Proceeding Paper

Calculation of the Vacuum Energy Density Using Zeta Function Regularization [†]

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Abstract: This paper presents a theoretical calculation of the vacuum energy density by summing the contributions of all quantum fields' vacuum states, which turns out to indicate that there seems to be a missing bosonic contribution in order to match the predictions of the current cosmological models and all observational data to date. The basis for this calculation is a new Zeta function regularization method used to tame the infinities present in the improper integrals of power functions. The paper also presents a few other contributions in the area of vacuum energy.

Keywords: vacuum energy; zero-point energy; zeta function regularization; riemman zeta function; divergent integrals; infinities; quantum harmonic oscillator; cosmological constant problem

1. Introduction

Much has been written on the Cosmological Constant (CC) problem and the vacuum catastrophe where we have the worst discrepancy between the theory and measurement of about a factor of more than 120 orders of magnitude; refer to [1,2] for a pedagogical overview. In this paper, we improve this discrepancy by more than half, which still remains very high, and propose four solutions, with one being an exciting prediction that can help resolve the vacuum catastrophe. We attack the heart of the problem, which is the divergent sums and integrals that have also plagued many other areas of Quantum Field Theory (QFT), using, as our tool, a new Zeta regularization technique to tame these infinities, without the use of an arbitrary momentum cutoff or renormalization parameter. Refer to [3] for a pedagogical overview of the Zeta function and the associated Zeta function regularization techniques with applications to physics.

2. Methods

The vacuum (zero-point) energy of a free (non-interacting) quantum field in flat Minkowski spacetime can be modeled as the sum of the zero-point energies of a set of an infinite number of quantum harmonic oscillators, with one for each normal mode given simply as $+\sum_k \frac{1}{2}\hbar\omega_k$ for bosonic fields and $-\sum_k \frac{1}{2}\hbar\omega_k$ for fermionic fields [4,5]. $\frac{1}{2}\hbar\omega_k$ represents the eigenvalues of the free Hamiltonian, $\omega_k = \sqrt{m^2 + k^2}$ in natural units, k is the wave number (with units of $1/\text{length}$), m is the particle mass associated with a specific field (assumed to be constant and not running), and $\hbar$ is the reduced Planck constant. The negative sign, in front of the infinite sum, is due to the negative energy solutions allowed in the fermionic fields that must follow the Pauli Exclusion principle by obeying the canonical anti-commutation relations.

The vacuum energy density (VED) of a single state of a given field (i.e., energy E per volume V) can be derived [4,5] (Section 10.8) to be

$$\rho = \frac{E}{V} = \pm\frac{1}{V}\sum_k \frac{1}{2}\hbar\omega_k = \cdots = \pm\frac{1}{4\pi^2}\int_0^\infty \sqrt{m^2 + k^2}\,k^2\,dk \tag{1}$$

and by reintroducing $\hbar$ and c in (1) and, considering all the states of a field, the VED of a field is given by

$$\rho = g \frac{1}{4\pi^2} \int_0^\infty \sqrt{(mc^2)^2 + (\hbar kc)^2} k^2 dk \tag{2}$$

where g is the degeneracy factor that includes a sign factor $(-1)^{2j}$, a spin factor $2j + 1$ for massive fields and 2 for massless fields, a factor of 3 for fields with a color charge, and a factor of 2 for fields that have an antiparticle distinct from their particle. Note also that c is the speed of light in the vacuum and j is the particle spin.

The most straightforward way of regularizing the integral in (1) and (2) is to introduce an Ultraviolet (UV) momentum cutoff K after which the integral can be evaluated to be

$$\int_0^K \sqrt{m^2 + k^2} k^2 dk = \frac{K^4}{4} + \frac{m^2 K^2}{4} + \frac{m^4}{32} - \frac{m^4}{16} \log\left(\frac{m^2}{4K^2}\right) + O\left(\frac{1}{K^2}\right) \tag{3}$$

Now, by imposing the UV cutoff at the Planck scale, $K = K_{PL} = \sqrt{\hbar c / G} \approx 10^{19}$ GeV (where G is the gravitational constant), the dominant term in (3), which is quartic in the momentum cutoff, provides a value of order 10^{76} GeV^4, which, when compared with the observational value of order 10^{-47} GeV^4 [6], is off by 123 orders of magnitude! This is the origin of the famous ≈ 120 order of magnitude discrepancy cited in the literature [1,2,5].

It is to be noted that the regularization in (3) was first introduced by Pauli [4] who also provided the corresponding 3 polynomial-in-mass and 1 logarithmic-in-mass conditions needed to be met in order to obtain a vanishing VED:

$$\sum_i g_i m_i^4 = 0 \qquad \sum_i g_i m_i^2 = 0 \qquad \sum_i g_i = 0 \qquad \sum_i g_i m_i^4 \log\left(\frac{m_i^2}{\mu^2}\right) = 0 \tag{4}$$

where index i represents different particle species and μ is an arbitrary parameter. We will show that, using our zeta regularization method, we can simplify this by eliminating 3 of the conditions, leaving only the key quartic polynomial-in-mass condition.

We begin our approach by writing Equation (2) as

$$\rho = g \frac{\hbar c}{4\pi^2} \int_0^\infty \sqrt{\left(\frac{mc}{\hbar}\right)^2 + k^2} k^2 dk = b \int_0^\infty \sqrt{a^2 + k^2} k^2 dk = b \int_0^\infty a\sqrt{1 + \left(\frac{k}{a}\right)^2} k^2 dk \tag{5}$$

where $a = \frac{mc}{\hbar}$ (with units of 1/length) and $b = g\frac{\hbar c}{4\pi^2}$. Now, by introducing a dimensionless change in variable, $x = k/a$ and $dx = dk/a$, we can turn the integral in (5) into a 'pure' integral and, hence, ready for Zeta function regularization:

$$\rho = b \int_0^\infty a\sqrt{1 + x^2}(ax)^2 a\, dx = a^4 b \int_0^\infty \sqrt{1 + x^2} x^2 dx \tag{6}$$

In order to apply the Zeta regularization results obtained in Appendix A, we need to perform a Maclaurin series expansion of the integrand in Equation (6) with respect to the dimensionless parameter x. Let us define

$$f(x) = x^2\sqrt{1 + x^2} \tag{7}$$

Now, the Maclaurin series expansion of (7) can be obtained as

$$f(x) = x^2 + \frac{x^4}{2} - \frac{x^6}{8} + \frac{x^8}{16} - \frac{5x^{10}}{128} + \frac{7x^{12}}{256} - \frac{21x^{14}}{1024} + O(x^{16}) \tag{8}$$

At first glance, when substituting (8) as the integrand in (6), the integral in (6) looks highly divergent and is essentially a sum of divergent, even powered, polynomial integrals. However, the finite values for these integrals can be assigned using a new Zeta regularization technique (see Appendix A). These integral values are $\int_0^\infty x^2 dx = -\frac{1}{12}$,

$\int_0^\infty x^4 dx = -\frac{1}{30}$, $\int_0^\infty x^6 dx = -\frac{1}{56}$, $\int_0^\infty x^8 dx = -\frac{1}{90}$, $\int_0^\infty x^{10} dx = -\frac{1}{132}$, $\int_0^\infty x^{12} dx = -\frac{1}{182}$, etc. A finite zeta-regularized value can hence be obtained for the divergent integral in (6) from the converging series below calculated up to the first 15 terms

$$\int_0^\infty \sqrt{1+x^2}\,x^2 dx = -\frac{1}{12} - \frac{1}{2}\frac{1}{30} + \frac{1}{8}\frac{1}{56} - \frac{1}{16}\frac{1}{90} + \frac{5}{128}\frac{1}{132} - \frac{7}{256}\frac{1}{182} + \cdots + \frac{185725}{33554432}\frac{1}{992} = -0.09826 \tag{9}$$

so we can obtain the following simple expression for (6):

$$\rho = -0.09826\,a^4 b = -g\frac{0.09826\,m^4 c^5}{4\pi^2\hbar^3} = -g\frac{0.002489\,m^4 c^5}{\hbar^3} \equiv -0.002489\,g\,m^4 \quad (Natural\ Units) \tag{10}$$

Now, the sign of the vacuum energy density in (2) is represented by the $(-1)^{2j}$ sign factor that dictates that the vacuum energy contribution is positive for the bosonic fields (i.e., $j = 1$) and negative for fermionic fields (i.e., $j = 1/2$), but only if one assumes that the integral in (2) is a positive number tending to plus infinity (i.e., we obtain a positive value for the integral if we use a UV cut-off, which is what is typically done in the literature). However, this is not what we obtain using our Zeta regularization technique where we find and assign a *negative* value to the otherwise divergent integral $\int_0^\infty \sqrt{1+x^2}\,x^2 dx$ in (6); by doing so, we reverse the vacuum energy density signs for the fermionic fields (positive) and bosonic fields (negative). This is indeed an unexpected turn of events!

3. Results

The vacuum energy density contributions of the relativistic massless photon and gluon particles are considered to be zero and, hence, will not be further considered in the calculations that follow. This can also be concluded from the equation of state for a relativistic particle [7] (page 8). The total theoretical value of the vacuum energy density can be obtained by summing over the free massive quantum fields of the Standard Model (represented by the index i), with their associated particle masses [8]:

$$\rho_{total} = \sum_i \rho_i = -0.002489 \sum_i g_i m_i^4 \tag{11}$$

from which we can clearly conclude now that the only condition required to have a vanishing ρ_{total} is the first one in (4), $\sum_i g_i m_i^4 = 0$.

The vacuum energy density values are tabulated below where the leptons' and quarks' masses (6 fields each) are lumped together. We note that the total theoretical value of the vacuum energy density *does not diverge* and is positive but is still quite large. In fact, we obtain a positive value of $\rho_{total} = 5.18 \times 10^{44}$ J/m^3 (= 2.49×10^7 GeV^4) and a corresponding CC value of $\Lambda = 4.17 \times 10^{-12}$ eV^2 using $\rho_{vac} = \frac{\Lambda}{8\pi G}$, [2] (Equation (14)).

This result is a significant improvement over the widely reported theoretical estimate that is about 120 orders of magnitude higher compared with the value obtained from the large-scale cosmological observations of approximately 5.26×10^{-10} J/m^3 = 2.52×10^{-47} GeV^4 ($\Lambda = (4.24 \pm 0.11) \times 10^{-66}$ eV^2) [6].

Quantum Fields	Degeneracy Factor g sign × spin × color × antiparticle	Mass (eV/c^2)	ρ (J/m^3)	ρ (GeV^4)
Quarks (×6): fermionic	$-1 \times (2 \times 1/2 + 1) \times 3 \times 2 = -12$	178.31×10^9	$+5.54 \times 10^{44}$	$+2.66 \times 10^7$
Leptons (×6): fermionic	$-1 \times (2 \times 1/2 + 1) \times 1 \times 2 = -4$	1.90×10^9	$+2.07 \times 10^{36}$	$+0.1$
W: bosonic	$1 \times (2 \times 1 + 1) \times 1 \times 2 = 6$	80.38×10^9	-1.30×10^{43}	-6.23×10^5
Z: bosonic	$1 \times (2 \times 1 + 1) \times 1 \times 1 = 3$	91.19×10^9	-1.08×10^{43}	-5.16×10^5
Higgs: bosonic	$1 \times (2 \times 0 + 1) \times 1 \times 1 = 1$	125.10×10^9	-1.27×10^{43}	-6.10×10^5
Total			$+5.18 \times 10^{44}$	$+2.49 \times 10^7$

4. Conclusions

In summary, we highlight that the results presented in this paper help improve the vacuum catastrophe or CC problem by reducing the widely reported discrepancy of more than 120 orders of magnitude to about 55 without requiring the introduction of any arbitrary cutoff or renormalization parameter as is typically performed in the literature. Therefore, this approach allows for the contribution of all particle momentum from 0 to infinity that one could argue will maintain any symmetry that may be present in the dynamics of VED and which would be otherwise lost when an arbitrary parameter is introduced. It is, however, clear that there is not yet a convincing resolution to this long standing CC problem. Of course, the VED contributions would cancel out to zero in a universe with exact supersymmetry because of the opposite signs between the boson and fermion contributions in (1). However, there is no evidence of any such supersymmetry in our universe and no hint of any supersymmetric particle at the energy scales probed so far at the LHC.

We do propose four possible reasons to explain this new discrepancy, which exists between our theoretical result and the observational value, as being wholly or partly due to:

(1) After carefully considering the order of magnitude of our newly calculated VED (10^{44} J/m^3 or 10^7 GeV^4), we make a startling observation that we have obtained a value in the right energy scale, which is suggestive of at least one missing heavy boson (e.g., a heavy cousin of the Higgs boson). For example, if we assume a scalar boson (with $g = 1$), then, using (10), its mass would be about **316** GeV/c^2. If we assume a vector boson (e.g., say $g = 3$ for a Z-prime boson), then its mass would be about **240** GeV/c^2. Of course, it is entirely possible that we are also missing a massive fermionic (dark matter) particle that, if detected and of consequential mass, would further push up the mass of the missing boson;
(2) The contribution of the gravitational field or the effect of spacetime curvature on VED;
(3) The effect of interacting quantum fields on VED;
(4) The effect of the running mass of particles on VED.

In conclusion, we list the key contributions of the paper:

(1) VED is finite without the need of an arbitrary UV momentum cut-off or a renormalization parameter, and is found to have a simple closed-form (10) quartic in particle mass;
(2) Only one of Pauli's four conditions is necessary and sufficient to zero out the VED;
(3) The vacuum energy contributions of the fermionic fields are positive and those of bosonic fields are negative.

Appendix A. Improper Integrals of Power Functions

Let us define the μ function as the improper integral of a power function with exponent p

$$\mu(p) = \int_0^\infty x^p dx \tag{A1}$$

with $p \in \mathbb{N}^0$, where $\mathbb{N}^0 = \{0, 1, 2, \cdots\}$.

The μ function can be interpreted as the natural extension of the Riemann zeta function where the discrete sum is replaced by a continuous integral. The above improper integral

can be equivalently written as the following infinite series by splitting the limits of the integration into successive integer numbers. That is,

$$\mu(p) = \sum_{n=1}^{\infty} \Delta(p, n) \tag{A2a}$$

where

$$\Delta(p, n) = \int_{n-1}^{n} x^p dx = \frac{1}{p+1}\left(n^{p+1} - (n-1)^{p+1}\right) \tag{A2b}$$

is the definite integral of the power function over limits $n - 1$ and n. In what follows, we will show that the divergent series in the RHS of (A2a) can converge to a finite result.

One can write the binomial polynomial expansion of $(n - 1)^{p+1}$ in (A2b) as [9] (Equation (2))

$$(n-1)^{p+1} = \sum_{k=0}^{p+1} \binom{p+1}{k} n^k (-1)^{p+1-k}, \tag{A3a}$$

where

$$\binom{t}{s} = \frac{t!}{s!(t-s)!} \tag{A3b}$$

is the binomial coefficient, with $\binom{t}{t} = \binom{t}{0} = 1$. Substituting (A3a) into (A2b) provides

$$\begin{aligned}
\Delta(p, n) &= \frac{1}{p+1}\left(n^{p+1} - \sum_{k=0}^{p+1} \binom{p+1}{k} n^k (-1)^{p+1-k}\right) \\
&= \frac{1}{p+1}\left(n^{p+1} - \sum_{k=0}^{p} \binom{p+1}{k} n^k (-1)^{p-k}(-1)^1 - \binom{p+1}{p+1} n^{p+1}(-1)^{p+1-(p+1)}\right) \\
&= \frac{1}{p+1}\sum_{k=0}^{p} \binom{p+1}{k} n^k (-1)^{p-k}
\end{aligned} \tag{A4}$$

Thus, (A2a) can be written as

$$\mu(p) = \frac{1}{p+1}\sum_{n=1}^{\infty}\sum_{k=0}^{p} \binom{p+1}{k}(-1)^{p-k} n^k = \frac{1}{p+1}\left(\sum_{k=0}^{p} \binom{p+1}{k}(-1)^{p-k}\sum_{n=1}^{\infty} n^k\right) \tag{A5}$$

The last summation in RHS of (A5) can be written in terms of the Reimann zeta function $\zeta(-k) = \sum_{n=1}^{\infty} n^k$ and, therefore, we arrive at

$$\mu(p) = \frac{1}{p+1}\sum_{k=0}^{p} \binom{p+1}{k}(-1)^{p-k}\zeta(-k) \qquad \forall p \in \mathbb{N}^0 \tag{A6}$$

For integers $k \geq 0$, the zeta function is related to Bernoulli numbers by [10] (Equation (66))

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1} \tag{A7}$$

Moreover, from definition (A3b), one can verify that the successive binomial coefficients hold the following useful identity

$$\binom{p+1}{k} = \frac{k+1}{p+2}\binom{p+2}{k+1} \tag{A8}$$

Finally, upon substituting the relevant terms from (A7) and (A8) into (A6), one can equivalently rewrite the latter equation in the following simple form:

$$\mu(p) = \frac{1}{p+1} \sum_{k=0}^{p} \frac{k+1}{p+2} \binom{p+2}{k+1} (-1)^{p-k} (-1)^k \frac{B_{k+1}}{k+1} = \frac{(-1)^p}{(p+1)(p+2)} \sum_{k=0}^{p} \binom{p+2}{k+1} B_{k+1}$$

$$= \frac{(-1)^p}{(p+1)(p+2)} \sum_{k=1}^{p+1} \binom{p+2}{k} B_k \qquad \forall p \in \mathbb{N}^0 \tag{A9}$$

Furthermore, the Bernoulli numbers satisfy the following property [11] (Equation (34)):

$$\sum_{k=0}^{l-1} \binom{l}{k} B_k = 0 \tag{A10}$$

Given that $B_0 = 1$ and using a change of variable $l = p + 2$, one can equivalently write (A10) as

$$\sum_{k=1}^{p+1} \binom{p+2}{k} B_k + \binom{p+2}{0} B_0 = 0$$

$$\sum_{k=1}^{p+1} \binom{p+2}{k} B_k = -1 \tag{A11}$$

Finally, upon substituting (A11) in (A9), we arrive at the explicit expression for the μ function:

$$\mu(p) = \int_0^{\infty} x^p dx \stackrel{\text{ac}}{=} \frac{(-1)^{p+1}}{(p+1)(p+2)} \qquad \forall p \in \mathbb{N}^0 \tag{A12}$$

This surprising and simple finite solution should be viewed as an alternative, which may be useful in certain cases, to an otherwise divergent result. Note that the = sign in (A12) should be considered to only be meaningful in the sense of analytical continuation (ac).

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