = K \rho_* e^{-\frac{c^2}{K} \left[ 1 - \left( \frac{2X}{c^2} \right)^{1/2} \right]}. \quad (330)$$

In the nonrelativistic limit, using Equation (189), we recover Equation (77).

### 5.3. Polytropic Equation of State of Type III

The polytropic equation of state of type III reads [149,156]

$$P = K\rho^\gamma. \quad (331)$$

Using Equation (170), the SF potential is

$$V_{\text{tot}}(\rho) = \frac{1}{2}\rho c^2 + \frac{K}{\gamma-1}\rho^\gamma. \quad (332)$$

It is similar to the Tsallis free energy. In the nonrelativistic limit, we recover Equations (67) and (68) [recalling Equation (130)]. Actually, Equations (331) and (332) coincide with Equations (67) and (68). The SF potential  $V(\rho)$  corresponds to a pure power-law. We note that  $P = (\gamma - 1)V$ .

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$P = \frac{K}{\rho}, \quad V_{\text{tot}} = \frac{1}{2}\rho c^2 - \frac{K}{2\rho}, \quad (333)$$

as found in [149,156]. We recover the potential from Equation (254) for the reason explained in Section 5.3.1.

(ii) For  $\gamma = 2$  (BEC), we obtain

$$P = K\rho^2, \quad V_{\text{tot}} = \frac{1}{2}\rho c^2 + K\rho^2. \quad (334)$$

The SF potential  $V(\rho) = K\rho^2$  from Equation (334) with  $K = 2\pi a_s \hbar^2 / m^3$  corresponds to the standard  $|\varphi|^4$  potential of a BEC [149,152,156].

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we obtain

$$P = K, \quad V_{\text{tot}} = \frac{1}{2}\rho c^2 - K. \quad (335)$$

The SF potential  $V(\rho) = -K$  from Equation (335) is constant. Using  $K = -\rho_\Lambda c^2$ , we see that  $V(\rho) = \rho_\Lambda c^2$  is equal to the cosmological density [156].

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$P = K\rho^3, \quad V_{\text{tot}} = \frac{1}{2}\rho c^2 + \frac{1}{2}K\rho^3. \quad (336)$$

The SF potential  $V(\rho) = K\rho^3$  from Equation (336) corresponds to the  $|\varphi|^6$  potential of a BEC.

(v) The case  $\gamma = 1$ , corresponding to a linear equation of state

$$P = K\rho, \quad (337)$$

must be treated specifically. Using Equation (170), the SF potential is

$$V_{\text{tot}}(\rho) = \frac{1}{2}\rho c^2 + K\rho \left[ \ln\left(\frac{\rho}{\rho_*}\right) - 1 \right]. \quad (338)$$

It is similar to the Boltzmann free energy. In the nonrelativistic limit, we recover Equations (73) and (74) [recalling Equation (130)]. Actually, Equations (337) and (338) coincide with Equations (73) and (74). The potential (338) was first obtained in [134,156].

### 5.3.1. Determination of $\epsilon$ and $P(\epsilon)$

The energy density is determined by Equation (214) with Equation (332). This yields

$$\epsilon = \rho c^2 + \frac{\gamma+1}{\gamma-1} K \rho^\gamma. \quad (339)$$

The pressure is determined by Equation (215) with Equation (332). This returns Equation (331). Eliminating  $\rho$  between Equations (331) and (339), we obtain  $P(\epsilon)$  under the inverse form  $\epsilon(P)$  as

$$\epsilon = \left(\frac{P}{K}\right)^{1/\gamma} c^2 + \frac{\gamma+1}{\gamma-1} P. \quad (340)$$

In the nonrelativistic limit, using  $\epsilon \sim \rho c^2$ , we recover Equation (67).

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$\epsilon = \rho c^2, \quad (341)$$

$$P = \frac{K c^2}{\epsilon}. \quad (342)$$

This returns the Chaplygin gas of type I (see Section 5.1). Therefore, the Chaplygin gas models of type I and III coincide.

(ii) For  $\gamma = 2$  (BEC), we obtain

$$\epsilon = \rho c^2 + 3K\rho^2, \quad (343)$$

$$\epsilon = \sqrt{\frac{P}{K}} c^2 + 3P, \quad (344)$$

$$\rho = \frac{-c^2 \pm \sqrt{c^4 + 12K\epsilon}}{6K}, \quad (345)$$

$$P = \frac{1}{36K} \left[ -c^2 \pm \sqrt{c^4 + 12K\epsilon} \right]^2. \quad (346)$$

This equation of state was first obtained in [152] (see also [149,150,156]).

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we obtain

$$P = K, \quad \epsilon = \rho c^2 - K. \quad (347)$$

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$\epsilon = \rho c^2 + 2K\rho^3, \quad (348)$$

$$\epsilon = \left(\frac{P}{K}\right)^{1/3} c^2 + 2P. \quad (349)$$

This is a third degree equation which can be solved by standard means to obtain  $P(\epsilon)$ .

(v) For  $\gamma = 1$ , the energy density is determined by Equation (214) with Equation (338). This yields

$$\epsilon = \rho c^2 + 2K\rho \ln\left(\frac{\rho}{\rho_*}\right) - K\rho. \quad (350)$$

The pressure is determined by Equation (215) with Equation (338). This returns Equation (337). Eliminating  $\rho$  between Equations (337) and (350), we obtain  $P(\epsilon)$  under the inverse form  $\epsilon(P)$  as

$$\epsilon = \frac{P}{K}c^2 + 2P \ln\left(\frac{P}{K\rho_*}\right) - P. \quad (351)$$

In the nonrelativistic limit, using  $\epsilon \sim \rho c^2$ , we recover Equation (73).

*Remark:* For the index  $\gamma = 1/2$ , we can inverse Equation (340) to obtain

$$P = \frac{3K^2}{2c^2} \pm K\sqrt{\frac{9K^2}{4c^4} + \frac{\epsilon}{c^2}}. \quad (352)$$

For the index  $\gamma = 3/2$ , Equation (339) becomes

$$\epsilon = \rho c^2 + 5K\rho^{3/2}. \quad (353)$$

This is a third degree equation for  $\sqrt{\rho_m}$  which can be solved by standard means. One can then obtain  $P(\epsilon)$  explicitly.

### 5.3.2. Determination of $\rho_m$ , $P(\rho_m)$ and $u(\rho_m)$

The rest-mass density and the internal energy are determined by Equations (218) and (220) with Equation (332). We get

$$\rho_m = \rho \sqrt{1 + \frac{2\gamma}{\gamma-1} \frac{K}{c^2} \rho^{\gamma-1}}, \quad (354)$$

$$u = \rho c^2 + \frac{\gamma+1}{\gamma-1} K \rho^\gamma - \rho_m(\rho) c^2. \quad (355)$$

Equations (331), (354) and (355) define  $P(\rho_m)$  and  $u(\rho_m)$  in parametric form with parameter  $\rho$ . In the nonrelativistic limit, using  $\rho_m \sim \rho[1 + \frac{\gamma}{\gamma-1} \frac{K}{c^2} \rho^{\gamma-1}]$ , we recover Equations (67) and (68) [recalling Equation (29)].

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$\rho_m c^2 = \sqrt{(\rho c^2)^2 + K c^2}, \quad (356)$$

$$\rho c^2 = \sqrt{(\rho_m c^2)^2 - K c^2}, \quad (357)$$

$$P = \frac{K c^2}{\sqrt{(\rho_m c^2)^2 - K c^2}}, \quad (358)$$

$$u = \sqrt{(\rho_m c^2)^2 - K c^2} - \rho_m c^2. \quad (359)$$

(ii) For  $\gamma = 2$  (BEC),  $\rho$  is determined by a cubic equation

$$\frac{4K}{c^2} \rho^3 + \rho^2 - \rho_m^2 = 0, \quad (360)$$

which can be solved by standard means. The internal energy is given by

$$u = \rho c^2 + 3K\rho^2 - \rho_m(\rho) c^2. \quad (361)$$

One can then obtain  $P(\rho_m)$  and  $u(\rho_m)$  explicitly.

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we obtain

$$P = K, \quad \rho_m = \rho, \quad u = -K. \quad (362)$$

We note that the rest-mass density coincides with the pseudo rest-mass density (one has  $\rho_m = \rho$ ).

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$\rho_m = \rho \sqrt{1 + \frac{3K}{c^2} \rho^2}, \quad (363)$$

$$\rho = \left( -\frac{c^2}{6K} \pm \frac{c^2}{6K} \sqrt{1 + \frac{12K}{c^2} \rho_m^2} \right)^{1/2}, \quad (364)$$

$$P = K \left( -\frac{c^2}{6K} \pm \frac{c^2}{6K} \sqrt{1 + \frac{12K}{c^2} \rho_m^2} \right)^{3/2}, \quad (365)$$

$$u = c^2 \left( -\frac{c^2}{6K} \pm \frac{c^2}{6K} \sqrt{1 + \frac{12K}{c^2} \rho_m^2} \right)^{1/2} \left( \frac{2}{3} \pm \frac{1}{3} \sqrt{1 + \frac{12K}{c^2} \rho_m^2} \right) - \rho_m c^2. \quad (366)$$

(v) For  $\gamma = 1$  the rest-mass density and the internal energy are determined by Equations (218) and (220) with Equation (338). This gives

$$\rho_m = \rho \sqrt{1 + \frac{2K}{c^2} \ln\left(\frac{\rho}{\rho_*}\right)}, \quad (367)$$

$$u = \rho c^2 + 2K\rho \ln\left(\frac{\rho}{\rho_*}\right) - K\rho - \rho_m(\rho)c^2. \quad (368)$$

Equations (337), (367) and (368) determine  $P(\rho_m)$  and  $u(\rho_m)$  in parametric form with parameter  $\rho$ . In the nonrelativistic limit, using  $\rho_m \sim \rho[1 + \frac{K}{c^2} \ln(\rho/\rho_*)]$ , we recover Equations (73) and (74) [recalling Equation (29)].

### 5.3.3. Lagrangian $\mathcal{L}(X)$

The Lagrangian  $\mathcal{L}(X)$  is determined by Equation (221) with Equation (332). We get

$$X = \frac{1}{2}c^2 + \frac{K\gamma}{\gamma-1}\rho^{\gamma-1}. \quad (369)$$

This relation can be reversed to give

$$\rho^{\gamma-1} = -\frac{\gamma-1}{2\gamma} \frac{c^2}{K} \left( 1 - \frac{2X}{c^2} \right). \quad (370)$$

We then obtain

$$\mathcal{L}(X) = P = K \left[ -\frac{\gamma-1}{2\gamma} \frac{c^2}{K} \left( 1 - \frac{2X}{c^2} \right) \right]^{\frac{\gamma}{\gamma-1}}. \quad (371)$$

In the nonrelativistic limit, using Equation (189), we recover Equation (71). Actually, Equation (371) coincides with Equation (71) with  $c^2/2 - X$  in place of  $x$ . Interestingly, the Lagrangian (371) corresponds to the Lagrangian introduced heuristically in [164] in



relation to the GCG [see their Equation (33)]. In [164] it was obtained from a heuristic relativistic Lagrangian

$$L = - \int \left[ \rho \dot{\theta} + \rho c^2 \sqrt{1 + \frac{2K}{\gamma - 1} \frac{1}{\rho^{1-\gamma} c^2}} \sqrt{1 + \frac{(\nabla \theta)^2}{c^2}} \right] d\mathbf{r} \quad (372)$$

which generalizes the Lagrangian from Equation (272). Our approach provides therefore a justification of the Lagrangian (371) from a more rigorous relativistic theory. We note that this Lagrangian differs from the Lagrangian (267) introduced in [95] except for the particular index  $\gamma = -1$  corresponding to the Chaplygin gas (see below). It is also different from the Lagrangian (324) even for  $\gamma = -1$ . This is an effect of the inequivalence between the equations of state of types I, II and III.

(i) For  $\gamma = -1$  (Chaplygin gas), we obtain

$$\mathcal{L}(X) = P = K \sqrt{\frac{c^2}{-K} \left( 1 - \frac{2X}{c^2} \right)}, \quad (373)$$

like in Equation (268).

(ii) For  $\gamma = 2$  (BEC), we obtain

$$\mathcal{L}(X) = P = K \left[ \frac{c^2}{-4K} \left( 1 - \frac{2X}{c^2} \right) \right]^2. \quad (374)$$

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), the k-essence Lagrangian is constant.

(iv) For  $\gamma = 3$  (superfluid), we obtain

$$\mathcal{L}(X) = P = K \left[ \frac{c^2}{-3K} \left( 1 - \frac{2X}{c^2} \right) \right]^{3/2}. \quad (375)$$

(v) For  $\gamma = 1$ , the Lagrangian  $\mathcal{L}(X)$  is determined by Equation (221) with Equation (338). We get

$$X = \frac{1}{2} c^2 + K \ln \left( \frac{\rho}{\rho_*} \right). \quad (376)$$

This relation can be reversed to give

$$\rho = \rho_* e^{-\frac{c^2}{2K} \left( 1 - \frac{2X}{c^2} \right)}. \quad (377)$$

We then obtain

$$\mathcal{L}(X) = P = K \rho_* e^{-\frac{c^2}{2K} \left( 1 - \frac{2X}{c^2} \right)}. \quad (378)$$

In the nonrelativistic limit, using Equation (189), we recover Equation (77). Actually, Equation (378) coincides with Equation (77) with  $c^2/2 - X$  in place of  $x$ .

## 6. Logotropes

In this section, we apply the general results of Section 4 to the case of a logotropic equation of state.

### 6.1. Logotropic Equation of State of Type I

The logotropic equation of state of type I reads [118]

$$P = A \ln \left( \frac{\epsilon}{\epsilon_*} \right). \quad (379)$$

### 6.1.1. Determination of $\rho_m$ , $P(\rho_m)$ and $u(\rho_m)$

The rest-mass density and the internal energy are determined by Equations (194) and (195) with the equation of state (379) yielding

$$\ln \rho_m = \int \frac{d\epsilon}{A \ln\left(\frac{\epsilon}{\epsilon_*}\right) + \epsilon}, \quad (380)$$

$$u = \epsilon - \rho_m(\epsilon)c^2. \quad (381)$$

Equations (379)–(381) determine  $P(\rho_m)$  and  $u(\rho_m)$  in parametric form with parameter  $\epsilon$ . Unfortunately, the integral in Equation (380) cannot be calculated analytically.

### 6.1.2. Determination of $\rho$ , $P(\rho)$ and $V_{\text{tot}}(\rho)$

The pseudo rest-mass density and the SF potential are determined by Equations (199) and (200) with the equation of state (379) yielding

$$\ln \rho = \int \frac{1 - \frac{A}{\epsilon}}{\epsilon + A \ln\left(\frac{\epsilon}{\epsilon_*}\right)} d\epsilon, \quad (382)$$

$$V_{\text{tot}} = \frac{1}{2} \left[ \epsilon - A \ln\left(\frac{\epsilon}{\epsilon_*}\right) \right]. \quad (383)$$

They can also be determined by Equations (201) and (202) with Equation (379) yielding

$$\ln \rho = \int \frac{\frac{\epsilon_*}{A} e^{P/A} - 1}{\epsilon_* e^{P/A} + P} dP, \quad (384)$$

$$V_{\text{tot}} = \frac{1}{2} (\epsilon_* e^{P/A} - P). \quad (385)$$

Equations (379) and (382)–(385) determine  $P(\rho)$  [or  $\rho(P)$ ] and  $V_{\text{tot}}(\rho)$  in parametric form with parameter  $\epsilon$  or  $P$ . Unfortunately, the integrals in Equations (382) and (384) cannot be calculated analytically.

### 6.1.3. Lagrangian $\mathcal{L}(X)$

The Lagrangian  $\mathcal{L}(X)$  is determined by Equation (203) or Equation (204) with Equation (379) yielding

$$\ln X = 2 \int \frac{dP}{\epsilon_* e^{P/A} + P} \quad (386)$$

or

$$\ln X = 2 \int \frac{\frac{A}{\epsilon}}{\epsilon + A \ln\left(\frac{\epsilon}{\epsilon_*}\right)} d\epsilon. \quad (387)$$

These equations determine  $P(X)$ , thus  $\mathcal{L}(X)$ . Unfortunately, the integrals in Equations (386) and (387) cannot be calculated analytically.

## 6.2. Logotropic Equation of State of Type II

The logotropic equation of state of type II reads [118]

$$P = A \ln\left(\frac{\rho_m}{\rho_*}\right). \quad (388)$$

Using Equation (A165), the internal energy is

$$u = -A \ln\left(\frac{\rho_m}{\rho_*}\right) - A. \quad (389)$$

In the nonrelativistic limit, using  $\rho_m \sim \rho$ , we recover Equations (99) and (100) [recalling Equation (29)]. Actually, Equations (388) and (389) coincide with Equations (99) and (100) with  $\rho_m$  in place of  $\rho$ .

#### 6.2.1. Determination of $\epsilon$ and $P(\epsilon)$

The energy density is determined by Equation (205) with Equation (389). We obtain

$$\epsilon = \rho_m c^2 - A \ln\left(\frac{\rho_m}{\rho_*}\right) - A. \quad (390)$$

The pressure is determined by Equation (206) with Equation (389). This returns Equation (388). Eliminating  $\rho_m$  between Equations (388) and (390) we obtain  $P(\epsilon)$  under the inverse form  $\epsilon(P)$  as

$$\epsilon = \rho_* c^2 e^{P/A} - P - A. \quad (391)$$

In the nonrelativistic limit, using  $\epsilon \sim \rho c^2$ , we recover Equation (99).

#### 6.2.2. Determination of $\rho$ , $P(\rho)$ and $V_{\text{tot}}(\rho)$

The pseudo rest-mass density and the SF potential are determined by Equations (209) and (210) with Equation (389). We get

$$\rho = \frac{\rho_m^2}{\rho_m - \frac{A}{c^2}}, \quad (392)$$

$$V_{\text{tot}} = \frac{1}{2} \left[ \rho_m c^2 - 2A \ln\left(\frac{\rho_m}{\rho_*}\right) - A \right]. \quad (393)$$

Equation (392) can be inverted to give

$$\rho_m = \frac{\rho \pm \sqrt{\rho^2 - \frac{4A\rho}{c^2}}}{2}. \quad (394)$$

Combined with Equations (388) and (393) we explicitly obtain  $P(\rho)$  and  $V_{\text{tot}}(\rho)$  under the form

$$P = A \ln\left(\frac{\rho \pm \sqrt{\rho^2 - \frac{4A\rho}{c^2}}}{2\rho_*}\right), \quad (395)$$

$$V_{\text{tot}} = \frac{\rho \pm \sqrt{\rho^2 - \frac{4A\rho}{c^2}}}{4} c^2 - A \ln\left(\frac{\rho \pm \sqrt{\rho^2 - \frac{4A\rho}{c^2}}}{2\rho_*}\right) - \frac{A}{2}. \quad (396)$$

In the nonrelativistic limit, using  $\rho_m \sim \rho(1 - A/\rho c^2)$  and Equation (130), we recover Equations (99) and (100).

#### 6.2.3. Lagrangian $\mathcal{L}(X)$

The Lagrangian  $\mathcal{L}(X)$  is determined by Equations (213), (388) and (389). We get

$$X = \frac{c^2}{2} \left( 1 - \frac{A}{\rho_m c^2} \right)^2. \quad (397)$$

This relation can be inverted to give

$$\rho_m = \frac{A/c^2}{1 - \left(\frac{2X}{c^2}\right)^{1/2}}. \quad (398)$$

We then obtain

$$\mathcal{L}(X) = P = -A \ln \left[ \frac{\rho_* c^2}{A} \left( 1 - \sqrt{\frac{2X}{c^2}} \right) \right]. \quad (399)$$

In the nonrelativistic limit, using Equation (189), we recover Equation (103).

*Remark:* Starting from Equation (324), taking the limit  $\gamma \rightarrow 0$ ,  $K \rightarrow +\infty$  with  $K\gamma = A$  constant, and proceeding as in Equation (105), we obtain Equation (399) up to an additional constant. More generally, we can recover in the same manner the other equations of this section.

### 6.3. Logotropic Equation of State of Type III

The logotropic equation of state of type III reads [116]

$$P = A \ln \left( \frac{\rho}{\rho_*} \right). \quad (400)$$

Using Equation (170), the SF potential is

$$V_{\text{tot}}(\rho) = \frac{1}{2}\rho c^2 - A \ln \left( \frac{\rho}{\rho_*} \right) - A. \quad (401)$$

In the nonrelativistic limit, we recover Equations (99) and (100) [recalling Equation (130)]. Actually, Equations (400) and (401) coincide with Equations (99) and (100). The SF potential  $V(\rho)$  is logarithmic.

#### 6.3.1. Determination of $\epsilon$ and $P(\epsilon)$

The energy density is determined by Equations (214) with Equation (401). This yields

$$\epsilon = \rho c^2 - A \ln \left( \frac{\rho}{\rho_*} \right) - 2A. \quad (402)$$

The pressure is determined by Equation (215) with Equation (401). This returns Equation (400). Eliminating  $\rho$  between Equations (400) and (402), we obtain  $P(\epsilon)$  under the inverse form  $\epsilon(P)$  as

$$\epsilon = \rho_* c^2 e^{P/A} - P - 2A. \quad (403)$$

In the nonrelativistic limit, using  $\epsilon \sim \rho c^2$ , we recover Equation (99).

#### 6.3.2. Determination of $\rho_m$ , $P(\rho_m)$ and $u(\rho_m)$

The rest-mass density and the internal energy are determined by Equations (218) and (220) with Equation (401). We get

$$\rho_m = \sqrt{\rho \left( \rho - \frac{2A}{c^2} \right)}, \quad (404)$$

$$u = \rho c^2 - A \ln \left( \frac{\rho}{\rho_*} \right) - 2A - \rho_m(\rho) c^2. \quad (405)$$

Equation (404) can be inverted to give

$$\rho = \frac{A}{c^2} + \sqrt{\frac{A^2}{c^4} + \rho_m^2}. \quad (406)$$

Combined with Equations (400) and (405) we explicitly obtain  $P(\rho_m)$  and  $u(\rho_m)$  under the form

$$P = A \ln \left( \frac{A}{\rho_* c^2} + \sqrt{\frac{A^2}{\rho_*^2 c^4} + \frac{\rho_m^2}{\rho_*^2}} \right), \quad (407)$$

$$u = \sqrt{A^2 + \rho_m^2 c^4} - A \ln \left( \frac{A}{\rho_* c^2} + \sqrt{\frac{A^2}{\rho_*^2 c^4} + \frac{\rho_m^2}{\rho_*^2}} \right) - A - \rho_m c^2. \quad (408)$$

In the nonrelativistic limit, using  $\rho_m \simeq \rho - A/c^2$ , we recover Equations (99) and (100) [recalling Equation (29)].

### 6.3.3. Lagrangian $\mathcal{L}(X)$

The Lagrangian  $\mathcal{L}(X)$  is determined by Equation (221) with Equation (401). We get

$$X = \frac{1}{2}c^2 - \frac{A}{\rho}. \quad (409)$$

This equation can be reversed to give

$$\rho = \frac{A}{\frac{1}{2}c^2 - X}. \quad (410)$$

We then obtain

$$\mathcal{L}(X) = P = -A \ln \left[ \frac{\rho_* c^2}{2A} \left( 1 - \frac{2X}{c^2} \right) \right]. \quad (411)$$

In the nonrelativistic limit, using Equation (189), we recover Equation (103). Actually, Equation (411) coincides with Equation (103) with  $c^2/2 - X$  in place of  $x$ . Using Equation (175) or Equation (178), we obtain the equation of motion

$$D_\mu \left[ \frac{\partial^\mu \theta}{1 - \frac{1}{c^2} \partial_\nu \theta \partial^\nu \theta} \right] = 0. \quad (412)$$

*Remark:* Starting from Equation (371), taking the limit  $\gamma \rightarrow 0$ ,  $K \rightarrow +\infty$  with  $K\gamma = A$  constant, and proceeding as in Equation (105), we obtain Equation (411) up to an additional constant. More generally, we can recover in the same manner the other equations of this section.

## 7. Conclusions

In this paper, we have shown that the equation of state of a relativistic barotropic fluid could be specified in different manners depending on whether the pressure  $P$  is expressed in terms of the energy density  $\epsilon$  (model I), the rest-mass density  $\rho_m$  (model II), or the pseudo rest-mass density  $\rho$  (model III). In model II, specifying the equation of state  $P(\rho_m)$  is equivalent to specifying the internal energy  $u(\rho_m)$ . In model III, specifying the equation of state  $P(\rho)$  is equivalent to specifying the potential  $V(\rho)$  of the complex SF to which the fluid is associated in the TF limit. In the nonrelativistic limit, these three formulations coincide.

We have shown how these different models are connected to each other. We have established general equations allowing us to determine  $[\epsilon, P(\epsilon)]$ ,  $[\rho_m, P(\rho_m), u(\rho_m)]$  and  $[\rho, P(\rho), V(\rho)]$  once an equation of state is specified under the form I, II or III.

In model III, we have determined the hydrodynamic representation of a complex SF with a potential  $V(|\phi|^2)$  and the form of its Lagrangian. In the TF approximation, we can use the Bernoulli equation to obtain a reduced Lagrangian of the form  $\mathcal{L}(X)$  with  $X = \frac{1}{2}\partial_\mu\theta\partial^\mu\theta$ , where  $\theta$  is the phase of the SF. This is a k-essence Lagrangian whose expression is determined by the potential of the complex SF. We have established general equations allowing us to obtain  $\mathcal{L}(X)$  once an equation of state is specified under the form I, II or III.

For illustration, we have applied our formalism to polytropic, isothermal and logotropic equations of state of type I, II and III that have been proposed as UDM models. We have recovered previously obtained results, and we have derived new results. For example, we have established the general analytical expression of the k-essence Lagrangian of polytropic and isothermal equations of state of type I, II and III. For  $\gamma = -1$  (Chaplygin gas), the models of type I and III are equivalent and return the Born-Infeld action, while the model of type II leads to a different action. We have also established the general analytical expression of the k-essence Lagrangian associated with a logotrope of type II and III (the k-essence Lagrangian associated with a logotrope of type I cannot be obtained analytically).

In a future contribution [165], we will apply our general formalism to more complicated equations of state which can be viewed as a superposition of polytropic, isothermal (linear) and logotropic equations of state.

The mixed equation of state of type I generically reads

$$P = K\left(\frac{\epsilon}{c^2}\right)^\gamma + \alpha\epsilon - \epsilon_\Lambda + A \ln\left(\frac{\epsilon}{\epsilon_P}\right), \quad (413)$$

where we can add several polytropic terms with different indices  $\gamma$ . More specifically, we can consider generalized polytropic models of type I of the form

$$P = -(\alpha + 1)\epsilon\left(\frac{\epsilon}{\epsilon_P}\right)^{1/|n_e|} + \alpha\epsilon - (\alpha + 1)\epsilon\left(\frac{\epsilon_\Lambda}{\epsilon}\right)^{1/|n_l|}, \quad (414)$$

or

$$P = -(\alpha + 1)\frac{\epsilon^2}{\epsilon_P} + \alpha\epsilon - (\alpha + 1)\epsilon_\Lambda. \quad (415)$$

Polytropic, isothermal (linear) and logotropic equations of state of type I have been studied in the context of relativistic stars [151,154,155,157–161] and cosmology [50,86,95,97–99,122]. Mixed models of type I of the form of Equations (413)–(415) have been introduced and studied in cosmology in Refs. [97–99,166–170]. In particular, the equations of state (414) and (415) describe the early inflation, the intermediate decelerating expansion, and the late accelerating expansion of the universe in a unified manner [168,170].

The mixed equation of state of type II generically reads

$$P = K\rho_m^\gamma + \alpha\rho_m c^2 - \rho_\Lambda c^2 + A \ln\left(\frac{\rho_m}{\rho_P}\right), \quad (416)$$

where we can add several polytropic terms with different indices  $\gamma$ . It is associated with an internal energy of the form

$$u = \frac{K}{\gamma - 1}\rho_m^\gamma + \alpha\rho_m c^2 \left[ \ln\left(\frac{\rho_m}{\rho_*}\right) - 1 \right] + \rho_\Lambda c^2 - A \ln\left(\frac{\rho_m}{\rho_P}\right) - A. \quad (417)$$

Polytropic, isothermal (linear) and logotropic equations of state of type II have been studied in the context of relativistic stars [154,162] and cosmology [118,122,163]. It is often assumed that DM is pressureless ( $P = 0$ ) so that  $\alpha = 0$ . However, a nonvanishing value of  $\alpha$  can account for thermal effects as in [171–173]. In that case  $\alpha c^2 = k_B T/m$ . Mixed models of type II of the form of Equations (416) and (417) have been introduced and studied in Refs. [118,163].

The mixed equation of state of type III generically reads

$$P = K\rho^\gamma + \alpha\rho c^2 - \rho_\Lambda c^2 + A \ln\left(\frac{\rho}{\rho_P}\right), \quad (418)$$

where we can add several polytropic terms with different indices  $\gamma$ . It is associated with a complex SF potential of the form

$$V_{\text{tot}} = \frac{1}{2}\rho c^2 + \frac{K}{\gamma-1}\rho^\gamma + \alpha\rho c^2 \left[ \ln\left(\frac{\rho}{\rho_*}\right) - 1 \right] + \rho_\Lambda c^2 - A \ln\left(\frac{\rho}{\rho_P}\right) - A, \quad (419)$$

where we recall that  $\rho = (m/\hbar)^2 |\phi|^2$ . Polytropic, isothermal (linear) and logotropic equations of state of type III have been studied in the context of relativistic stars [152,154,155] and cosmology [116,122,149,153,156]. Mixed models of type III of the form of Equations (418) and (419) have been introduced and studied in Refs. [116,156].

The aim of the present paper was to develop a general SF theory that can be applied to different situations of physical, astrophysical, and cosmological interest. BECs with repulsive or attractive self-interactions have many applications in condensed matter physics [140]. On the other hand, self-gravitating BECs can describe boson stars [152,174,175], axion stars [176], and DM halos [177]. A SF can also be used to describe the primordial inflation [34] or the dark energy [24]. This SF is respectively called inflaton or quintessence. A single SF called vacuumon may even drive the whole evolution of the universe from its early inflation to its late accelerating expansion [170]. Recently, using the general formalism developed in this paper, we have specifically studied polytropic and logotropic equations of state of type III in Refs. [156] and [116] respectively. They are associated with a complex SF possessing a power-law or a logarithmic potential. Depending on the value of the polytropic index  $\gamma$  and on the sign of the polytropic constant  $K$ , the polytropic equation of state of type III can generate different models of universe which are either always expanding or oscillating. However, as discussed in detail in [156], only a few models are consistent with the observations. A viable model corresponds to a polytropic index  $\gamma = 2$  and a positive polytropic constant  $K > 0$ . It is associated with a repulsive  $|\phi|^4$  SF potential. In that case, the universe undergoes a stiff matter era in the slow oscillation regime followed, in the fast oscillation regime, by a dark radiation era (due to the self-interaction of the SF), and finally a DM (matterlike) era [149,153]. However, this model does not account for the present acceleration of the universe. This could be remedied by considering a mixed equation of state of the form

$$P = K\rho^2 - \rho_\Lambda c^2, \quad (420)$$

corresponding to a SF potential [116]

$$V_{\text{tot}} = \frac{m^2 c^2}{2\hbar^2} |\phi|^2 + \frac{Km^4}{\hbar^4} |\phi|^4 + \rho_\Lambda c^2 \quad (421)$$

including a cosmological constant  $\rho_\Lambda c^2$ . The energy density reads

$$\epsilon = \rho c^2 + 3K\rho^2 + \rho_\Lambda c^2. \quad (422)$$

The associated equation of state of type I is

$$P = \frac{1}{36K} \left[ -c^2 + \sqrt{c^4 + 12K(\epsilon - \rho_\Lambda c^2)} \right]^2 - \rho_\Lambda c^2 \quad (423)$$

and the associated k-essence Lagrangian is

$$\mathcal{L}(X) = \frac{c^4}{16K} \left( 1 - \frac{2X}{c^2} \right)^2 - \rho_\Lambda c^2. \quad (424)$$

This model describes a universe undergoing, in the fast oscillation regime, a dark radiation era, a DM (matterlike) era, and a DE era. The  $|\varphi|^4$  model studied in [149,153,156] is recovered for  $\rho_\Lambda = 0$  and the  $\Lambda$ CDM model (in its complex SF interpretation [116]) is recovered for  $K = 0$ . On the other hand, polytropic models with  $\gamma \leq 0$  and  $K < 0$  (including the Chaplygin gas model and the logotropic model) display, in the fast oscillation regime, a DM (matterlike) era followed by a DE era. These models can account for the evolution of the cosmological background but fail to reproduce the formation of structures and the matter power spectrum unless  $\gamma$  is extremely close to  $\gamma = 0$ , returning the  $\Lambda$ CDM model. If we use a two-fluid representation of these models (see Appendix D.3), we can correctly describe not only the cosmological background but also the formation of structures and the matter power spectrum. However, in that case, we lose the original interest of UDM models (see the Remark at the end of Appendix D.3). In conclusion, our general formalism allows us to deal with various situations. It may help us selecting the most interesting models and rule out the others.

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## Appendix A. General Identities for a Nonrelativistic Cold Gas

The first principle of thermodynamics for a nonrelativistic gas can be written as

$$d\left(\frac{u}{\rho}\right) = -Pd\left(\frac{1}{\rho}\right) + Td\left(\frac{s}{\rho}\right), \quad (\text{A1})$$

where  $u$  is the density of internal energy,  $s$  the density of entropy,  $\rho = nm$  the mass density,  $P$  the pressure, and  $T$  the temperature. For a cold ( $T = 0$ ) or isentropic ( $s/\rho = \text{cst}$ ) gas, Equation (A1) reduces to

$$d\left(\frac{u}{\rho}\right) = -Pd\left(\frac{1}{\rho}\right) = \frac{P}{\rho^2}d\rho. \quad (\text{A2})$$

Introducing the enthalpy per particle

$$h = \frac{P + u}{\rho}, \quad (\text{A3})$$

we get

$$du = h d\rho \quad \text{and} \quad dh = \frac{dP}{\rho}. \quad (\text{A4})$$

For a barotropic gas for which  $P = P(\rho)$ , the foregoing equations can be written as

$$P(\rho) = -\frac{d(u/\rho)}{d(1/\rho)} = \rho^2 \left[ \frac{u(\rho)}{\rho} \right]' = \rho u'(\rho) - u(\rho), \quad (\text{A5})$$

$$c_s^2 = P'(\rho) = \rho u''(\rho), \quad h(\rho) = \frac{P(\rho) + u(\rho)}{\rho}, \quad (\text{A6})$$

$$h(\rho) = u'(\rho), \quad h'(\rho) = \frac{P'(\rho)}{\rho}, \quad (\text{A7})$$



where  $c_s^2 = P'(\rho)$  is the squared speed of sound. Equation (A5) determines the equation of state  $P(\rho)$  as a function of the internal energy  $u(\rho)$ . Inversely, the internal energy is determined by the equation of state according to the relation

$$u(\rho) = \rho \int \frac{P(\rho)}{\rho^2} d\rho, \quad (\text{A8})$$

which is the solution of the differential equation

$$\rho \frac{du}{d\rho} - u(\rho) = P(\rho). \quad (\text{A9})$$

Comparing Equation (A8) with Equation (28), we see that the potential  $V(\rho)$  represents the density of internal energy:

$$u(\rho) = V(\rho). \quad (\text{A10})$$

We then have

$$P(\rho) = \rho^2 \left[ \frac{V(\rho)}{\rho} \right]' = \rho V'(\rho) - V(\rho), \quad (\text{A11})$$

$$c_s^2 = P'(\rho) = \rho V''(\rho), \quad h(\rho) = \frac{P(\rho) + V(\rho)}{\rho}, \quad (\text{A12})$$

$$h(\rho) = V'(\rho), \quad h'(\rho) = \frac{P'(\rho)}{\rho}, \quad (\text{A13})$$

$$V(\rho) = \rho \int \frac{P(\rho)}{\rho^2} d\rho. \quad (\text{A14})$$

*Remark:* The first principle of thermodynamics can be written as

$$du = Tds + \mu dn, \quad (\text{A15})$$

where  $\mu$  is the local chemical potential. This can be viewed as the variational principle ( $\delta s/k_B - \beta \delta u + \alpha \delta n = 0$  with  $\beta = 1/k_B T$  and  $\alpha = \mu/k_B T$ ) associated with the maximization of the entropy density  $s$  at fixed energy density  $u$  and particle density  $n$  [178]. Combined with the Gibbs-Duhem relation [178]

$$s = \frac{u + P - \mu n}{T}, \quad (\text{A16})$$

we obtain Equation (A1) and

$$sdT - dP + nd\mu = 0. \quad (\text{A17})$$

If  $T = \text{cst}$ , then  $dP = nd\mu$ . For  $T = 0$ , the foregoing equations reduce to

$$du = \mu dn, \quad \mu = \frac{u + P}{n}, \quad dP = nd\mu, \quad (\text{A18})$$

which are equivalent to Equations (A3) and (A4) with  $\mu = mh$ . Therefore, the enthalpy  $h(\mathbf{r})$  is equal to the local chemical potential  $\mu(\mathbf{r})$  by unit of mass:  $h(\mathbf{r}) = \mu(\mathbf{r})/m$ .

## Appendix B. K-Essence Lagrangian of a Real SF

### Appendix B.1. General Results

We consider a relativistic real SF  $\varphi(x^\mu) = \varphi(x, y, z, t)$  characterized by the action

$$S = \int \mathcal{L} \sqrt{-g} d^4x, \quad (\text{A19})$$

where  $\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$  is the Lagrangian density and  $g = \det(g_{\mu\nu})$  is the determinant of the metric tensor. The Lagrangian of a relativistic real SF  $\varphi$  is usually written in the canonical form

$$\mathcal{L} = X - V(\varphi), \quad (\text{A20})$$

where

$$X = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi \quad (\text{A21})$$

is the kinetic energy and  $V$  is the potential energy<sup>31</sup>. In that case, all the physics of the problem is contained in the potential term. However, some authors have proposed to take  $V = 0$  and modify the kinetic term. This leads to a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(X) \quad (\text{A22})$$

that is called a k-essence Lagrangian [38]. In that case, the physics of the problem is encapsulated in the noncanonical kinetic term  $\mathcal{L}(X)$ <sup>32</sup>. Equation (A22) is a pure k-essence Lagrangian. More general Lagrangians

$$\mathcal{L} = \mathcal{L}(X, \varphi) \quad (\text{A23})$$

can depend both on  $X$  and  $\varphi$  [40,41]. The particular forms  $\mathcal{L} = V(\varphi)F(X)$  and  $\mathcal{L} = F(X) - V(\varphi)$  have been specifically introduced in Refs. [40] and [179–181] respectively.

The least action principle  $\delta S = 0$ , which is equivalent to the Euler-Lagrange equation

$$D_\mu \left[ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right] - \frac{\partial \mathcal{L}}{\partial \varphi} = 0, \quad (\text{A24})$$

yields the equation of motion

$$D_\mu \left( \frac{\partial \mathcal{L}}{\partial X} \partial^\mu \varphi \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. \quad (\text{A25})$$

For the Lagrangian (A22), it reduces to

$$D_\mu [\mathcal{L}'(X) \partial^\mu \varphi] = 0. \quad (\text{A26})$$

For the Lagrangian (A22) the current is given by

$$J^\mu = - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}, \quad (\text{A27})$$

yielding

$$J_\mu = -\mathcal{L}'(X) \partial_\mu \varphi. \quad (\text{A28})$$

Equation (A26) can then be written as  $D_\mu J^\mu = 0$ . It can therefore be viewed as a continuity equation expressing the local conservation of the charge (or the local conservation of the boson number) given by  $Q = \frac{e}{mc} \int J^0 \sqrt{-g} d^3x$ , i.e.,

$$Q = - \frac{e}{mc} \int \mathcal{L}'(X) \partial^0 \varphi \sqrt{-g} d^3x. \quad (\text{A29})$$

In the present context, the conservation of the charge (or boson number) arises from the invariance of the Lagrangian density under the constant shift  $\varphi \rightarrow \varphi + \text{cst}$  of the SF (Noether theorem)<sup>33</sup>. Introducing the quadrivelocity

$$u_\mu = -\frac{\partial_\mu \varphi}{\sqrt{2X}}c, \quad (\text{A30})$$

which satisfies by construction the identity  $u_\mu u^\mu = c^2$ , we get

$$J_\mu = \mathcal{L}'(X)\sqrt{2X}\frac{u_\mu}{c}. \quad (\text{A31})$$

We can therefore rewrite the continuity equation as

$$D_\mu \left[ \mathcal{L}'(X)\sqrt{2X}u^\mu \right] = 0. \quad (\text{A32})$$

The rest-mass density  $\rho_m$  is defined by

$$J_\mu = \rho_m u_\mu, \quad (\text{A33})$$

and the continuity equation can be written as

$$D_\mu (\rho_m u^\mu) = 0. \quad (\text{A34})$$

Comparing Equations (A31) and (A32) with Equations (A33) and (A34), we find that the rest-mass density is given by

$$\rho_m = \mathcal{L}'(X)\sqrt{2X}\frac{1}{c}. \quad (\text{A35})$$

The energy-momentum tensor is given by Equation (115). It satisfies the conservation law  $D_\mu T^{\mu\nu} = 0$ . For a real SF we have

$$T_\mu^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \varphi)} \partial_\mu \varphi - g_\mu^\nu \mathcal{L}. \quad (\text{A36})$$

The energy-momentum tensor associated with the Lagrangian (A23) is

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial X} \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \mathcal{L}. \quad (\text{A37})$$

Introducing the quadrivelocity from Equation (A30) we get

$$T_{\mu\nu} = 2X \frac{\partial \mathcal{L}}{\partial X} \frac{u_\mu u_\nu}{c^2} - g_{\mu\nu} \mathcal{L}. \quad (\text{A38})$$

The energy-momentum tensor (A37) can be written under the perfect fluid form

$$T_{\mu\nu} = (\epsilon + P) \frac{u_\mu u_\nu}{c^2} - P g_{\mu\nu}, \quad (\text{A39})$$

where  $\epsilon$  is the energy density and  $P$  is the pressure, provided that we make the identifications

$$P = \mathcal{L} \quad \text{and} \quad \epsilon + P = 2X \frac{\partial \mathcal{L}}{\partial X}. \quad (\text{A40})$$

As a result, the pressure and the energy density associated with the Lagrangian (A23) are given by

$$P = \mathcal{L}(X, \varphi), \quad (\text{A41})$$

$$\epsilon = 2X \frac{\partial P}{\partial X} - P. \quad (\text{A42})$$

The Lagrangian plays the role of an effective pressure. If the Lagrangian satisfies the condition  $X\partial P/\partial X \ll P$  for some range of  $X$  and  $\varphi$ , then the equation of state is  $P \simeq -\epsilon$  (vacuum energy) and we have an inflationary solution [182]. On the other hand, for the Lagrangian  $\mathcal{L} = V(\varphi)X$  corresponding to  $P \propto X$  i.e.,  $X\partial P/\partial X = P$  we obtain the stiff equation of state  $P = \epsilon$ . In that case, the equation of motion (A25) becomes  $D_\mu(V(\varphi)\partial^\mu\varphi) - \frac{1}{2}V'(\varphi)\partial_\mu\varphi\partial^\mu\varphi = 0$ . It reduces to  $\square\varphi = 0$  when  $\mathcal{L} = AX$ .

The equation of state parameter and the squared speed of sound are given by [41]

$$w = \frac{P}{\epsilon} = \frac{P}{2X \frac{\partial P}{\partial X} - P} \quad (\text{A43})$$

and

$$c_s^2 = \frac{\frac{\partial P}{\partial X}}{\frac{\partial \epsilon}{\partial X}} c^2 = \frac{\frac{\partial P}{\partial X}}{\frac{\partial P}{\partial X} + 2X \frac{\partial^2 P}{\partial X^2}} c^2. \quad (\text{A44})$$

We note that  $c_s \simeq c$  if  $2X\partial^2 P/\partial X^2 \ll \partial P/\partial X$ . This is the case in particular for the Lagrangian  $\mathcal{L} = V(\varphi)X$  discussed above for which  $c_s = c$  exactly.

In the general case, we have  $P = P(X, \varphi)$  and  $\epsilon = \epsilon(X, \varphi)$  so that the fluid is not necessarily barotropic. However, for a  $k$ -essence SF described by a Lagrangian of the form of Equation (A22), we have  $P = P(X)$  and  $\epsilon = \epsilon(X)$  implying  $P = P(\epsilon)$ . In that case, the fluid is barotropic and  $c_s^2 = P'(\epsilon)c^2$ . On the other hand, using Equations (A35), (A41), (A42) and (A163), we find that the enthalpy is given by

$$h = c\sqrt{2X}. \quad (\text{A45})$$

*Remark:* For the Lagrangian  $\mathcal{L} = V(\varphi)X^{(\alpha+1)/2\alpha}$ , we obtain the linear equation of state  $P = \alpha\epsilon$ . This includes stiff matter ( $\alpha = 1$ ;  $\mathcal{L} = V(\varphi)X$ ), radiation ( $\alpha = 1/3$ ;  $\mathcal{L} = V(\varphi)X^2$ ) and a cosmological constant ( $\alpha = -1$ ;  $\mathcal{L} = V(\varphi)$ ). The equation of motion (A25) becomes

$$\frac{\alpha+1}{2\alpha} D_\mu \left[ V(\varphi) X^{\frac{1-\alpha}{2\alpha}} \partial^\mu \varphi \right] - V'(\varphi) X^{\frac{\alpha+1}{2\alpha}} = 0. \quad (\text{A46})$$

It reduces to

$$D_\mu \left( X^{\frac{1-\alpha}{2\alpha}} \partial^\mu \varphi \right) = 0 \quad (\text{A47})$$

when  $\mathcal{L} = AX^{(\alpha+1)/2\alpha}$ . For the Lagrangian  $\mathcal{L} = V(\varphi) \ln X$  (which can be viewed as a limit of the Lagrangian  $V(\varphi)X^{(\alpha+1)/2\alpha}$  for  $\alpha \rightarrow -1$  and  $V \rightarrow +\infty$  with  $(\alpha+1)V$  finite), we obtain  $P = 2V(\varphi) - \epsilon$  which reduces to the affine equation of state  $P = 2A - \epsilon$  [98,99] when  $V(\varphi) = A$ . In that case, the equation of motion (A25) becomes

$$D_\mu \left[ \frac{V(\varphi)}{X} \partial^\mu \varphi \right] - V'(\varphi) \ln X = 0. \quad (\text{A48})$$

It reduces to

$$D_\mu \left( \frac{1}{X} \partial^\mu \varphi \right) = 0 \quad (\text{A49})$$

when  $\mathcal{L} = A \ln X$ .

### Appendix B.2. Canonical SF

The Lagrangian of a real canonical SF is

$$\mathcal{L} = \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi), \quad (\text{A50})$$

where the first term is the kinetic energy and the second term is minus the potential energy. It is of the form  $\mathcal{L} = X - V(\varphi)$ . The least action principle  $\delta S = 0$ , which is equivalent to the Euler-Lagrange Equation (A24), leads to the KG equation

$$\square\varphi + \frac{dV}{d\varphi} = 0, \quad (\text{A51})$$

where  $\square = D_\mu\partial^\mu$  is the d'Alembertian. A canonical real SF does not conserve the charge.

The energy-momentum tensor (A36) associated with the canonical Lagrangian (A50) is

$$T_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\mathcal{L}. \quad (\text{A52})$$

Repeating the procedure of Appendix B.1 we find that the energy density and the pressure are given by

$$\epsilon = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi), \quad (\text{A53})$$

$$P = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi). \quad (\text{A54})$$

Since  $\epsilon = X + V(\varphi)$  and  $P = X - V(\varphi)$ , we find that  $P = \epsilon - 2V(\varphi)$ ,  $w = [X - V(\varphi)]/[X + V(\varphi)]$  and  $c_s = c$ . For a canonical SF, the speed of sound is equal to the speed of light.

When  $X \gg V$ , we obtain  $\epsilon = X$  and  $P = X$  leading to the equation of state  $P = \epsilon$  corresponding to stiff matter. This is the so-called kination regime [183]. This regime is achieved in particular when  $V = 0$ . In that case, the Lagrangian  $\mathcal{L} = X$  describes a noninteracting massless SF and the KG equation reduces to  $\square\varphi = 0$ . When  $X \ll V$ , we obtain  $\epsilon = V$  and  $P = -V$  leading to the equation of state  $P = -\epsilon$  corresponding to the vacuum energy. This regime is achieved in particular when  $\varphi = \varphi_0$  is constant ( $X = 0$ ) and lies at an extremum of the potential ( $V'(\varphi_0) = 0$ ). In cosmology, this equation of state leads to a de Sitter era where  $\epsilon = V(\varphi_0)$  is constant and the scale factor increases exponentially rapidly with time as  $a \propto \exp[(8\pi G\epsilon/3c^2)^{1/2}t]$ . The condition  $X \ll V$  corresponds to the slow-roll regime [182].

When  $V(\varphi) = V_0$  is constant, the Lagrangian  $\mathcal{L} = X - V_0$  describes a massless SF in the presence of a cosmological constant ( $\epsilon_\Lambda = V_0$ ). In that case,  $\epsilon = X + V_0$  and  $P = X - V_0$  leading to the affine equation of state  $P = \epsilon - 2V_0 = \epsilon - 2\epsilon_\Lambda$  [98]. In cosmology, when  $V_0 > 0$ , this equation of state generically leads to a stiff matter era followed by a de Sitter era (or a de Sitter era alone when  $X = 0$ , i.e.,  $\varphi = \text{cst}$ ). When  $V_0 = 0$  we recover the Lagrangian  $\mathcal{L} = X$  of a free massless SF. In that case,  $\epsilon = X$  and  $P = X$  leading to the stiff equation of state  $P = \epsilon$ . In cosmology, it describes a pure stiff matter era.

*Remark:* The Lagrangian of a particle of mass  $m$  and position  $q(t)$  in Newtonian mechanics is  $L = (1/2)m\dot{q}^2 - V(q)$ . Its impulse is  $p = \partial L/\partial\dot{q} = m\dot{q}$  and its energy is  $E = p\dot{q} - L = \dot{q}\partial L/\partial\dot{q} - L = (1/2)m\dot{q}^2 + V(q)$ , i.e.,  $E = p^2/(2m) + V(q)$ . Its equation of motion is given by the Euler-Lagrange equation  $(d/dt)(\partial L/\partial\dot{q}) - \partial L/\partial q = 0$  yielding  $m\ddot{q} = -V'(q)$ . The Lagrangian equations of a canonical SF are similar to the Lagrangian equations of a nonrelativistic particle in which the SF  $\varphi(x^\mu)$  plays the role of  $q(t)$  and the SF potential  $V(\varphi)$  the role of  $V(q)$ .

### Appendix B.3. Tachyonic SF

The Lagrangian of a real tachyonic SF is

$$\mathcal{L} = -V(\varphi) \sqrt{1 - \frac{1}{c^2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi}. \quad (\text{A55})$$

This corresponds to the Born-Infeld Lagrangian (5) multiplied by  $V(\varphi)$ . It is of the form  $\mathcal{L} = -V(\varphi) \sqrt{1 - 2X/c^2}$ . This Lagrangian was introduced by Sen [77–79] in the context of string theory and  $d$ -branes and further discussed in [80–87]. The relation to k-essence fields was made in [81,83,87]. The least action principle  $\delta S = 0$ , which is equivalent to the Euler-Lagrange Equation (A24), leads to the equation of motion

$$D_\mu \left[ \frac{V(\varphi)/c^2}{\sqrt{1 - \frac{1}{c^2} \partial_\mu \varphi \partial^\mu \varphi}} \partial^\mu \varphi \right] + V'(\varphi) \sqrt{1 - \frac{1}{c^2} \partial_\mu \varphi \partial^\mu \varphi} = 0 \quad (\text{A56})$$

or, equivalently,

$$D_\mu \partial^\mu \varphi + \frac{D_\mu \partial_\nu \varphi}{1 - \partial_\mu \varphi \partial^\mu \varphi} \partial^\mu \varphi \partial^\nu \varphi + (\ln V)' c^2 = 0. \quad (\text{A57})$$

A real tachyonic SF does not conserve the charge.

The energy-momentum tensor (A36) associated with the tachyonic Lagrangian (A55) is

$$T_{\mu\nu} = \frac{V(\varphi)/c^2}{\sqrt{1 - \frac{1}{c^2} \partial_\mu \varphi \partial^\mu \varphi}} \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} \mathcal{L}. \quad (\text{A58})$$

Repeating the procedure of Appendix B.1 we find that the energy density and the pressure are given by

$$\epsilon = \frac{V(\varphi)}{\sqrt{1 - \frac{1}{c^2} \partial_\mu \varphi \partial^\mu \varphi}}, \quad (\text{A59})$$

$$P = -V(\varphi) \sqrt{1 - \frac{1}{c^2} \partial_\mu \varphi \partial^\mu \varphi}. \quad (\text{A60})$$

Since  $\epsilon = V(\varphi)/\sqrt{1 - 2X/c^2}$  and  $P = -V(\varphi)\sqrt{1 - 2X/c^2}$ , we find that  $P = -V(\varphi)^2/\epsilon$ ,  $w = -(1 - 2X/c^2)$  and  $c_s^2 = (1 - 2X/c^2)c^2 = -wc^2$ .

When  $V(\varphi) = V_0$  is constant, the Lagrangian (A55) reduces to the Born-Infeld Lagrangian (5) and we obtain  $\epsilon = V_0/\sqrt{1 - 2X/c^2}$  and  $P = -V_0\sqrt{1 - 2X/c^2}$  leading to the Chaplygin equation of state  $P = -V_0^2/\epsilon$  (inversely, the Chaplygin equation of state  $P = -Ac^2/\epsilon$  leads to the Born-Infeld Lagrangian (5) corresponding to Equation (A55) with  $V(\varphi) = V_0$  constant). Therefore, the Chaplygin gas can be considered as the simplest tachyon model where the tachyon field is associated with a purely kinetic Lagrangian. The relation between the tachyonic Lagrangian with a constant potential (reducing to the Born-Infeld Lagrangian) and the Chaplygin gas [50] was first made by [81]. The fact that the Chaplygin gas is associated with the Born-Infeld Lagrangian was understood by [58,59,70,71]. The relation between the Chaplygin gas, the Born-Infeld Lagrangian, k-essence Lagrangians and tachyon fields were further discussed in [86,88,89,94,184].

*Remark:* The Lagrangian of a particle of mass  $m$  and position  $q(t)$  in special relativity is  $L = -mc^2 \sqrt{1 - \dot{q}^2/c^2}$ . Its impulse is  $p = m\dot{q}/\sqrt{1 - \dot{q}^2/c^2}$  and its energy is  $E = mc^2/\sqrt{1 - \dot{q}^2/c^2}$  implying  $E^2 = p^2 c^2 + m^2 c^4$ . Its equation of motion is given by the Euler-Lagrange equation  $(d/dt)(\partial L/\partial \dot{q}) - \partial L/\partial q = 0$  yielding  $(d/dt)(m\dot{q}/\sqrt{1 - \dot{q}^2/c^2}) + m'(q)c^2 \sqrt{1 - \dot{q}^2/c^2} = 0$ . The Lagrangian equations of a tachyonic SF are similar to the

Lagrangian equations of a relativistic particle in which the SF  $\varphi(x^\mu)$  plays the role of  $q(t)$  and the SF potential  $V(\varphi)$  the role of the mass  $m$  which may depend on  $q$  in the general case.

#### Appendix B.4. Nonrelativistic Limit

The action of a nonrelativistic real SF is

$$S = \int \mathcal{L} d^3x dt, \quad (\text{A61})$$

where  $\mathcal{L} = \mathcal{L}(\varphi, \dot{\varphi}, \nabla\varphi)$  is the Lagrangian density. We consider a pure k-essence Lagrangian of the form

$$\mathcal{L} = \mathcal{L}(x), \quad (\text{A62})$$

where

$$x = \dot{\varphi} + \frac{1}{2}(\nabla\varphi)^2. \quad (\text{A63})$$

More general k-essence Lagrangians

$$\mathcal{L} = \mathcal{L}(x, \varphi) \quad (\text{A64})$$

can depend both on  $x$  and  $\varphi$ . The least action principle  $\delta S = 0$ , which is equivalent to the Euler-Lagrange equation

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \right) + \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial \nabla \varphi} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0, \quad (\text{A65})$$

yields the equation of motion

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial x} \right) + \nabla \cdot \left( \frac{\partial \mathcal{L}}{\partial x} \nabla \varphi \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. \quad (\text{A66})$$

For the Lagrangian (A62), it reduces to

$$\frac{\partial}{\partial t} [\mathcal{L}'(x)] + \nabla \cdot [\mathcal{L}'(x) \nabla \varphi] = 0. \quad (\text{A67})$$

For the Lagrangian (A62) the current is given by

$$J^\mu = -\frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)}. \quad (\text{A68})$$

It determines the mass density

$$\rho \equiv -\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = -\mathcal{L}'(x) \quad (\text{A69})$$

and the mass flux

$$\mathbf{J} \equiv -\frac{\partial \mathcal{L}}{\partial (\nabla \varphi)} = -\mathcal{L}'(x) \nabla \varphi. \quad (\text{A70})$$

Equation (A67) can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (\text{A71})$$

It expresses the local conservation of mass. This conservation law is associated with the invariance of the Lagrangian density under the transformation  $\varphi \rightarrow \varphi + \text{cst}$  (Noether theorem). Introducing the velocity

$$\mathbf{u} = \nabla \varphi, \quad (\text{A72})$$

we get

$$\mathbf{J} = \rho \mathbf{u}, \quad (\text{A73})$$

and the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (\text{A74})$$

The energy-momentum tensor is given by

$$T_{\mu}^{\nu} = \partial_{\mu} \varphi \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} \varphi)} - \mathcal{L} \delta_{\mu}^{\nu}. \quad (\text{A75})$$

The local conservation of energy and impulse can be written as

$$\frac{\partial T_{00}}{\partial t} - \partial_i T_{0i} = 0, \quad (\text{A76})$$

$$-\frac{\partial T_{i0}}{\partial t} + \partial_j T_{ij} = 0. \quad (\text{A77})$$

For the Lagrangian (A64) we obtain the energy density

$$T_{00} \equiv \dot{\varphi} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x} \dot{\varphi} - \mathcal{L}(x), \quad (\text{A78})$$

the momentum density

$$-T_{i0} \equiv -\partial_i \varphi \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = -\frac{\partial \mathcal{L}}{\partial x} \partial_i \varphi, \quad (\text{A79})$$

the energy flux

$$-T_{0i} \equiv \dot{\varphi} \frac{\partial \mathcal{L}}{\partial (\partial_i \varphi)} = \frac{\partial \mathcal{L}}{\partial x} \dot{\varphi} \partial_i \varphi, \quad (\text{A80})$$

and the momentum fluxes (stress tensor)

$$T_{ij} \equiv -\partial_i \varphi \frac{\partial \mathcal{L}}{\partial (\partial_j \varphi)} + \mathcal{L} \delta_{ij} = -\frac{\partial \mathcal{L}}{\partial x} \partial_i \varphi \partial_j \varphi + \mathcal{L} \delta_{ij}. \quad (\text{A81})$$

Introducing the velocity from Equation (A72), we get

$$T_{ij} = -\frac{\partial \mathcal{L}}{\partial x} u_i u_j + \mathcal{L} \delta_{ij}. \quad (\text{A82})$$

The energy-momentum tensor  $T_{ij}$  can be written under the perfect fluid form

$$T_{ij} = \rho u_i u_j + P \delta_{ij} \quad (\text{A83})$$

provided that we make the identifications

$$P = \mathcal{L}(x, \varphi) \quad (\text{A84})$$



and

$$\rho = -\frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial P}{\partial x}. \quad (\text{A85})$$

The equation of state parameter and the squared speed of sound are given by

$$w = \frac{P}{\rho c^2} = -\frac{P}{\frac{\partial P}{\partial x} c^2}, \quad (\text{A86})$$

$$c_s^2 = \frac{\frac{\partial P}{\partial x}}{\frac{\partial \rho}{\partial x}} = -\frac{\frac{\partial P}{\partial x}}{\frac{\partial^2 P}{\partial x^2}}. \quad (\text{A87})$$

Using Equations (A84) and (A85), we can rewrite Equation (A66) and Equations (A78)–(A80) as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) + \frac{\partial \mathcal{L}}{\partial \varphi} = 0, \quad (\text{A88})$$

$$T_{00} = -\rho \dot{\varphi} - \mathcal{L}(x), \quad (\text{A89})$$

$$-T_{i0} = \rho u_i, \quad (\text{A90})$$

$$-T_{0i} = -\rho \dot{\varphi} u_i. \quad (\text{A91})$$

For the Lagrangian (A62), Equation (A88) reduces to the continuity Equation (A74). In that case, the momentum density is equal to the mass flux:  $-T_{i0} = J_i$ . On the other hand, using Equation (A63), the energy density can be written as

$$T_{00} = \frac{1}{2} \rho \mathbf{u}^2 + V(\rho), \quad (\text{A92})$$

where we have defined the potential  $V(\rho)$  by the Legendre transform  $V(\rho) = -\rho x - \mathcal{L}(x)$ . Using Equation (A85), we get  $V'(\rho) = -x$ . Then, using Equation (A84), we obtain  $P(\rho) = \rho V'(\rho) - V(\rho)$  returning Equations (26) and (58). Therefore,  $V(\rho)$  coincides with the potential introduced in Section 2. Similarly, the energy flux can be written as

$$-T_{0i} = \rho \left[ \frac{1}{2} \mathbf{u}^2 + V'(\rho) \right] u_i, \quad (\text{A93})$$

where  $h(\rho) = V'(\rho)$  is the enthalpy. These results are consistent with the results obtained in Appendix I when  $\hbar = 0$ .

*Remark:* Equations (A84) and (A85) are the counterparts of Equations (A41) and (A42) in the relativistic case. Indeed, using Equations (189) and  $\epsilon \sim \rho c^2$  valid in the nonrelativistic limit, Equations (A41) and (A42) imply  $P = \mathcal{L}(x, \varphi)$  and

$$\rho \sim \frac{\epsilon}{c^2} \sim -\frac{\partial P}{\partial x} \sim -\frac{\partial \mathcal{L}}{\partial x}, \quad (\text{A94})$$

returning Equations (A84) and (A85).

### Appendix B.5. Cosmological Evolution

We now consider a spatially homogeneous real SF described by a k-essence Lagrangian in an expanding universe. In that case<sup>34</sup>

$$X = \frac{1}{2}\dot{\phi}^2. \quad (\text{A95})$$

On the other hand, the energy-momentum tensor is diagonal  $T_\mu^\nu = \text{diag}(\epsilon, -P, -P, -P)$ <sup>35</sup> using Equation (A39) with  $u^0 = c$  and  $u^i = 0$ , we get  $T_0^0 = \epsilon$  and  $T_i^i = -P$ . The energy density and the pressure of the SF are given by

$$\epsilon = T_0^0 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L}, \quad P = -T_i^i = \mathcal{L}, \quad (\text{A96})$$

returning Equations (A41) and (A42).

For a Lagrangian density of the form  $\mathcal{L} = \mathcal{L}(X, \phi)$ , the equation of motion (A25) of the SF becomes

$$\left( \frac{\partial \mathcal{L}}{\partial X} + 2X \frac{\partial^2 \mathcal{L}}{\partial X^2} \right) \ddot{\phi} + 3H \frac{\partial \mathcal{L}}{\partial X} \dot{\phi} + 2X \frac{\partial^2 \mathcal{L}}{\partial X \partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \quad (\text{A97})$$

This equation is equivalent to the energy conservation Equation (A108). Indeed, taking the time derivative of  $\epsilon$  from Equation (A42) and substituting the result into Equation (A108) we get

$$\left( \frac{\partial \mathcal{L}}{\partial X} + 2X \frac{\partial^2 \mathcal{L}}{\partial X^2} \right) \dot{X} + \left( 2X \frac{\partial^2 \mathcal{L}}{\partial X \partial \phi} - \frac{\partial \mathcal{L}}{\partial \phi} \right) \dot{\phi} + 6HX \frac{\partial \mathcal{L}}{\partial X} = 0. \quad (\text{A98})$$

Recalling Equation (A95), we obtain Equation (A97). We can check that this equation returns Equations (A116) and (A140) for a canonical and a tachyonic SF respectively.

For a Lagrangian density of the form  $\mathcal{L} = V(\phi)F(X)$ , Equation (A97) reduces to

$$(F_X + 2XF_{XX})\ddot{\phi} + 3HF_X\dot{\phi} + (2XF_X - F)\frac{V'}{V} = 0. \quad (\text{A99})$$

In the particular case  $\mathcal{L} = V(\phi)X$ , corresponding to a stiff equation of state [see the comment after Equation (A41)], we get

$$\ddot{\phi} + 3H\dot{\phi} + X\frac{V'}{V} = 0. \quad (\text{A100})$$

For a Lagrangian density of the form  $\mathcal{L} = F(X) - V(\phi)$ , Equation (A97) reduces to

$$(F_X + 2XF_{XX})\ddot{\phi} + 3HF_X\dot{\phi} + V'(\phi) = 0. \quad (\text{A101})$$

For a pure k-essence Lagrangian  $\mathcal{L} = \mathcal{L}(X)$ , Equation (A97) reduces to

$$(\mathcal{L}' + 2X\mathcal{L}'')\ddot{\phi} + 3H\mathcal{L}'\dot{\phi} = 0. \quad (\text{A102})$$

We also have [see Equation (A98)]

$$(\mathcal{L}' + 2X\mathcal{L}'')\frac{dX}{da} + \frac{6}{a}X\mathcal{L}' = 0. \quad (\text{A103})$$

This equation integrates to give

$$\sqrt{X}\mathcal{L}'(X) = \frac{k}{a^3}. \quad (\text{A104})$$

Using Equation (A35), we see that this equation is equivalent to the conservation of the rest-mass:  $\rho_m \propto a^{-3}$ . Equation (A104) was first obtained by Chimento [185] and Scherrer [186] but they did not realize the relation with the rest-mass density. Our approach provides therefore a physical interpretation of their result.

### Appendix C. Equation of State of Type I

In this Appendix, we consider a barotropic fluid described by an equation of state of type I where the pressure  $P = P(\epsilon)$  is specified as a function of the energy density. We show that, in a cosmological context, it is possible to associate to this fluid a real SF with a potential  $V(\varphi)$  which is fully determined by the equation of state. As an illustration, we determine the real SF potential associated with a polytropic equation of state of type I.

#### Appendix C.1. Friedmann Equations

If we consider an expanding homogeneous background and adopt the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, the Einstein field equations reduce to the Friedmann equations

$$H^2 = \frac{8\pi G}{3c^2}\epsilon, \quad (\text{A105})$$

$$2\dot{H} + 3H^2 = -\frac{8\pi G}{c^2}P, \quad (\text{A106})$$

where  $H = \dot{a}/a$  is the Hubble parameter and  $a(t)$  is the scale factor. To obtain Equation (A105), we have assumed that the universe is flat ( $k = 0$ ) in agreement with the inflation paradigm [5] and the observations of the cosmic microwave background (CMB) [1,2]. On the other hand, we have set the cosmological constant to zero ( $\Lambda = 0$ ) since dark energy can be taken into account in the equation of state  $P(\epsilon)$  or in the SF potential  $V(\varphi)$  (quintessence). Equation (A106) can also be written as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(3P + \epsilon), \quad (\text{A107})$$

showing that the expansion of the universe is decelerating when  $P > -\epsilon/3$  and accelerating when  $P < -\epsilon/3$ .

Using Equations (A105) and (A106), we obtain the energy conservation equation

$$\frac{d\epsilon}{dt} + 3H(\epsilon + P) = 0. \quad (\text{A108})$$

This equation can be directly deduced from the conservation of the energy-momentum tensor  $D_\mu T^{\mu\nu} = 0$  which results from the Bianchi identities. The energy density decreases with the scale factor when  $P > -\epsilon$  and increases with the scale factor when  $P < -\epsilon$ . The latter case corresponds to a phantom behavior.

For a given equation of state  $P(\epsilon)$  we can solve Equation (A108) to get

$$\ln a = -\frac{1}{3} \int \frac{d\epsilon}{\epsilon + P(\epsilon)}. \quad (\text{A109})$$

This equation determines  $\epsilon(a)$ . We can then solve the Friedmann Equation (A105) to obtain the temporal evolution of the scale factor  $a(t)$ .

A polytropic equation of state of type I is defined by

$$P = K \left( \frac{\epsilon}{c^2} \right)^\gamma \quad \text{with} \quad \gamma = 1 + 1/n. \quad (\text{A110})$$

Assuming  $1 + (K/c^2)(\epsilon/c^2)^{1/n} \geq 0$ , i.e.,  $P \geq -\epsilon$  corresponding to a nonphantom universe<sup>36</sup>, the energy conservation Equation (A108) can be integrated into [97,98]

$$\epsilon = \frac{\rho_* c^2}{[(a/a_*)^{3/n} \mp 1]^n}, \quad (\text{A111})$$

where  $\rho_* = (c^2/|K|)^n$  and  $a_*$  is a constant of integration. The upper sign corresponds to  $K > 0$  and the lower sign to  $K < 0$ .

The case  $\gamma = 1$ , corresponding to a linear equation of state

$$P = \alpha \epsilon \quad (\text{A112})$$

with  $\alpha = K/c^2$ , must be treated specifically. In that case, the solution of Equation (A108) can be written as

$$\epsilon = \frac{\rho_* c^2}{(a/a_*)^{3(1+\alpha)}}, \quad (\text{A113})$$

where  $\rho_* a_*^{3(1+\alpha)}$  is a constant of integration.

*Remark:* Unfortunately, for the logotropic equation of state of type I [see Equation (379)], the energy conservation equation

$$\ln a = -\frac{1}{3} \int \frac{d\epsilon}{\epsilon + A \ln(\epsilon/\epsilon_*)} \quad (\text{A114})$$

cannot be integrated explicitly.

### Appendix C.2. Canonical SF

We consider a spatially homogeneous real canonical SF in an expanding universe with a Lagrangian

$$L = \frac{1}{2} \dot{\phi}^2 - V(\phi). \quad (\text{A115})$$

It evolves according to the KG equation [see Equation (A97)]

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0 \quad (\text{A116})$$

coupled to the Friedmann Equation (A105). The SF tends to run down the potential towards lower energies while experiencing a Hubble friction. The energy-momentum tensor is diagonal  $T_\mu^\nu = \text{diag}(\epsilon, -P, -P, -P)$ . The energy density and the pressure of the SF are given by [see Equation (A96)]

$$\epsilon = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (\text{A117})$$

$$P = \frac{1}{2} \dot{\phi}^2 - V(\phi). \quad (\text{A118})$$

We note that, here,  $V$  represents the total SF potential including the rest-mass term. When the kinetic term dominates we obtain the stiff equation of state  $P = \epsilon$ . When the potential term dominates, we obtain the equation of state  $P = -\epsilon$  corresponding to the vacuum energy. We can easily check that the KG Equation (A116) with Equations (A117) and (A118) implies the energy conservation Equation (A108) (see Appendix G). Inversely, the energy conservation Equation (A108) with Equations (A117) and (A118) implies the KG Equation (A116). The equation of state parameter  $w = P/\epsilon$  is given by

$$w = \frac{\frac{1}{2} \dot{\phi}^2 - V(\phi)}{\frac{1}{2} \dot{\phi}^2 + V(\phi)}. \quad (\text{A119})$$

It satisfies  $-1 \leq w \leq 1$ . The speed of sound is equal to the speed of light ( $c_s = c$ ) [see Equation (A44)].

Using standard techniques [83,187–189], we can obtain the canonical SF potential associated with a given equation of state of type I as follows [98]. From Equations (A117) and (A118), we get

$$\dot{\varphi}^2 = (w + 1)\epsilon. \quad (\text{A120})$$

Then, using  $\dot{\varphi} = (d\varphi/da)Ha$  and the Friedmann Equation (A105), we find that the relation between the SF and the scale factor is given by

$$\frac{d\varphi}{da} = \left( \frac{3c^4}{8\pi G} \right)^{1/2} \frac{\sqrt{1+w}}{a}. \quad (\text{A121})$$

We note that  $\varphi$  is a monotonic function of  $a$ . We have selected the solution + for which  $\varphi$  increases with  $a$ . On the other hand, according to Equations (A117) and (A118), we have

$$V = \frac{1}{2}(1 - w)\epsilon. \quad (\text{A122})$$

Therefore, the potential  $V(\varphi)$  of the canonical SF is determined in parametric form by the equations

$$\varphi(a) = \left( \frac{3c^4}{8\pi G} \right)^{1/2} \int \sqrt{1+w(a)} \frac{da}{a}, \quad (\text{A123})$$

$$V(a) = \frac{1}{2}[1 - w(a)]\epsilon(a). \quad (\text{A124})$$

We note that  $\varphi$  is defined up to an additive constant.

The canonical SF potential corresponding to a polytropic equation of state of type I [see Equation (A105)] has been determined in Section 8.1. of [98]. It is given by

$$V = \frac{1}{2}\rho_* c^2 \frac{\cosh^2 \psi + 1}{\cosh^{\frac{2\gamma}{\gamma-1}} \psi} \quad (K < 0), \quad (\text{A125})$$

$$V = \frac{1}{2}\rho_* c^2 \frac{\sinh^2 \psi - 1}{\sinh^{\frac{2\gamma}{\gamma-1}} \psi} \quad (K > 0), \quad (\text{A126})$$

where

$$\psi = \left( \frac{8\pi G}{3c^4} \right)^{1/2} \frac{3}{2}(\gamma - 1)\varphi. \quad (\text{A127})$$

The relation between the scale factor and the SF is

$$\frac{a}{a_*} = \sinh^{\frac{2}{3(\gamma-1)}} \psi \quad (K < 0), \quad (\text{A128})$$

$$\frac{a}{a_*} = \cosh^{\frac{2}{3(\gamma-1)}} \psi \quad (K > 0). \quad (\text{A129})$$

These expressions are valid for  $\psi \geq 0$ .

(i) For  $\gamma = -1$  (Chaplygin gas), we get

$$V = \frac{1}{2}\rho_* c^2 \left( \cosh \psi + \frac{1}{\cosh \psi} \right) \quad (K < 0), \quad (\text{A130})$$

$$V = \frac{1}{2}\rho_*c^2\left(\sinh\psi - \frac{1}{\sinh\psi}\right) \quad (K > 0), \quad (\text{A131})$$

with  $\rho_* = \sqrt{|K|/c^2}$ . This SF potential was first obtained in [50].

(ii) For  $\gamma = 2$  (BEC), we get

$$V = \frac{1}{2}\rho_*c^2\frac{\cosh^2\psi + 1}{\cosh^4\psi} \quad (K < 0), \quad (\text{A132})$$

$$V = \frac{1}{2}\rho_*c^2\frac{\sinh^2\psi - 1}{\sinh^4\psi} \quad (K > 0), \quad (\text{A133})$$

with  $\rho_* = c^2/|K|$ . This SF potential was first obtained in [98].

(iii) For  $\gamma = 0$  ( $\Lambda$ CDM model), we get

$$V = \frac{1}{2}\rho_*c^2(\cosh^2\psi + 1) \quad (K < 0), \quad (\text{A134})$$

$$V = \frac{1}{2}\rho_*c^2(\sinh^2\psi - 1) \quad (K > 0), \quad (\text{A135})$$

with  $\rho_* = |K|/c^2$ . This SF potential was first obtained in [86] and rediscovered independently in [98].

(iv) For  $\gamma = 3$  (superfluid), we get

$$V = \frac{1}{2}\rho_*c^2\frac{\cosh^2\psi + 1}{\cosh^3\psi} \quad (K < 0), \quad (\text{A136})$$

$$V = \frac{1}{2}\rho_*c^2\frac{\sinh^2\psi - 1}{\sinh^3\psi} \quad (K > 0), \quad (\text{A137})$$

with  $\rho_* = \sqrt{c^2/|K|}$ . To our knowledge, this SF potential is new.

(v) For  $\gamma = 1$ , we get

$$V(\varphi) = \frac{1}{2}\rho_*c^2(1 - \alpha)e^{-3\sqrt{\alpha+1}\left(\frac{8\pi G}{3c^4}\right)^{1/2}\varphi}. \quad (\text{A138})$$

This exponential potential was obtained in [83] but it appeared in earlier works on inflation and quintessence [26,27,33,190,191]. In that case, the relation between the scale factor and the SF is

$$\varphi(a) = \left(\frac{3c^4}{8\pi G}\right)^{1/2} \sqrt{1 + \alpha} \ln\left(\frac{a}{a_*}\right). \quad (\text{A139})$$

For  $\alpha = 1$  (stiff matter), we find that  $V(\varphi) = 0$ . On the other hand, for  $\alpha = -1$  (vacuum energy), we find that  $\dot{\varphi} = 0$  so that  $\varphi = \varphi_0$  is constant. This is consistent with the equation of motion (A116) provided that  $V'(\varphi_0) = 0$ . Therefore,  $\varphi_0$  must be at an extremum of the potential  $V(\varphi)$ . In that case,  $\epsilon = V(\varphi_0) = V_0$  and  $P = -V(\varphi_0) = -V_0$ , yielding  $P = -\epsilon$ . Note that the SF potential  $V(\varphi)$  is not necessarily constant but it must have an extremum at  $\varphi_0$  such that  $V_0 > 0$ .

### Appendix C.3. Tachyonic SF

We consider a spatially homogeneous real tachyonic SF [80–87] in an expanding universe with a Lagrangian

$$L = -V(\varphi)\sqrt{1 - \dot{\varphi}^2}. \quad (\text{A140})$$

It evolves according to the equation [see Equation (A97)]

$$\frac{\ddot{\varphi}}{1 - \dot{\varphi}^2} + 3H\dot{\varphi} + \frac{1}{V} \frac{dV}{d\varphi} = 0 \quad (\text{A141})$$

coupled to the Friedmann Equation (A105). The SF tends to run down the potential towards lower energies while experiencing a Hubble friction. The energy-momentum tensor is diagonal  $T_\mu^\nu = \text{diag}(\epsilon, -P, -P, -P)$ . The density and the pressure of the SF are given by [see Equation (A96)]

$$\epsilon = \frac{V(\varphi)}{\sqrt{1 - \dot{\varphi}^2}}, \quad (\text{A142})$$

$$P = -V(\varphi)\sqrt{1 - \dot{\varphi}^2}. \quad (\text{A143})$$

We can easily check that the equation of motion (A141) with Equations (A142) and (A143) implies the energy conservation Equation (A108) (see Appendix G). Inversely, the energy conservation Equation (A108) with Equations (A142) and (A143) implies the equation of motion (A141). The equation of state parameter  $w = P/\epsilon$  is given by

$$w = \dot{\varphi}^2 - 1. \quad (\text{A144})$$

It satisfies  $-1 \leq w \leq 0$ . The squared speed of sound is given by  $c_s^2/c^2 = 1 - \dot{\varphi}^2 = -w$  [see Equation (A44)]. It satisfies  $0 \leq c_s^2/c^2 \leq 1$ .

Using standard techniques [83,187–189], we can obtain the tachyonic SF potential associated with a given equation of state of type I as follows [98]. From Equations (A142) and (A143), we obtain

$$\dot{\varphi}^2 = 1 + w. \quad (\text{A145})$$

Using  $\dot{\varphi} = (d\varphi/da)Ha$ , and the Friedmann Equation (A105), we get

$$\frac{d\varphi}{da} = \left( \frac{3c^4}{8\pi G} \right)^{1/2} \frac{\sqrt{1+w}}{\sqrt{\epsilon a}}. \quad (\text{A146})$$

We note that  $\varphi$  is a monotonic function of  $a$ . We have selected the solution + for which  $\varphi$  increases with  $a$ . On the other hand, from Equations (A142) and (A143), we have

$$V^2 = -w\epsilon^2. \quad (\text{A147})$$

Therefore, the potential  $V(\varphi)$  of the tachyonic SF is determined in parametric form by the equations

$$\varphi(a) = \left( \frac{3c^4}{8\pi G} \right)^{1/2} \int \frac{\sqrt{1+w(a)}}{\sqrt{\epsilon(a)}} \frac{da}{a}, \quad (\text{A148})$$

$$V(a) = \sqrt{-w(a)\epsilon(a)}. \quad (\text{A149})$$

The tachyonic SF potential corresponding to a polytropic equation of state of type I [see Equation (A105)] has been determined in Section 8.2. of [98]. It is defined only for  $K < 0$ . It is given in parametric form by

$$V = \frac{\rho_* c^2}{(x^2 + 1)^{\frac{1+\gamma}{2(\gamma-1)}}}, \quad (\text{A150})$$

$$\psi = \int (x^2 + 1)^{\frac{2-\gamma}{2(\gamma-1)}} dx, \quad (\text{A151})$$

where we have introduced the variable

$$\psi = \sqrt{\rho_* c^2} \left( \frac{8\pi G}{3c^4} \right)^{1/2} \frac{3}{2} (\gamma - 1) \varphi. \quad (\text{A152})$$

The relation between the scale factor and the SF is given by Equation (A151) with

$$x = \left( \frac{a}{a_*} \right)^{\frac{3}{2}(\gamma-1)}. \quad (\text{A153})$$

The integral in Equation (A151) can be expressed in terms of hypergeometric functions. Simple analytical expressions can be obtained in special cases.

(i) For  $\gamma = -1$  and  $K < 0$  (Chaplygin gas), we find that  $V(\varphi) = \rho_* c^2$  with  $\rho_* = \sqrt{|K|/c^2}$ . In that case, the potential is constant [86]. This leads to the Born-Infeld Lagrangian (see Appendix B.3).

(ii) For  $\gamma = 2$  and  $K < 0$  (BEC), we get

$$V(\psi) = \frac{\rho_* c^2}{(\psi^2 + 1)^{3/2}} \quad (\text{A154})$$

with  $\rho_* = c^2/|K|$ . We have  $a/a_* = \psi^{2/3}$  with  $\psi \geq 0$ . This potential was first obtained in [98].

(iii) For  $\gamma = 0$  and  $K < 0$  ( $\Lambda$ CDM model), we get

$$V(\psi) = \frac{\rho_* c^2}{\cos \psi} \quad (\text{A155})$$

with  $\rho_* = |K|/c^2$ . We have  $a/a_* = 1/\tan(\psi)^{2/3}$  with  $0 \leq \psi \leq \pi/2$ . This potential was first obtained in [86] and rediscovered independently in [98].

(iv) For  $\gamma = 3$  and  $K < 0$  (superfluid), it is not possible to obtain explicit expressions.

(v) For  $\gamma = 1$  and  $-1 < \alpha < 0$ , we get

$$V(\varphi) = \frac{\sqrt{-\alpha}}{1+\alpha} \frac{c^4}{6\pi G} \frac{1}{\varphi^2}. \quad (\text{A156})$$

This inverse square law potential was first obtained in [83,87]. In that case, the relation between the scale factor and the SF is

$$\varphi = \frac{2}{3} \frac{1}{\sqrt{\rho_* c^2}} \left( \frac{3c^4}{8\pi G} \right)^{1/2} \frac{1}{\sqrt{1+\alpha}} \left( \frac{a}{a_*} \right)^{3(1+\alpha)/2}. \quad (\text{A157})$$

For  $\alpha = -1$  (vacuum energy), we find that  $\dot{\varphi} = 0$  so that  $\varphi = \varphi_0$  is constant. This is consistent with the equation of motion (A141) provided that  $V'(\varphi_0) = 0$ . Therefore,  $\varphi_0$  must be at an extremum of the potential  $V(\varphi)$ . In that case,  $\epsilon = V(\varphi_0) = V_0$  and



$P = -V(\varphi_0) = -V_0$ , yielding  $P = -\epsilon$ . Note that the SF potential  $V(\varphi)$  is not necessarily constant but it must have an extremum at  $\varphi_0$  such that  $V_0 > 0$ .

*Remark:* For  $\gamma = 1/2$  and  $K < 0$ , we get

$$V(\psi) = \frac{\rho_* c^2}{(1 - \psi^2)^{3/2}} \quad (\text{A158})$$

with  $\rho_* = (|K|/c^2)^2$ . The relation between the scale factor and the SF is  $(a/a_*)^{3/4} = \sqrt{1 - \psi^2}/\psi$  with  $0 \leq \psi \leq 1$ . This potential was first obtained in [98].

#### Appendix D. Equation of State of Type II

In this Appendix, we consider a barotropic fluid described by an equation of state of type II where the pressure  $P = P(\rho_m)$  is specified as a function of the rest-mass density. After recalling general results, we apply this equation of state to a cosmological context.

##### Appendix D.1. General Results

The first principle of thermodynamics for a relativistic gas can be written as

$$d\left(\frac{\epsilon}{\rho_m}\right) = -Pd\left(\frac{1}{\rho_m}\right) + Td\left(\frac{s}{\rho_m}\right), \quad (\text{A159})$$

where

$$\epsilon = \rho_m c^2 + u(\rho_m) \quad (\text{A160})$$

is the energy density including the rest-mass energy density  $\rho_m c^2$  (where  $\rho_m = nm$  is the rest-mass density) and the internal energy density  $u(\rho_m)$ ,  $s$  is the density of entropy,  $P$  is the pressure, and  $T$  is the temperature. We assume that  $Td(s/\rho_m) = 0$ . This corresponds to cold ( $T = 0$ ) or isentropic ( $s/\rho_m = \text{cst}$ ) gases. In that case, Equation (A159) reduces to

$$d\left(\frac{\epsilon}{\rho_m}\right) = -Pd\left(\frac{1}{\rho_m}\right) = \frac{P}{\rho_m^2} d\rho_m. \quad (\text{A161})$$

This equation can be rewritten as

$$\frac{d\epsilon}{d\rho_m} = \frac{P + \epsilon}{\rho_m}, \quad (\text{A162})$$

where the term in the right hand side is the enthalpy  $h$ . We have

$$h = \frac{P + \epsilon}{\rho_m}, \quad h = \frac{d\epsilon}{d\rho_m}, \quad dh = \frac{dP}{\rho_m}. \quad (\text{A163})$$

Equation (A161) can be integrated into

$$\epsilon = \rho_m c^2 + \rho_m \int \frac{P(\rho_m)}{\rho_m^2} d\rho_m \quad (\text{A164})$$

establishing that

$$u(\rho_m) = \rho_m \int \frac{P(\rho_m)}{\rho_m^2} d\rho_m. \quad (\text{A165})$$

This equation determines the internal energy density as a function of the equation of state  $P(\rho_m)$ . Inversely, the equation of state is determined by the internal energy density  $u(\rho_m)$  from the relation

$$P(\rho_m) = -\frac{d(u/\rho_m)}{d(1/\rho_m)} = \rho_m^2 \left[ \frac{u(\rho_m)}{\rho_m} \right]' = \rho_m u'(\rho_m) - u(\rho_m). \quad (\text{A166})$$

We note that

$$P'(\rho_m) = \rho_m u''(\rho_m). \quad (\text{A167})$$

The squared speed of sound is

$$c_s^2 = P'(\epsilon) c^2 = \frac{\rho_m \epsilon''(\rho_m)}{\epsilon'(\rho_m)} c^2 = \frac{\rho_m u''(\rho_m) c^2}{c^2 + u'(\rho_m)}. \quad (\text{A168})$$

*Remark:* The first principle of thermodynamics can be written as

$$d\epsilon = Tds + \mu dn, \quad (\text{A169})$$

where  $\mu$  is the local chemical potential. This can be viewed as the variational principle ( $\delta s/k_B - \beta \delta \epsilon + \alpha \delta n = 0$  with  $\beta = 1/k_B T$  and  $\alpha = \mu/k_B T$ ) associated with the maximization of the entropy density  $s$  at fixed energy density  $\epsilon$  and particle density  $n$  [178]. Combined with the Gibbs-Duhem relation [178]

$$s = \frac{\epsilon + P - \mu n}{T}, \quad (\text{A170})$$

we obtain Equation (A159) and

$$sdT - dP + nd\mu = 0. \quad (\text{A171})$$

If  $T = \text{cst}$ , then  $dP = nd\mu$ . For  $T = 0$ , the foregoing equations reduce to

$$d\epsilon = \mu dn, \quad \mu = \frac{\epsilon + P}{n}, \quad dP = nd\mu, \quad (\text{A172})$$

which are equivalent to Equation (A163) with  $\mu = mh$ . Therefore, the enthalpy  $h(\mathbf{r})$  is equal to the local chemical potential  $\mu(\mathbf{r})$  by unit of mass:  $h(\mathbf{r}) = \mu(\mathbf{r})/m$ .

#### Appendix D.2. Cosmology

Let us apply these equations in a cosmological context, namely for a spatially homogeneous fluid in an expanding background. Combining the energy conservation Equation (A108) with Equation (A162), we obtain

$$\frac{d\rho_m}{dt} + 3H\rho_m = 0. \quad (\text{A173})$$

This equation expresses the conservation of the particle number (or rest-mass). It can be integrated into  $\rho_m \propto a^{-3}$ . Inserting this relation into Equation (A160), we see that  $\rho_m$  represents DM while  $u$  represents DE<sup>37</sup>. This decomposition provides therefore a simple (and nice) interpretation of DM and DE in terms of a single dark fluid [118,119]. DM corresponds to the rest-mass energy density of the dark fluid and DE corresponds to its internal energy density. Owing to this interpretation, we can write

$$\rho_m c^2 = \frac{\Omega_{m,0} \epsilon_0}{a^3} \quad (\text{A174})$$

and

$$\epsilon = \frac{\Omega_{m,0}\epsilon_0}{a^3} + u\left(\frac{\Omega_{m,0}\epsilon_0}{c^2 a^3}\right), \quad (\text{A175})$$

where  $\epsilon_0$  is the present energy density of the universe and  $\Omega_{m,0}$  is the present proportion of DM. For given  $P(\rho_m)$  or  $u(\rho_m)$  we can get  $\epsilon(a)$  from Equation (A175). We can then solve the Friedmann Equation (A105) to obtain the temporal evolution of the scale factor  $a(t)$ .

*Remark:* Equations (A160) and (A166) determine the equation of state  $P = P(\epsilon)$ . As a result, we can obtain Equation (A174) directly from Equations (A160), (A166) and the energy conservation Equation (A108). Indeed, combining these equations we obtain Equation (A173) which integrates to give Equation (A174).

### Appendix D.3. Two-Fluid Model

In the model of type II, we have a single dark fluid with an equation of state  $P = P(\rho_m)$ . Still, the energy density (A160) is the sum of two terms, a rest-mass density term  $\rho_m$  which mimics DM and an internal energy term  $u(\rho_m)$  which mimics DE. It is interesting to consider a two-fluid model which leads to the same results as the single dark fluid model, at least for what concerns the evolution of the homogeneous background. In this two-fluid model, one fluid corresponds to pressureless DM with an equation of state  $P_m = 0$  and a density  $\rho_m c^2 = \Omega_{m,0}\epsilon_0/a^3$  determined by the energy conservation equation for DM, and the other fluid corresponds to DE with an equation of state  $P_{de}(\epsilon_{de})$  and an energy density  $\epsilon_{de}(a)$  determined by the energy conservation equation for DE. We can obtain the equation of state of DE yielding the same results as the one-fluid model by taking

$$P_{de} = P(\rho_m), \quad \epsilon_{de} = u(\rho_m), \quad (\text{A176})$$

and eliminating  $\rho_m$  from these two relations. In other words, the equation of state  $P_{de}(\epsilon_{de})$  of DE in the two-fluid model corresponds to the relation  $P(u)$  in the single fluid model. Explicit examples of the correspondance between the one and two-fluid models are given below. Although the one and two-fluid models are equivalent for the evolution of the homogeneous background, they may differ for what concerns the formation of the large-scale structures of the Universe and for inhomogeneous systems in general.

In the two-fluid model associated with the Chaplygin gas of type I (or III), the DE has an equation of state

$$P_{de} = \frac{2Kc^2\epsilon_{de}}{\epsilon_{de}^2 - Kc^2}, \quad (\text{A177})$$

which is obtained by eliminating  $\rho_m$  between Equations (235) and (236), and by identifying  $P(u)$  with  $P_{de}(\epsilon_{de})$ .

In the two-fluid model associated with the BEC of type I, the DE has an equation of state

$$P_{de} = \frac{4K\epsilon_{de}^2}{\left[-\frac{K\epsilon_{de}}{c^2} \pm \sqrt{\left(\frac{K\epsilon_{de}}{c^2}\right)^2 + 4K\epsilon_{de}}\right]^2}, \quad (\text{A178})$$

which is obtained by eliminating  $\rho_m$  between Equations (238) and (239), and by identifying  $P(u)$  with  $P_{de}(\epsilon_{de})$ .

In the two-fluid model associated with the  $\Lambda$ CDM model, the DE has an equation of state

$$P_{de} = -\epsilon_{de}. \quad (\text{A179})$$

In the two-fluid model associated with a polytrope of type II, the DE has an equation of state

$$P_{\text{de}} = (\gamma - 1)\epsilon_{\text{de}}, \quad (\text{A180})$$

which is obtained by eliminating  $\rho_m$  between Equations (277) and (278), and by identifying  $P(u)$  with  $P_{\text{de}}(\epsilon_{\text{de}})$ .

In the two-fluid model associated with a logotrope of type II, the DE has an equation of state [120]

$$P_{\text{de}} = -\epsilon_{\text{de}} - A, \quad (\text{A181})$$

which is obtained by eliminating  $\rho_m$  between Equations (388) and (389), and by identifying  $P(u)$  with  $P_{\text{de}}(\epsilon_{\text{de}})$ .

By integrating the energy conservation equation for DE with these equations of state  $P_{\text{de}}(\epsilon_{\text{de}})$  we recover the DE density  $\epsilon_{\text{de}}(a)$  given by  $u(\Omega_{m,0}\epsilon_0/c^2 a^3)$  [see Equation (A175)].

*Remark:* Using a sort of inverse approach, we have shown that UDM models like the polytropic and logotropic models are equivalent, for what concerns the evolution of the cosmological background, to a two-fluid model made of pressureless DM and DE. In certain cases, we have determined the corresponding equation of state of DE analytically. As far as we know, this inverse approach has not been developed before. It is of interest because, in the two-fluid model, there is no problem with the formation of structures since DM is pressureless. This approach therefore solves the problem discussed at the end of the introduction. It is a bit unconventional to pass from a one fluid model to a two-fluid model since, originally, the one fluid model aimed at a unification of DM and DE but this procedure is well defined and yields new types of equations of state for DE that may be of interest.

## Appendix E. Equation of State of Type III

In this Appendix, we consider a barotropic fluid described by an equation of state of type III where the pressure  $P = P(\rho)$  is specified as a function of the pseudo rest-mass density. As explained in Section 3.3 this hydrodynamic description arises naturally when considering a complex SF with a potential  $V(|\phi|^2)$  in the TF approximation. Here, we consider the case of a spatially homogeneous complex SF in an expanding background.

### Appendix E.1. General Results

Let us first establish general results that are valid beyond the TF approximation.

We consider a spatially homogeneous complex SF in an expanding universe with a Lagrangian

$$L = \frac{1}{2c^2}|\dot{\phi}|^2 - V_{\text{tot}}(\phi). \quad (\text{A182})$$

Its cosmological evolution obtained from the least action principle ( $\delta S = 0$ ) is governed by the KGF equations

$$\frac{1}{c^2} \frac{d^2 \phi}{dt^2} + \frac{3H}{c^2} \frac{d\phi}{dt} + 2 \frac{dV_{\text{tot}}}{d|\phi|^2} \phi = 0, \quad (\text{A183})$$

$$H^2 = \frac{8\pi G}{3c^2} \epsilon. \quad (\text{A184})$$

The energy density  $\epsilon(t)$  and the pressure  $P(t)$  of the SF are given by

$$\epsilon \equiv T_0^0 = \frac{\partial L}{\partial \dot{\phi}} \dot{\phi} + \frac{\partial L}{\partial \phi^*} \phi^* - L, \quad P \equiv -T_i^i = L, \quad (\text{A185})$$

yielding

$$\epsilon = \frac{1}{2c^2} \left| \frac{d\varphi}{dt} \right|^2 + V_{\text{tot}}(|\varphi|^2), \quad (\text{A186})$$

$$P = \frac{1}{2c^2} \left| \frac{d\varphi}{dt} \right|^2 - V_{\text{tot}}(|\varphi|^2). \quad (\text{A187})$$

When the kinetic term dominates we obtain the stiff equation of state  $P = \epsilon$ . When the potential term dominates, we obtain the equation of state  $P = -\epsilon$  corresponding to the vacuum energy.

In the following, we use the hydrodynamic representation of the SF (see Section 3.3 and [149]). The Lagrangian is

$$L = \frac{1}{2c^2} \rho \dot{\theta}^2 + \frac{\hbar^2}{8m^2 \rho c^2} \dot{\rho}^2 - V_{\text{tot}}(\rho). \quad (\text{A188})$$

The energy density  $\epsilon(t)$  and the pressure  $P(t)$  of the SF are given by

$$\epsilon \equiv T_0^0 = \frac{\partial L}{\partial \dot{\theta}} \dot{\theta} + \frac{\partial L}{\partial \dot{\rho}} \dot{\rho} - L, \quad P \equiv -T_i^i = L, \quad (\text{A189})$$

yielding

$$\epsilon = \frac{1}{2c^2} \rho \dot{\theta}^2 + \frac{\hbar^2}{8m^2 \rho c^2} \dot{\rho}^2 + V_{\text{tot}}(\rho), \quad (\text{A190})$$

$$P = \frac{1}{2c^2} \rho \dot{\theta}^2 + \frac{\hbar^2}{8m^2 \rho c^2} \dot{\rho}^2 - V_{\text{tot}}(\rho). \quad (\text{A191})$$

The equation  $D_\nu T^{\mu\nu} = 0$  leads to the energy conservation equation

$$\frac{d\epsilon}{dt} + 3H(\epsilon + P) = 0. \quad (\text{A192})$$

This equation can also be obtained from the KG Equation (A183) with Equations (A186) and (A187) (see Appendix G). Inversely, the energy conservation Equation (A192) with Equations (A186) and (A187) implies the KG Equation (A183).

The equation  $D_\mu J^\mu = 0$ , which is equivalent to the continuity Equation (133), can be written as

$$\frac{d}{dt} (E_{\text{tot}} \rho a^3) = 0, \quad (\text{A193})$$

where

$$E_{\text{tot}} = \hbar \omega = -m\dot{\theta} = -\dot{S}_{\text{tot}} \quad (\text{A194})$$

is the energy of the SF ( $\omega = -\dot{\Theta}$  with  $\Theta = m\theta/\hbar$  is its pulsation). Equation (A193) expresses the conservation of the charge of the complex SF (or equivalently the conservation of the boson number). It can be written as

$$\rho E_{\text{tot}} = \frac{Q m^2 c^2}{a^3}, \quad (\text{A195})$$

where  $Q = Ne$  is a constant of integration representing the charge of the SF which is proportional to the boson number  $N$  [149,150,153,192–195]. Indeed, according to Equation (123), the charge of the SF is defined by<sup>38</sup>

$$Q = \frac{1}{mc} \int J^0 \sqrt{-g} d^3x, \quad (\text{A196})$$

where  $J^0$  is the time component of the quadricurrent  $J^\mu = -\rho\partial^\mu\theta$  (see Section 3.2). For a spatially homogeneous SF in an expanding background, we have

$$J^0 = -\frac{1}{c}\rho\dot{\theta} = \frac{\hbar}{mc}\rho\omega = \frac{1}{mc}\rho E_{\text{tot}} \quad (\text{A197})$$

and  $Q = J^0 a^3 / mc = \rho E_{\text{tot}} a^3 / m^2 c^2$  yielding Equation (A195).

The quantum Hamilton-Jacobi (or Bernoulli) Equation (134) takes the form

$$E_{\text{tot}}^2 = \hbar^2 \frac{1}{\sqrt{\rho}} \frac{d^2 \sqrt{\rho}}{dt^2} + 3H\hbar^2 \frac{1}{\sqrt{\rho}} \frac{d\sqrt{\rho}}{dt} + 2m^2 c^2 V'_{\text{tot}}(\rho). \quad (\text{A198})$$

Finally, we have established in the general case (see Section 3.2) that the rest-mass density is given by

$$\rho_m = \frac{\rho}{c} \sqrt{\partial_\mu \theta \partial^\mu \theta}. \quad (\text{A199})$$

For a spatially homogeneous SF in an expanding background, we get

$$\rho_m = -\frac{\rho}{c} \partial_0 \theta = -\frac{1}{c^2} \rho \dot{\theta} = \frac{\hbar}{mc^2} \rho \omega = \frac{1}{mc^2} \rho E_{\text{tot}}. \quad (\text{A200})$$

Using Equation (A200), Equations (A193) and (A195) can be rewritten as

$$\frac{d\rho_m}{dt} + 3H\rho_m = 0 \quad (\text{A201})$$

and

$$\rho_m = \frac{Qm}{a^3}, \quad (\text{A202})$$

respectively. Equations (A201) and (A202) can also be obtained from the first law of thermodynamics for a cold fluid ( $T = 0$ ) in a homogeneous background (see Appendix D). They express the conservation of the particle number. Inversely, Equation (A200) can be directly obtained from Equation (A195) using Equation (A202). Comparing Equations (A197) and (A200), we note that

$$\rho_m = \frac{J^0}{c}. \quad (\text{A203})$$

This relation is not generally valid (see Section 3.2). In the present case, it arises from the general identity  $J^\mu = \rho_m u^\mu$  and the fact that  $u^\mu = c\delta_0^\mu$  since the fluid (SF) is static in the expanding background.

#### Appendix E.2. TF Approximation

In the TF approximation ( $\hbar \rightarrow 0$ ), the Lagrangian (A188) reduces to

$$L = \frac{1}{2c^2} \rho \dot{\theta}^2 - V_{\text{tot}}(\rho), \quad (\text{A204})$$

the energy density  $\epsilon(t)$  and the pressure  $P(t)$  of the SF reduce to

$$\epsilon = \frac{1}{2c^2} \rho \dot{\theta}^2 + V_{\text{tot}}(\rho), \quad (\text{A205})$$

$$P = \frac{1}{2c^2} \rho \dot{\theta}^2 - V_{\text{tot}}(\rho), \quad (\text{A206})$$

and the quantum Hamilton-Jacobi (or Bernoulli) Equation (A198) reduces to<sup>39</sup>

$$E_{\text{tot}}^2 = 2m^2c^2V'_{\text{tot}}(\rho). \quad (\text{A207})$$

Combining Equations (A195) and (A207), we obtain

$$\frac{Qmc}{a^3} = \rho \sqrt{2V'_{\text{tot}}(\rho)}. \quad (\text{A208})$$

This equation determines the relation between the pseudo rest-mass density  $\rho$  and the scale factor  $a$ . On the other hand, according to Equations (A200) and (A207), the rest-mass density is given by

$$\rho_m = \frac{\rho}{c} \sqrt{2V'_{\text{tot}}(\rho)}. \quad (\text{A209})$$

Finally, according to Equations (A194), (A205) and (A206) the energy density and the pressure of the SF in the TF approximation are given by

$$\epsilon = \rho V'_{\text{tot}}(\rho) + V_{\text{tot}}(\rho), \quad (\text{A210})$$

$$P = \rho V'_{\text{tot}}(\rho) - V_{\text{tot}}(\rho). \quad (\text{A211})$$

Equation (A211) determines the equation of state  $P(\rho)$  as a function of the SF potential  $V_{\text{tot}}(\rho)$ . Inversely, the SF potential is determined by the equation of state according to the relation

$$V_{\text{tot}}(\rho) = \rho \int \frac{P(\rho)}{\rho^2} d\rho. \quad (\text{A212})$$

Equations (A209)–(A212) are always valid in the TF approximation even for inhomogeneous systems (see Section 3.3). For given  $P(\rho)$  or  $V_{\text{tot}}(\rho)$ , we can obtain  $\rho(a)$  from Equation (A208) and  $\epsilon(a)$  from Equation (A210). We can then solve the Friedmann Equation (A184) with  $\epsilon(a)$  to obtain the temporal evolution of the scale factor  $a(t)$ . Actually, since it is not always possible to invert Equation (A208), it is better to proceed differently (see [149]). Taking the logarithmic derivative of Equation (A208), we get

$$\frac{\dot{a}}{a} = -\frac{1}{3} \frac{\dot{\rho}}{\rho} \left[ 1 + \frac{\rho V''_{\text{tot}}(\rho)}{2V'_{\text{tot}}(\rho)} \right]. \quad (\text{A213})$$

Then, using Equations (A184) and (A210), we obtain

$$\frac{c^2}{24\pi G} \left( \frac{\dot{\rho}}{\rho} \right)^2 = \frac{\rho V'_{\text{tot}}(\rho) + V_{\text{tot}}(\rho)}{\left[ 1 + \frac{\rho V''_{\text{tot}}(\rho)}{2V'_{\text{tot}}(\rho)} \right]^2}. \quad (\text{A214})$$

For given  $V_{\text{tot}}(\rho)$ , Equation (A214) is just a first order differential equation which can be solved by a simple integration.

*Remark:* Equations (A210) and (A211) determine the equation of state  $P = P(\epsilon)$ . As a result, we can obtain Equation (A208) directly from Equations (A210) and (A211) and the energy conservation Equation (A192). Indeed, combining these equations we obtain

$$[2V'_{\text{tot}}(\rho) + \rho V''_{\text{tot}}(\rho)] \frac{d\rho}{dt} = -6H\rho V'_{\text{tot}}(\rho). \quad (\text{A215})$$

leading to

$$\int \frac{2V'_{\text{tot}}(\rho) + \rho V''_{\text{tot}}(\rho)}{\rho V'_{\text{tot}}(\rho)} = -6 \ln a. \quad (\text{A216})$$

Equation (A216) integrates to give Equation (A208).

## Appendix F. Analogies and Differences between $u$ and $V$

For a relativistic fluid of type II, we have established the identities (see Appendix D)

$$\epsilon = \rho_m c^2 + u(\rho_m), \quad (\text{A217})$$

$$P = \rho_m u'(\rho_m) - u(\rho_m), \quad (\text{A218})$$

$$u(\rho_m) = \rho_m \int \frac{P(\rho_m)}{\rho_m^2} d\rho_m, \quad (\text{A219})$$

where  $\rho_m$  is the rest-mass density and  $u$  is the internal energy.

For a relativistic fluid of type III, we have established the identities (see Section 3.3)

$$\epsilon = \rho c^2 + \rho V'(\rho) + V(\rho), \quad (\text{A220})$$

$$P = \rho V'(\rho) - V(\rho), \quad (\text{A221})$$

$$V(\rho) = \rho \int \frac{P(\rho)}{\rho^2} d\rho, \quad (\text{A222})$$

where  $\rho$  is the pseudo rest-mass density and  $V$  is the potential of the complex SF.

We note that Equations (A221) and (A222) are identical to Equations (A218) and (A219) with  $\rho$  instead of  $\rho_m$  and  $V$  instead of  $u$ . In general, the variables  $\rho$  and  $V$  are different from the variables  $\rho_m$  and  $u$ . However, they coincide in the nonrelativistic limit. For a nonrelativistic complex SF (BEC), Equations (A217)–(A222) reduce to

$$\epsilon \sim \rho c^2, \quad (\text{A223})$$

$$P = \rho V'(\rho) - V(\rho), \quad (\text{A224})$$

$$V(\rho) = \rho \int \frac{P(\rho)}{\rho^2} d\rho, \quad (\text{A225})$$

where  $\rho = \rho_m$  is the mass density and  $V = u$  is the potential of the SF or the internal energy of the corresponding barotropic fluid (see Appendix A).

## Appendix G. Energy Conservation Equation for a SF

### Appendix G.1. Complex SF

We consider a spatially homogeneous complex SF in an expanding background (see Appendix E). Taking the time derivative of the energy density given by Equation (A186), we get

$$\frac{d\epsilon}{dt} = \frac{1}{2c^2} \frac{d^2\varphi}{dt^2} \frac{d\varphi^*}{dt} + V'_{\text{tot}}(|\varphi|^2) \frac{d\varphi}{dt} \varphi^* + \text{c.c.} \quad (\text{A226})$$



Using the KG Equation (A183), we obtain after simplification

$$\frac{d\epsilon}{dt} = -\frac{3H}{c^2} \left| \frac{d\varphi}{dt} \right|^2. \quad (\text{A227})$$

From Equations (A186) and (A187) we have

$$\epsilon + P = \frac{1}{c^2} \left| \frac{d\varphi}{dt} \right|^2. \quad (\text{A228})$$

Combining Equations (A227) and (A228), we obtain the energy conservation Equation (A192). Inversely, from Equations (A186), (A187) and (A192), we can directly derive the KG Equation (A183).

### Appendix G.2. Real Canonical SF

We consider a spatially homogeneous real canonical SF in an expanding background (see Appendix C.2). Taking the time derivative of the energy density given by Equation (A117), we get

$$\frac{d\epsilon}{dt} = \frac{d^2\varphi}{dt^2} \frac{d\varphi}{dt} + V'(\varphi) \frac{d\varphi}{dt}. \quad (\text{A229})$$

Using the KG Equation (A116), we obtain after simplification

$$\frac{d\epsilon}{dt} = -3H\dot{\varphi}^2. \quad (\text{A230})$$

From Equations (A117) and (A118) we have

$$\epsilon + P = \dot{\varphi}^2. \quad (\text{A231})$$

Combining Equations (A230) and (A231), we obtain the energy conservation Equation (A108). Inversely, from Equations (A108), (A117) and (A118) we can directly derive the KG Equation (A116).

### Appendix G.3. Real Tachyonic SF

We consider a spatially homogeneous real tachyonic SF in an expanding background (see Appendix C.3). Taking the time derivative of the energy density given by Equation (A142), we get

$$\frac{d\epsilon}{dt} = \frac{V'(\varphi)}{\sqrt{1-\dot{\varphi}^2}} \dot{\varphi} + \frac{V(\varphi)}{(1-\dot{\varphi}^2)^{3/2}} \dot{\varphi} \ddot{\varphi}. \quad (\text{A232})$$

Using the field Equation (A141), we obtain after simplification

$$\frac{d\epsilon}{dt} = -3H \frac{V(\varphi)}{\sqrt{1-\dot{\varphi}^2}} \dot{\varphi}^2. \quad (\text{A233})$$

From Equations (A142) and (A143) we have

$$\epsilon + P = \frac{V(\varphi)}{\sqrt{1-\dot{\varphi}^2}} \dot{\varphi}^2. \quad (\text{A234})$$

Combining Equations (A233) and (A234), we obtain the energy conservation Equation (A108). Inversely, from Equations (A108), (A142) and (A143), we can directly derive the field Equation (A141).

## Appendix H. Some Studies Devoted to Polytopic and Logotropic Equations of State of Type I, II and III

In this Appendix, we briefly mention studies devoted to polytopic and logotropic equations of state of type I, II and III in the context of stars, DM halos and cosmology.

The study of nonrelativistic stars described by a polytopic equation of state dates back to the paper of Lane [196]. Isothermal stars were first considered by Zöllner [197]. A very complete study of polytopic and isothermal stars is presented in the books of Emden [198] and Chandrasekhar [139]. Nonrelativistic logotropic stars were studied by McLaughlin and Pudritz [199]. The logotropic equation of state was applied to DM halos by Chavanis [116,118,122,143].

General relativistic stars described by a polytopic equation of state of type I were first considered by Tooper [151]. Polytopes of type I with index  $\gamma = 2$  were specifically studied by Chavanis and Harko [154,155] in relation to general relativistic BEC stars (however, this is not the correct equation of state for these systems—see below). General relativistic stars described by a linear equation of state, extending the models of Newtonian isothermal stars, were studied by Chandrasekhar [157] (see also [158–161] and references therein). Cosmological models based on a polytopic equation of state of type I with an arbitrary index  $\gamma$  were studied in [97–99]. The specific index  $\gamma = -1$  corresponds to the Chaplygin gas [50,86] and the indices  $-1 \leq \gamma \leq 0$  correspond to the GCG [95]. A cosmological model based on the logotropic equation of state of type I was studied in Appendix B of [116] (see also [122]).

General relativistic stars described by a polytopic equation of state of type II were first considered by Tooper [162]. Polytopes of type II with index  $\gamma = 2$  were specifically studied by Chavanis [154] and Latifah et al. [200] in relation to general relativistic BEC stars (however, this is not the correct equation of state for these systems—see below). Cosmological models based on a polytopic equation of state of type II with an arbitrary index  $\gamma$  were studied in [163] (the index  $\gamma = 2$  of a BEC is specifically treated in the main text of [163] and the case of a general index is treated in Appendix D of [163]). A cosmological model based on the logotropic equation of state of type II was studied in [118] (see also [122]).

General relativistic stars described by a polytopic equation of state of type III were studied by Colpi et al. [152] and Chavanis and Harko [154,155] for the particular index  $\gamma = 2$  corresponding to BECs. This is the hydrodynamic representation, valid in the TF regime, of a complex SF with a repulsive  $|\varphi|^4$  self-interaction described by the KGE equations [152]. Therefore, a polytopic equation of state of type III with index  $\gamma = 2$ , leading to the equation of state (346), is the correct equation of state of a relativistic BEC with a quartic self-interaction in the TF regime. Cosmological models based on a polytopic equation of state of type III with an arbitrary index  $\gamma$  were studied in [149,156] (the index  $\gamma = 2$  of a BEC is specifically treated in the main text of [149] and the case of an arbitrary index is treated in Appendix I of [149]). This is the hydrodynamic representation, valid in the TF regime or in the fast oscillation regime, of a complex SF with an algebraic potential  $|\varphi|^{2\gamma}$  described by the KGE equations (the  $|\varphi|^4$  potential [153] corresponds to  $\gamma = 2$ ). A cosmological model based on the logotropic equation of state of type III has been studied recently in [116] (see also [122]).

## Appendix I. Conservation Laws for a Nonrelativistic SF

In this Appendix, we establish the local conservation laws of mass, impulse and energy for a nonrelativistic SF (see Section 2).

### Appendix I.1. Conservation Laws in Terms of Hydrodynamic Variables

The equation of continuity (20) can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \quad (\text{A235})$$

where  $\rho$  is the mass density and

$$\mathbf{J} = \rho \mathbf{u} \quad (\text{A236})$$

is the density current. This equation expresses the local conservation of mass  $M = \int \rho \, d\mathbf{r}$ .

Using the continuity Equation (20), the quantum Euler Equation (24) can be rewritten as

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P + \frac{\rho}{m} \nabla Q = \mathbf{0}. \quad (\text{A237})$$

On the other hand, the quantum force can be written under the form (see Section 2.5 of [136])

$$-\frac{\rho}{m} \partial_i Q = -\partial_j P_{ij}^Q, \quad (\text{A238})$$

where the anisotropic quantum pressure tensor  $P_{ij}^Q$  is given by

$$P_{ij}^Q = -\frac{\hbar^2}{4m^2} \rho \partial_i \partial_j \ln \rho = \frac{\hbar^2}{4m^2} \left( \frac{1}{\rho} \partial_i \rho \partial_j \rho - \partial_i \partial_j \rho \right) \quad (\text{A239})$$

or, alternatively, by

$$P_{ij}^Q = \frac{\hbar^2}{4m^2} \left( \frac{1}{\rho} \partial_i \rho \partial_j \rho - \delta_{ij} \Delta \rho \right). \quad (\text{A240})$$

Substituting Equation (A238) into Equation (A237), we obtain

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \nabla(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P + \partial_j P_{ij}^Q = \mathbf{0}. \quad (\text{A241})$$

Introducing the momentum density

$$-T_{i0} = \rho u_i, \quad (\text{A242})$$

we can rewrite Equation (A241) as

$$-\frac{\partial T_{i0}}{\partial t} + \partial_j T_{ij} = 0, \quad (\text{A243})$$

where

$$T_{ij} = \rho u_i u_j + P \delta_{ij} + P_{ij}^Q \quad (\text{A244})$$

is the stress tensor. Using Equations (A239) and (A240), we get

$$\begin{aligned} T_{ij} &= \rho u_i u_j + P \delta_{ij} - \frac{\hbar^2}{4m^2} \rho \partial_i \partial_j \ln \rho \\ &= \rho u_i u_j + P \delta_{ij} + \frac{\hbar^2}{4m^2} \left( \frac{1}{\rho} \partial_i \rho \partial_j \rho - \partial_i \partial_j \rho \right) \end{aligned} \quad (\text{A245})$$

or, alternatively,

$$T_{ij} = \rho u_i u_j + \left[ P(\rho) - \frac{\hbar^2}{4m^2} \Delta \rho \right] \delta_{ij} + \frac{\hbar^2}{4m^2} \frac{1}{\rho} \partial_i \rho \partial_j \rho. \quad (\text{A246})$$

Equation (A243) expresses the local conservation of the momentum  $\mathbf{P} = \int \rho \mathbf{u} \, d\mathbf{r}$ .

Introducing the energy density

$$T_{00} = \rho e = \rho \frac{\mathbf{u}^2}{2} + \rho \frac{Q}{m} + V(\rho) \quad (\text{A247})$$

and combining the equation of continuity (20) and the quantum Euler Equation (24), we obtain (see Appendix E of [136])

$$\frac{\partial}{\partial t}(\rho e) + \nabla \cdot (\rho e \mathbf{u}) + \nabla \cdot (P \mathbf{u}) + \nabla \cdot \mathbf{J}_Q = 0, \quad (\text{A248})$$

where

$$\mathbf{J}_Q = \frac{\hbar^2}{4m^2} \rho \frac{\partial \nabla \ln \rho}{\partial t} \quad (\text{A249})$$

is the quantum current. Introducing the energy current

$$\begin{aligned} -T_{0i} &= \rho e \mathbf{u} + P \mathbf{u} + \mathbf{J}_Q \\ &= \rho \left[ \frac{\mathbf{u}^2}{2} + \frac{Q}{m} + \frac{V(\rho) + P}{\rho} \right] \mathbf{u} + \mathbf{J}_Q \\ &= \rho \left[ \frac{\mathbf{u}^2}{2} + \frac{Q}{m} + V'(\rho) \right] \mathbf{u} + \mathbf{J}_Q, \end{aligned} \quad (\text{A250})$$

where we have used Equation (26) to obtain the last equality, we can rewrite Equation (A248) as

$$\frac{\partial T_{00}}{\partial t} - \partial_i T_{0i} = 0. \quad (\text{A251})$$

This equation expresses the local conservation of energy  $E = \int \rho e d\mathbf{r}$ . We also recall that  $h(\rho) = V'(\rho)$  is the enthalpy.

For classical systems ( $\hbar = 0$ ), or for BECs in the TF limit, the foregoing equations reduce to

$$T_{00} = \rho e = \rho \frac{\mathbf{u}^2}{2} + V(\rho), \quad (\text{A252})$$

$$-T_{0i} = \rho \left[ \frac{\mathbf{u}^2}{2} + V'(\rho) \right] \mathbf{u}, \quad (\text{A253})$$

$$-T_{i0} = \rho \mathbf{u}, \quad (\text{A254})$$

$$T_{ij} = \rho u_i u_j + P \delta_{ij}. \quad (\text{A255})$$

*Remark:* We note that  $T_{0i} \neq T_{i0}$  because the theory is not Lorentz invariant. By contrast,  $T_{ij} = T_{ji}$  because the theory is invariant against spatial rotations. We also note that the momentum density is equal to the mass flux:

$$-T_{i0} = \rho \mathbf{u} = \mathbf{J}. \quad (\text{A256})$$

### Appendix I.2. Conservation Laws in Terms of the Wave Function

Using Equations (17)–(19), the density current (A236) can be expressed in terms of the wave function as

$$\mathbf{J} = \frac{\hbar}{2im}(\psi^* \nabla \psi - \psi \nabla \psi^*). \quad (\text{A257})$$

As a result, the equation of continuity (A235) takes the form

$$\frac{\partial |\psi|^2}{\partial t} + \frac{\hbar}{2im} \nabla \cdot (\psi^* \nabla \psi - \psi \nabla \psi^*) = 0. \quad (\text{A258})$$

Similarly, the momentum density (A242) and the stress tensor (A246) can be written in terms of the wave function as (see Appendix A of [136])

$$-T_{i0} = \mathbf{J} = \frac{\hbar}{2im}(\psi^* \nabla \psi - \psi \nabla \psi^*) \quad (\text{A259})$$

and

$$T_{ij} = \frac{\hbar^2}{m^2} \text{Re} \left( \frac{\partial \psi}{\partial x_i} \frac{\partial \psi^*}{\partial x_j} \right) + \left[ P(|\psi|^2) - \frac{\hbar^2}{4m^2} \Delta |\psi|^2 \right] \delta_{ij} \quad (\text{A260})$$

or, alternatively,

$$T_{ij} = \frac{\hbar^2}{2m^2} \text{Re} \left( \frac{\partial \psi}{\partial x_i} \frac{\partial \psi^*}{\partial x_j} - \psi \frac{\partial^2 \psi^*}{\partial x_i \partial x_j} \right) + P \delta_{ij}. \quad (\text{A261})$$

Finally, the energy density (A247) and the energy current (A250) can be written in terms of the wave function as

$$T_{00} = \frac{\hbar^2}{2m^2} |\nabla \psi|^2 + V(|\psi|^2) \quad (\text{A262})$$

and

$$-T_{0i} = \left[ \frac{\hbar^2}{2m^2} \frac{|\nabla \psi|^2}{|\psi|^2} + V'(|\psi|^2) \right] \mathbf{J} + \mathbf{J}_Q \quad (\text{A263})$$

with

$$\mathbf{J}_Q = \frac{\hbar^2}{4m^2} |\psi|^2 \frac{\partial \nabla \ln |\psi|^2}{\partial t}. \quad (\text{A264})$$

### Appendix I.3. Conservation Laws from the Lagrangian Expressed in Terms of the Wave Function

The current of a complex SF is given by

$$J^\mu = \frac{m}{i\hbar} \left[ \psi \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} - \psi^* \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi^*)} \right]. \quad (\text{A265})$$

For the Lagrangian (33) we obtain the mass density

$$J_0 = |\psi|^2 = \rho \quad (\text{A266})$$

and the mass flux

$$\mathbf{J} = -\frac{i\hbar}{2m}(\psi^* \nabla \psi - \psi \nabla \psi^*). \quad (\text{A267})$$

The energy-momentum tensor of a complex SF is given by

$$T_{\mu}^{\nu} = \partial_{\mu}\psi \frac{\partial \mathcal{L}}{\partial(\partial_{\nu}\psi)} + \partial_{\mu}\psi^{*} \frac{\partial \mathcal{L}}{\partial(\partial_{\nu}\psi^{*})} - \mathcal{L}\delta_{\mu}^{\nu}. \quad (\text{A268})$$

For the Lagrangian (33) we obtain the energy density

$$T_{00} = \frac{\hbar^2}{2m^2} |\nabla\psi|^2 + V(|\psi|^2), \quad (\text{A269})$$

the momentum density

$$-T_{i0} = -\frac{i\hbar}{2m} (\psi^{*}\nabla\psi - \psi\nabla\psi^{*}) = \mathbf{J}, \quad (\text{A270})$$

the energy flux

$$-T_{0i} = -\frac{\hbar^2}{2m^2} (\psi\nabla\psi^{*} + \psi^{*}\nabla\psi), \quad (\text{A271})$$

and the momentum fluxes (stress tensor)

$$T_{ij} = \frac{\hbar^2}{m^2} \text{Re}(\partial_i\psi\partial_j\psi^{*}) + \mathcal{L}\delta_{ij}. \quad (\text{A272})$$

These are their general expressions. If we use the GP Equation (16), which is obtained after extremizing the action, we can rewrite Equation (A271) as Equation (A263). Similarly, if we use the expression (46) of the Lagrangian which relies on the GP Equation (16), we can rewrite Equation (A272) as Equation (A260). In this manner, we recover the equations of Appendix I.2 (up to terms that vanish by integration).

*Remark:* The energy density is

$$\begin{aligned} T_{00} &= \psi \frac{\partial \mathcal{L}}{\partial \dot{\psi}} + \psi^{*} \frac{\partial \mathcal{L}}{\partial \dot{\psi}^{*}} - \mathcal{L} \\ &= \pi \dot{\psi} + \pi^{*} \dot{\psi}^{*} - \mathcal{L} \\ &= \frac{i\hbar}{2m} (\psi^{*} \dot{\psi} - \psi \dot{\psi}^{*}) - \mathcal{L}, \end{aligned} \quad (\text{A273})$$

where  $\pi = \partial \mathcal{L} / \partial \dot{\psi} = \frac{i\hbar}{2m} \psi^{*}$  is the conjugate momentum to  $\psi$ . This leads to Equations (35) and (36). On the other hand, Equation (37) can be rewritten in the form of Hamilton equations

$$\frac{\partial \psi}{\partial t} = \frac{1}{2} \frac{\delta H}{\delta \pi}, \quad \frac{\partial \pi}{\partial t} = -\frac{1}{2} \frac{\delta H}{\delta \psi}. \quad (\text{A274})$$

They are equivalent to the GP Equation (16) and its complex conjugate.

#### Appendix I.4. Conservation Laws from the Lagrangian Expressed in Terms of Hydrodynamic Variables

The current of a complex SF in its hydrodynamic representation is given by

$$J^{\mu} = -m \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}S)}. \quad (\text{A275})$$

For the Lagrangian (40) we obtain the mass density

$$J_0 = \rho \quad (\text{A276})$$

and the mass flux

$$\mathbf{J} = \rho \frac{\nabla S}{m} = \rho \mathbf{u}. \quad (\text{A277})$$

The energy-momentum tensor of a complex SF in its hydrodynamic representation is given by

$$T_{\mu}^{\nu} = \partial_{\mu} \rho \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} \rho)} + \partial_{\mu} S \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} S)} - \mathcal{L} \delta_{\mu}^{\nu}. \quad (\text{A278})$$

For the Lagrangian (40) we obtain the energy density

$$\begin{aligned} T_{00} &= \frac{\rho}{2m^2} (\nabla S)^2 + \frac{\hbar^2}{8m^2} \frac{(\nabla \rho)^2}{\rho} + V(\rho) \\ &= \frac{1}{2} \rho \mathbf{u}^2 + \rho \frac{Q}{m} + V(\rho), \end{aligned} \quad (\text{A279})$$

the momentum density

$$-T_{i0} = \frac{\rho}{m} \nabla S = \rho \mathbf{u} = \mathbf{J}, \quad (\text{A280})$$

the energy flux

$$-T_{0i} = -\frac{\partial \rho}{\partial t} \frac{\hbar^2}{4m^2} \frac{\nabla \rho}{\rho} - \frac{\partial S}{\partial t} \frac{\rho}{m^2} \nabla S, \quad (\text{A281})$$

and the momentum fluxes

$$\begin{aligned} T_{ij} &= \frac{\hbar^2}{4m^2} \frac{1}{\rho} \partial_i \rho \partial_j \rho + \frac{\rho}{m^2} \partial_i S \partial_j S + \mathcal{L} \delta_{ij} \\ &= \rho u_i u_j + \frac{\hbar^2}{4m^2} \frac{1}{\rho} \partial_i \rho \partial_j \rho + \mathcal{L} \delta_{ij}. \end{aligned} \quad (\text{A282})$$

These are their general expressions. If we use the quantum Hamilton-Jacobi (or Bernoulli) Equation (21), which is obtained after extremizing the action, we can rewrite Equation (A281) as Equation (A250). Similarly, if we use the expression (46) of the Lagrangian which relies on the quantum Hamilton-Jacobi (or Bernoulli) Equation (21), we can rewrite Equation (A282) as Equation (A246). In this manner, we recover the equations of Appendix I.1 (up to terms that vanish by integration).

*Remark:* The energy density is

$$T_{00} = \dot{S} \frac{\partial \mathcal{L}}{\partial \dot{S}} - \mathcal{L} = \pi \dot{S} - \mathcal{L} = -\frac{\rho}{m} \dot{S} - \mathcal{L}, \quad (\text{A283})$$

where  $\pi = \partial \mathcal{L} / \partial \dot{S} = -\rho/m$  is the conjugate momentum to  $S$ . This leads to Equations (43) and (44). On the other hand, Equation (45) can be rewritten in the form of Hamilton equations

$$\frac{\partial S}{\partial t} = \frac{\delta H}{\delta \pi}, \quad \frac{\partial \pi}{\partial t} = -\frac{\delta H}{\delta S}. \quad (\text{A284})$$

They are equivalent to the continuity Equation (20) and to the quantum Hamilton-Jacobi (or Bernoulli) Equation (21).

## Appendix J. Conservation Laws for a Relativistic Complex SF

In this Appendix, we establish the local conservation laws of boson number (charge), impulse and energy for a relativistic complex SF (see Section 3). For simplicity, we consider a flat metric and a static background.

### Appendix J.1. Conservation Laws from the Lagrangian Expressed in Terms of the Wave Function

The density Lagrangian of a complex SF is [see Equation (110)]

$$\mathcal{L} = \frac{1}{2c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 - \frac{1}{2} |\nabla \varphi|^2 - V_{\text{tot}}(|\varphi|^2). \quad (\text{A285})$$

The components of the current (120) or (121) are

$$J^0 = -\frac{m}{2i\hbar c} \left( \varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t} \right), \quad (\text{A286})$$

$$J^i = \frac{m}{2i\hbar} (\varphi^* \partial_i \varphi - \varphi \partial_i \varphi^*). \quad (\text{A287})$$

The local conservation of the boson number (charge) can be written as

$$\frac{1}{c} \frac{\partial J^0}{\partial t} + \partial_i J^i = 0. \quad (\text{A288})$$

The components of the energy-momentum tensor (116) are

$$T^{00} = \frac{1}{2c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 + \frac{1}{2} |\nabla \varphi|^2 + V_{\text{tot}}(|\varphi|^2), \quad (\text{A289})$$

$$T^{0i} = -\frac{1}{2c} \frac{\partial \varphi^*}{\partial t} \partial_i \varphi - \frac{1}{2c} \frac{\partial \varphi}{\partial t} \partial_i \varphi^*, \quad (\text{A290})$$

$$T^{ij} = \frac{1}{2} \partial_i \varphi^* \partial_j \varphi + \frac{1}{2} \partial_i \varphi \partial_j \varphi^* + \delta_{ij} \mathcal{L}. \quad (\text{A291})$$

We note that  $T^{ii} = |\nabla \varphi|^2 + 3\mathcal{L}$ . The conservation of the energy-momentum tensor [see Equation (118)] can be written as

$$\frac{1}{c} \frac{\partial T^{00}}{\partial t} + \partial_i T^{0i} = 0, \quad (\text{A292})$$

$$\frac{1}{c} \frac{\partial T^{i0}}{\partial t} + \partial_j T^{ij} = 0. \quad (\text{A293})$$

The Euler-Lagrange Equation (112) yields the KG equation

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi + 2 \frac{dV_{\text{tot}}}{d|\varphi|^2} \varphi = 0 \quad (\text{A294})$$

which involves the d'Alembertian operator  $\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta$ .

In the homogeneous case, the foregoing equations reduce to

$$\mathcal{L} = \frac{1}{2c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 - V_{\text{tot}}(|\varphi|^2), \quad (\text{A295})$$



$$T^{00} = \frac{1}{2c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 + V_{\text{tot}}(|\varphi|^2), \quad (\text{A296})$$

$$T^{0i} = 0, \quad (\text{A297})$$

$$T^{ij} = \delta_{ij} \mathcal{L}, \quad (\text{A298})$$

$$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} + 2 \frac{dV_{\text{tot}}}{d|\varphi|^2} \varphi = 0. \quad (\text{A299})$$

The energy-momentum tensor is diagonal  $T^{\mu\nu} = \text{diag}(\epsilon, P, P, P)$ . The energy density and the pressure are given by

$$\epsilon = \frac{1}{2c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 + V_{\text{tot}}(|\varphi|^2), \quad (\text{A300})$$

$$P = \frac{1}{2c^2} \left| \frac{\partial \varphi}{\partial t} \right|^2 - V_{\text{tot}}(|\varphi|^2). \quad (\text{A301})$$

To obtain the nonrelativistic limit, we first make the Klein transformation (14) and use Equation (111). We get

$$\mathcal{L} = \frac{\hbar^2}{2m^2 c^2} \left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{i\hbar}{2m} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \frac{\hbar^2}{2m^2} |\nabla \psi|^2 - V(|\psi|^2), \quad (\text{A302})$$

$$J^0 = -\frac{\hbar}{2icm} \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} - \frac{2imc^2}{\hbar} |\psi|^2 \right), \quad (\text{A303})$$

$$J^i = \frac{\hbar}{2im} (\psi^* \partial_i \psi - \psi \partial_i \psi^*), \quad (\text{A304})$$

$$T^{00} = \frac{\hbar^2}{2m^2 c^2} \left| \frac{\partial \psi}{\partial t} \right|^2 + \frac{\hbar^2}{2m^2} |\nabla \psi|^2 + c^2 |\psi|^2 + V(|\psi|^2) + \frac{i\hbar}{2m} \left( \frac{\partial \psi}{\partial t} \psi^* - \psi \frac{\partial \psi^*}{\partial t} \right), \quad (\text{A305})$$

$$T^{0i} = -\frac{\hbar^2}{2m^2 c} \left( \frac{\partial \psi^*}{\partial t} \partial_i \psi + \frac{\partial \psi}{\partial t} \partial_i \psi^* \right) - \frac{i\hbar c}{2m} (\psi^* \partial_i \psi - \psi \partial_i \psi^*), \quad (\text{A306})$$

$$T^{ij} = \frac{\hbar^2}{m^2} \text{Re}(\partial_i \psi \partial_j \psi^*) + \delta_{ij} \mathcal{L}. \quad (\text{A307})$$

If we take the limit  $c \rightarrow +\infty$  in Equation (A302) we recover Equation (33). If we divide Equation (A303) by  $c$  and take the limit  $c \rightarrow +\infty$ , we obtain  $J^0/c = |\psi|^2$  leading to Equation (A266). Equation (A304) is equivalent to Equation (A267). To leading order, Equation (A305) gives  $T^{00} \sim \rho c^2$ . If we subtract the rest mass term  $cJ^0$  (see Ref. [145]) and take the limit  $c \rightarrow +\infty$  in Equation (A305), we recover Equation (A269). If we multiply or divide Equation (A306) by  $c$  and consider the terms that are independent of  $c$  (see Ref. [145]), we get

$$\frac{T^{0i}}{c} = -\frac{i\hbar}{2m} (\psi^* \partial_i \psi - \psi \partial_i \psi^*), \quad (\text{A308})$$

$$T^{0i} c = -\frac{\hbar^2}{2m^2} \left( \frac{\partial \psi^*}{\partial t} \partial_i \psi + \frac{\partial \psi}{\partial t} \partial_i \psi^* \right). \quad (\text{A309})$$

This returns Equations (A270) and (A271). Equation (A307) returns Equation (A272). Finally, the KG Equation (A294) becomes

$$i\hbar \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2mc^2} \frac{\partial^2 \psi}{\partial t^2} + \frac{\hbar^2}{2m} \Delta \psi - m \frac{dV}{d|\psi|^2} \psi = 0. \quad (\text{A310})$$

In the nonrelativistic limit  $c \rightarrow +\infty$ , we recover the GP Equation (16).

*Remark:* We note that the energy-momentum tensor is symmetric in relativity theory ( $T^{\mu\nu} = T^{\nu\mu}$ ) while it is not symmetric in Newtonian theory ( $T_{0i} \neq T_{i0}$ ). This is because space and time are not treated on equal footing in Newtonian theory. This is also why we have to consider the two terms  $T^{0i}/c$  and  $T^{0i}c$  individually in the nonrelativistic limit [see Equations (A308) and (A309)].

## Appendix J.2. Conservation Laws from the Lagrangian Expressed in Terms of Hydrodynamic Variables

The density Lagrangian of a complex SF in its hydrodynamic representation is [see Equation (129)]

$$\mathcal{L} = \frac{\hbar^2}{8\rho m^2 c^2} \left( \frac{\partial \rho}{\partial t} \right)^2 + \frac{1}{2c^2} \rho \left( \frac{\partial \theta}{\partial t} \right)^2 - \frac{\hbar^2}{8\rho m^2} (\nabla \rho)^2 - \frac{1}{2} \rho (\nabla \theta)^2 - V_{\text{tot}}(\rho). \quad (\text{A311})$$

The components of the current (143) are

$$J^0 = -\frac{\rho}{c} \frac{\partial \theta}{\partial t}, \quad J^i = \rho \partial_i \theta, \quad (\text{A312})$$

and they satisfy the conservation of boson number (charge) from Equation (A288).

The components of the energy-momentum tensor (140) are

$$T^{00} = \frac{\hbar^2}{8\rho m^2 c^2} \left( \frac{\partial \rho}{\partial t} \right)^2 + \frac{1}{2c^2} \rho \left( \frac{\partial \theta}{\partial t} \right)^2 + \frac{\hbar^2}{8\rho m^2} (\nabla \rho)^2 + \frac{1}{2} \rho (\nabla \theta)^2 + V_{\text{tot}}(\rho), \quad (\text{A313})$$

$$T^{0i} = -\frac{1}{c} \rho \frac{\partial \theta}{\partial t} \partial_i \theta - \frac{\hbar^2}{4\rho m^2 c} \frac{\partial \rho}{\partial t} \partial_i \rho, \quad (\text{A314})$$

$$T^{ij} = \frac{\hbar^2}{4\rho m^2} \partial_i \rho \partial_j \rho + \rho \partial_i \theta \partial_j \theta + \delta_{ij} \mathcal{L}, \quad (\text{A315})$$

and they satisfy the conservation of impulse and energy from Equations (A292) and (A293).

The Euler-Lagrange Equations (131) and (132) yield the continuity equation

$$\frac{1}{c^2} \frac{\partial}{\partial t} \left( \rho \frac{\partial \theta}{\partial t} \right) - \nabla \cdot (\rho \nabla \theta) = 0 \quad (\text{A316})$$

and the quantum Hamilton-Jacobi (or Bernoulli) equation

$$\frac{1}{2c^2} \left( \frac{\partial \theta}{\partial t} \right)^2 - \frac{1}{2} (\nabla \theta)^2 - \frac{Q}{m} - V'_{\text{tot}}(\rho) = 0, \quad (\text{A317})$$

with the quantum potential

$$Q \equiv \frac{\hbar^2}{2m} \frac{\square \sqrt{\rho}}{\sqrt{\rho}} = \frac{\hbar^2}{4\rho m c^2} \frac{\partial^2 \rho}{\partial t^2} - \frac{\hbar^2}{8\rho^2 m c^2} \left( \frac{\partial \rho}{\partial t} \right)^2 - \frac{\hbar^2}{4\rho m} \Delta \rho + \frac{\hbar^2}{8\rho^2 m} (\nabla \rho)^2. \quad (\text{A318})$$

According to Equation (136), the pseudo energy and the pseudo velocity are

$$v^0 = v_0 = -\frac{1}{c} \frac{\partial \theta}{\partial t}, \quad v^i = -v_i = \partial_i \theta. \quad (\text{A319})$$

Recalling that  $\theta = S_{\text{tot}}/m$ , we get

$$E_{\text{tot}} = -\frac{\partial S_{\text{tot}}}{\partial t}, \quad \mathbf{v} = \frac{\nabla S_{\text{tot}}}{m}. \quad (\text{A320})$$

We also have

$$J^0 = \frac{\rho E_{\text{tot}}}{mc}, \quad \mathbf{J} = \rho \mathbf{v}. \quad (\text{A321})$$

The continuity Equation (A316) and the quantum Hamilton-Jacobi (or Bernoulli) Equation (A317) can be rewritten as

$$\frac{\partial}{\partial t} \left( \rho \frac{E_{\text{tot}}}{mc^2} \right) + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (\text{A322})$$

and

$$\frac{E_{\text{tot}}^2}{2m^2c^2} - \frac{\mathbf{v}^2}{2} - \frac{Q}{m} - V'_{\text{tot}}(\rho) = 0. \quad (\text{A323})$$

Taking the gradient of Equation (A323), we obtain the Euler equation

$$\frac{E_{\text{tot}}}{mc^2} \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{m} \nabla Q - \frac{1}{\rho} \nabla P. \quad (\text{A324})$$

In the homogeneous case, the foregoing equations reduce to

$$\mathcal{L} = \frac{\hbar^2}{8\rho m^2 c^2} \left( \frac{d\rho}{dt} \right)^2 + \frac{1}{2c^2} \rho \left( \frac{d\theta}{dt} \right)^2 - V_{\text{tot}}(\rho), \quad (\text{A325})$$

$$T^{00} = \frac{\hbar^2}{8\rho m^2 c^2} \left( \frac{d\rho}{dt} \right)^2 + \frac{1}{2c^2} \rho \left( \frac{d\theta}{dt} \right)^2 + V_{\text{tot}}(\rho), \quad (\text{A326})$$

$$T^{0i} = 0, \quad (\text{A327})$$

$$T^{ij} = \delta_{ij} \mathcal{L}, \quad (\text{A328})$$

$$\frac{d}{dt} \left( \rho \frac{d\theta}{dt} \right) = 0, \quad (\text{A329})$$

$$\frac{1}{2c^2} \left( \frac{d\theta}{dt} \right)^2 - \frac{Q}{m} - V'_{\text{tot}}(\rho) = 0, \quad (\text{A330})$$

with the quantum potential

$$Q \equiv \frac{\hbar^2}{2mc^2 \sqrt{\rho}} \frac{d^2 \sqrt{\rho}}{dt^2} = \frac{\hbar^2}{4\rho mc^2} \frac{d^2 \rho}{dt^2} - \frac{\hbar^2}{8\rho^2 mc^2} \left( \frac{d\rho}{dt} \right)^2. \quad (\text{A331})$$

The energy-momentum tensor is diagonal  $T^{\mu\nu} = \text{diag}(\epsilon, P, P, P)$ . The energy and the pressure are given by

$$\epsilon = \frac{\hbar^2}{8\rho m^2 c^2} \left( \frac{d\rho}{dt} \right)^2 + \frac{1}{2c^2} \rho \left( \frac{d\theta}{dt} \right)^2 + V_{\text{tot}}(\rho), \quad (\text{A332})$$

$$P = \frac{\hbar^2}{8\rho m^2 c^2} \left( \frac{d\rho}{dt} \right)^2 + \frac{1}{2c^2} \rho \left( \frac{d\theta}{dt} \right)^2 - V_{\text{tot}}(\rho). \quad (\text{A333})$$

To obtain the nonrelativistic limit, we first make the Klein transformation (see Section 3.5)

$$m\theta = S - mc^2 t, \quad (\text{A334})$$

and use Equation (130). We get

$$\begin{aligned} \mathcal{L} = & \frac{\hbar^2}{8\rho m^2 c^2} \left( \frac{\partial \rho}{\partial t} \right)^2 + \frac{\rho}{2m^2 c^2} \left( \frac{\partial S}{\partial t} \right)^2 - \frac{\rho}{m} \frac{\partial S}{\partial t} \\ & - \frac{\hbar^2}{8\rho m^2} (\nabla \rho)^2 - \frac{\rho}{2m^2} (\nabla S)^2 - V(\rho), \end{aligned} \quad (\text{A335})$$

$$J^0 = -\frac{\rho}{mc} \frac{\partial S}{\partial t} + \rho c, \quad (\text{A336})$$

$$J^i = \rho \frac{\partial_i S}{m} = \rho u^i, \quad (\text{A337})$$

$$\begin{aligned} T^{00} = & \frac{\hbar^2}{8\rho m^2 c^2} \left( \frac{\partial \rho}{\partial t} \right)^2 + \frac{\rho}{2m^2 c^2} \left( \frac{\partial S}{\partial t} \right)^2 + \rho c^2 - \frac{\rho}{m} \frac{\partial S}{\partial t} \\ & + \frac{\hbar^2}{8\rho m^2} (\nabla \rho)^2 + \frac{\rho}{2m^2} (\nabla S)^2 + V(\rho), \end{aligned} \quad (\text{A338})$$

$$T^{0i} = \rho \frac{\partial_i S}{m} c - \frac{\rho}{m^2 c} \frac{\partial S}{\partial t} \partial_i S - \frac{\hbar^2}{4\rho m^2 c} \frac{\partial \rho}{\partial t} \partial_i \rho, \quad (\text{A339})$$

$$T^{ij} = \frac{\hbar^2}{4\rho m^2} \partial_i \rho \partial_j \rho + \frac{\rho}{m^2} \partial_i S \partial_j S + \delta_{ij} \mathcal{L}. \quad (\text{A340})$$

If take the limit  $c \rightarrow +\infty$  in Equation (A335) we recover Equation (40). If we divide Equation (A336) by  $c$  and take the limit  $c \rightarrow +\infty$ , we obtain  $J^0/c = \rho$  leading to Equation (A276). Equation (A337) is equivalent to Equation (A277). To leading order, Equation (A338) gives  $T^{00} \sim \rho c^2$ . If we subtract the rest mass term  $cJ^0$  (see Ref. [145]) and take the limit  $c \rightarrow +\infty$  in Equation (A338), we recover Equation (A279). If we multiply or divide Equation (A339) by  $c$  and consider the terms that are independent of  $c$  (see Ref. [145]) we get

$$\frac{T^{0i}}{c} = \rho \frac{\partial_i S}{m}, \quad (\text{A341})$$

$$T^{0i} c = -\frac{\rho}{m^2} \frac{\partial S}{\partial t} \partial_i S - \frac{\hbar^2}{4\rho m^2} \frac{\partial \rho}{\partial t} \partial_i \rho. \quad (\text{A342})$$

This returns Equations (A280) and (A281). Equation (A340) returns Equation (A282). Finally, the continuity Equation (A316) and the quantum Hamilton-Jacobi (or Bernoulli) Equation (A317) become

$$-\frac{1}{mc^2} \frac{\partial}{\partial t} \left( \rho \frac{\partial S}{\partial t} \right) + \frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho \frac{\nabla S}{m} \right) = 0 \quad (\text{A343})$$

and

$$-\frac{1}{2mc^2} \left( \frac{\partial S}{\partial t} \right)^2 + \frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + Q + mV'(\rho) = 0. \quad (\text{A344})$$

Using the identities

$$S_{\text{tot}} = S - mc^2 t, \quad E_{\text{tot}} = E + mc^2, \quad (\text{A345})$$

$$E = -\frac{\partial S}{\partial t}, \quad \mathbf{u} = \frac{\nabla S}{m}, \quad (\text{A346})$$

$$J^0 = \frac{\rho E}{mc} + \rho c, \quad \mathbf{J} = \rho \mathbf{u}, \quad (\text{A347})$$

they can be rewritten as

$$\frac{1}{mc^2} \frac{\partial}{\partial t} (\rho E) + \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (\text{A348})$$

and

$$-\frac{E^2}{2mc^2} - E + \frac{1}{2} m \mathbf{u}^2 + Q + mV'(\rho) = 0. \quad (\text{A349})$$

Taking the gradient of Equation (A349) we obtain the Euler equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \frac{1}{m} \nabla Q + \frac{1}{2m^2 c^2} \nabla (E^2). \quad (\text{A350})$$

In the limit  $c \rightarrow +\infty$ , we recover Equations (20), (21) and (24).

## Notes

- <sup>1</sup> For simplicity of presentation we ignore the contribution of baryonic matter in the present discussion. We also assume that the universe is spatially flat in agreement with the inflation paradigm [5] and the measurements of the CMB anisotropies [1,2].
- <sup>2</sup> SFs have been used in a variety of inflationary models [34] to describe the transition from the exponential (de Sitter) expansion of the early universe to a decelerated expansion (radiation era). It was therefore natural to try to understand the present acceleration of the universe, which has an exponential behaviour too, in terms of SFs [35,36]. However, one has to deal now with the opposite situation, i.e., describing the transition from a decelerated expansion (matter era) to an exponential (de Sitter) expansion.
- <sup>3</sup> A class of k-essence models [38,39] has been claimed to solve the coincidence problem by linking the onset of DE domination to the epoch of DM domination.
- <sup>4</sup> To explain the accelerated expansion taking place today, the universe must be dominated by a fluid of negative pressure violating the strong energy condition, i.e.,  $P < -\epsilon/3$ . For a fluid with a linear equation of state  $P = \omega\epsilon$ , where  $\omega$  is a constant, we need  $\omega < -1/3$  to have an acceleration. But, in that case,  $c_s^2 = \omega c^2 < 0$ , yielding instabilities at small scales [18]. This is the usual problem for a fluid description of domain walls ( $\omega = -2\epsilon/3$ ) and cosmic strings ( $\omega = -\epsilon/3$ ). Quintessence models with a standard kinetic term do not have this problem because the speed of sound is equal to the speed of light [51]. K-essence models [37–39] with a nonstandard kinetic term are different in this respect [41], but they still have a positive squared speed of sound (note that  $c_s$  can exceed the speed of light in that case).
- <sup>5</sup> Negative pressures arise in different domains of physics such as exchange forces in atoms, stripe states in the quantum Hall effect, Bose-Einstein condensates with an attractive self-interaction etc.
- <sup>6</sup> A  $d$ -brane is a  $d$ -dimensional extended object. For example, a ( $d = 2$ )-brane is a membrane.  $d$ -branes arise in string theory for the following reason. Just as the action of a relativistic point particle is proportional to the world line it follows, the action of a relativistic string is proportional to the area of the sheet that it traces by traveling through spacetime. The close connection

between a relativistic membrane [( $d = 2$ )-brane] in three spatial dimensions and planar fluid mechanics was known to J. Goldstone (unpublished) and developed by Hoppe and Bordemann [66,70] (the Chaplygin equation of state  $P = -A/\rho$  appears explicitly in [66] and the Born-Infeld Lagrangian associated with the action of a membrane appears explicitly in [70]). These results were generalized to arbitrary  $d$ -branes by Jackiw and Polychronakos [58]. The same Lagrangian appears as the leading term in Sundrum's [75] effective field theory approach to large extra dimensions. The Born-Infeld Lagrangian can be viewed as a k-essence Lagrangian involving a nonstandard kinetic term. The Chaplygin equation of state is obtained from the stress-energy tensor  $T_{\mu\nu}$  derived from this Lagrangian. Therefore, the Chaplygin gas is the hydrodynamical description of a SF with the Born-Infeld Lagrangian. A more general k-essence model is the string theory inspired tachyon Lagrangian with a potential  $V(\theta)$  [76–87]. It can be shown that every tachyon condensate model can be interpreted as a  $3 + 1$  brane moving in a  $4 + 1$  bulk [88,89]. The Born-Infeld Lagrangian is recovered when  $V(\theta)$  is replaced by a constant [81].

This generalization was mentioned by Kamenshchik et al. [50], Bilic et al. [71] and Gorini et al. [94], and was specifically worked out by Bento et al. [95]. Equation (7) can be viewed as a polytropic equation of state  $P = K(\epsilon/c^2)^\gamma$  with a polytropic index  $\gamma = -\alpha$  and a polytropic constant  $K = -A$ . A further generalization of the GCG model has been proposed. It has an equation of state

$$P = \omega\epsilon - \frac{A}{(\epsilon/c^2)^\alpha}, \quad (6)$$

where  $\omega$  is a constant. This is called the Modified Chaplygin Gas (MCG) model [96]. It can be viewed as the sum of a linear equation of state and a polytropic equation of state. This generalized polytropic equation of state has been studied in detail in a cosmological framework in Refs. [97–99]. The potential  $V(\varphi)$  of its real SF representation generalizing the result of Kamenshchik et al. [50] has been determined explicitly in these papers.

If a solution to these problems cannot be provided, this would appear as an evidence for an independent origin of DM and DE (i.e., they are two distinct substances) [103].

This procedure is not well-defined mathematically because it yields an infinite additional constant  $K \rightarrow +\infty$ . This constant disappears if we take the gradient of the pressure as in [118]. However, in general, an infinite constant term remains. Therefore, the above procedure simply suggests a connection between the polytropic and logotropic equations of state, but this connection is rather subtle.

There is no DM and no DE in the logotropic model, just a single dark fluid. Its rest mass density mimics DM and its internal energy mimics DE [118,119]. In that case,  $\Omega_{\text{dm},0}$  represents the constant that appears in the asymptotic expression  $\epsilon/\epsilon_0 \sim \Omega_{\text{dm},0}/a^3$  of the energy density versus scale factor relation for  $a \ll 1$ .

We could also consider the case of self-gravitating BECs. In that case, one has to introduce a mean field gravitational potential  $\Phi(\mathbf{r}, t)$  in the GP equation which is produced by the particles themselves through a Poisson equation (see Ref. [126] for more details).

We note that the potential is defined from the pressure up to a term of the form  $A\rho$ , where  $A$  is a constant. If we add a term  $A\rho$  in the potential  $V$ , we do not change the pressure. On the other hand, if we add a constant term  $C$  in the potential  $V$ , this adds a term  $-C$  in the pressure. However, for nonrelativistic systems, this constant term has no observable effect since only the gradient of the pressure matters.

Similarly, the invariance of the Lagrangian with respect to spatial translation implies the conservation of linear momentum. In relativity theory, these apparently separate conservation laws are aspects of a single conservation law, that of the energy-momentum tensor (see Section 3).

The variable  $\theta$  is related to the phase (angle)  $\Theta = S/\hbar$  of the SF by  $\theta = \hbar\Theta/m$ .

In our framework, the limit  $\hbar \rightarrow 0$  corresponds to the TF approximation where the quantum potential can be neglected.

The condition  $V'' > 0$  is necessary for local stability since  $c_s^2 = P'(\rho) = \rho V''(\rho)$ . When the system is subjected to a potential  $\Phi$ , the function  $F$  determines the relation between the density and the potential at equilibrium (see Section 3.4 of [136]).

For that reason, the polytropic equation of state (67) is also called the GCG [95].

We stress that  $\rho$  is not the rest-mass density  $\rho_m = nm$  (see below). It is only in the nonrelativistic regime  $c \rightarrow +\infty$  that  $\rho$  coincides with the rest-mass density  $\rho_m$ .

The variable  $\theta$  is related to the phase (angle)  $\Theta = S_{\text{tot}}/\hbar$  of the SF by  $\theta = \hbar\Theta/m$ . In the nonrelativistic limit, we shall denote the variable  $\theta$  by  $\theta_{\text{NR}}$ .

The pseudo quadrivelocity  $v_\mu$  does not satisfy  $v_\mu v^\mu = c^2$  so it is not guaranteed to be always timelike. Nevertheless,  $v_\mu$  can be introduced as a convenient notation.

It differs from the pseudo quadrivelocity  $v_\mu$  introduced in Equation (136).

We note that

$$\epsilon = \frac{u^\mu u^\nu}{c^2} T_{\mu\nu}. \quad (164)$$

- We note that the potential is defined from the pressure up to a term of the form  $A\rho$ , where  $A$  is a constant. If we add a term  $A\rho$  in the potential  $V$ , we do not change the pressure  $P$  but we introduce a term  $2A\rho$  in the energy density. On the other hand, if we add a constant term  $C$  in the potential  $V$  (cosmological constant), this adds a term  $-C$  in the pressure and a term  $+C$  in the energy density.
- We cannot directly take the limit  $\hbar \rightarrow 0$  in the KG equation. This is why we have to average over the oscillations. Alternatively, we can directly take the limit  $\hbar \rightarrow 0$  in the hydrodynamic equations associated with the KG equation. This is equivalent to the WKB method.
- Substituting Equations (17) and (126) into Equation (14), we obtain  $S_{\text{tot}} = S - mc^2 t$  which is equivalent to Equation (185).
- We note that the expression of  $V(\rho)$  for an equation of state  $P(\rho)$  of type III coincides with the expression of  $u(\rho_m)$  for an equation of state  $P(\rho_m)$  of type II of the same functional form provided that we make the replacements  $u \rightarrow V$  and  $\rho_m \rightarrow \rho$  (see Appendix F).
- The true equation of state of a relativistic BEC is given by Equation (331) or (346) corresponding to a polytrope of type III with index  $\gamma = 2$  [149,152–156]. However, in order to have a unified terminology throughout the paper, we shall always associate the polytropic index  $\gamma = 2$  to a BEC even if this association is not quite correct for models of type I and II in the relativistic regime (see Appendix H). As explained in Appendix F, all the models coincide in the nonrelativistic limit.
- The linear equation of state (227) with  $\alpha = \Gamma - 1$  corresponds to the ultrarelativistic limit of the equation of state (286) associated with a polytrope of type II with index  $\Gamma$ . Indeed, for a polytrope  $P = K\rho_m^\Gamma$ , Equation (286) yields  $P \sim (\Gamma - 1)\epsilon$  in the ultrarelativistic limit. The index  $\Gamma = 4/3$  corresponds to  $\alpha = 1/3$  (radiation) and the index  $\Gamma = 2$  corresponds to  $\alpha = 1$  (stiff matter).
- The linear equation of state (227) with  $\alpha = (\Gamma - 1)/(\Gamma + 1)$  corresponds to the ultrarelativistic limit of the equation of state (340) associated with a polytrope of type III with index  $\Gamma$ . Indeed, for a polytrope  $P = K\rho^\Gamma$ , Equation (340) yields  $P \sim [(\Gamma - 1)/(\Gamma + 1)]\epsilon$  in the ultrarelativistic limit. The index  $\Gamma = 2$  corresponds to  $\alpha = 1/3$  (radiation) and the index  $\Gamma = \infty$  corresponds to  $\alpha = 1$  (stiff matter).
- Equation (264) is similar to the Tsallis free energy density of index  $q = (1 + \alpha)/(1 - \alpha)$ . Comments similar to those following Equation (248) apply to the present situation.
- Note that  $V$  represents here the total potential including the rest mass term. For brevity, we write  $V$  instead of  $V_{\text{tot}}$ .
- K-essence Lagrangians were initially introduced to describe inflation (k-inflation) [40,41]. They were later used to described dark energy [37–39]. K-essence Lagrangians can also be obtained from a canonical complex SF in the TF limit  $\hbar \rightarrow 0$  [71,89]. In that case, the real SF  $\varphi$  corresponds to the action (phase)  $\theta$  of the complex SF (see Section 3.4).
- Note that the more general Lagrangian (A23) does not conserve the charge (or the boson number).
- In this Appendix and in Appendices C.2 and C.3,  $t$  stands for  $ct$ .
- In a homogeneous universe,
- The case of a phantom universe is treated in [99].
- According to Equations (A161) and (A164), the rest-mass energy density (DM) corresponds to a sort of “constant” of integration.
- We have taken  $e = 1$  so that the charge of the SF coincides with the boson number.
- For a spatially homogeneous SF, it is shown in Ref. [149] that the TF approximation is equivalent to the fast oscillation approximation  $\omega \gg H$ .

## References

- Planck Collaboration; Ade, P.A.R.; Aghanim, N.; Armitage-Caplan, C.; Arnaud, M.; Ashdown, M.; Atrio-Barandela, F.; Aumont, J.; Baccigalupi, C.; Banday, A. J.; Barreiro, R. B.; et al. Planck 2013 results. XVI. Cosmological parameters. *Astron. Astrophys.* **2014**, *571*, A16.
- Planck Collaboration; Ade, P. A. R.; Aghanim, N.; Arnaud, M.; Ashdown, M.; Aumont, J.; Baccigalupi, C.; Banday, A. J.; Barreiro, R. B.; Bartlett, J. G.; Bartolo, N.; et al. Planck 2015 results. XIII. Cosmological parameters. *Astron. Astrophys.* **2016**, *594*, A13.
- Sahni, V.; Starobinsky, A. The Case for a Positive Cosmological  $\Lambda$ -Term. *Int. J. Mod. Phys. D* **2000**, *9*, 373. [\[CrossRef\]](#)
- Carroll, S.M. The Cosmological Constant. *Living Rev. Relativ.* **2001**, *4*, 1. [\[CrossRef\]](#)
- Guth, A.H. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D* **1981**, *23*, 347. [\[CrossRef\]](#)
- Weinberg, S. The cosmological constant problem. *Rev. Mod. Phys.* **1989**, *61*, 1. [\[CrossRef\]](#)
- Padmanabhan, T. Cosmological constant—the weight of the vacuum. *Phys. Rep.* **2003**, *380*, 235. [\[CrossRef\]](#)
- Steinhardt, P. In *Critical Problems in Physics*; Fitch, V.L., Marlow, D.R., Eds.; Princeton University Press: Princeton, NJ, USA, 1997.
- Zlatev, I.; Wang, L.; Steinhardt, P.J. Quintessence, Cosmic Coincidence, and the Cosmological Constant. *Phys. Rev. Lett.* **1999**, *82*, 896. [\[CrossRef\]](#)
- Moore, B.; Quinn, T.; Governato, F.; Stadel, J.; Lake, G. Cold collapse and the core catastrophe. *MNRAS* **1999**, *310*, 1147. [\[CrossRef\]](#)
- Kauffmann, G.; White, S.D.M.; Guiderdoni, B. The formation and evolution of galaxies within merging dark matter haloes. *Mon. Not. R. Astron. Soc.* **1993**, *264*, 201. [\[CrossRef\]](#)
- Klypin, A.; Kravtsov, A.V.; Valenzuela, O. Where Are the Missing Galactic Satellites? *Astrophys. J.* **1999**, *522*, 82. [\[CrossRef\]](#)



13. Kamionkowski, M.; Liddle, A.R. The Dearth of Halo Dwarf Galaxies: Is There Power on Short Scales? *Phys. Rev. Lett.* **2000**, *84*, 4525. [[CrossRef](#)] [[PubMed](#)]
14. Boylan-Kolchin, M.; Bullock, J.S.; Kaplinghat, M. Too big to fail? The puzzling darkness of massive Milky Way subhaloes. *MNRAS* **2011**, *415*, L40. [[CrossRef](#)]
15. Bullock, J.S.; Boylan-Kolchin, M. Small-Scale Challenges to the  $\Lambda$ CDM Paradigm. *Ann. Rev. Astron. Astrophys.* **2017**, *55*, 343. [[CrossRef](#)]
16. Vilenkin, A.; Shellard, E.P.S. *Cosmic String and Topological Defects*; Cambridge University Press: Cambridge, UK, 1995.
17. Kamionkowski, M.; Toumbas, N. A Low-Density Closed Universe. *Phys. Rev. Lett.* **1996**, *77*, 587. [[CrossRef](#)]
18. Fabris, J.C.; Martin, J. Amplification of density perturbations in fluids with negative pressure. *Phys. Rev. D* **1997**, *55*, 5205. [[CrossRef](#)]
19. Fabris, J.C.; de Borja Gonçalves, S.V. Perturbations in a stringlike fluid. *Phys. Rev. D* **1997**, *56*, 6128. [[CrossRef](#)]
20. Spergel, D.; Pen, U.L. Cosmology in a String-Dominated Universe. *Astrophys. J.* **1997**, *491*, L67. [[CrossRef](#)]
21. Turner, M.S.; White, M. CDM models with a smooth component. *Phys. Rev. D* **1997**, *56*, R4439. [[CrossRef](#)]
22. Silveira, V.; Waga, I. Cosmological properties of a class of  $\Lambda$  decaying cosmologies. *Phys. Rev. D* **1997**, *56*, 4625. [[CrossRef](#)]
23. Chiba, T.; Sugiyama, N.; Nakamura, T. Cosmology with x-matter. *Mon. Not. R. Astron. Soc.* **1997**, *289*, L5. [[CrossRef](#)]
24. Caldwell, R.R.; Dave, R.; Steinhardt, P.J. Cosmological Imprint of an Energy Component with General Equation of State. *Phys. Rev. Lett.* **1998**, *80*, 1582. [[CrossRef](#)]
25. Peebles, P.J.E.; Ratra, B. Cosmology with a Time-Variable Cosmological "Constant". *Astrophys. J.* **1988**, *325*, L17. [[CrossRef](#)]
26. Ratra, B.; Peebles, P.J.E. Cosmological consequences of a rolling homogeneous scalar field. *Phys. Rev. D* **1988**, *37*, 3406. [[CrossRef](#)] [[PubMed](#)]
27. Wetterich, C. Cosmology and the fate of dilatation symmetry. *Nucl. Phys. B* **1988**, *302*, 668. [[CrossRef](#)]
28. Wetterich, C. An asymptotically vanishing time-dependent cosmological "constant". *Astron. Astrophys.* **1995**, *301*, 321.
29. Frieman, J.; Hill, C.; Stebbins, A.; Waga, I. Cosmology with Ultralight Pseudo Nambu-Goldstone Bosons. *Phys. Rev. Lett.* **1995**, *75*, 2077. [[CrossRef](#)]
30. Coble, K.; Dodelson, S.; Frieman, J.A. Dynamical  $\Lambda$  models of structure formation. *Phys. Rev. D* **1997**, *55*, 1851. [[CrossRef](#)]
31. Ferreira, P.G.; Joyce, M. Structure Formation with a Self-Tuning Scalar Field. *Phys. Rev. Lett.* **1997**, *79*, 4740. [[CrossRef](#)]
32. Copeland, E.J.; Liddle, A.R.; Wands, D. Exponential potentials and cosmological scaling solutions. *Phys. Rev. D* **1998**, *57*, 4686. [[CrossRef](#)]
33. Ferreira, P.G.; Joyce, M. Cosmology with a primordial scaling field. *Phys. Rev. D* **1998**, *58*, 023503. [[CrossRef](#)]
34. Linde, A. *Particle Physics and Inflationary Cosmology*; Harwood: Chur, Switzerland, 1990.
35. Starobinsky, A. How to determine an effective potential for a variable cosmological term. *JETP Lett.* **1998**, *68*, 757. [[CrossRef](#)]
36. Saini, T.D.; Raychaudhury, S.; Sahni, V.; Starobinsky, A. Reconstructing the Cosmic Equation of State from Supernova Distances. *Phys. Rev. Lett.* **2000**, *85*, 1162. [[CrossRef](#)] [[PubMed](#)]
37. Chiba, T.; Okabe, T.; Yamaguchi, M. Kinetically driven quintessence. *Phys. Rev. D* **2000**, *62*, 023511. [[CrossRef](#)]
38. Armendariz-Picon, C.; Mukhanov, V.; Steinhardt, P.J. Dynamical Solution to the Problem of a Small Cosmological Constant and Late-Time Cosmic Acceleration. *Phys. Rev. Lett.* **2000**, *85*, 4438. [[CrossRef](#)]
39. Armendariz-Picon, C.; Mukhanov, V.; Steinhardt, P.J. Essentials of k-essence. *Phys. Rev. D* **2001**, *63*, 103510. [[CrossRef](#)]
40. Armendariz-Picon, C.; Damour, T.; Mukhanov, V. k-Inflation. *Phys. Lett. B* **1999**, *458*, 209. [[CrossRef](#)]
41. Garriga, J.; Mukhanov, V. Perturbations in k-inflation. *Phys. Lett. B* **1999**, *458*, 219. [[CrossRef](#)]
42. Caldwell, R.R. A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state. *Phys. Lett. B* **2002**, *545*, 23. [[CrossRef](#)]
43. Caldwell, R.R.; Kamionkowski, M.; Weinberg, N. Phantom Energy: Dark Energy with  $w < -1$  Causes a Cosmic Doomsday. *Phys. Rev. Lett.* **2003**, *91*, 071301.
44. Bronstein, M.P. On the expanding universe. *Phys. Z. Sowjetunion* **1933**, *3*, 73.
45. Bertolami, O. Brans-Dicke Cosmology with a Scalar Field Dependent Cosmological Term. *Fortschr. Phys.* **1986**, *34*, 829.
46. Bertolami, O. Time-dependent cosmological term. *Nuovo Cimento* **1986**, *93*, 36. [[CrossRef](#)]
47. Özer, M.; Taha, M.O. A model of the universe free of cosmological problems. *Nucl. Phys. B* **1987**, *287*, 776. [[CrossRef](#)]
48. Brax, P.; Martin, J. Quintessence and supergravity. *Phys. Lett. B* **1999**, *468*, 40. [[CrossRef](#)]
49. Steinhardt, P.J.; Wang, L.; Zlatev, I. Cosmological tracking solutions. *Phys. Rev. D* **1999**, *59*, 123504. [[CrossRef](#)]
50. Kamenshchik, A.; Moschella, U.; Pasquier, V. An alternative to quintessence. *Phys. Lett. B* **2001**, *511*, 265. [[CrossRef](#)]
51. Grishchuk, L.P. Density perturbations of quantum-mechanical origin and anisotropy of the microwave background. *Phys. Rev. D* **1994**, *50*, 7154. [[CrossRef](#)]
52. Makler, M.; Oliveira, S.Q.; Waga, I. Constraints on the generalized Chaplygin gas from supernovae observations. *Phys. Lett. B* **2003**, *555*, 1. [[CrossRef](#)]
53. Chaplygin, S. Gas jets. *Sci. Mem. Moscow Univ. Math. Phys.* **1904**, *21*, 1.
54. Tsien, H.S. Two-Dimensional Subsonic Flow of Compressible Fluids. *J. Aeron. Sci.* **1939**, *6*, 399. [[CrossRef](#)]
55. von Kármán, T. Compressibility Effects in Aerodynamics. *J. Aeron. Sci.* **1941**, *8*, 337. [[CrossRef](#)]
56. Stanyukovich, K. *Unsteady Motion of Continuous Media*; Pergamon: Oxford, UK, 1960.



57. Bazeia, D.; Jackiw, R. Nonlinear Realization of a Dynamical Poincaré Symmetry by a Field-Dependent Diffeomorphism. *Ann. Phys.* **1998**, *270*, 246. [[CrossRef](#)]
58. Jackiw, R.; Polychronakos, A.P. Fluid Dynamical Profiles and Constants of Motion from d-Branes. *Commun. Math. Phys.* **1999**, *207*, 107. [[CrossRef](#)]
59. Jackiw, R. *Lectures on Fluid Dynamics. A Particle Theorist's View of Supersymmetric, Non-Abelian, Noncommutative Fluid Mechanics and d-Branes*; Springer: New York, NY, USA, 2002.
60. Hassaïne, M.; Horváthy, P.A. Field-Dependent Symmetries of a Non-relativistic Fluid Model. *Ann. Phys.* **2000**, *282*, 218. [[CrossRef](#)]
61. Hoppe, J. Supermembranes in 4 Dimensions. *arXiv* **1993**, arXiv:hep-th/9311059.
62. Bazeia, D. Galileo invariant system and the motion of relativistic d-branes. *Phys. Rev. D* **1999**, *59*, 085007. [[CrossRef](#)]
63. Jackiw, R.; Polychronakos, A.P. Supersymmetric fluid mechanics. *Phys. Rev. D* **2000**, *62*, 085019. [[CrossRef](#)]
64. Bergner, Y.; Jackiw, R. Integrable supersymmetric fluid mechanics from superstrings. *Phys. Lett. A* **2000**, *284*, 146. [[CrossRef](#)]
65. Hassaïne, M. Supersymmetric Chaplygin gas. *Phys. Lett. A* **2001**, *290*, 157. [[CrossRef](#)]
66. Bordemann, M.; Hoppe, J. The dynamics of relativistic membranes. Reduction to 2-dimensional fluid dynamics. *Phys. Lett. B* **1993**, *317*, 315. [[CrossRef](#)]
67. Jevicki, A. Light-front partons and dimensional reduction in relativistic field theory. *Phys. Rev. D* **1998**, *57*, R5955. [[CrossRef](#)]
68. Ogawa, N. Remark on the classical solution of the Chaplygin gas as d-branes. *Phys. Rev. D* **2000**, *62*, 085023. [[CrossRef](#)]
69. Born, M.; Infeld, L. Foundations of the New Field Theory. *Proc. R. Soc. A* **1934**, *144*, 425. [[CrossRef](#)]
70. Hoppe, J. Some classical solutions of relativistic membrane equations in 4 space-time dimensions. *Phys. Lett. B* **1994**, *329*, 10. [[CrossRef](#)]
71. Bilic, N.; Tupper, G.B.; Viollier, R.D. Unification of dark matter and dark energy: the inhomogeneous Chaplygin gas. *Phys. Lett. B* **2002**, *535*, 17. [[CrossRef](#)]
72. Polchinski, J. *String Theory*; Cambridge University Press: Cambridge, UK, 1998.
73. Rubakov, V.A. Large and infinite extra dimensions. *Phys. Usp.* **2001**, *44*, 871. [[CrossRef](#)]
74. Nambu, Y. Lectures on the Copenhagen Summer Symposium 1970. *unpublished*.
75. Sundrum, R. Effective field theory for a three-brane universe. *Phys. Rev. D* **1999**, *59*, 085009. [[CrossRef](#)]
76. Garousi, M.R. Tachyon couplings on non-BPS D-branes and Dirac-Born-Infeld action. *Nucl. Phys. B* **2000**, *584*, 284. [[CrossRef](#)]
77. Sen, A. Rolling Tachyon. *J. High Energy Phys.* **2002**, *04*, 048. [[CrossRef](#)]
78. Sen, A. Tachyon Matter. *J. High Energy Phys.* **2002**, *07*, 065. [[CrossRef](#)]
79. Sen, A. Field Theory of Tachyon Matter. *Mod. Phys. Lett. A* **2002**, *17*, 1797. [[CrossRef](#)]
80. Gibbons, G. Cosmological evolution of the rolling tachyon. *Phys. Lett. B* **2002**, *537*, 1. [[CrossRef](#)]
81. Frolov, A.; Kofman, L.; Starobinsky, A. Prospects and problems of tachyon matter cosmology. *Phys. Lett. B* **2002**, *545*, 8. [[CrossRef](#)]
82. Felder, G.; Kofman, L.; Starobinsky, A. Caustics in Tachyon Matter and Other Born-Infeld Scalars. *JHEP* **2002**, *09*, 026. [[CrossRef](#)]
83. Padmanabhan, T. Accelerated expansion of the universe driven by tachyonic matter. *Phys. Rev. D* **2002**, *66*, 021301. [[CrossRef](#)]
84. Padmanabhan, T.; Choudhury, T.R. Can the clustered dark matter and the smooth dark energy arise from the same scalar field? *Phys. Rev. D* **2002**, *66*, 081301. [[CrossRef](#)]
85. Bagla, J.S.; Jassal, H.K.; Padmanabhan, T. Cosmology with tachyon field as dark energy. *Phys. Rev. D* **2003**, *67*, 063504. [[CrossRef](#)]
86. Gorini, V.; Kamenshchik, A.; Moschella, U.; Pasquier, V. Tachyons, scalar fields, and cosmology. *Phys. Rev. D* **2004**, *69*, 123512. [[CrossRef](#)]
87. Feinstein, A. Power-law inflation from the rolling tachyon. *Phys. Rev. D* **2002**, *66*, 063511. [[CrossRef](#)]
88. Bilic, N.; Tupper, G.B.; Viollier, R.D. Chaplygin gas cosmology-unification of dark matter and dark energy. *J. Phys. A* **2007**, *40*, 6877. [[CrossRef](#)]
89. Bilic, N.; Tupper, G.B.; Viollier, R.D. Cosmological tachyon condensation. *Phys. Rev. D* **2009**, *80*, 023515. [[CrossRef](#)]
90. Randall, L.; Sundrum, R. An Alternative to Compactification. *Phys. Rev. Lett.* **1999**, *83*, 4690. [[CrossRef](#)]
91. Banados, M.; Henneaux, M.; Teitelboim, C.; Zanelli, J. Geometry of the 2 + 1 black hole. *Phys. Rev. D* **1993**, *48*, 1506. [[CrossRef](#)]
92. Hawking, S.; Page, D. Thermodynamics of black holes in anti-de Sitter space. *Commun. Math. Phys.* **1983**, *87*, 577. [[CrossRef](#)]
93. Kamenshchik, A.; Moschella, U.; Pasquier, V. Chaplygin-like gas and branes in black hole bulks. *Phys. Lett. B* **2000**, *487*, 7. [[CrossRef](#)]
94. Gorini, V.; Kamenshchik, A.; Moschella, U. Can the Chaplygin gas be a plausible model for dark energy? *Phys. Rev. D* **2003**, *67*, 063509. [[CrossRef](#)]
95. Bento, M.C.; Bertolami, O.; Sen, A.A. Generalized Chaplygin gas, accelerated expansion, and dark-energy-matter unification. *Phys. Rev. D* **2002**, *66*, 043507. [[CrossRef](#)]
96. Benaoum, H.B. Accelerated Universe from Modified Chaplygin Gas and Tachyonic Fluid. *arXiv* **2002**, arXiv:hep-th/0205140.
97. Chavanis, P.H. Models of universe with a polytropic equation of state : I. The early universe. *Eur. Phys. J. Plus* **2014**, *129*, 38. [[CrossRef](#)]
98. Chavanis, P.H. Models of universe with a polytropic equation of state : II. The late universe. *Eur. Phys. J. Plus* **2014**, *129*, 222. [[CrossRef](#)]
99. Chavanis, P.H. Models of universe with a polytropic equation of state: III. The phantom universe. *arXiv* **2012**, arXiv:1208.1185.
100. Bilic, N.; Lindebaum, R.J.; Tupper, G.B.; Viollier, R.D. Nonlinear evolution of dark matter and dark energy in the Chaplygin-gas cosmology. *J. Cosmol. Astropart. Phys.* **2004**, *11*, 008. [[CrossRef](#)]

101. Neves, R.; Vas, C. Brane dynamics, the polytropic gas and conformal bulk fields. *Phys. Lett. B* **2003**, *568*, 153. [[CrossRef](#)]
102. Avelino, P.P.; Beca, L.M.G.; de Carvalho, J.P.M.; Martins, C.J.A.P. The  $\Lambda$ CDM limit of the generalized Chaplygin gas scenario. *J. Cosmol. Astropart. Phys.* **2003**, *9*, 2. [[CrossRef](#)]
103. Sandvik, H.B.; Tegmark, M.; Zaldarriaga, M.; Waga, I. The end of unified dark matter? *Phys. Rev. D* **2004**, *69*, 123524. [[CrossRef](#)]
104. Fabris, J.C.; Gonçalves, S.V.B.; de Sá Ribeiro, R. Generalized Chaplygin Gas with  $\alpha = 0$  and the  $\Lambda$ CDM Cosmological Model. *Gen. Relat. Grav.* **2004**, *36*, 211. [[CrossRef](#)]
105. Bento, M.C.; Bertolami, O.; Sen, A.A. WMAP constraints on the generalized Chaplygin gas model. *Phys. Lett. B* **2003**, *575*, 172. [[CrossRef](#)]
106. Bento, M.C.; Bertolami, O.; Sen, A.A. Generalized Chaplygin gas and cosmic microwave background radiation constraints. *Phys. Rev. D* **2003**, *67*, 063003. [[CrossRef](#)]
107. Bento, M.C.; Bertolami, O.; Sen, A.A. Generalized Chaplygin Gas Model: Dark Energy-Dark Matter Unification and CMBR Constraints. *Gen. Relat. Grav.* **2003**, *35*, 2063. [[CrossRef](#)]
108. Carturan, D.; Finelli, F. Cosmological effects of a class of fluid dark energy models. *Phys. Rev. D* **2003**, *68*, 103501. [[CrossRef](#)]
109. Amendola, L.; Finelli, F.; Burigana, C.; Carturan, D. WMAP and the generalized Chaplygin gas. *JCAP* **2003**, *7*, 5. [[CrossRef](#)]
110. Fabris, J.C.; Gonçalves, S.V.B.; de Souza, P.E. Fitting the Supernova Type Ia Data with the Chaplygin Gas. *arXiv* **2002**, arXiv:astro-ph/0207430.
111. Dev, A.; Alcaniz, J.S.; Jain, D. Cosmological consequences of a Chaplygin gas dark energy. *Phys. Rev. D* **2003**, *67*, 023515. [[CrossRef](#)]
112. Alcaniz, J.S.; Jain, D.; Dev, A. High-redshift objects and the generalized Chaplygin gas. *Phys. Rev. D* **2003**, *67*, 043514. [[CrossRef](#)]
113. Colistete, R.; Fabris, J.C. Gonçalves, S.V.B.; de Souza, P.E. Bayesian Analysis of the Chaplygin Gas and Cosmological Constant Models Using the SNe Ia Data. *Int. J. Mod. Phys. D* **2004**, *13*, 669. [[CrossRef](#)]
114. Silva, P.T.; Bertolami, O. Expected Constraints on the Generalized Chaplygin Equation of State from Future Supernova Experiments and Gravitational Lensing Statistics. *Astrophys. J.* **2003**, *599*, 829.
115. Bean, R.; Doré, O. Are Chaplygin gases serious contenders for the dark energy? *Phys. Rev. D* **2003**, *68*, 023515. [[CrossRef](#)]
116. Chavanis, P.H. New logotropic model based on a complex scalar field with a logarithmic potential. *Phys. Rev. D* **2022**, *106*, 063525. [[CrossRef](#)]
117. Abdullah, A.; El-Zant, A.A.; Ellithi, A. The growth of fluctuations in Chaplygin gas cosmologies: A nonlinear Jeans scale for unified dark matter. *arXiv* **2021**, arXiv:2108.03260.
118. Chavanis, P.H. Is the Universe logotropic? *Eur. Phys. J. Plus* **2015**, *130*, 130. [[CrossRef](#)]
119. Chavanis, P.H. The Logotropic Dark Fluid as a unification of dark matter and dark energy. *Phys. Lett. B* **2016**, *758*, 59. [[CrossRef](#)]
120. Chavanis, P.H.; Kumar, S. Comparison between the Logotropic and  $\Lambda$ CDM models at the cosmological scale. *J. Cosmol. Astropart. Phys.* **2017**, *5*, 18. [[CrossRef](#)]
121. Chavanis, P.H. New predictions from the logotropic model. *Phys. Dark Univ.* **2019**, *24*, 100271. [[CrossRef](#)]
122. Chavanis, P.H. Predictions from the logotropic model: The universal surface density of dark matter halos and the present proportions of dark matter and dark energy. *Phys. Dark Univ.* **2022**, *37*, 101098. [[CrossRef](#)]
123. Donato, F.; Gentile, G.; Salucci, P.; Frigerio Martins, C.; Wilkinson, M. I.; Gilmore, G.; Grebel, E. K.; Koch, A.; Wyse, R. A constant dark matter halo surface density in galaxies. *Mon. Not. R. Astron. Soc.* **2009**, *397*, 1169. [[CrossRef](#)]
124. Ferreira, V.M.C.; Avelino, P.P. New limit on logotropic unified dark energy models. *Phys. Lett. B* **2017**, *770*, 213. [[CrossRef](#)]
125. Chavanis, P.H. Laboratoire de Physique Théorique, Université de Toulouse, CNRS, UPS, Toulouse, France. *in preparation*.
126. Chavanis, P.H. Mass-radius relation of Newtonian self-gravitating Bose-Einstein condensates with short-range interactions. I. Analytical results. *Phys. Rev. D* **2011**, *84*, 043531. [[CrossRef](#)]
127. Gross, E.P. Classical theory of boson wave fields. *Ann. Phys.* **1958**, *4*, 57. [[CrossRef](#)]
128. Gross, E.P. Structure of a quantized vortex in boson systems. *Nuovo Cimento* **1961**, *20*, 454. [[CrossRef](#)]
129. Gross, E.P. Hydrodynamics of a Superfluid Condensate. *J. Math. Phys.* **1963**, *4*, 195. [[CrossRef](#)]
130. Pitaevskii, L.P. Vortex Lines in an Imperfect Bose Gas. *Sov. Phys. JETP* **1961**, *13*, 451.
131. Suárez, A.; Chavanis, P.H. Hydrodynamic representation of the Klein-Gordon-Einstein equations in the weak field limit. *J. Phys. Conf. Ser.* **2015**, *654*, 012008. [[CrossRef](#)]
132. Chavanis, P.H.; Matos, T. Covariant theory of Bose-Einstein condensates in curved spacetimes with electromagnetic interactions: The hydrodynamic approach. *Eur. Phys. J. Plus* **2017**, *132*, 30. [[CrossRef](#)]
133. Chavanis, P.H. Phase transitions between dilute and dense axion stars. *Phys. Rev. D* **2018**, *98*, 023009. [[CrossRef](#)]
134. Chavanis, P.H. Quantum tunneling rate of dilute axion stars close to the maximum mass. *Phys. Rev. D* **2020**, *102*, 083531. [[CrossRef](#)]
135. Madelung, E. Quantentheorie in hydrodynamischer Form. *Zeit. F. Phys.* **1927**, *40*, 322. [[CrossRef](#)]
136. Chavanis, P.H. Dissipative self-gravitating Bose-Einstein condensates with arbitrary nonlinearity as a model of dark matter halos. *Eur. Phys. J. Plus* **2017**, *132*, 248. [[CrossRef](#)]
137. Eckart, C. The Electrodynamics of Material Media. *Phys. Rev.* **1938**, *54*, 920. [[CrossRef](#)]
138. Schakel, A. Effective Field Theory of Ideal-Fluid Hydrodynamics. *Mod. Phys. Lett. B* **1996**, *10*, 999. [[CrossRef](#)]
139. Chandrasekhar, S. *An Introduction to the Theory of Stellar Structure*; University of Chicago Press: Chicago, IL, USA, 1939.
140. Pethick, C.J.; Smith, H. *Bose-Einstein Condensation in Dilute Gases*; Cambridge University Press: Cambridge, UK, 2002.

141. Ferreira, E.G.M. Ultra-light dark matter. *Astron. Astrophys. Rev.* **2021**, *29*, 7. [[CrossRef](#)]
142. Bekenstein, J.; Milgrom, M. Does the missing mass problem signal the breakdown of Newtonian gravity? *Astrophys. J.* **1984**, *286*, 7. [[CrossRef](#)]
143. Chavanis, P.H.; Sire, C. Logotropic distributions *Physica A* **2007**, *375*, 140. [[CrossRef](#)]
144. Noether, E. Invariante Variationsprobleme. *Nachr. D. Königl. Gesellsch. D. Wiss. Zu Göttingen, Math-phys. Klasse* **1918**, *40*, 235–237.
145. Landau, L.D.; Lifshitz, E.M. *Fluid Mechanics*; Pergamon Press: London, UK, 1959.
146. de Broglie, L. La mécanique ondulatoire et la structure atomique de la matière et du rayonnement. *J. Phys.* **1927**, *8*, 225. [[CrossRef](#)]
147. de Broglie, L. Sur le rôle des ondes continues  $\Psi$  en mécanique ondulatoire. *Compt. Rend. Acad. Sci. Paris* **1927**, *185*, 380.
148. de Broglie, L. Corpuscules et ondes  $\Psi$ . *Compt. Rend. Acad. Sci. Paris* **1927**, *185*, 1118.
149. Suárez, A.; Chavanis, P.H. Cosmological evolution of a complex scalar field with repulsive or attractive self-interaction. *Phys. Rev. D* **2017**, *95*, 063515. [[CrossRef](#)]
150. Suárez, A.; Chavanis, P.H. Hydrodynamic representation of the Klein-Gordon-Einstein equations in the weak field limit: General formalism and perturbations analysis. *Phys. Rev. D* **2015**, *92*, 023510. [[CrossRef](#)]
151. Tooper, R.F. General Relativistic Polytropic Fluid Spheres. *Astrophys. J.* **1964**, *140*, 434. [[CrossRef](#)]
152. Colpi, M.; Shapiro, S.L.; Wasserman, I. Boson stars: Gravitational equilibria of self-interacting scalar fields. *Phys. Rev. Lett.* **1986**, *57*, 2485. [[CrossRef](#)] [[PubMed](#)]
153. Li, B.; Rindler-Daller, T.; Shapiro, P.R. Cosmological constraints on Bose-Einstein-condensed scalar field dark matter. *Phys. Rev. D* **2014**, *89*, 083536. [[CrossRef](#)]
154. Chavanis, P.H. Partially relativistic self-gravitating Bose-Einstein condensates with a stiff equation of state. *Eur. Phys. J. Plus* **2015**, *130*, 181. [[CrossRef](#)]
155. Chavanis, P.H.; Harko, T. Bose-Einstein condensate general relativistic stars. *Phys. Rev. D* **2012**, *86*, 064011. [[CrossRef](#)]
156. Chavanis, P.H. Cosmological models based on a complex scalar field with a power-law potential associated with a polytropic equation of state. *Phys. Rev. D* **2022**, *106*, 043502. [[CrossRef](#)]
157. Chandrasekhar, S. A limiting case of relativistic equilibrium. In *General Relativity; Papers in Honour of Synge, J.L.; O' Raifeartaigh, L., Ed.*; Oxford University Press: Oxford, UK, 1972.
158. Yabushita, S. Pulsational instability of isothermal gas spheres within the frame-work of general relativity. *Mon. Not. R. Astron. Soc.* **1973**, *165*, 17. [[CrossRef](#)]
159. Yabushita, S. On the analogy between neutron star models and isothermal gas spheres and their general relativistic instability. *Mon. Not. R. Astron. Soc.* **1974**, *167*, 95. [[CrossRef](#)]
160. Chavanis, P.H. Gravitational instability of finite isothermal spheres in general relativity. Analogy with neutron stars. *Astron. Astrophys.* **2002**, *381*, 709. [[CrossRef](#)]
161. Chavanis, P.H. Relativistic stars with a linear equation of state: analogy with classical isothermal spheres and black holes. *Astron. Astrophys.* **2008**, *483*, 673. [[CrossRef](#)]
162. Tooper, R.F. Adiabatic Fluid Spheres in General Relativity. *Astrophys. J.* **1965**, *142*, 1541. [[CrossRef](#)]
163. Chavanis, P.H. Cosmology with a stiff matter era. *Phys. Rev. D* **2015**, *92*, 103004. [[CrossRef](#)]
164. Banerjee, R.; Ghosh, S.; Kulkarni, S. Remarks on the generalized Chaplygin gas. *Phys. Rev. D* **2007**, *75*, 025008. [[CrossRef](#)]
165. Chavanis, P.H. Laboratoire de Physique Théorique, Université de Toulouse, CNRS, Toulouse, France. *in preparation*.
166. Chavanis, P.H. A simple model of universe describing the early inflation and the late accelerated expansion in a symmetric manner. *AIP Conf. Proc.* **2013**, *1548*, 75.
167. Chavanis, P.H. A Cosmological model based on a quadratic equation of state unifying vacuum energy, radiation, and dark energy. *J. Gravity* **2013**, *2013*, 682451. [[CrossRef](#)]
168. Chavanis, P.H. A Cosmological Model Describing the Early Inflation, the Intermediate Decelerating Expansion, and the Late Accelerating Expansion of the Universe by a Quadratic Equation of State. *Universe* **2015**, *1*, 357. [[CrossRef](#)]
169. Chavanis, P.H. A simple model of universe with a polytropic equation of state. *J. Phys. Conf. Ser.* **2018**, *1030*, 012009. [[CrossRef](#)]
170. Chavanis, P.H. A Real Scalar Field Unifying the Early Inflation and the Late Accelerating Expansion of the Universe through a Quadratic Equation of State: The Vacuumon. *Universe* **2022**, *8*, 92. [[CrossRef](#)]
171. Chavanis, P.H.; Lemou, M.; Méhats, F. Models of dark matter halos based on statistical mechanics: The classical King model. *Phys. Rev. D* **2015**, *91*, 063531. [[CrossRef](#)]
172. Chavanis, P.H.; Lemou, M.; Méhats, F. Models of dark matter halos based on statistical mechanics: The fermionic King model. *Phys. Rev. D* **2015**, *92*, 123527. [[CrossRef](#)]
173. Chavanis, P.H. Predictive model of BEC dark matter halos with a solitonic core and an isothermal atmosphere. *Phys. Rev. D* **2019**, *100*, 083022. [[CrossRef](#)]
174. Kaup, D.J. Klein-Gordon Geon. *Phys. Rev.* **1968**, *172*, 1331. [[CrossRef](#)]
175. Ruffini, R.; Bonazzola, S. Systems of Self-Gravitating Particles in General Relativity and the Concept of an Equation of State. *Phys. Rev.* **1969**, *187*, 1767. [[CrossRef](#)]
176. Braaten, E.; Zhang, H. The physics of axion stars. *Rev. Mod. Phys.* **2019**, *91*, 041002. [[CrossRef](#)]
177. Hui, L.; Ostriker, J.; Tremaine, S.; Witten, E. Ultralight scalars as cosmological dark matter. *Phys. Rev. D* **2017**, *95*, 043541. [[CrossRef](#)]

178. Chavanis, P.H. Statistical mechanics of self-gravitating systems in general relativity : I. The quantum Fermi gas. *Eur. Phys. J. Plus* **2020**, *135*, 290. [[CrossRef](#)]
179. Mukhanov, V.; Vikman, A. Enhancing the tensor-to-scalar ratio in simple inflation. *J. Cosmol. Astropart. Phys.* **2006**, *2*, 4. [[CrossRef](#)]
180. Babichev, E. Global topological k-defects. *Phys. Rev. D* **2006**, *74*, 085004. [[CrossRef](#)]
181. Fang, W.; Lu, H.Q.; Huang, Z.G. Cosmologies with a general non-canonical scalar field. *Class. Quantum Grav.* **2007**, *24*, 3799. [[CrossRef](#)]
182. Mukhanov, V. *Physical Foundations of Cosmology*; Cambridge University Press: Cambridge, UK, 2012.
183. Joyce, M. Electroweak baryogenesis and the expansion rate of the Universe. *Phys. Rev. D* **1997**, *55*, 1875. [[CrossRef](#)]
184. Gorini, V.; Kamenshchik, A.; Moschella, U.; Pasquier, V.; Starobinsky, A. Stability properties of some perfect fluid cosmological models. *Phys. Rev. D* **2005**, *72*, 103518. [[CrossRef](#)]
185. Chimento, L.P. Extended tachyon field, Chaplygin gas, and solvable k-essence cosmologies. *Phys. Rev. D* **2004**, *69*, 123517. [[CrossRef](#)]
186. Scherrer, R.J. Purely Kinetic k Essence as Unified Dark Matter. *Phys. Rev. Lett.* **2004**, *93*, 011301. [[CrossRef](#)]
187. Ellis, G.F.R.; Madsen, M.S. Exact scalar field cosmologies. *Class. Quantum Grav.* **1991**, *8*, 667. [[CrossRef](#)]
188. Copeland, E.J.; Sami, M.; Tsujikawa, S. Dynamics of Dark Energy. *Int. J. Mod. Phys. D* **2006**, *15*, 1753. [[CrossRef](#)]
189. Bamba, K.; Capozziello, S.; Nojiri, S.; Odintsov, S.D. Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests. *Astrophys. Space Sci.* **2012**, *342*, 155. [[CrossRef](#)]
190. Halliwell, J.J. Scalar fields in cosmology with an exponential potential. *Phys. Lett. B* **1987**, *185*, 341. [[CrossRef](#)]
191. Lucchin, F.; Matarrese, S. Power-law inflation. *Phys. Rev. D* **1985**, *32*, 1316. [[CrossRef](#)]
192. Arbey, A.; Lesgourgues, J.; Salati, P. Cosmological constraints on quintessential halos. *Phys. Rev. D* **2002**, *65*, 083514. [[CrossRef](#)]
193. Gu, J.-A.; Hwang, W.-Y.P. Can the quintessence be a complex scalar field? *Phys. Lett. B* **2001**, *517*, 1. [[CrossRef](#)]
194. Kasuya, S. Difficulty of a spinning complex scalar field to be dark energy. *Phys. Lett. B* **2001**, *515*, 121. [[CrossRef](#)]
195. Boyle, L.A.; Caldwell, R.R.; Kamionkowski, M. Spintessence! New models for dark matter and dark energy. *Phys. Lett. B* **2002**, *545*, 17. [[CrossRef](#)]
196. Lane, H.J. On the Theoretical Temperature of the Sun; under the Hypothesis of a Gaseous Mass maintaining its Volume by its Internal Heat, and depending on the Laws of Gases as known to Terrestrial Experiment. *Am. J. Sci.* **1870**, *50*, 57. [[CrossRef](#)]
197. Zöllner, E. *Über die stabilität kosmischer Massen*; Universitätsbibliothek Leipzig: Leipzig, Germany, 1871.
198. Emden, R. *Gaskugeln*; B.G. Teubne: Leipzig, Germany, 1907.
199. McLaughlin, D.; Pudritz, R. A Model for the Internal Structure of Molecular Cloud Cores. *Astrophys. J.* **1996**, *469*, 194. [[CrossRef](#)]
200. Latifah, S.; Sulaksono, A.; Mart, T. Bosons star at finite temperature. *Phys. Rev. D* **2014**, *90*, 127501. [[CrossRef](#)]