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Binomial Sum Relations Involving Fibonacci and Lucas Numbers

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Abstract: In this paper, we provide a first systematic treatment of binomial sum relations involving (generalized) Fibonacci and Lucas numbers. The paper introduces various classes of relations involving (generalized) Fibonacci and Lucas numbers and different kinds of binomial coefficients. We also present some novel relations between sums with two and three binomial coefficients. In the course of exploration, we rediscover a few isolated results existing in the literature, commonly presented as problem proposals.

Keywords: binomial coefficient; central binomial coefficient; Fibonacci number; Lucas number; Horadam sequence; recurrence relation

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1. Introduction and Motivation

The literature on Fibonacci numbers is immensely rich. There exist dozens of articles and problem proposals dealing with binomial sums involving these sequences as (weighted) summands. We attempt to give a short survey, not claiming completeness. The following binomial sums have been studied (X_n stands for a (weighted) Fibonacci or Lucas number, alternating or non-alternating, or a product of them):

- Standard form and variants of it [1–9]

$$\sum_{k=0}^{n}\binom{n}{k}X_k;$$

- Forms coming from the Waring formula and studied by Gould [10,11], for instance,

$$\sum_{k=0}^{n-1}\frac{n}{n-k}\binom{n-k}{k}X_k;$$

- Forms introduced by Filipponi [12]

$$\sum_{k=0}^{n}\binom{2n-k-1}{k}X_k;$$

- Forms introduced by Jennings [13]

$$\sum_{k=0}^{n}\binom{n+k}{2k}X_k;$$

- Forms introduced by Kilic and Ionascu [14]

$$\sum_{k=0}^{n}\binom{2n}{n+k}X_k;$$

- Forms studied recently by Bai, Chu and Guo [15]

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{2n}{n-2k}X_k \quad\text{and}\quad \sum_{k=0}^{\lfloor n/2\rfloor}\binom{2n+2}{n-2k}X_k;$$

- Forms studied by the authors in the recent paper [16]

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}X_k;$$

- Forms studied by the authors in the recent paper [17]

$$\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}X_k \quad\text{and}\quad \sum_{k=0}^{n}\frac{k}{n+k}\binom{n+k}{n-k}X_k.$$

We note that

$$\sum_{k=0}^{n}\binom{n+k}{2k}X_k=\sum_{k=0}^{n}\frac{n}{n+k}\binom{n+k}{n-k}X_k+\sum_{k=0}^{n}\frac{k}{n+k}\binom{n+k}{n-k}X_k.$$

These sums can also be extended to Fibonacci and Lucas polynomials which possess important applications, among others, in cryptography [18] and in numerical analysis [19].

Let $(W_j(a,b;p,q))_{j\geq 0}$ be the Horadam sequence [20] defined for all non-negative integers j by the recurrence

$$W_0=a,\ \ W_1=b;\quad W_j=pW_{j-1}-qW_{j-2},\quad j\geq 2, \tag{1}$$

where a, b, p and q are arbitrary complex numbers, with $p\neq 0$ and $q\neq 0$. An extension of the definition of (W_j) to negative subscripts is provided by writing the recurrence relation as $W_{-j}=\frac{1}{q}(pW_{-j+1}-W_{-j+2})$.

Two important cases of (W_j) are the Lucas sequences of the first kind, $(U_j(p,q))=(W_j(0,1;p,q))$, and of the second kind, $(V_j(p,q))=(W_j(2,p;p,q))$, so that

$$U_0=0,\ \ U_1=1,\quad U_j=pU_{j-1}-qU_{j-2},\quad j\geq 2;$$

and

$$V_0=2,\ \ V_1=p,\quad V_j=pV_{j-1}-qV_{j-2},\quad j\geq 2.$$

The most well-known Lucas sequences are the Fibonacci sequence $(F_j)=(U_j(1,-1))$ and the sequence of Lucas numbers $(L_j)=(V_j(1,-1))$.

The Binet formulas for sequences (U_j), (V_j) and (W_j) in the non-degenerate case, $p^2-4q>0$, are

$$U_j=\frac{\tau^j-\sigma^j}{\Delta},\qquad V_j=\tau^j+\sigma^j,\qquad W_j=A\tau^j+B\sigma^j\,, \tag{2}$$

with $\Delta=\sqrt{p^2-4q}$, $A=\frac{b-a\sigma}{\Delta}$ and $B=\frac{a\tau-b}{\Delta}$, where $\tau=\tau(p,q)=\frac{p+\Delta}{2}$ and $\sigma=\sigma(p,q)=\frac{p-\Delta}{2}$ are the distinct zeros of the characteristic polynomial x^2-px+q of the Horadam sequence (1).

The Binet formulas for the Fibonacci and Lucas numbers are

$$F_j = \frac{\alpha^j - \beta^j}{\sqrt{5}}, \qquad L_j = \alpha^j + \beta^j, \tag{3}$$

where $\alpha = \tau(1,-1) = (1+\sqrt{5})/2$ is the golden ratio and $\beta = \sigma(1,-1) = -1/\alpha$.

The sequences $(F_n)_{n\ge 0}$ and $(L_n)_{n\ge 0}$ are indexed in the On-Line Encyclopedia of Integer Sequences [21] as entries A000045 and A000032, respectively. For more information on them, we recommend the books by Koshy [22] and Vajda [23], among others.

In this paper, we provide a first systematic treatment of binomial sum relations involving (generalized) Fibonacci and Lucas numbers. It is motivated by some isolated results we found in the literature, published as problem proposals by Leonard Carlitz. We cover various classes of relations involving (generalized) Fibonacci and Lucas numbers and different kinds of binomial coefficients. We also present some novel relations between sums with two and three binomial coefficients.

We will make use of the following known results.

Lemma 1. *If a, b, c and d are rational numbers and λ is an irrational number, then*

$$a + b\,\lambda = c + d\,\lambda \qquad \Longleftrightarrow \qquad a = c, \quad b = d.$$

The next three lemmas can be obtained from Binet's formulas (2).

Lemma 2. *For any integer s,*

$$q^s + \tau^{2s} = \tau^s V_s, \qquad q^s - \tau^{2s} = -\Delta \tau^s U_s, \tag{4}$$

$$q^s + \sigma^{2s} = \sigma^s V_s, \qquad q^s - \sigma^{2s} = \Delta \sigma^s U_s. \tag{5}$$

In particular,

$$(-1)^s + \alpha^{2s} = \alpha^s L_s, \qquad (-1)^s - \alpha^{2s} = -\sqrt{5}\alpha^s F_s, \tag{6}$$

$$(-1)^s + \beta^{2s} = \beta^s L_s, \qquad (-1)^s - \beta^{2s} = \sqrt{5}\beta^s F_s. \tag{7}$$

Lemma 3. *Let r and d be any integers. Then*

$$V_{r+s} - \tau^r V_s = -\Delta \sigma^s U_r, \tag{8}$$

$$V_{r+s} - \sigma^r V_s = \Delta \tau^s U_r, \tag{9}$$

$$U_{r+s} - \tau^r U_s = \sigma^s U_r, \tag{10}$$

$$U_{r+s} - \sigma^r U_s = \tau^s U_r. \tag{11}$$

In particular, [5],

$$L_{r+s} - L_r \alpha^s = -\sqrt{5}\beta^r F_s, \qquad L_{r+s} - L_r \beta^s = \sqrt{5}\alpha^r F_s, \tag{12}$$

$$F_{r+s} - F_r \alpha^s = \beta^r F_s, \qquad F_{r+s} - F_r \beta^s = \alpha^r F_s. \tag{13}$$

Lemma 4. *For any integer j,*

$$A\tau^j - B\sigma^j = \frac{W_{j+1} - qW_{j-1}}{\Delta}, \tag{14}$$

$$A\sigma^j + B\tau^j = q^j W_{-j}. \tag{15}$$

2. Relations from a Classical Polynomial Identity

The first binomial sum relations follow from the next classical polynomial identity, which we state in the next lemma.

Lemma 5 ([24]). *If x is a complex variable and m and n are non-negative integers, then*

$$\sum_{k=0}^{n}\binom{m-n+k}{k}(1+x)^{n-k}x^k=\sum_{k=0}^{n}\binom{m+1}{k}x^k,\quad x\neq -1. \tag{16}$$

According to Gould [24], identity (16) is due to Laplace. In addition, we note that the binomial theorem is a special case of (16), which occurs at $m=n-1$.

Using

$$\binom{-n}{k}=(-1)^k\binom{n+k-1}{k}$$

and replacing m by $m-1$, we have the equivalent and useful form of Lemma 5:

$$\sum_{k=0}^{n}(-1)^k\binom{n-m}{k}(1+x)^{n-k}x^k=\sum_{k=0}^{n}\binom{m}{k}x^k,\quad x\neq -1.$$

Theorem 1. *If r, s and t are any integers and m and n are non-negative integers, then*

$$\sum_{k=0}^{n}\binom{m-n+k}{k}U_{r+s}^kU_s^{n-k}W_{t+r(n-k)}=\sum_{k=0}^{n}(-q^s)^{n-k}\binom{m+1}{k}U_{r+s}^kU_r^{n-k}W_{t-s(n-k)}.$$

Proof. Set $x=-U_{r+s}/(U_r\sigma^s)$ in (16), use (11), and multiply through by τ^t, obtaining

$$\sum_{k=0}^{n}\binom{m-n+k}{k}U_{r+s}^kU_s^{n-k}\tau^{r(n-k)+t}=(-1)^t\sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}U_{r+s}^kU_r^{n-k}\sigma^{s(n-k)-t}.$$

Similarly, setting $x=-U_{r+s}/(U_r\tau^s)$ in (16), using (10), and multiplying through by σ^t yields

$$\sum_{k=0}^{n}\binom{m-n+k}{k}U_{r+s}^kU_s^{n-k}\sigma^{r(n-k)+t}=(-1)^t\sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}U_{r+s}^kU_r^{n-k}\tau^{s(n-k)-t}.$$

The results follow by combining these identities according to the Binet formulas (2) and Lemma 4. □

In particular,

$$\sum_{k=0}^{n}\binom{m-n+k}{k}U_{r+s}^kU_s^{n-k}V_{r(n-k)+t}=q^t\sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}U_{r+s}^kU_r^{n-k}V_{s(n-k)-t},$$
$$\sum_{k=0}^{n}\binom{m-n+k}{k}U_{r+s}^kU_s^{n-k}U_{r(n-k)+t}=q^t\sum_{k=0}^{n}(-1)^{n-k+1}\binom{m+1}{k}U_{r+s}^kU_r^{n-k}U_{s(n-k)-t};$$

with the special cases

$$\sum_{k=0}^{n}\binom{m-n+k}{k}F_{r+s}^kF_s^{n-k}L_{r(n-k)+t}=(-1)^t\sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}F_{r+s}^kF_r^{n-k}L_{s(n-k)-t}, \tag{17}$$
$$\sum_{k=0}^{n}\binom{m-n+k}{k}F_{r+s}^kF_s^{n-k}F_{r(n-k)+t}=(-1)^{t-1}\sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}F_{r+s}^kF_r^{n-k}F_{s(n-k)-t}. \tag{18}$$

Corollary 1. *If r, s and t are any integers and n is a non-negative integer, then*

$$\sum_{k=0}^{n}(-q^s)^{n-k}\binom{n+1}{k}U_{r+s}^kU_r^{n-k}W_{t-s(n-k)}=\sum_{k=0}^{n}U_{r+s}^kU_s^{n-k}W_{t+r(n-k)}.$$

Proof. Set $m=n$ in Theorem 1. □

In particular,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+1}{k}U_{r+s}^{k}U_{r}^{n-k}V_{s(n-k)-t}=\frac{1}{q^{t}}\sum_{k=0}^{n}U_{r+s}^{k}U_{s}^{n-k}V_{r(n-k)+t},$$
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+1}{k}U_{r+s}^{k}U_{r}^{n-k}U_{s(n-k)-t}=-\frac{1}{q^{t}}\sum_{k=0}^{n}U_{r+s}^{k}U_{s}^{n-k}U_{r(n-k)+t};$$

with

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+1}{k}F_{r+s}^{k}F_{r}^{n-k}L_{s(n-k)-t}=(-1)^{t}\sum_{k=0}^{n}F_{r+s}^{k}F_{s}^{n-k}L_{r(n-k)+t},$$
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n+1}{k}F_{r+s}^{k}F_{r}^{n-k}F_{s(n-k)-t}=(-1)^{t-1}\sum_{k=0}^{n}F_{r+s}^{k}F_{s}^{n-k}F_{r(n-k)+t}.$$

Corollary 2. *If r, s and t are any integers and n is a non-negative integer, then*

$$\sum_{k=0}^{n}(-q^{s})^{n-k}\binom{n}{k}U_{r+s}^{k}U_{r}^{n-k}W_{t-s(n-k)}=U_{s}^{n}W_{t+rn}.$$

Proof. Set $m=n-1$ in Theorem 1. □

In particular,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}U_{r+s}^{k}U_{r}^{n-k}V_{s(n-k)-t}=\frac{U_{s}^{n}V_{rn+t}}{q^{t}},$$
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}U_{r+s}^{k}U_{r}^{n-k}U_{s(n-k)-t}=-\frac{U_{s}^{n}U_{rn+t}}{q^{t}};$$

with the special cases

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F_{r+s}^{k}F_{r}^{n-k}L_{s(n-k)-t}=(-1)^{t}F_{s}^{n}L_{rn+t}, \tag{19}$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}F_{r+s}^{k}F_{r}^{n-k}F_{s(n-k)-t}=(-1)^{t+1}F_{s}^{n}F_{rn+t}. \tag{20}$$

We mention that identities (19) and (20) exhibit strong similarities to those derived by Hoggatt, Phillips and Leonard in [25].

Corollary 3. *If m and n are non-negative integers and r is any integer, then*

$$\sum_{k=0}^{n}(-q^{r})^{k}\binom{m-n+k}{k}W_{r(n-2k)}=\sum_{k=0}^{n}(-q^{r})^{k}\binom{m+1}{k}V_{r}^{n-k}W_{-rk}.$$

Proof. Make the substitutions $r\mapsto 2r$, $s\mapsto -r$ and $t\mapsto -rn$ in Theorem 1 and simplify. □

In particular,

$$\sum_{k=0}^{n}(-q^{r})^{k}\binom{m-n+k}{k}V_{r(n-2k)}=\sum_{k=0}^{n}(-1)^{k}\binom{m+1}{k}V_{r}^{n-k}V_{rk},$$
$$\sum_{k=0}^{n}(-q^{r})^{k}\binom{m-n+k}{k}U_{r(n-2k)}=\sum_{k=0}^{n}(-1)^{k+1}\binom{m+1}{k}V_{r}^{n-k}U_{rk};$$

with the special cases

$$\sum_{k=0}^{n}(-1)^{k(r+1)}\binom{m-n+k}{k}L_{r(n-2k)}=\sum_{k=0}^{n}(-1)^{k}\binom{m+1}{k}L_r^{n-k}L_{rk},$$
$$\sum_{k=0}^{n}(-1)^{k(r+1)}\binom{m-n+k}{k}F_{r(n-2k)}=\sum_{k=0}^{n}(-1)^{k+1}\binom{m+1}{k}L_r^{n-k}F_{rk}.$$

By making appropriate substitutions in Theorem 1, many new sum relations can be established. For example, setting $r=1$, $s=-2$, and $t=1$ (or $r=-1$, $s=2$ and $t=-1$) in (17) gives

$$\sum_{k=0}^{n}(-1)^{k}\binom{m-n+k}{k}L_{n-k+1}=\sum_{k=0}^{n}(-1)^{k}\binom{m+1}{k}L_{2(n-k)+1}$$

which at $m=2n$ gives

$$\sum_{k=0}^{n}(-1)^{k}\binom{n+k}{k}L_{n-k+1}=\sum_{k=0}^{n}(-1)^{k}\binom{2n+1}{k}L_{2(n-k)+1}. \tag{21}$$

The corresponding Fibonacci sums from (18) are of exactly the same structure

$$\sum_{k=0}^{n}(-1)^{k}\binom{m-n+k}{k}F_{n-k+1}=\sum_{k=0}^{n}(-1)^{k}\binom{m+1}{k}F_{2(n-k)+1}$$

with the special case

$$\sum_{k=0}^{n}(-1)^{k}\binom{n+k}{k}F_{n-k+1}=\sum_{k=0}^{n}(-1)^{k}\binom{2n+1}{k}F_{2(n-k)+1}. \tag{22}$$

Another example is the relation

$$\sum_{k=0}^{n}(-1)^{k}\binom{m-n+k}{k}L_{n-3k}=\sum_{k=0}^{n}(-1)^{k}\binom{m+1}{k}2^{n-k}L_{2k}$$

which at $m=2n$ gives

$$\sum_{k=0}^{n}(-1)^{k}\binom{n+k}{k}L_{n-3k}=\sum_{k=0}^{n}(-1)^{k}\binom{2n+1}{k}2^{n-k}L_{2k}, \tag{23}$$

and its Fibonacci counterparts:

$$\sum_{k=0}^{n}(-1)^{k}\binom{m-n+k}{k}F_{n-3k}=\sum_{k=1}^{n}(-1)^{k-1}\binom{m+1}{k}2^{n-k}F_{2k},$$
$$\sum_{k=0}^{n}(-1)^{k}\binom{n+k}{k}F_{n-3k}=\sum_{k=1}^{n}(-1)^{k-1}\binom{2n+1}{k}2^{n-k}F_{2k}.$$

Theorem 2. *If m and n are non-negative integers and r,s are any integers, then*

$$\sum_{k=0}^{n}(-q^s)^{k}\binom{m-n+k}{k}U_r^kU_s^{n-k}W_{r(n-k)-sk+t}=\sum_{k=0}^{n}(-q^s)^{k}\binom{m+1}{k}U_r^kU_{r+s}^{n-k}W_{t-sk}.$$

Proof. Set $x=-U_r\sigma^s/U_{r+s}$ in (16), use (11), and multiply through by τ^t, obtaining

$$\sum_{k=0}^{n}(-1)^{k}\binom{m-n+k}{k}q^{sk}U_r^kU_s^{n-k}\tau^{rn-(r+s)k+t}=q^t\sum_{k=0}^{n}(-1)^{k}\binom{m+1}{k}U_r^kU_{r+s}^{n-k}\sigma^{sk-t}.$$

Similarly, setting $x = -U_r\tau^s/U_{r+s}$ in (16), using (13), and multiplying through by σ^t yields

$$\sum_{k=0}^{n}(-1)^k\binom{m-n+k}{k}q^{sk}U_r^kU_s^{n-k}\sigma^{rn-(r+s)k+t} = q^t\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}U_r^kU_{r+s}^{n-k}\tau^{sk-t}.$$

Now, the result follows immediately upon combining according to the Binet formulas (2). $\square$

In particular,

$$\sum_{k=0}^{n}(-q^s)^k\binom{m-n+k}{k}U_r^kU_s^{n-k}V_{rn-(r+s)k+t} = q^t\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}U_r^kU_{r+s}^{n-k}V_{sk-t}, \tag{24}$$

$$\sum_{k=0}^{n}(-q^s)^k\binom{m-n+k}{k}U_r^kU_s^{n-k}U_{rn-(r+s)k+t} = -q^t\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}U_r^kU_{r+s}^{n-k}U_{sk-t}; \tag{25}$$

with the special cases

$$\sum_{k=0}^{n}(-1)^{k(s-1)}\binom{m-n+k}{k}F_r^kF_s^{n-k}L_{rn-(r+s)k+t} = \sum_{k=0}^{n}(-1)^{k(s-1)}\binom{m+1}{k}F_r^kF_{r+s}^{n-k}L_{t-sk},$$
$$\sum_{k=0}^{n}(-1)^{k(s-1)}\binom{m-n+k}{k}F_r^kF_s^{n-k}F_{rn-(r+s)k+t} = \sum_{k=0}^{n}(-1)^{k(s-1)}\binom{m+1}{k}F_r^kF_{r+s}^{n-k}F_{t-sk}.$$

Corollary 4. *If n is a non-negative integer and r and s are any integers, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}U_r^kU_{r+s}^{n-k}V_{sk-t} = \frac{1}{q^t}\sum_{k=0}^{n}(-q^s)^kU_r^kU_s^{n-k}V_{rn-(r+s)k+t},$$
$$\sum_{k=0}^{n}(-1)^k\binom{n+1}{k}U_r^kU_{r+s}^{n-k}U_{sk-t} = -\frac{1}{q^t}\sum_{k=0}^{n}(-q^s)^kU_r^kU_s^{n-k}U_{rn-(r+s)k+t}.$$

Proof. Set $m = n$ in (24) and (25). $\square$

In particular,

$$\sum_{k=0}^{n}(-1)^{k(s-1)}\binom{n+1}{k}F_r^kF_{r+s}^{n-k}L_{t-sk} = \sum_{k=0}^{n}(-1)^{k(s-1)}F_r^kF_s^{n-k}L_{rn-(r+s)k+t},$$
$$\sum_{k=0}^{n}(-1)^{k(s-1)}\binom{n+1}{k}F_r^kF_{r+s}^{n-k}F_{t-sk} = \sum_{k=0}^{n}(-1)^{k(s-1)}F_r^kF_s^{n-k}F_{rn-(r+s)k+t}.$$

We mention that setting $m = n-1$ in Theorem 2 gives again Corollary 2.

Corollary 5. *If m and n are non-negative integers and r is any integer, then*

$$\sum_{k=0}^{n}\binom{m-n+k}{k}V_r^kV_{r(n-k)} = \sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}V_r^kV_{r(n-k)},$$
$$\sum_{k=0}^{n}\binom{m-n+k}{k}V_r^kU_{r(n-k)} = \sum_{k=0}^{n}(-1)^{n-k+1}\binom{m+1}{k}V_r^kU_{r(n-k)}.$$

Proof. Make the substitutions $r \mapsto 2r$, $s \mapsto -r$, $t \mapsto -rn$ in (24), (25) and simplify. $\square$

In particular,

$$\sum_{k=0}^{n}\binom{m-n+k}{k}L_r^kL_{r(n-k)}=\sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}L_r^kL_{r(n-k)},$$
$$\sum_{k=0}^{n}\binom{m-n+k}{k}L_r^kF_{r(n-k)}=\sum_{k=0}^{n}(-1)^{n-k+1}\binom{m+1}{k}L_r^kF_{r(n-k)}.$$

Theorem 3. *If m and n are non-negative integers and s,t are integers, then*

$$\begin{aligned}&\sum_{k=0}^{2n}\binom{m-2n+k}{k}2^{2n-k}V_s^kW_{s(2n-k)+t}\\&\quad=W_t\sum_{k=0}^{n}\binom{m+1}{2k}\Delta^{2(n-k)}U_s^{2(n-k)}V_s^{2k}\\&\qquad+(W_{t+1}-qW_{t-1})\sum_{k=1}^{n}\binom{m+1}{2k-1}\Delta^{2(n-k)}U_s^{2(n-k)+1}V_s^{2k-1},\end{aligned}$$
$$\begin{aligned}&\sum_{k=0}^{2n-1}\binom{m-2n+k+1}{k}2^{2n-k-1}V_s^kW_{s(2n-k-1)+t}\\&\quad=W_t\sum_{k=1}^{n}\binom{m+1}{2k-1}\Delta^{2(n-k)}U_s^{2(n-k)}V_s^{2k-1}\\&\qquad+(W_{t+1}-qW_{t-1})\sum_{k=0}^{n-1}\binom{m+1}{2k}\Delta^{2(n-k-1)}U_s^{2(n-k)-1}V_s^{2k}.\end{aligned}$$

Proof. Set $x=V_s/(\Delta U_s)$ in (16) and multiply through by τ^t to obtain

$$\begin{aligned}&\sum_{k=0}^{n}\binom{m-n+k}{k}2^{n-k}\tau^{s(n-k)+t}V_s^k\\&\quad=\tau^t\sum_{k=0}^{\lfloor n/2\rfloor}\binom{m+1}{2k}\Delta^{n-2k}U_s^{n-2k}V_s^{2k}+\tau^t\sum_{k=1}^{\lceil n/2\rceil}\binom{m+1}{2k-1}\Delta^{n-2k+1}U_s^{n-2k+1}V_s^{2k-1}.\end{aligned}\tag{26}$$

Similarly, set $x=-V_s/(\Delta U_s)$ in (16) and multiply through by σ^t to obtain

$$\begin{aligned}&(-1)^n\sum_{k=0}^{n}\binom{m-n+k}{k}2^{n-k}\sigma^{s(n-k)+t}V_s^k\\&\quad=\sigma^t\sum_{k=0}^{\lfloor n/2\rfloor}\binom{m+1}{2k}\Delta^{n-2k}U_s^{n-2k}V_s^{2k}-\sigma^t\sum_{k=1}^{\lceil n/2\rceil}\binom{m+1}{2k-1}\Delta^{n-2k+1}U_s^{n-2k+1}V_s^{2k-1}.\end{aligned}\tag{27}$$

Combine (26) and (27) according to the Binet formula while making use also of (14). Consider the cases $n\mapsto 2n$ and $n\mapsto 2n-1$, in turn. □

In particular,

$$\begin{aligned}&\sum_{k=0}^{2n}\binom{m-2n+k}{k}2^{2n-k}V_s^kV_{s(2n-k)+t}\\&\quad=V_t\sum_{k=0}^{n}\binom{m+1}{2k}\Delta^{2(n-k)}U_s^{2(n-k)}V_s^{2k}+U_t\sum_{k=1}^{n}\binom{m+1}{2k-1}\Delta^{2(n-k+1)}U_s^{2(n-k)+1}V_s^{2k-1},\end{aligned}\tag{28}$$

$$\begin{aligned}&\sum_{k=0}^{2n}\binom{m-2n+k}{k}2^{2n-k}V_s^kU_{s(2n-k)+t}\\&\quad=U_t\sum_{k=0}^{n}\binom{m+1}{2k}\Delta^{2(n-k)}U_s^{2(n-k)}V_s^{2k}+V_t\sum_{k=1}^{n}\binom{m+1}{2k-1}\Delta^{2(n-k)}U_s^{2(n-k)+1}V_s^{2k-1},\end{aligned}\tag{29}$$

$$\sum_{k=0}^{2n-1}\binom{m-2n+k+1}{k}2^{2n-k-1}V_s^kV_{s(2n-k-1)+t} \\ = U_t\sum_{k=0}^{n-1}\binom{m+1}{2k}\Delta^{2(n-k)}U_s^{2(n-k)-1}V_s^{2k} + V_t\sum_{k=1}^{n}\binom{m+1}{2k-1}\Delta^{2(n-k)}U_s^{2(n-k)}V_s^{2k-1} \tag{30}$$

and

$$\sum_{k=0}^{2n-1}\binom{m-2n+k+1}{k}2^{2n-k-1}V_s^kU_{s(2n-k-1)+t} \\ = V_t\sum_{k=0}^{n-1}\binom{m+1}{2k}\Delta^{2(n-k-1)}U_s^{2(n-k)-1}V_s^{2k} + U_t\sum_{k=1}^{n}\binom{m+1}{2k-1}\Delta^{2(n-k)}U_s^{2(n-k)}V_s^{2k-1} \tag{31}$$

with the special cases

$$\sum_{k=0}^{2n}\binom{m-2n+k}{k}2^{2n-k}L_s^kL_{s(2n-k)+t} \\ = L_t\sum_{k=0}^{n}\binom{m+1}{2k}5^{n-k}L_s^{2k}F_s^{2(n-k)} + F_t\sum_{k=1}^{n}\binom{m+1}{2k-1}5^{n-k+1}L_s^{2k-1}F_s^{2(n-k)+1},$$

$$\sum_{k=0}^{2n}\binom{m-2n+k}{k}2^{2n-k}L_s^kF_{s(2n-k)+t} \\ = F_t\sum_{k=0}^{n}\binom{m+1}{2k}5^{n-k}L_s^{2k}F_s^{2(n-k)} + L_t\sum_{k=1}^{n}\binom{m+1}{2k-1}5^{n-k}L_s^{2k-1}F_s^{2(n-k)+1},$$

$$\sum_{k=0}^{2n-1}\binom{m-2n+k+1}{k}2^{2n-k-1}L_s^kL_{s(2n-k-1)+t} \\ = F_t\sum_{k=0}^{n-1}\binom{m+1}{2k}5^{n-k}L_s^{2k}F_s^{2(n-k)-1} + L_t\sum_{k=1}^{n}\binom{m+1}{2k-1}5^{n-k}L_s^{2k-1}F_s^{2(n-k)},$$

$$\sum_{k=0}^{2n-1}\binom{m-2n+k+1}{k}2^{2n-k-1}L_s^kF_{s(2n-k-1)+t} \\ = L_t\sum_{k=0}^{n-1}\binom{m+1}{2k}5^{n-k-1}L_s^{2k}F_s^{2(n-k)-1} + F_t\sum_{k=1}^{n}\binom{m+1}{2k-1}5^{n-k}L_s^{2k-1}F_s^{2(n-k)}.$$

Note that in (28)–(31), we used

$$U_{t+1}-qU_{t-1}=V_t, \qquad V_{t+1}-qV_{t-1}=\Delta^2U_t. \tag{32}$$

Lemma 6. *If x is a complex variable and m,n are non-negative integers, then*

$$\sum_{k=0}^{n}\binom{m-n+k}{k}x^k = \sum_{k=0}^{n}\binom{m+1}{k}x^k(1-x)^{n-k} \tag{33}$$

Proof. Use the transformation $\frac{x}{1+x}\mapsto x$ in (16). □

Theorem 4. *If m and n are non-negative integers and s,t are integers, then*

$$L_t\sum_{k=0}^{n}\binom{m-2n+2k}{2k}\frac{F_s^{2(n-k)}L_s^{2k}}{5^k} - F_t\sum_{k=1}^{n}\binom{m-2n+2k-1}{2k-1}\frac{F_s^{2(n-k)+1}L_s^{2k-1}}{5^{k-1}} \\ = \left(\frac{4}{5}\right)^n\sum_{k=0}^{2n}(-1)^k\binom{m+1}{k}\frac{L_s^kL_{s(2n-k)+t}}{2^k}, \tag{34}$$

$$F_t\sum_{k=0}^{n}\binom{m-2n+2k}{2k}\frac{F_s^{2(n-k)}L_s^{2k}}{5^k}-L_t\sum_{k=1}^{n}\binom{m-2n+2k-1}{2k-1}\frac{F_s^{2(n-k)+1}L_s^{2k-1}}{5^k}=\left(\frac{4}{5}\right)^n\sum_{k=0}^{2n}(-1)^k\binom{m+1}{k}\frac{L_s^kF_{s(2n-k)+t}}{2^k},\tag{35}$$

$$L_t\sum_{k=1}^{n}\binom{m-2n+2k}{2k-1}\frac{F_s^{2(n-k)}L_s^{2k-1}}{5^k}-F_t\sum_{k=0}^{n-1}\binom{m-2n+2k+1}{2k}\frac{F_s^{2(n-k)-1}L_s^{2k}}{5^k}=\left(\frac{4}{5}\right)^n\sum_{k=0}^{2n-1}(-1)^{k-1}\binom{m+1}{k}\frac{L_s^kL_{s(2n-k-1)+t}}{2^{k+1}},\tag{36}$$

$$F_t\sum_{k=1}^{n}\binom{m-2n+2k}{2k-1}\frac{F_s^{2(n-k)}L_s^{2k-1}}{5^k}-L_t\sum_{k=0}^{n-1}\binom{m-2n+2k+1}{2k}\frac{F_s^{2(n-k)-1}L_s^{2k}}{5^{k+1}}=\left(\frac{4}{5}\right)^n\sum_{k=0}^{2n-1}(-1)^{k-1}\binom{m+1}{k}\frac{L_s^kF_{s(2n-k-1)+t}}{2^{k+1}}.\tag{37}$$

Proof. Set $x=L_s/(\sqrt{5}F_s)$ in (33) to obtain

$$\sum_{k=0}^{n}\binom{m-n+k}{k}(\sqrt{5})^{n-k}F_s^{n-k}L_s^k=\sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}2^{n-k}L_s^k\beta^{s(n-k)};$$

so that

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{m-n+2k}{2k}\frac{F_s^{n-2k}L_s^{2k}}{(\sqrt{5})^{2k}}+\sum_{k=1}^{\lceil n/2\rceil}\binom{m-n+2k-1}{2k-1}\frac{F_s^{n-2k+1}L_s^{2k-1}}{(\sqrt{5})^{2k-1}}=\left(\frac{2}{\sqrt{5}}\right)^n\sum_{k=0}^{n}(-1)^{n-k}\binom{m+1}{k}\frac{L_s^k\beta^{s(n-k)}}{2^k}.\tag{38}$$

Writing $2n$ for n in (38) (after multiplying through by β^t) and comparing the coefficients of $\sqrt{5}$ produces (34) and (35). Writing $2n-1$ for n gives (36) and (37). □

Corollary 6. *If m and n are non-negative integers and s is an integer, then*

$$\sum_{k=0}^{n}\binom{m-2n+2k}{2k}5^{n-k}F_s^{2(n-k)}L_s^{2k}=\sum_{k=0}^{2n}(-1)^k\binom{m+1}{k}2^{2n-k-1}L_s^kL_{s(2n-k)},$$
$$\sum_{k=1}^{n}\binom{m-2n+2k-1}{2k-1}5^{n-k}F_s^{2(n-k)+1}L_s^{2k-1}=\sum_{k=0}^{2n}(-1)^{k+1}\binom{m+1}{k}2^{2n-k-1}L_s^kF_{s(2n-k)},$$
$$\sum_{k=1}^{n}\binom{m-2n+2k}{2k-1}5^{n-k}F_s^{2(n-k)}L_s^{2k-1}=\sum_{k=0}^{2n-1}(-1)^{k+1}\binom{m+1}{k}2^{2n-k-2}L_s^kL_{s(2n-k-1)},$$
$$\sum_{k=0}^{n-1}\binom{m-2n+2k+1}{2k}5^{n-k-1}F_s^{2(n-k)-1}L_s^{2k}=\sum_{k=0}^{2n-1}(-1)^k\binom{m+1}{k}2^{2n-k-2}L_s^kF_{s(2n-k-1)}.$$

Corollary 7. *If n is a non-negative integer and s is any integer, then*

$$\begin{aligned}
\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}2^{2n-1-k}L_s^kL_{s(2n-k)} &= 5^nF_s^{2n},\\
\sum_{k=0}^{2n}(-1)^{k+1}\binom{2n+1}{k}2^{2n-k-1}L_s^kF_{s(2n-k)} &= \sum_{k=1}^{n}5^{n-k}F_s^{2(n-k)+1}L_s^{2k-1},\\
\sum_{k=0}^{2n-1}(-1)^{k+1}\binom{2n}{k}2^{2n-k-2}L_s^kL_{s(2n-k-1)} &= \sum_{k=1}^{n}5^{n-k}F_s^{2(n-k)}L_s^{2k-1},\\
\sum_{k=0}^{2n-1}(-1)^{k}\binom{2n}{k}2^{2n-k-2}L_s^kF_{s(2n-k-1)} &= \sum_{k=0}^{n-1}5^{n-k-1}F_s^{2(n-k)-1}L_s^{2k}.
\end{aligned}$$

3. Relations from a Recent Identity by Alzer

In 2015, Alzer [26], building on the work of Aharonov and Elias [27], studied the polynomial

$$P_n(x) = (1-x)^{n+1}\sum_{k=0}^{n}\binom{n+k}{k}x^k, \qquad x \in \mathbb{C}. \tag{39}$$

Among other things, he showed that

$$P_n(x) = 1 - x + (1-2x)\sum_{k=0}^{n-1}\binom{2k+1}{k}x^{k+1}(1-x)^{k+1}. \tag{40}$$

Such a polynomial identity immediately offers many appealing Fibonacci and Lucas sum relations as can been seen from the next series of theorems.

Theorem 5. *For each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}F_{n+1-k} = 1 - 2\sum_{k=0}^{n-1}(-1)^k\binom{2k+1}{k}, \tag{41}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}L_{n+1-k} = 1. \tag{42}$$

Proof. Set $x = \alpha$ and $x = \beta$ in (39) and (40), respectively, and combine according to the Binet Formulas (3). □

Comparing (22) with (41), and (21) with (42), we find

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}F_{2(n-k)+1} = 1 - 2\sum_{k=0}^{n-1}(-1)^k\binom{2k+1}{k},$$

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}L_{2(n-k)+1} = 1.$$

Theorem 6. *For each non-negative integer n, we have the relations*

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n+k}{k}F_{n+2k+1} &= (-1)^n - \sum_{k=0}^{n-1}(-1)^{n-k}\binom{2k+1}{k}F_{3(k+2)},\\
\sum_{k=0}^{n}\binom{n+k}{k}L_{n+2k+1} &= (-1)^n - \sum_{k=0}^{n-1}(-1)^{n-k}\binom{2k+1}{k}L_{3(k+2)}.
\end{aligned}$$

Proof. Set $x = \alpha^2$ and $x = \beta^2$ in (39) and (40), respectively, and combine according to the Binet Formulas (3). □

The next theorem generalizes Theorem 5.

Theorem 7. *For non-negative integers n and m, we have the relations*

$$\sum_{k=0}^{n}(-1)^{mk}\binom{n+k}{k}\frac{F_{m(n+1-k)}}{L_m^k}=F_mL_m^n\left(1+2\sum_{k=0}^{n-1}\frac{(-1)^{m(k+1)}}{L_m^{2(k+1)}}\binom{2k+1}{k}\right),$$
$$\sum_{k=0}^{n}(-1)^{mk}\binom{n+k}{k}\frac{L_{m(n+1-k)}}{L_m^k}=L_m^{n+1}. \tag{43}$$

Proof. Set $x=\alpha^m/L_m$ and $x=\beta^m/L_m$ in (39) and (40), respectively, and combine according to the Binet formulas. □

When $m=1$, then Theorem 7 reduces to Theorem 5. As additional examples, we state the next relations:

$$\sum_{k=0}^{n}\binom{n+k}{k}2^{-k}=2^n,$$

which also appears in Alzer's paper [26] as Equation (1.4), and

$$\sum_{k=0}^{n}\binom{n+k}{k}\frac{F_{2(n+1-k)}}{3^k}=3^n\left(1+2\sum_{k=0}^{n-1}\binom{2k+1}{k}\frac{1}{9^{k+1}}\right),$$
$$\sum_{k=0}^{n}\binom{n+k}{k}\frac{L_{2(n+1-k)}}{3^k}=3^{n+1}.$$

Theorem 8. *For non-negative integer n and any integers m and t, we have the relations*

$$q^{mn}\sum_{k=0}^{n}\binom{n+k}{k}\frac{W_{mk+t}}{V_m^k}$$
$$=V_m^nW_{mn+t}-V_m^nU_m\left(W_{m(n+1)+t+1}-qW_{m(n+1)+t-1}\right)\sum_{k=0}^{n-1}\binom{2k+1}{k}\frac{q^{mk}}{V_m^{2(k+1)}}.$$

Proof. Set $x=\tau^m/V_m$ and $x=\sigma^m/V_m$ in (39) and (40), respectively, and combine according to the Binet formulas, while making use also of Lemma 4. □

In particular,

$$q^{mn}\sum_{k=0}^{n}\binom{n+k}{k}\frac{U_{mk+t}}{V_m^k}$$
$$=V_m^nU_{mn+t}-V_m^nU_m\left(U_{m(n+1)+t+1}-qU_{m(n+1)+t-1}\right)\sum_{k=0}^{n-1}\binom{2k+1}{k}\frac{q^{mk}}{V_m^{2(k+1)}},$$
$$q^{mn}\sum_{k=0}^{n}\binom{n+k}{k}\frac{V_{mk+t}}{V_m^k}$$
$$=V_m^nV_{mn+t}-V_m^nU_m\left(V_{m(n+1)+t+1}-qV_{m(n+1)+t-1}\right)\sum_{k=0}^{n-1}\binom{2k+1}{k}\frac{q^{mk}}{V_m^{2(k+1)}};$$

with the special cases

$$\sum_{k=0}^{n}\binom{n+k}{k}\frac{F_{mk+t}}{L_m^k}=(-1)^{mn}L_m^nF_{mk+t}-L_m^nF_mL_{m(n+1)+t}\sum_{k=0}^{n-1}\binom{2k+1}{k}\frac{(-1)^{m(n-k)}}{L_m^{2(k+1)}},$$
$$\sum_{k=0}^{n}\binom{n+k}{k}\frac{L_{mk+t}}{L_m^k}=(-1)^{mn}L_m^nL_{mk+t}-5L_m^nF_mF_{m(n+1)+t}\sum_{k=0}^{n-1}\binom{2k+1}{k}\frac{(-1)^{m(n-k)}}{L_m^{2(k+1)}}.$$

Theorem 9. *For each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}\frac{F_{2(n+1)-k}}{2^k}=2^n-\sum_{k=0}^{n-1}(-1)^k\binom{2k+1}{k}2^{n-2k-1}F_{k+2},$$
$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}\frac{L_{2(n+1)-k}}{2^k}=3\cdot 2^n-\sum_{k=0}^{n-1}(-1)^k\binom{2k+1}{k}2^{n-2k-1}L_{k+2}.$$

Proof. Set $x=\alpha/2$ and $x=\beta/2$ in (39) and (40), respectively, and combine according to the Binet formulas. □

Theorem 10. *For each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}F_{2(n+1)+k}=1-\sum_{k=0}^{n-1}(-1)^k\binom{2k+1}{k}F_{3(k+2)},$$
$$\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}L_{2(n+1)+k}=3-\sum_{k=0}^{n-1}(-1)^k\binom{2k+1}{k}L_{3(k+2)}.$$

Proof. Set $x=1/\alpha$ and $x=1/\beta$ in (39) and (40), respectively, and combine according to the Binet formulas. □

Remark 1. *Combining Theorem 6 with Theorem 10 gives the relations*

$$\sum_{k=0}^{n}\binom{n+k}{k}F_{n+1+2k}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{k}F_{2(n+1)+k},$$
$$\sum_{k=0}^{n}\binom{n+k}{k}L_{n+1+2k}=2(-1)^n+\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{k}L_{2(n+1)+k}.$$

Theorem 11. *For each non-negative integer n, we have the relations*

$$3^n\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}F_{2(n+1+2k)}=1-\sum_{k=0}^{n-1}(-3)^k\binom{2k+1}{k}\left(3F_{6k+8}+F_{6k+10}\right),$$
$$3^n\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}L_{2(n+1+2k)}=3-\sum_{k=0}^{n-1}(-3)^k\binom{2k+1}{k}\left(3L_{6k+8}+L_{6k+10}\right).$$

Proof. Set $x=-\alpha^4$ and $x=-\beta^4$ in (39) and (40), respectively, and combine according to the Binet formulas. □

The last Theorem in this set involves mixed identities.

Theorem 12. *For each non-negative integer n, we have the relations*

$$2^n\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}F_{2(n+1)+3k}=1-\sum_{k=0}^{n-1}(-2)^k\binom{2k+1}{k}L_{5k+8},$$
$$2^n\sum_{k=0}^{n}(-1)^k\binom{n+k}{k}L_{2(n+1)+3k}=3-5\sum_{k=0}^{n-1}(-2)^k\binom{2k+1}{k}F_{5k+8}.$$

Proof. Set $x=-\alpha^3$ and $x=-\beta^3$ in (39) and (40), respectively, and combine according to the Binet formulas. □

As a final remark in this section, we note that some of the identities presented in this section follow also from the following lemma.

Lemma 7 ([24] Identities 6.22, 6.23)**.** *If x is a complex variable and m, n are non-negative integers, then*

$$\sum_{k=0}^{n}\binom{n+k}{k}\big((1-x)^{n+1}x^{k}+x^{n+1}(1-x)^{k}\big)=1,\qquad x\neq 0, \tag{44}$$

$$\sum_{k=0}^{n}\binom{n+k}{k}\big((1-x)^{n+1}+x^{n+1-k}(1-x)^{k}\big)=x^{n+1},\qquad x\neq 0,\ x\neq 1. \tag{45}$$

For instance, identity (42) is an immediate consequence of (44) at $x=\alpha$. Also, (43) follows easily from (44).

4. Relations Involving Two Central Binomial Coefficients

Lemma 8. *Let x be a complex variable. Then*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}x^{2k}=\sum_{k=0}^{n}\binom{n}{k}^{2}(1+x)^{2k}(1-x)^{2(n-k)}. \tag{46}$$

Proof. From Riordan's book [28], it is known that for the polynomial

$$A_n(t)=\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}t^{k}$$

we have the relation

$$A_n\big((2t-1)^2\big)=4^n\sum_{k=0}^{n}\binom{n}{k}^{2}t^{2k}(1-t)^{2(n-k)}.$$

Set $x=2t-1$ and simplify. □

Theorem 13. *For each integer r and each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}F_{2k+r}=\sum_{k=0}^{n}\binom{n}{k}^{2}F_{6k-2n+r},$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}L_{2k+r}=\sum_{k=0}^{n}\binom{n}{k}^{2}L_{6k-2n+r}.$$

Proof. Set $x=\alpha$ and $x=\beta$ in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas (3). □

Theorem 14. *For each integer r and each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}F_{4k+r}=F_{2n+r}\sum_{k=0}^{n}\binom{n}{k}^{2}5^{k},$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}L_{4k+r}=L_{2n+r}\sum_{k=0}^{n}\binom{n}{k}^{2}5^{k}.$$

Proof. Set $x=\alpha^2$ and $x=\beta^2$ in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

We note the following particular results:

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}F_{2(2k-n)}=0,$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}L_{2(2k-n)}=2\sum_{k=0}^{n}\binom{n}{k}^{2}5^{k}.$$

Theorem 15. *For each integer r and each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}4^{n-k}F_{2k+r}=\sum_{k=0}^{n}\binom{n}{k}^{2}5^{k}F_{6k-4n+r},$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}4^{n-k}L_{2k+r}=\sum_{k=0}^{n}\binom{n}{k}^{2}5^{k}L_{6k-4n+r}.$$

Proof. Set $x=\alpha/2$ and $x=\beta/2$ in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

Theorem 16. *For each integer r and each non-negative integer n, we have the relations*

$$F_r\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}5^{k}4^{n-k}=\sum_{k=0}^{n}\binom{n}{k}^{2}F_{6(2k-n)+r},$$
$$L_r\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}5^{k}4^{n-k}=\sum_{k=0}^{n}\binom{n}{k}^{2}L_{6(2k-n)+r}.$$

Proof. Set $x=\sqrt{5}/2$ and $x=-\sqrt{5}/2$ in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

Theorem 17. *For each integer r and each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}F_{6k+r}=4^{n}\sum_{k=0}^{n}\binom{n}{k}^{2}F_{2(n+k)+r},$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}L_{6k+r}=4^{n}\sum_{k=0}^{n}\binom{n}{k}^{2}L_{2(n+k)+r}.$$

Proof. Set $x=\alpha^3$ and $x=\beta^3$ in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

Theorem 18. *For each integer r and each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}F_{8k+r}=F_{4n+r}\sum_{k=0}^{n}\binom{n}{k}^{2}9^{k}5^{n-k},$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}L_{8k+r}=L_{4n+r}\sum_{k=0}^{n}\binom{n}{k}^{2}9^{k}5^{n-k}.$$

Proof. Set $x=\alpha^4$ and $x=\beta^4$ in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

In particular,

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}F_{4(2k-n)}=0$$

and

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}L_{4(2k-n)}=2\sum_{k=0}^{n}\binom{n}{k}^{2}9^{k}5^{n-k}.$$

We proceed with some identities involving an additional parameter.

Theorem 19. *If n is a non-negative integer and r, s are any integers, then*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}q^{2s(n-k)}W_{4ks+r}=W_{2ns+r}\sum_{k=0}^{n}\binom{n}{k}^2\Delta^{2k}V_s^{2(n-k)}U_s^{2k}.$$

Proof. Set $x=\sigma^s/\tau^s$ and $x=\tau^s/\sigma^s$, in turn, in Lemma 8, multiply through by τ^r and σ^r, respectively, and combine according to the Binet formulas. □

In particular,

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}q^{2s(n-k)}U_{4ks+r}=U_{2ns+r}\sum_{k=0}^{n}\binom{n}{k}^2\Delta^{2k}V_s^{2(n-k)}U_s^{2k},$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}q^{2s(n-k)}V_{4ks+r}=V_{2ns+r}\sum_{k=0}^{n}\binom{n}{k}^2\Delta^{2k}V_s^{2(n-k)}U_s^{2k};$$

with the special cases

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}F_{4sk+r}=F_{2ns+r}\sum_{k=0}^{n}\binom{n}{k}^2(5F_s^2)^{n-k}L_s^{2k},\tag{47}$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}L_{4sk+r}=L_{2ns+r}\sum_{k=0}^{n}\binom{n}{k}^2(5F_s^2)^{n-k}L_s^{2k}.\tag{48}$$

Remark 2. *Note that Theorems 14 and 18 are particular cases of (47) and (48) at $s=1$ and $s=2$, respectively.*

Theorem 20. *For integers r and $s\geq 1$, and each non-negative integer n, we have the relations*

$$F_r\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}\left(5F_s^2\right)^kL_s^{2(n-k)}=4^n\sum_{k=0}^{n}\binom{n}{k}^2F_{2s(2k-n)+r},$$
$$L_r\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}\left(5F_s^2\right)^kL_s^{2(n-k)}=4^n\sum_{k=0}^{n}\binom{n}{k}^2L_{2s(2k-n)+r}.$$

Proof. Set $x=\sqrt{5}F_s/L_s$ and $x=-\sqrt{5}F_s/L_s$ in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

Theorem 21. *For integers r and $s\geq 1$, and each non-negative integer n, we have the relations*

$$F_r\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}(5F_s^2)^{n-k}L_s^{2k}=4^n\sum_{k=0}^{n}\binom{n}{k}^2F_{2s(2k-n)+r},$$
$$L_r\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}(5F_s^2)^{n-k}L_s^{2k}=4^n\sum_{k=0}^{n}\binom{n}{k}^2L_{2s(2k-n)+r}.$$

Proof. Set $x=L_s/(\sqrt{5}F_s)$ and $x=-L_s/(\sqrt{5}F_s)$ in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

Theorem 22. *For each integer r and each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}5^{n-k}F_{6k+r}=4^{n}\sum_{k=0}^{n}\binom{n}{k}^{2}4^{k}F_{2k+r},$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}5^{n-k}L_{6k+r}=4^{n}\sum_{k=0}^{n}\binom{n}{k}^{2}4^{k}L_{2k+r}.$$

Proof. Set $x=\alpha^3/\sqrt{5}$ and $x=\beta^3/\sqrt{5}$, in turn, in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas, using also the fact that $\sqrt{5}-\alpha^3=-2$ and $\sqrt{5}+\alpha^3=4\alpha$. □

Theorem 23. *For each integer r, and each non-negative integer n, we have the relations*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}9^{n-k}F_{6k+r}=4^{n}\sum_{k=0}^{n}\binom{n}{k}^{2}5^{k}F_{4k-2n+r},$$
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}9^{n-k}L_{6k+r}=4^{n}\sum_{k=0}^{n}\binom{n}{k}^{2}5^{k}L_{4k-2n+r}.$$

Proof. Set $x=\alpha^3/3$ and $x=\beta^3/3$, in turn, in Lemma 8, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas, using also the fact that $3-\alpha^3=2\beta$ and $3+\alpha^3=2\sqrt{5}\alpha$. □

Theorem 24. *If n is a non-negative integer and r,s are any integers, then*

$$\sum_{k=0}^{n}\binom{n}{k}^{2}q^{2s(n-k)}W_{4sk+r}=\frac{W_{2sn+r}}{4^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}\Delta^{2k}V_s^{2(n-k)}U_s^{2k}.$$

Proof. Set $x=\Delta U_s/V_s$ in Lemma 8 and multiply through by σ^r. Repeat for $x=-\Delta U_s/V_s$ and multiply through by τ^r. Now, combine the resulting equations using the Binet formula. □

In particular,

$$\sum_{k=0}^{n}\binom{n}{k}^{2}q^{2s(n-k)}U_{4sk+r}=\frac{U_{2sn+r}}{4^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}\Delta^{2k}V_s^{2(n-k)}U_s^{2k},$$
$$\sum_{k=0}^{n}\binom{n}{k}^{2}q^{2s(n-k)}V_{4sk+r}=\frac{V_{2sn+r}}{4^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}\Delta^{2k}V_s^{2(n-k)}U_s^{2k};$$

with the special cases

$$\sum_{k=0}^{n}\binom{n}{k}^{2}F_{4sk+r}=\frac{F_{2sn+r}}{4^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}5^{k}L_s^{2(n-k)}F_s^{2k},$$
$$\sum_{k=0}^{n}\binom{n}{k}^{2}L_{4sk+r}=\frac{L_{2sn+r}}{4^n}\sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}5^{k}L_s^{2(n-k)}F_s^{2k}.$$

5. Another Class of Identities with Squared Binomial Coefficients

Lemma 9 ([29])**.** *If n is a non-negative integer and x is any complex variable, then*

$$\sum_{k=0}^{n}\binom{n}{k}^{2}x^{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(x-1)^{n-k}. \tag{49}$$

Theorem 25. *Let r, s and m be arbitrary integers with $r \neq 0$. Then, for each non-negative integer n, we have the relations*

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{k(s+1)} \binom{n}{k}^2 \left(\frac{F_s}{F_r}\right)^k F_{(r+s)k+m} \\ = \sum_{k=0}^{n} (-1)^{(s+1)(n-k)} \binom{n}{k}\binom{n+k}{k} \left(\frac{F_{r+s}}{F_r}\right)^{n-k} F_{s(n-k)+m},\end{aligned} \tag{50}$$

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{k(s+1)} \binom{n}{k}^2 \left(\frac{F_s}{F_r}\right)^k L_{(r+s)k+m} \\ = \sum_{k=0}^{n} (-1)^{(s+1)(n-k)} \binom{n}{k}\binom{n+k}{k} \left(\frac{F_{r+s}}{F_r}\right)^{n-k} L_{s(n-k)+m}.\end{aligned} \tag{51}$$

Proof. Set $x = -F_s\beta^r/(F_r\alpha^s)$ and $x = -F_s\alpha^r/(F_r\beta^s)$, respectively, in Lemma 9, and use Lemma 3. Multiply through by α^m and β^m, respectively, and combine according to the Binet formulas. □

Corollary 8. *For each integer m and each non-negative integer n, we have*

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{k+m} = (-1)^{m+1} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{n+k}{k} F_{n-k-m},$$
$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{k+m} = (-1)^{m} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}\binom{n+k}{k} L_{n-k-m}.$$

Proof. Set $r = 2$ and $s = -1$ in Theorem 25. □

The case $m = 0$ in Corollary 8 was proposed by Carlitz as a problem in the Fibonacci Quarterly [30] (with a typo).

Corollary 9. *For each integer m and each non-negative integer n, we have*

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{2k+m} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} F_{n-k+m},$$
$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{2k+m} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} L_{n-k+m}.$$

Proof. Set $r = s = 1$ in Theorem 25. □

The case $m = 0$ in Corollary 9 was proposed by Carlitz as another problem in the Fibonacci Quarterly [31].

Corollary 10. *For each integer m and each non-negative integer n, we have*

$$\sum_{k=0}^{n} \binom{n}{k}^2 F_{3k+m} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} 2^{n-k} F_{n-k+m},$$
$$\sum_{k=0}^{n} \binom{n}{k}^2 L_{3k+m} = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} 2^{n-k} L_{n-k+m}.$$

Proof. Set $r = 2$ and $s = 1$ in Theorem 25. □

Corollary 11. *For each integer m and each non-negative integer n, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2F_{k+m}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}F_{2(n-k)+m},$$
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2L_{k+m}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}L_{2(n-k)+m}.$$

Proof. Set $r=-1$ and $s=2$ in Theorem 25. □

Corollary 12. *For each integer r and each non-negative integer n, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2F_{3k+m}=\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k}\binom{n+k}{k}F_{2(n-k)+m},$$
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2L_{3k+m}=\sum_{k=0}^{n}(-2)^{n-k}\binom{n}{k}\binom{n+k}{k}L_{2(n-k)+m}.$$

Proof. Set $r=1$ and $s=2$ in Theorem 25. □

Corollary 13. *For each integer r and each non-negative integer n, we have*

$$\sum_{k=0}^{n}(-1)^{(r+1)k}\binom{n}{k}^2F_{2rk+m}=\sum_{k=0}^{n}(-1)^{(r+1)(n-k)}\binom{n}{k}\binom{n+k}{k}L_r^{n-k}F_{r(n-k)+m},$$
$$\sum_{k=0}^{n}(-1)^{(r+1)k}\binom{n}{k}^2L_{2rk+m}=\sum_{k=0}^{n}(-1)^{(r+1)(n-k)}\binom{n}{k}\binom{n+k}{k}L_r^{n-k}L_{r(n-k)+m}.$$

Proof. Set $s=r$ in Theorem 25. □

Corollary 14. *For each integer r and each non-negative integer n, we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2L_r^kF_{3rk+m}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}(L_{2r}+(-1)^r)^{n-k}F_{2r(n-k)+m},$$
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2L_r^kL_{3rk+m}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}(L_{2r}+(-1)^r)^{n-k}L_{2r(n-k)+m}.$$

Proof. Set $s=2r$ in Theorem 25 and make use of $F_{3r}/F_r=L_{2r}+(-1)^r$. □

Theorem 26. *For each non-negative integers r, m and n, we have the relations*

$$\sum_{k=0}^{n}\binom{n}{k}^2L_m^{n-k}F_{mk+r}=(-1)^{r-1}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}L_m^kF_{m(n-k)-r},$$
$$\sum_{k=0}^{n}\binom{n}{k}^2L_m^{n-k}L_{mk+r}=(-1)^{r}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}L_m^kL_{m(n-k)-r}.$$

Proof. Set $x=\alpha^m/L_m$ and $x=\beta^m/L_m$ in Lemma 9, multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

When $m = 1$, then Theorem 26 reduces to Corollary 8. When $m = 2$, then

$$\sum_{k=0}^{n}\binom{n}{k}^2 3^{n-k}F_{2k+r} = (-1)^{n-r-1}\sum_{k=0}^{n}(-3)^k\binom{n}{k}\binom{n+k}{k}F_{2(n-k)-r},$$
$$\sum_{k=0}^{n}\binom{n}{k}^2 3^{n-k}L_{2k+r} = (-1)^{n-r}\sum_{k=0}^{n}(-3)^k\binom{n}{k}\binom{n+k}{k}L_{2(n-k)-r}.$$

Theorem 27. *For each non-negative integer n, any odd integer m and any integer r, we have the relations*

$$\sum_{k=0}^{n}\binom{n}{k}^2 F_{2mk+r} = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}L_m^{n-k}F_{m(n-k)+r},$$
$$\sum_{k=0}^{n}\binom{n}{k}^2 L_{2mk+r} = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}L_m^{n-k}L_{m(n-k)+r}.$$

Proof. Set $x = \alpha^{2m}$ and $x = \beta^{2m}$, m odd, in Lemma 9, and use the facts that $\alpha^{2m} - 1 = \alpha^m L_m$ and $\beta^{2m} - 1 = \beta^m L_m$. Multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

When $m = 1$, then Theorem 27 reduces to Corollary 9.

We conclude this section with a sort of inverse relation compared to those from Theorem 25.

Theorem 28. *For each non-negative integer n and any integers m, r and s, we have the relations*

$$\sum_{k=0}^{n}(-1)^{sk}\binom{n}{k}^2\left(\frac{F_{r+s}}{F_r}\right)^k F_{sk+m} = \sum_{k=0}^{n}(-1)^{s(n-k)}\binom{n}{k}\binom{n+k}{k}\left(\frac{F_s}{F_r}\right)^{n-k}F_{(r+s)(n-k)+m},$$
$$\sum_{k=0}^{n}(-1)^{sk}\binom{n}{k}^2\left(\frac{F_{r+s}}{F_r}\right)^k L_{sk+m} = \sum_{k=0}^{n}(-1)^{s(n-k)}\binom{n}{k}\binom{n+k}{k}\left(\frac{F_s}{F_r}\right)^{n-k}L_{(r+s)(n-k)+m}.$$

Proof. Set $x = F_{r+s}/(\alpha^s F_r)$ and $x = F_{r+s}/(\beta^s F_r)$, respectively, in Lemma 9, and use Lemma 3. Multiply through by α^m and β^m, respectively, and combine according to the Binet formulas. □

6. More Identities with Two Binomial Coefficients

Lemma 10 ([32] Identity (3.17))**.** *If n is a non-negative integer, m is any real number and x is any complex variable, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{n}(x-1)^k = \sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}x^k. \tag{52}$$

In particular,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}(x-1)^k = \sum_{k=0}^{n}\binom{n}{k}^2 x^k \tag{53}$$

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}(x-1)^k = \binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}x^k. \tag{54}$$

Proof. The first particular case is obvious. The second follows upon setting $m = n - 1/2$ in (52) and using $\binom{n-1/2}{k} = \frac{\binom{2n}{n}\binom{n}{k}}{2^{2k}\binom{2(n-k)}{n-k}}$ with $0 \le k \le n$ (see [32] Identity (Z.45)). □

Remark 3. *Comparing* (49) *with* (53)*, we immediately obtain an "identity" of the form*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}(x-1)^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(x-1)^{n-k}.$$

Such an identity does not contain any new information as the identities can be trivially transformed into each other by reindexing

$$\sum_{k=0}^{n}a_k=\sum_{k=0}^{n}a_{n-k}.$$

This shows that the binomial sum relations from the previous section are actually special instances of those derived now.

Theorem 29. *Let r, s and p be arbitrary integers with $r \neq 0$, and let m be any real number. Then, for all non-negative integer n, we have the relations*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{(s+1)k}\binom{n}{k}\binom{m+n-k}{n}\left(\frac{F_{r+s}}{F_r}\right)^k F_{sk+p}&\\ =\sum_{k=0}^{n}(-1)^{(s+1)k}\binom{n}{k}\binom{m}{k}\left(\frac{F_s}{F_r}\right)^k F_{(r+s)k+p},&\end{aligned}\tag{55}$$

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{(s+1)k}\binom{n}{k}\binom{m+n-k}{n}\left(\frac{F_{r+s}}{F_r}\right)^k L_{sk+p}&\\ =\sum_{k=0}^{n}(-1)^{(s+1)k}\binom{n}{k}\binom{m}{k}\left(\frac{F_s}{F_r}\right)^k L_{(r+s)k+p}.&\end{aligned}\tag{56}$$

In particular,

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{(s+1)k}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}\left(\frac{F_{r+s}}{F_r}\right)^k F_{sk+p}&\\ =\binom{2n}{n}\sum_{k=0}^{n}(-1)^{(s+1)k}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}\left(\frac{F_s}{F_r}\right)^k F_{(r+s)k+p},&\\ \sum_{k=0}^{n}(-1)^{(s+1)k}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}\left(\frac{F_{r+s}}{F_r}\right)^k L_{sk+p}&\\ =\binom{2n}{n}\sum_{k=0}^{n}(-1)^{(s+1)k}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}\left(\frac{F_s}{F_r}\right)^k L_{(r+s)k+p}.&\end{aligned}$$

Proof. Set $x=-F_s\beta^r/(F_r\alpha^s)$ and $x=-F_s\alpha^r/(F_r\beta^s)$, respectively, in (52), and use Lemma 3. Multiply through by α^p and β^p, respectively, and combine according to the Binet formulas. □

Corollary 15. *If n is a non-negative integer, m is any real number and p is any integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{n}F_{p-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}F_{p+k},$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{n}L_{p-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}L_{p+k}.$$

In particular,

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}F_{p-k}=\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}F_{p+k},$$
$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}L_{p-k}=\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}L_{p+k}.$$

Corollary 16. *If n is a non-negative integer, m is any real number and p is any integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{n}F_{k+p}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}F_{2k+p},$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{n}L_{k+p}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}L_{2k+p}.$$

In particular,

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}F_{k+p}=\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}F_{2k+p},$$
$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}L_{k+p}=\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}L_{2k+p}.$$

Corollary 17. *If n is a non-negative integer, m is any real number and p is any integer, then*

$$\sum_{k=0}^{n}(\pm 2)^k\binom{n}{k}\binom{m+n-k}{n}F_{(\frac{3\mp 1}{2})k+p}=\sum_{k=0}^{n}(\pm 1)^k\binom{n}{k}\binom{m}{k}F_{3k+p},\tag{57}$$
$$\sum_{k=0}^{n}(\pm 2)^k\binom{n}{k}\binom{m+n-k}{n}L_{(\frac{3\mp 1}{2})k+p}=\sum_{k=0}^{n}(\pm 1)^k\binom{n}{k}\binom{m}{k}L_{3k+p}.\tag{58}$$

In particular,

$$\sum_{k=0}^{n}(\pm 2)^k\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}F_{(\frac{3\mp 1}{2})k+p}=\binom{2n}{n}\sum_{k=0}^{n}(\pm 1)^k\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}F_{3k+p},$$
$$\sum_{k=0}^{n}(\pm 2)^k\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}L_{(\frac{3\mp 1}{2})k+p}=\binom{2n}{n}\sum_{k=0}^{n}(\pm 1)^k\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}L_{3k+p}.$$

Corollary 18. *If n is a non-negative integer, m is any real number and p is any integer, then*

$$\sum_{k=0}^{n}(-3)^k\binom{n}{k}\binom{m+n-k}{n}F_{2k+p}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m}{k}F_{4k+p},$$
$$\sum_{k=0}^{n}(-3)^k\binom{n}{k}\binom{m+n-k}{n}L_{2k+p}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m}{k}L_{4k+p}.$$

In particular,

$$\sum_{k=0}^{n}(-3)^k\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}F_{2k+p}=\binom{2n}{n}\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}F_{4k+p},$$
$$\sum_{k=0}^{n}(-3)^k\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}L_{2k+p}=\binom{2n}{n}\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}L_{4k+p}.$$

Theorem 30. *If n is a non-negative integer, m is any real number and s, r are any integers, then*

$$\sum_{k=0}^{n}(-1)^{k(s-1)}\binom{n}{k}\binom{m+n-k}{n}\frac{F_{r-sk}}{L_s^k}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{F_{sk+r}}{L_s^k},$$
$$\sum_{k=0}^{n}(-1)^{k(s-1)}\binom{n}{k}\binom{m+n-k}{n}\frac{L_{r-sk}}{L_s^k}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\frac{L_{sk+r}}{L_s^k}.$$

In particular,

$$\sum_{k=0}^{n}(-1)^{k(s-1)}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}\frac{F_{r-sk}}{L_s^k}=\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}\frac{F_{sk+r}}{L_s^k},$$
$$\sum_{k=0}^{n}(-1)^{k(s-1)}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}\frac{L_{r-sk}}{L_s^k}=\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}\frac{L_{sk+r}}{L_s^k}.$$

Proof. Set $x=\alpha^s/L_s$ and $x=\beta^s/L_s$ in (52), multiply through by α^r and β^r, respectively, and combine according to the Binet formulas. □

Theorem 31. *If n is a non-negative integer, m is any real number, r is any integer and s is an odd integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{n}L_s^kF_{sk+r}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}F_{2sk+r},$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+n-k}{n}L_s^kL_{sk+r}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}L_{2sk+r}.$$

In particular,

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}L_s^kF_{sk+r}=\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}F_{2sk+r},$$
$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}L_s^kL_{sk+r}=\binom{2n}{n}\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{\binom{2(n-k)}{n-k}}2^{2(n-k)}L_{2sk+r}.$$

Proof. Set $x=\alpha^{2s}$ in (52) and use the fact that $\alpha^{2s}-1=\alpha^sL_s$ if s is an odd integer. □

Theorem 32. *Let n be a non-negative integer, m be a real number and r, s, p and t be any integers. Then*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+n-k}{n}\frac{L_{r+s}^k}{L_r^k}L_{s(p-k)+t}&=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{m}{2k}\frac{5^kF_s^{2k}}{L_r^{2k}}L_{sp-2k(r+s)+t}\\&+(-1)^r\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\binom{m}{2k-1}\frac{5^kF_s^{2k-1}}{L_r^{2k-1}}F_{sp-(2k-1)(r+s)+t},\end{aligned}\tag{59}$$

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+n-k}{n}\frac{L_{r+s}^k}{L_r^k}F_{s(p-k)+t}&=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{m}{2k}\frac{5^kF_s^{2k}}{L_r^{2k}}F_{sp-2k(r+s)+t}\\&+(-1)^r\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\binom{m}{2k-1}\frac{5^{k-1}F_s^{2k-1}}{L_r^{2k-1}}L_{sp-(2k-1)(r+s)+t}.\end{aligned}\tag{60}$$

Proof. Set $x = \sqrt{5}\beta^r F_s/(L_r\alpha^s)$ in (52) and use (12) to obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+n-k}{n}L_r^{p-k}L_{r+s}^k\alpha^{s(p-k)+t}$$
$$=\sum_{k=0}^{n}(-1)^{kr}\binom{n}{k}\binom{m}{k}(\sqrt{5})^k L_r^{p-k}F_s^k\alpha^{sp-(r+s)k+t},$$

from which the results follow. □

Corollary 19. *Let n be a non-negative integer, let m be a real number and let s and p be any integers. Then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+n-k}{n}\frac{2^{k-1}L_{sk}}{L_s^k}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{m}{2k}\frac{5^kF_s^{2k}}{L_s^{2k}},$$
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m+n-k}{n}\frac{2^{k-1}F_{sk}}{L_s^k}=-\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}\binom{m}{2k-1}\frac{5^{k-1}F_s^{2k-1}}{L_s^{2k-1}}.$$

Proof. Set $t=-sp$ and $r=-s$ in (59) and (60). □

Corollary 20. *Let n be a non-negative integer and let s and p be any integers. Then*

$$\sum_{k=0}^{n}(-2)^k\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}\frac{L_{sk}}{L_s^k}=\binom{2n}{n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{\binom{n}{2k}^2}{\binom{2(n-2k)}{n-2k}}\frac{5^k2^{2(n-2k)+1}F_s^{2k}}{L_s^{2k}},$$
$$\sum_{k=0}^{n}(-2)^k\frac{\binom{n}{k}\binom{2(2n-k)}{2n-k}\binom{2n-k}{n}}{\binom{2(n-k)}{n-k}}\frac{F_{sk}}{L_s^k}=-\binom{2n}{n}\sum_{k=1}^{\lceil n/2\rceil}\frac{\binom{n}{2k-1}^2}{\binom{2(n-2k+1)}{n-2k+1}}\frac{2^{2(n-2k+1)+1}5^{k-1}F_s^{2k-1}}{L_s^{2k-1}}.$$

7. Still Other Classes of Identities with Two Binomial Coefficients

Lemma 11 ([32] Identity (3.18))**.** *If n is a non-negative integer, m is any real number and x, y are any complex variables, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}(x-y)^{n-k}y^k=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}x^{n-k}y^k. \tag{61}$$

Theorem 33. *If n is a non-negative integer, m is any real number and r, s and t are any integers, then*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}q^{rk}U_r^{n-k}U_s^kW_{t-(r+s)k+sn}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}q^{rk}U_{r+s}^{n-k}U_s^kW_{t-rk}.$$

Proof. Set $x=U_{r+s}$ and $y=\tau^rU_s$ in (61) and multiply by σ^t to obtain

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}\sigma^{s(n-k)+t}\tau^{rk}U_r^{n-k}U_s^k=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\sigma^t\tau^{rk}U_{r+s}^{n-k}U_s^k. \tag{62}$$

Similarly obtain

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}\sigma^{rk}\tau^{s(n-k)+t}U_r^{n-k}U_s^k=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\sigma^{rk}\tau^tU_{r+s}^{n-k}U_s^k. \tag{63}$$

Combine (62) and (63) using the Binet formula. □

In particular,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}q^{rk}U_r^{n-k}U_s^kV_{t-(r+s)k+sn}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}q^{rk}U_{r+s}^{n-k}U_s^kV_{t-rk} \tag{64}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}q^{rk}U_r^{n-k}U_s^kU_{t-(r+s)k+sn}=\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}q^{rk}U_{r+s}^{n-k}U_s^kU_{t-rk};$$

with the special cases

$$\sum_{k=0}^{n}(-1)^{rk}\binom{n}{k}\binom{m+k}{k}F_r^{n-k}F_s^kL_{t-(r+s)k+sn}=\sum_{k=0}^{n}(-1)^{rk}\binom{n}{k}\binom{m}{k}F_{r+s}^{n-k}F_s^kL_{t-rk},$$
$$\sum_{k=0}^{n}(-1)^{rk}\binom{n}{k}\binom{m+k}{k}F_r^{n-k}F_s^kF_{t-(r+s)k+sn}=\sum_{k=0}^{n}(-1)^{rk}\binom{n}{k}\binom{m}{k}F_{r+s}^{n-k}F_s^kF_{t-rk}.$$

Lemma 12 ([32] Identity (3.84))**.** *If n is a non-negative integer and x is any complex variable, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}2^{n-2k}x^k=\sum_{k=0}^{n}(-1)^k\binom{2(n-k)}{n-k}(x-1)^k.$$

Lemma 13 ([33,34])**.** *If n is a non-negative integer and x is any complex variable, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}(1+x)^k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k}x^k.$$

Next, we present an obvious extension of (33) and some associated Fibonacci–Lucas sums.

Lemma 14. *Let x and y be complex variables, let m and n be non-negative integers and let r be any integer. Then*

$$\sum_{k=0}^{n}(-1)^k\binom{m-n+k}{k}\binom{n-k}{r}x^{n-k-r}y^k=\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}(x-y)^{n-k-r}y^k. \tag{65}$$

Theorem 34. *If m and n are non-negative integers and s, r and t are any integers, then*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{m-n+k}{k}\binom{n-k}{r}q^{s(n-k-r)}W_{2sk+t}\\=\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}V_s^{n-k-r}W_{s(n+k-r)+t}.\end{aligned} \tag{66}$$

Proof. Choose $x=q^s$ and $y=\tau^{2s}$ in (65), use (4), and multiply through by τ^t to obtain

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{m-n+k}{k}\binom{n-k}{r}q^{s(n-k-r)}\tau^{2sk+t}\\=\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}\tau^{s(n+k-r)+t}V_s^{n-k-r}.\end{aligned} \tag{67}$$

Similarly obtain

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{m-n+k}{k}\binom{n-k}{r}q^{s(n-k-r)}\sigma^{2sk+t}\\=\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}\sigma^{s(n+k-r)+t}V_s^{n-k-r},\end{aligned}\tag{68}$$

The result follows from (67), (68) and the Binet formula. □

In particular,

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{m-n+k}{k}\binom{n-k}{r}q^{s(n-k-r)}U_{2sk+t}\\=\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}V_s^{n-k-r}U_{s(n+k-r)+t},\\\sum_{k=0}^{n}(-1)^k\binom{m-n+k}{k}\binom{n-k}{r}q^{s(n-k-r)}V_{2sk+t}\\=\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}V_s^{n-k-r}V_{s(n+k-r)+t};\end{aligned}$$

with the special cases

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{k(s+1)}\binom{m-n+k}{k}\binom{n-k}{r}L_{2sk+t}\\=\frac{(-1)^{s(n-r)}}{L_s^r}\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}L_s^{n-k}L_{s(n+k-r)+t},\\\sum_{k=0}^{n}(-1)^{k(s+1)}\binom{m-n+k}{k}\binom{n-k}{r}F_{2sk+t}\\=\frac{(-1)^{s(n-r)}}{L_s^r}\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}L_s^{n-k}F_{s(n+k-r)+t}.\end{aligned}$$

Theorem 35. *Let m, n, r, s and t be integers with $n\geq 0$. If n and r have the same parity, then*

$$\begin{aligned}&W_t\sum_{k=0}^{\lfloor n/2\rfloor}\binom{m-n+2k}{2k}\binom{n-2k}{r}\Delta^{n-2k-r}U_s^{n-2k-r}V_s^{2k}\\&\quad-(W_{t+1}-qW_{t-1})\sum_{k=1}^{\lceil n/2\rceil}\binom{m-n+2k-1}{2k-1}\binom{n-2k+1}{r}\Delta^{n-2k-r}U_s^{n-2k-r+1}V_s^{2k-1}\\&\quad=\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}2^{n-k-r}V_s^kW_{s(n-k-r)+t},\end{aligned}$$

while if n and r have different parities, then

$$\begin{aligned}&(W_{t+1}-qW_{t-1})\sum_{k=0}^{\lfloor n/2\rfloor}\binom{m-n+2k}{2k}\binom{n-2k}{r}\Delta^{n-2k-r-1}U_s^{n-2k-r}V_s^{2k}\\&\quad-W_t\sum_{k=1}^{\lceil n/2\rceil}\binom{m-n+2k-1}{2k-1}\binom{n-2k+1}{r}\Delta^{n-2k+1-r}U_s^{n-2k-r+1}V_s^{2k-1}\\&\quad=\sum_{k=0}^{n}(-1)^k\binom{m+1}{k}\binom{n-k}{r}2^{n-k-r}V_s^kW_{s(n-k-r)+t}.\end{aligned}$$

Proof. Set $y = V_s$ and $x = \Delta U_s$ in (65), then use Lemma 4 and the summation identity $\sum_{j=0}^{n} f_j = \sum_{j=0}^{\lfloor n/2 \rfloor} f_{2j} + \sum_{j=1}^{\lceil n/2 \rceil} f_{2j-1}$. □

In particular, if n and r have the same parity, then

$$\begin{aligned} U_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} U_s^{n-2k-r} V_s^{2k} \Delta^{n-2k-r} \\ - V_t \sum_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r} U_s^{n-2k-r+1} V_s^{2k-1} \\ = \sum_{k=0}^{n} (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k U_{s(n-k-r)+t}, \end{aligned} \tag{69}$$

$$\begin{aligned} V_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} \Delta^{n-2k-r} U_s^{n-2k-r} V_s^{2k} \\ - U_t \sum_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r+2} U_s^{n-2k-r+1} V_s^{2k-1} \\ = \sum_{k=0}^{n} (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k V_{s(n-k-r)+t}; \end{aligned} \tag{70}$$

while if n and r have different parities, then

$$\begin{aligned} V_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} \Delta^{n-2k-r-1} U_s^{n-2k-r} V_s^{2k} \\ - U_t \sum_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r+1} U_s^{n-2k-r+1} V_s^{2k-1} \\ = \sum_{k=0}^{n} (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k U_{s(n-k-r)+t}, \end{aligned} \tag{71}$$

and

$$\begin{aligned} U_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} \Delta^{n-2k-r+1} U_s^{n-2k-r} V_s^{2k} \\ - V_t \sum_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} \Delta^{n-2k-r+1} U_s^{n-2k-r+1} V_s^{2k-1} \\ = \sum_{k=0}^{n} (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} V_s^k V_{s(n-k-r)+t}; \end{aligned} \tag{72}$$

with the special cases:

$$\begin{aligned} L_t \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k} \binom{n-2k}{r} 5^{(n-r)/2-k} F_s^{n-2k-r} L_s^{2k} \\ - F_t \sum_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1} \binom{n-2k+1}{r} 5^{(n-r)/2-k+1} F_s^{n-2k-r+1} L_s^{2k-1} \\ = \sum_{k=0}^{n} (-1)^k \binom{m+1}{k} \binom{n-k}{r} 2^{n-k-r} L_s^k L_{s(n-k-r)+t}, \end{aligned}$$

$$
\begin{aligned}
F_t \sum_{k=0}^{\lfloor n/2 \rfloor} & \binom{m-n+2k}{2k}\binom{n-2k}{r} 5^{(n-r)/2-k} F_s^{n-2k-r} L_s^{2k} \\
& - L_t \sum_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1}\binom{n-2k+1}{r} 5^{(n-r)/2-k} F_s^{n-2k-r+1} L_s^{2k-1} \\
& \qquad = \sum_{k=0}^{n} (-1)^k \binom{m+1}{k}\binom{n-k}{r} 2^{n-k-r} L_s^k F_{s(n-k-r)+t};
\end{aligned}
$$

while if n and r have different parities, then

$$
\begin{aligned}
F_t \sum_{k=0}^{\lfloor n/2 \rfloor} & \binom{m-n+2k}{2k}\binom{n-2k}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r} L_s^{2k} \\
& - L_t \sum_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1}\binom{n-2k+1}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r+1} L_s^{2k-1} \\
& \qquad = \sum_{k=0}^{n} (-1)^k \binom{m+1}{k}\binom{n-k}{r} 2^{n-k-r} L_s^k L_{s(n-k-r)+t}, \\
L_t \sum_{k=0}^{\lfloor n/2 \rfloor} & \binom{m-n+2k}{2k}\binom{n-2k}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r} L_s^{2k} \\
& - F_t \sum_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1}\binom{n-2k+1}{r} 5^{(n-r-1)/2-k} F_s^{n-2k-r+1} L_s^{2k-1} \\
& \qquad = \sum_{k=0}^{n} (-1)^k \binom{m+1}{k}\binom{n-k}{r} 2^{n-k-r} L_s^k F_{s(n-k-r)+t}.
\end{aligned}
$$

Note that in simplifying (69)–(72), we used (32).

Corollary 21. *If m, n, r and s are integers, then*

$$
\begin{aligned}
& \sum_{k=0}^{n} (-1)^k \binom{m+1}{k}\binom{n-k}{r} 2^{n-k-r-1} L_s^k L_{s(n-k-r)} \\
& \quad = \begin{cases} \sum\limits_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k}\binom{n-2k}{r} 5^{(n-r)/2-k} F_s^{n-2k-r} L_s^{2k}, & \text{if } n-r \text{ is even;} \\ -\sum\limits_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1}\binom{n-2k+1}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r+1} L_s^{2k-1}, & \text{otherwise;} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{k=0}^{n} (-1)^k \binom{m+1}{k}\binom{n-k}{r} 2^{n-k-r-1} L_s^k F_{s(n-k-r)} \\
& \quad = \begin{cases} -\sum\limits_{k=1}^{\lceil n/2 \rceil} \binom{m-n+2k-1}{2k-1}\binom{n-2k+1}{r} 5^{(n-r)/2-k} F_s^{n-2k-r+1} L_s^{2k-1}, & \text{if } n-r \text{ is even;} \\ \sum\limits_{k=0}^{\lfloor n/2 \rfloor} \binom{m-n+2k}{2k}\binom{n-2k}{r} 5^{(n-r+1)/2-k} F_s^{n-2k-r} L_s^{2k}, & \text{otherwise.} \end{cases}
\end{aligned}
$$

8. Identities with Three Binomial Coefficients

Concerning identities with three binomial coefficients, some classical Fibonacci (Lucas) examples exist. For instance, Carlitz [35] presented the identities

$$\sum_{k=0}^{n}\binom{n}{k}^3 F_k = \sum_{2k\le n}\frac{(n+k)!}{(k!)^3(n-2k)!}F_{2n-3k}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}^3 L_k = \sum_{2k\le n}\frac{(n+k)!}{(k!)^3(n-2k)!}L_{2n-3k}.$$

In addition, Zeitlin [36,37] derived

$$\sum_{k=0}^{2n}\binom{2n}{k}^3 F_{2k} = F_{2n}\sum_{k=0}^{n}\frac{(2n+k)!}{(k!)^3(2n-2k)!}5^{n-k},$$

$$\sum_{k=0}^{2n}\binom{2n}{k}^3 L_{2k} = L_{2n}\sum_{k=0}^{n}\frac{(2n+k)!}{(k!)^3(2n-2k)!}5^{n-k}.$$

In his solution to Carlitz' proposal from above, Zeitlin [38] proved *mutatis mutandis* the identity

$$\sum_{k=0}^{n}\binom{n}{k}^3(-q)^{n-k}p^kW_k = \sum_{2k\le n}\frac{(n+k)!}{(k!)^3(n-2k)!}p^k(-q)^kW_{2n-3k}.$$

His results are based on the polynomial identity

$$\sum_{k=0}^{n}\binom{n}{k}^3 x^k = \sum_{2k\le n}\frac{(n+k)!}{(k!)^3(n-2k)!}x^k(x+1)^{n-2k}.$$

In this section, we provide more examples of this kind using "Zeitlin's identity" in its equivalent form given in the next lemma.

Lemma 15 ([32] Identity (6.7), [39])**.** *If n is a non-negative integer and x is any complex variable, then*

$$\sum_{k=0}^{n}\binom{n}{k}^3 x^k = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{n-k}{k}x^k(1+x)^{n-2k}. \tag{73}$$

Theorem 36. *If n is a non-negative integer and r and t are any integers, then*

$$\sum_{k=0}^{n}\binom{n}{k}^3 q^{rk}W_{r(n-2k)+t} = W_t\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{n-k}{k}q^{rk}V_r^{n-2k}.$$

Proof. Set $x=\tau^r/\sigma^r$ and $x=\sigma^r/\tau^r$, in turn, in (73). Combine according to the Binet formula. □

Corollary 22. *If n is a non-negative integer and r is any integer, then*

$$\sum_{k=0}^{n}\binom{n}{k}^3 q^{rk}U_{r(n-2k)} = 0.$$

In particular,

$$\sum_{k=0}^{n}\binom{n}{k}^3(-1)^{rk}F_{r(n-2k)} = 0.$$

Theorem 37. *If n is a non-negative integer and r, s and t are any integers, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 U_{r+s}^{n-k}U_s^kW_{t+rk}$$
$$=\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}\binom{2k}{k}\binom{n-k}{k}q^{s(n-2k)}U_{r+s}^kU_s^kU_r^{n-2k}W_{t+rk-s(n-2k)}.$$

Proof. Set $x=-\tau^rU_s/U_{r+s}$ and $x=-\sigma^rU_s/U_{r+s}$ in (73), in turn, bearing in mind (10) and (11). Combine the resulting equations using the Binet formula and Lemma 4. □

Corollary 23. *If n is a non-negative integer and r and t are any integers, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3 V_r^{n-k}W_{t+rk}=\sum_{k=0}^{n}(-1)^k\binom{n+k}{2k}\binom{2k}{k}\binom{n-k}{k}q^{r(n-2k)}V_r^kW_{t+r(3k-n)}.$$

9. Concluding Comments

In this paper, a first systematic study of sum relations involving (generalized) Fibonacci numbers and different kinds of binomial coefficients was provided. In particular, we have covered five different classes of relations. All results have their origins in polynomial relations involving one, two or three binomial coefficients. This study is by far not complete and additional relations can be added in the future. For instance, further identities with two binomial coefficients can be derived from the next lemma, which is a generalization of (16).

Lemma 16. *Let x and y be complex variables, let m and n be non-negative integers and let r be any integer. Then*

$$\sum_{k=0}^{n}\binom{m-n+k}{k}\binom{n-k}{r}(x+y)^{n-k-r}y^k=\sum_{k=0}^{n}\binom{m+1}{k}\binom{n-k}{r}x^{n-k-r}y^k.$$

In our future research projects, we will consider additional classes of relations. Currently, we work on binomial sums depending on the modulo 5 nature of the upper summation limit.

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