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A Note on Korn's Inequality in an N-Dimensional Context and a Global Existence Result for a Non-Linear Plate Model

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Abstract: In the first part of this article, we present a new proof for Korn's inequality in an n-dimensional context. The results are based on standard tools of real and functional analysis. For the final result, the standard Poincaré inequality plays a fundamental role. In the second text part, we develop a global existence result for a non-linear model of plates. We address a rather general type of boundary conditions and the novelty here is the more relaxed restrictions concerning the external load magnitude.

Keywords: Korn's inequality; global existence result; non-linear plate model

MSC: 35Q74; 35J58

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1. Introduction

In this article, we present a proof for Korn's inequality in $\mathbb{R}^n$. The results are based on standard tools of functional analysis and on the Sobolev spaces theory.

We emphasize that such a proof is relatively simple and easy to follow since it is established in a very transparent and clear fashion.

About the references, we highlight that related results in a three-dimensional context may be found in [1]. Other important classical results on Korn's inequality and concerning applications to models in elasticity may be found in [2–4].

Remark 1. *Generically, throughout the text we denote*

$$\|u\|_{0,2,\Omega} = \left(\int_\Omega |u|^2\,dx\right)^{1/2},\ \forall u \in L^2(\Omega),$$

and

$$\|u\|_{0,2,\Omega} = \left(\sum_{j=1}^n \|u_j\|_{0,2,\Omega}^2\right)^{1/2},\ \forall u = (u_1,\ldots,u_n) \in L^2(\Omega;\mathbb{R}^n).$$

Moreover,

$$\|u\|_{1,2,\Omega} = \left(\|u\|_{0,2,\Omega}^2 + \sum_{j=1}^n \|u_{x_j}\|_{0,2,\Omega}^2\right)^{1/2},\ \forall u \in W^{1,2}(\Omega),$$

where we shall also refer throughout the text to the well-known corresponding analogous norm for $u \in W^{1,2}(\Omega;\mathbb{R}^n)$.

At this point, we first introduce the following definition.

Definition 1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set. We say that $\partial\Omega$ is $\hat{C}^1$ if such a manifold is oriented and for each $x_0 \in \partial\Omega$, denoting $\hat{x} = (x_1,...,x_{n-1})$ for a local coordinate system compatible* *with the manifold $\partial\Omega$ orientation, there exist $r > 0$ and a function $f(x_1, ..., x_{n-1}) = f(\hat{x})$ such that*

$$W = \overline{\Omega} \cap B_r(x_0) = \{x \in B_r(x_0) \mid x_n \leq f(x_1, ..., x_{n-1})\}.$$

Moreover, $f(\hat{x})$ is a Lipschitz continuous function, so that

$$|f(\hat{x}) - f(\hat{y})| \leq C_1 |\hat{x} - \hat{y}|_2, \text{ on its domain,}$$

for some $C_1 > 0$. Finally, we assume

$$\left\{ \frac{\partial f(\hat{x})}{\partial x_k} \right\}_{k=1}^{n-1}$$

is classically defined, almost everywhere also on its concerning domain, so that $f \in W^{1,2}$.

Remark 2. *This mentioned set Ω is of a Lipschitzian type, so that we may refer to such a kind of sets as domains with a Lipschitzian boundary, or simply as Lipschitzian sets.*

At this point, we recall the following result found in [5], at page 222 in its Chapter 11.

Theorem 1. *Assume $\Omega \subset \mathbb{R}^n$ is an open bounded set, and that $\partial\Omega$ is $\hat{C}^1$. Let $1 \leq p < \infty$, and let V be a bounded open set such that $\Omega \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n),$$

such that for each $u \in W^{1,p}(\Omega)$ we have:

1. *$Eu = u$, a.e. in Ω,;*
2. *Eu has support in V;*
3. *$\|Eu\|_{1,p,\mathbb{R}^n} \leq C\|u\|_{1,p,\Omega}$, where the constant depends only on p, Ω, and V.*

Remark 3. *Considering the proof of such a result, the constant $C > 0$ may be also such that*

$$\|e_{ij}(Eu)\|_{0,2,V} \leq C(\|e_{ij}(u)\|_{0,2,\Omega} + \|u\|_{0,2,\Omega}),\ \forall u \in W^{1,2}(\Omega; \mathbb{R}^n),\ \forall i, j \in \{1, \ldots, n\},$$

for the operator $e : W^{1,2}(\Omega; \mathbb{R}^n) \to L^2(\Omega; \mathbb{R}^{n \times n})$ specified in the next theorem.

Finally, as the meaning is clear, we may simply denote $Eu = u$.

2. The Main Results, the Korn Inequalities

Our main result is summarized by the following theorem.

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with a $\hat{C}^1$ (Lipschitzian) boundary $\partial\Omega$.*

Define $e : W^{1,2}(\Omega; \mathbb{R}^n) \to L^2(\Omega; \mathbb{R}^{n \times n})$ by

$$e(u) = \{e_{ij}(u)\}$$

where

$$e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ \forall i, j \in \{1, \ldots, n\},$$

and where generically, we denote

$$u_{i,j} = \frac{\partial u_i}{\partial x_j},\ \forall i, j \in \{1, \cdots, n\}.$$

Define also,

$$\|e(u)\|_{0,2,\Omega} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \|e_{ij(u)}\|_{0,2,\Omega}^2\right)^{1/2}.$$

Let $L \in \mathbb{R}^+$ be such $V = [-L, L]^n$ is also such that $\overline{\Omega} \subset V^0$.
Under such hypotheses, there exists $C(\Omega, L) \in \mathbb{R}^+$ such that

$$\|u\|_{1,2,\Omega} \leq C(\Omega, L)(\|u\|_{0,2,\Omega} + \|e(u)\|_{0,2,\Omega}),\ \forall u \in W^{1,2}(\Omega; \mathbb{R}^n). \tag{1}$$

Proof. Suppose, to obtain contradiction, that the concerning claim does not hold.

Thus, we are assuming that there is no positive real constant $C = C(\Omega, L)$ such that (1) is valid.

In particular, $k = 1 \in \mathbb{N}$ is not such a constant C, so that there exists a function $u_1 \in W^{1,2}(\Omega; \mathbb{R}^n)$ such that

$$\|u_1\|_{1,2,\Omega} > 1\ (\|u_1\|_{0,2,\Omega} + \|e(u_1)\|_{0,2,\Omega}).$$

Similarly, $k = 2 \in \mathbb{N}$ is not such a constant C, so that there exists a function $u_2 \in W^{1,2}(\Omega; \mathbb{R}^n)$ such that

$$\|u_2\|_{1,2,\Omega} > 2\ (\|u_2\|_{0,2,\Omega} + \|e(u_2)\|_{0,2,\Omega}).$$

Hence, since no $k \in \mathbb{N}$ is such a constant C, reasoning inductively, for each $k \in \mathbb{N}$ there exists a function $u_k \in W^{1,2}(\Omega; \mathbb{R}^n)$ such that

$$\|u_k\|_{1,2,\Omega} > k(\|u_k\|_{0,2,\Omega} + \|e(u_k)\|_{0,2,\Omega}).$$

In particular, defining

$$v_k = \frac{u_k}{\|u_k\|_{1,2,\Omega}}$$

we obtain

$$\|v_k\|_{1,2,\Omega} = 1 > k(\|v_k\|_{0,2,\Omega} + \|e(v_k)\|_{0,2,\Omega}),$$

so that

$$(\|v_k\|_{0,2,\Omega} + \|e(v_k)\|_{0,2,\Omega}) < \frac{1}{k},\ \forall k \in \mathbb{N}.$$

From this we obtain

$$\|v_k\|_{0,2,\Omega} < \frac{1}{k},$$

and

$$\|e_{ij}(v_k)\|_{0,2,\Omega} < \frac{1}{k},\ \forall k \in \mathbb{N},$$

so that

$$\|v_k\|_{0,2,\Omega} \to 0,\ \text{as } k \to \infty,$$

and

$$\|e_{ij}(v_k)\|_{0,2,\Omega} \to 0,\ \text{as } k \to \infty.$$

In particular,

$$\|(v_k)_{j,j}\|_{0,2,\Omega} \to 0,\ \forall j \in \{1, \ldots, n\}.$$

At this point, we recall the following identity in the distributional sense, found in [3], page 12:

$$\partial_j(\partial_l v_i) = \partial_j e_{il}(v) + \partial_l e_{ij}(v) - \partial_i e_{jl}(v),\ \forall i, j, l \in \{1, \ldots, n\}. \tag{2}$$

Fix $j \in \{1, \ldots, n\}$ and observe that

$$\|(v_k)_j\|_{1,2,V} \leq C\|(v_k)_j\|_{1,2,\Omega},$$

so that

$$\frac{C}{\|(v_k)_j\|_{1,2,V}} \geq \frac{1}{\|(v_k)_j\|_{1,2,\Omega}},\ \forall k \in \mathbb{N}.$$

Hence,

$$\begin{aligned}
& \|(v_k)_j\|_{1,2,\Omega} \\
= & \sup_{\varphi \in C^1(\Omega)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(\Omega)} + \langle (v_k)_j, \varphi \rangle_{L^2(\Omega)} \ : \ \|\varphi\|_{1,2,\Omega} \leq 1 \right\} \\
= & \left\langle \nabla (v_k)_j, \nabla \left(\frac{(v_k)_j}{\|(v_k)_j\|_{1,2,\Omega}} \right) \right\rangle_{L^2(\Omega)} \\
& + \left\langle (v_k)_j, \left(\frac{(v_k)_j}{\|(v_k)_j\|_{1,2,\Omega}} \right) \right\rangle_{L^2(\Omega)} \\
\leq & C \left(\left\langle \nabla (v_k)_j, \nabla \left(\frac{(v_k)_j}{\|(v_k)_j\|_{1,2,V}} \right) \right\rangle_{L^2(V)} + \left\langle (v_k)_j, \left(\frac{(v_k)_j}{\|(v_k)_j\|_{1,2,V}} \right) \right\rangle_{L^2(V)} \right) \\
= & C \sup_{\varphi \in C^1_c(V)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(V)} + \langle (v_k)_j, \varphi \rangle_{L^2(V)} \ : \ \|\varphi\|_{1,2,V} \leq 1 \right\}.
\end{aligned} \tag{3}$$

Here, we recall that $C > 0$ is the constant concerning the extension Theorem 1. From such results and (2), we have that

$$\begin{aligned}
& \sup_{\varphi \in C^1(\Omega)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(\Omega)} + \langle (v_k)_j, \varphi \rangle_{L^2(\Omega)} \ : \ \|\varphi\|_{1,2,\Omega} \leq 1 \right\} \\
\leq & C \sup_{\varphi \in C^1_c(V)} \left\{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(V)} + \langle (v_k)_j, \varphi \rangle_{L^2(V)} \ : \ \|\varphi\|_{1,2,V} \leq 1 \right\} \\
= & C \sup_{\varphi \in C^1_c(V)} \Big\{ \langle e_{jl}(v_k), \varphi_{,l} \rangle_{L^2(V)} + \langle e_{jl}(v_k), \varphi_{,l} \rangle_{L^2(V)} \\
& -\langle e_{ll}(v_k), \varphi_{,j} \rangle_{L^2(V)} + \langle (v_k)_j, \varphi \rangle_{L^2(V)}, \ : \ \|\varphi\|_{1,2,V} \leq 1 \Big\}.
\end{aligned} \tag{4}$$

Therefore,

$$\begin{aligned}
& \|(v_k)_j\|_{(W^{1,2}(\Omega))} \\
= & \sup_{\varphi \in C^1(\Omega)} \{ \langle \nabla (v_k)_j, \nabla \varphi \rangle_{L^2(\Omega)} + \langle (v_k)_j, \varphi \rangle_{L^2(\Omega)} \ : \ \|\varphi\|_{1,2,\Omega} \leq 1 \} \\
\leq & C \left(\sum_{l=1}^n \left\{ \|e_{jl}(v_k)\|_{0,2,V} + \|e_{ll}(v_k)\|_{0,2,V} \right\} + \|(v_k)_j\|_{0,2,V} \right) \\
\leq & C_1 \left(\sum_{l=1}^n \left\{ \|e_{jl}(v_k)\|_{0,2,\Omega} + \|e_{ll}(v_k)\|_{0,2,\Omega} \right\} + \|(v_k)_j\|_{0,2,\Omega} \right) \\
< & \frac{C_2}{k},
\end{aligned} \tag{5}$$

for appropriate $C_1 > 0$ and $C_2 > 0$.

Summarizing,

$$\|(v_k)_j\|_{(W^{1,2}(\Omega))} < \frac{C_2}{k},\ \forall k \in \mathbb{N}.$$

From this we obtain

$$\|v_k\|_{1,2,\Omega} \to 0,\ \text{as } k \to \infty,$$

which contradicts
$$\|v_k\|_{1,2,\Omega} = 1, \ \forall k \in \mathbb{N}.$$

The proof is complete. □

Corollary 1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and connected set with a $\hat{C}^1$ boundary $\partial\Omega$. Define $e : W^{1,2}(\Omega;\mathbb{R}^n) \to L^2(\Omega;\mathbb{R}^{n\times n})$ by*

$$e(u) = \{e_{ij}(u)\}$$

where
$$e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ \forall i,j \in \{1,\ldots,n\}.$$

Define also,
$$\|e(u)\|_{0,2,\Omega} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \|e_{ij(u)}\|_{0,2,\Omega}^2\right)^{1/2}.$$

Let $L \in \mathbb{R}^+$ be such $V = [-L,L]^n$ is also such that $\overline{\Omega} \subset V^0$.
Moreover, define

$$\hat{H}_0 = \{u \in W^{1,2}(\Omega;\mathbb{R}^n) \ : \ u = \mathbf{0}, \ on \ \Gamma_0\},$$

where $\Gamma_0 \subset \partial\Omega$ is a measurable set such that the Lebesgue measure $m_{\mathbb{R}^{n-1}}(\Gamma_0) > 0$.

Assume also Γ_0 is such that for each $j \in \{1,\cdots,n\}$ and each $x = (x_1,\cdots,x_n) \in \Omega$ there exists $x_0 = ((x_0)_1,\cdots,(x_0)_n) \in \Gamma_0$ such that

$$(x_0)_l = x_l, \ \forall l \neq j, \ l \in \{1,\cdots,n\},$$

and the line
$$A_{x_0,x} \subset \overline{\Omega}$$

where
$$A_{x_0,x} = \{(x_1,\cdots,(1-t)(x_0)_j + tx_j,\cdots,x_n) \ : \ t \in [0,1]\}.$$

Under such hypotheses, there exists $C(\Omega,L) \in \mathbb{R}^+$ such that

$$\|u\|_{1,2,\Omega} \leq C(\Omega,L) \ \|e(u)\|_{0,2,\Omega}, \ \forall u \in \hat{H}_0.$$

Proof. Suppose, to obtain contradiction, that the concerning claim does not hold.

Hence, for each $k \in \mathbb{N}$ there exists $u_k \in \hat{H}_0$ such that

$$\|u_k\|_{1,2,\Omega} > k \ \|e(u_k)\|_{0,2,\Omega}.$$

In particular, defining
$$v_k = \frac{u_k}{\|u_k\|_{1,2,\Omega}}$$

similarly to the proof of the last theorem, we may obtain

$$\|(v_k)_{j,j}\|_{0,2,\Omega} \to 0, \text{ as } k \to \infty, \ \forall j \in \{1,\ldots,n\}.$$

From this, the hypotheses on Γ_0 and from the standard Poincaré inequality proof we obtain
$$\|(v_k)_j\|_{0,2,\Omega} \to 0, \text{ as } k \to \infty, \ \forall j \in \{1,\ldots,n\}.$$

Thus, also similarly as in the proof of the last theorem, we may infer that

$$\|v_k\|_{1,2,\Omega} \to 0, \text{ as } k \to \infty,$$

which contradicts
$$\|v_k\|_{1,2,\Omega} = 1, \ \forall k \in \mathbb{N}.$$

The proof is complete. □

3. An Existence Result for a Non-Linear Model of Plates

In the present section, as an application of the results on Korn's inequalities presented in the previous sections, we develop a new global existence proof for a Kirchhoff–Love thin plate model. Previous results on the existence of mathematical elasticity and related models may be found in [2–4].

At this point we start to describe the primal formulation.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set which represents the middle surface of a plate of thickness h. The boundary of Ω, which is assumed to be regular (Lipschitzian), is denoted by $\partial\Omega$. The vectorial basis related to the cartesian system $\{x_1, x_2, x_3\}$ is denoted by $(\mathbf{a}_\alpha, \mathbf{a}_3)$, where $\alpha = 1, 2$ (in general, Greek indices stand for 1 or 2), and where $\mathbf{a}_3$ is the vector normal to Ω, whereas $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal vectors parallel to Ω. Furthermore, $\mathbf{n}$ is the outward normal to the plate surface.

The displacements will be denoted by

$$\hat{\mathbf{u}} = \{\hat{u}_\alpha, \hat{u}_3\} = \hat{u}_\alpha \mathbf{a}_\alpha + \hat{u}_3 \mathbf{a}_3.$$

The Kirchhoff–Love relations are

$$\begin{aligned} &\hat{u}_\alpha(x_1, x_2, x_3) = u_\alpha(x_1, x_2) - x_3 w(x_1, x_2)_{,\alpha} \\ &\text{and } \hat{u}_3(x_1, x_2, x_3) = w(x_1, x_2). \end{aligned} \tag{6}$$

Here, $-h/2 \leq x_3 \leq h/2$ so that we have $u = (u_\alpha, w) \in U$ where

$$\begin{aligned} U &= \Big\{(u_\alpha, w) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{2,2}(\Omega), \\ &\qquad u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\} \\ &= W_0^{1,2}(\Omega; \mathbb{R}^2) \times W_0^{2,2}(\Omega). \end{aligned}$$

It is worth emphasizing that the boundary conditions here specified refer to a clamped plate.

We define the operator $\Lambda : U \to Y \times Y$, where $Y = Y^* = L^2(\Omega; \mathbb{R}^{2\times 2})$, by

$$\Lambda(u) = \{\gamma(u), \kappa(u)\},$$

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{w_{,\alpha} w_{,\beta}}{2},$$

$$\kappa_{\alpha\beta}(u) = -w_{,\alpha\beta}.$$

The constitutive relations are given by

$$N_{\alpha\beta}(u) = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u), \tag{7}$$

$$M_{\alpha\beta}(u) = h_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}(u), \tag{8}$$

where $\{H_{\alpha\beta\lambda\mu}\}$ and $\{h_{\alpha\beta\lambda\mu} = \frac{h^2}{12} H_{\alpha\beta\lambda\mu}\}$, are symmetric positive definite fourth-order tensors. From now on, we denote $\{\overline{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1}$ and $\{\overline{h}_{\alpha\beta\lambda\mu}\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}$.

Furthermore, $\{N_{\alpha\beta}\}$ denote the membrane force tensor and $\{M_{\alpha\beta}\}$ the moment one. The plate stored energy, represented by $(G \circ \Lambda) : U \to \mathbb{R}$, is expressed by

$$(G \circ \Lambda)(u) = \frac{1}{2} \int_{\Omega} N_{\alpha\beta}(u)\gamma_{\alpha\beta}(u)\, dx + \frac{1}{2} \int_{\Omega} M_{\alpha\beta}(u)\kappa_{\alpha\beta}(u)\, dx \tag{9}$$

and the external work, represented by $F : U \to \mathbb{R}$, is given by

$$F(u) = \langle w, P \rangle_{L^2(\Omega)} + \langle u_\alpha, P_\alpha \rangle_{L^2(\Omega)}, \tag{10}$$

where $P, P_1, P_2 \in L^2(\Omega)$ are external loads in the directions $\mathbf{a}_3$, $\mathbf{a}_1$, and $\mathbf{a}_2$, respectively. The potential energy, denoted by $J : U \to \mathbb{R}$, is expressed by

$$J(u) = (G \circ \Lambda)(u) - F(u)$$

Finally, we also emphasize from now on, as their meaning are clear, we may denote $L^2(\Omega)$ and $L^2(\Omega; \mathbb{R}^{2\times 2})$ simply by L^2, and the respective norms by $\| \cdot \|_2$. Moreover, derivatives are always understood in the distributional sense, $\mathbf{0}$ may denote the zero vector in appropriate Banach spaces, and the following and relating notations are used:

$$w_{,\alpha} = \frac{\partial w}{\partial x_\alpha},$$

$$w_{,\alpha\beta} = \frac{\partial^2 w}{\partial x_\alpha \partial x_\beta},$$

$$u_{\alpha,\beta} = \frac{\partial u_\alpha}{\partial x_\beta},$$

$$N_{\alpha\beta,1} = \frac{\partial N_{\alpha\beta}}{\partial x_1},$$

and

$$N_{\alpha\beta,2} = \frac{\partial N_{\alpha\beta}}{\partial x_2}.$$

4. On the Existence of a Global Minimizer

At this point, we present an existence result concerning the Kirchhoff–Love plate model. We start with the following two remarks.

Remark 4. *Let $\{P_\alpha\} \in L^\infty(\Omega; \mathbb{R}^2)$. We may easily obtain by appropriate Lebesgue integration $\{\tilde{T}_{\alpha\beta}\}$ symmetric and such that*

$$\tilde{T}_{\alpha\beta,\beta} = -P_\alpha, \text{ in } \Omega.$$

Indeed, extending $\{P_\alpha\}$ to zero outside Ω if necessary, we may set

$$\tilde{T}_{11}(x, y) = -\int_0^x P_1(\xi, y)\, d\xi,$$

$$\tilde{T}_{22}(x, y) = -\int_0^y P_2(x, \xi)\, d\xi,$$

and

$$\tilde{T}_{12}(x, y) = \tilde{T}_{21}(x, y) = 0, \text{ in } \Omega.$$

Thus, we may choose a $C > 0$ sufficiently big, such that

$$\{T_{\alpha\beta}\} = \{\tilde{T}_{\alpha\beta} + C\delta_{\alpha\beta}\}$$

is positive definite in Ω, so that

$$T_{\alpha\beta,\beta} = \tilde{T}_{\alpha\beta,\beta} = -P_\alpha,$$

where

$$\{\delta_{\alpha\beta}\}$$

is the Kronecker delta.

Therefore, for the kind of boundary conditions of the next theorem, we do not have any restriction for the $\{P_\alpha\}$ norm.

In summary, the next result is new and it is really a step forward concerning the previous one in Ciarlet [3]. We emphasize that this result and its proof through such a tensor $\{T_{\alpha\beta}\}$ are new, even though the final part of the proof is established through a standard procedure in the calculus of variations.

Finally, more details on the Sobolev spaces involved may be found in [5–8]. Related duality principles are addressed in [5,7,9].

At this point, we present the main theorem in this section.

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set with a Lipschitzian boundary denoted by $\partial\Omega = \Gamma$. Suppose $(G \circ \Lambda) : U \to \mathbb{R}$ is defined by*

$$G(\Lambda u) = G_1(\gamma(u)) + G_2(\kappa(u)), \ \forall u \in U,$$

where

$$G_1(\gamma u) = \frac{1}{2}\int_\Omega H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)\,dx,$$

and

$$G_2(\kappa u) = \frac{1}{2}\int_\Omega h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)\,dx,$$

where

$$\Lambda(u) = (\gamma(u), \kappa(u)) = (\{\gamma_{\alpha\beta}(u)\}, \{\kappa_{\alpha\beta}(u)\}),$$

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{w_{,\alpha}w_{,\beta}}{2},$$

$$\kappa_{\alpha\beta}(u) = -w_{,\alpha\beta},$$

and where

$$\begin{aligned} J(u) \quad = \quad & W(\gamma(u), \kappa(u)) - \langle P_\alpha, u_\alpha\rangle_{L^2(\Omega)} \\ & -\langle w, P\rangle_{L^2(\Omega)} - \langle P^t_\alpha, u_\alpha\rangle_{L^2(\Gamma_t)} \\ & -\langle P^t, w\rangle_{L^2(\Gamma_t)}, \end{aligned} \tag{11}$$

where,

$$\begin{aligned} U \quad = \quad & \{u = (u_\alpha, w) = (u_1, u_2, w) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{2,2}(\Omega) \ : \\ & u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0, \ on\ \Gamma_0\}, \end{aligned} \tag{12}$$

where $\partial\Omega = \Gamma_0 \cup \Gamma_t$ and the Lebesgue measures

$$m_\Gamma(\Gamma_0 \cap \Gamma_t) = 0,$$

and

$$m_\Gamma(\Gamma_0) > 0.$$

We also define

$$\begin{aligned} F_1(u) &= -\langle w, P\rangle_{L^2(\Omega)} - \langle u_\alpha, P_\alpha\rangle_{L^2(\Omega)} - \langle P^t_\alpha, u_\alpha\rangle_{L^2(\Gamma_t)} \\ &\quad -\langle P^t, w\rangle_{L^2(\Gamma_t)} + \langle \varepsilon_\alpha, u_\alpha^2\rangle_{L^2(\Gamma_t)} \\ &\equiv -\langle u, \mathbf{f}\rangle_{L^2} + \langle \varepsilon_\alpha, u_\alpha^2\rangle_{L^2(\Gamma_t)} \\ &\equiv -\langle u, \mathbf{f}_1\rangle_{L^2} - \langle u_\alpha, P_\alpha\rangle_{L^2(\Omega)} + \langle \varepsilon_\alpha, u_\alpha^2\rangle_{L^2(\Gamma_t)}, \end{aligned} \tag{13}$$

where

$$\langle u, \mathbf{f}_1\rangle_{L^2} = \langle u, \mathbf{f}\rangle_{L^2} - \langle u_\alpha, P_\alpha\rangle_{L^2(\Omega)},$$

$\varepsilon_\alpha > 0,\ \forall \alpha \in \{1,2\}$ *and*

$$\mathbf{f} = (P_\alpha, P) \in L^\infty(\Omega; \mathbb{R}^3).$$

Let $J : U \to \mathbb{R}$ *be defined by*

$$J(u) = G(\Lambda u) + F_1(u),\ \forall u \in U.$$

Assume there exists $\{c_{\alpha\beta}\} \in \mathbb{R}^{2\times 2}$ *such that* $c_{\alpha\beta} > 0,\ \forall \alpha, \beta \in \{1,2\}$ *and*

$$G_2(\kappa(u)) \geq c_{\alpha\beta}\|w_{,\alpha\beta}\|_2^2,\ \forall u \in U.$$

Under such hypotheses, there exists $u_0 \in U$ *such that*

$$J(u_0) = \min_{u \in U} J(u).$$

Proof. Observe that we may find $\mathbf{T}_\alpha = \{(T_\alpha)_\beta\}$ such that

$$div \mathbf{T}_\alpha = T_{\alpha\beta,\beta} = -P_\alpha,$$

and also such that $\{T_{\alpha\beta}\}$ is positive, definite, and symmetric (please see Remark 4).

Thus, defining

$$v_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2} w_{,\alpha} w_{,\beta}, \tag{14}$$

we obtain

$$\begin{aligned} J(u) &= G_1(\{v_{\alpha\beta}(u)\}) + G_2(\kappa(u)) - \langle u, \mathbf{f}\rangle_{L^2} + \langle \varepsilon_\alpha, u_\alpha^2\rangle_{L^2(\Gamma_t)} \\ &= G_1(\{v_{\alpha\beta}(u)\}) + G_2(\kappa(u)) + \langle T_{\alpha\beta,\beta}, u_\alpha\rangle_{L^2(\Omega)} - \langle u, \mathbf{f}_1\rangle_{L^2} + \langle \varepsilon_\alpha, u_\alpha^2\rangle_{L^2(\Gamma_t)} \\ &= G_1(\{v_{\alpha\beta}(u)\}) + G_2(\kappa(u)) - \left\langle T_{\alpha\beta}, \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2}\right\rangle_{L^2(\Omega)} \\ &\quad + \langle T_{\alpha\beta} n_\beta, u_\alpha\rangle_{L^2(\Gamma_t)} - \langle u, \mathbf{f}_1\rangle_{L^2} + \langle \varepsilon_\alpha, u_\alpha^2\rangle_{L^2(\Gamma_t)} \\ &= G_1(\{v_{\alpha\beta}(u)\}) + G_2(\kappa(u)) - \left\langle T_{\alpha\beta}, v_{\alpha\beta}(u) - \frac{1}{2} w_{,\alpha} w_{,\beta}\right\rangle_{L^2(\Omega)} - \langle u, \mathbf{f}_1\rangle_{L^2} + \langle \varepsilon_\alpha, u_\alpha^2\rangle_{L^2(\Gamma_t)} \\ &\quad + \langle T_{\alpha\beta} n_\beta, u_\alpha\rangle_{L^2(\Gamma_t)} \\ &\geq c_{\alpha\beta}\|w_{,\alpha\beta}\|_2^2 + \frac{1}{2}\langle T_{\alpha\beta}, w_{,\alpha} w_{,\beta}\rangle_{L^2(\Omega)} - \langle u, \mathbf{f}_1\rangle_{L^2} + \langle \varepsilon_\alpha, u_\alpha^2\rangle_{L^2(\Gamma_t)} + G_1(\{v_{\alpha\beta}(u)\}) \\ &\quad - \langle T_{\alpha\beta}, v_{\alpha\beta}(u)\rangle_{L^2(\Omega)} + \langle T_{\alpha\beta} n_\beta, u_\alpha\rangle_{L^2(\Gamma_t)}. \end{aligned} \tag{15}$$

From this, since $\{T_{\alpha\beta}\}$ is positive definite, clearly J is bounded below.

Let $\{u_n\} \in U$ be a minimizing sequence for J. Thus, there exists $\alpha_1 \in \mathbb{R}$ such that

$$\lim_{n\to\infty} J(u_n) = \inf_{u \in U} J(u) = \alpha_1.$$

From (15), there exists $K_1 > 0$ such that

$$\|(w_n)_{,\alpha\beta}\|_2 < K_1, \forall \alpha, \beta \in \{1,2\},\ n \in \mathbb{N}.$$

Therefore, there exists $w_0 \in W^{2,2}(\Omega)$ such that, up to a subsequence not relabeled,

$$(w_n)_{,\alpha\beta} \rightharpoonup (w_0)_{,\alpha\beta}, \text{ weakly in } L^2,$$

$\forall \alpha, \beta \in \{1,2\}$, as $n \to \infty$.

Moreover, also up to a subsequence not relabeled,

$$(w_n)_{,\alpha} \to (w_0)_{,\alpha}, \text{ strongly in } L^2 \text{ and } L^4, \tag{16}$$

$\forall \alpha, \in \{1,2\}$, as $n \to \infty$.

Furthermore, from (15), there exists $K_2 > 0$ such that,

$$\|(v_n)_{\alpha\beta}(u)\|_2 < K_2, \forall \alpha, \beta \in \{1,2\},\ n \in \mathbb{N},$$

and thus, from this, (14) and (16), we may infer that there exists $K_3 > 0$ such that

$$\|(u_n)_{\alpha,\beta} + (u_n)_{\beta,\alpha}\|_2 < K_3, \forall \alpha, \beta \in \{1,2\},\ n \in \mathbb{N}.$$

From this and Korn's inequality, there exists $K_4 > 0$ such that

$$\|u_n\|_{W^{1,2}(\Omega;\mathbb{R}^2)} \leq K_4,\ \forall n \in \mathbb{N}.$$

Therefore, up to a subsequence not relabeled, there exists $\{(u_0)_\alpha\} \in W^{1,2}(\Omega, \mathbb{R}^2)$, such that

$$(u_n)_{\alpha,\beta} + (u_n)_{\beta,\alpha} \rightharpoonup (u_0)_{\alpha,\beta} + (u_0)_{\beta,\alpha}, \text{ weakly in } L^2,$$

$\forall \alpha, \beta \in \{1,2\}$, as $n \to \infty$, and

$$(u_n)_\alpha \to (u_0)_\alpha, \text{ strongly in } L^2,$$

$\forall \alpha \in \{1,2\}$, as $n \to \infty$.

Moreover, the boundary conditions satisfied by the subsequences are also satisfied for w_0 and u_0 in a trace sense, so that

$$u_0 = ((u_0)_\alpha, w_0) \in U.$$

From this, up to a subsequence not relabeled, we obtain

$$\gamma_{\alpha\beta}(u_n) \rightharpoonup \gamma_{\alpha\beta}(u_0), \text{ weakly in } L^2,$$

$\forall \alpha, \beta \in \{1,2\}$, and

$$\kappa_{\alpha\beta}(u_n) \rightharpoonup \kappa_{\alpha\beta}(u_0), \text{ weakly in } L^2,$$

$\forall \alpha, \beta \in \{1,2\}$.

Therefore, from the convexity of G_1 in γ and G_2 in κ, we obtain

$$\begin{aligned} \inf_{u \in U} J(u) &= \alpha_1 \\ &= \liminf_{n \to \infty} J(u_n) \\ &\geq J(u_0). \end{aligned} \tag{17}$$

Thus,

$$J(u_0) = \min_{u \in U} J(u).$$

The proof is complete. □

5. Conclusions

In this article, we have developed a new proof for Korn's inequality in a specific n-dimensional context. In the second text part, we present a global existence result for a non-linear model of plates. Both results represent some new advances concerning the present literature. In particular, the results for Korn's inequality known so far are for a three-dimensional context such as in [1], for example, whereas we have here addressed a more general n-dimensional case.

In a future research, we intend to address more general models, including the corresponding results for manifolds in $\mathbb{R}^n$.

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