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# The Wiener–Hopf Equation with Probability Kernel and Submultiplicative Asymptotics of the Inhomogeneous Term

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**Abstract:** We consider the inhomogeneous Wiener–Hopf equation whose kernel is a nonarithmetic probability distribution with positive mean. The inhomogeneous term behaves like a submultiplicative function. We establish asymptotic properties of the solution to which the successive approximations converge. These properties depend on the asymptotics of the submultiplicative function.

**Keywords:** Wiener–Hopf equation; inhomogeneous equation; nonarithmetic probability distribution; positive mean; submultiplicative function; asymptotic behavior

**MSC:** 45E10; 60K05

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## 1. Introduction

The classical Wiener–Hopf equation has the form

$$z(x) = \int_0^\infty k(x-y)z(y)\,dy + g(x), \qquad x \ge 0,$$

or, equivalently,

$$z(x) = \int_{-\infty}^x z(x-y)k(y)\,dy + g(x), \qquad x \ge 0.$$

We shall consider the inhomogeneous generalized Wiener–Hopf equation

$$z(x) = \int_{-\infty}^x z(x-y)\,F(dy) + g(x), \qquad x \ge 0, \tag{1}$$

where $z$ is the function sought, $F$ is a given probability distribution on $\mathbb{R}$, and the inhomogeneous term $g$ is a known complex function. A probability distribution $G$ on $\mathbb{R}$ is called *nonarithmetic* if it is not concentrated on the set of points of the form $0, \pm\lambda, \pm 2\lambda, \ldots$ (see Section V.2, Definition 3 of [1]). Let $\mathbb{R}_+$ be the set of all nonnegative numbers and $\mathbb{R}_- := \mathbb{R} \setminus \mathbb{R}_+$ be the set of all negative numbers. For $c \in \mathbb{C}$, we assume that $c/\infty$ is equal to zero. The relation $a(x) \sim cb(x)$ as $x \to \infty$ means that $a(x)/b(x) \to c$ as $x \to \infty$; if $c = 0$, then $a(x) = o(b(x))$.

**Definition 1.** *A positive function $\varphi(x)$, $x \in \mathbb{R}$, is called submultiplicative if it is finite, Borel measurable, and satisfies the conditions: $\varphi(0) = 1$, $\varphi(x+y) \le \varphi(x)\,\varphi(y)$, $x, y \in \mathbb{R}$.*

The following properties are valid for submultiplicative functions defined on the whole line (Theorem 7.6.2) of [2]:

$$\begin{aligned} -\infty < r_- := \lim_{x\to-\infty} \frac{\log\varphi(x)}{x} &= \sup_{x<0} \frac{\log\varphi(x)}{x} \\ &\le \inf_{x>0} \frac{\log\varphi(x)}{x} = \lim_{x\to\infty} \frac{\log\varphi(x)}{x} =: r_+ < \infty. \end{aligned} \tag{2}$$

Here are some examples of submultiplicative function on $\mathbb{R}_+$: (i) $\varphi(x) = (x+1)^r$, $r > 0$; (ii) $\varphi(x) = \exp(cx^\beta)$, where $c > 0$ and $0 < \beta < 1$; and (iii) $\varphi(x) = \exp(\gamma x)$, where $\gamma \in \mathbb{R}$. In (i) and (ii), $r_+ = 0$, while in (iii), $r_+ = \gamma$. The product of a finite number of submultiplicative function is again a submultiplicative function.

In the present paper, we investigate the asymptotic behavior of the solution to Equation (1), where $F$ is a nonarithmetic probability distribution with finite positive mean $\mu := \int_{\mathbb{R}} x\, F(dx)$ and the function $g(x)$ is asymptotically equivalent (up to a constant factor) to a nondecreasing submultiplicative function $\varphi(x)$ tending to infinity as $x \to \infty$: $g(x) \sim c\varphi(x)$ as $x \to \infty$. In the main theorems (Theorems 2 and 3), $\varphi(x)$, $x \in \mathbb{R}_+$, is a nondecreasing submultiplicative function for which there exists $\lim_{x\to\infty} \varphi(x+y)/\varphi(x)$ for each $y \in \mathbb{R}$. If such a limit exists, then it is equal to $\exp(r_+ y)$.

Earlier [3], the asymptotic behavior of $z$ was studied in detail under the following assumptions: (i) $\mu \in (0, +\infty]$ and (ii) $g$ belong to either $g \in L_1(0,\infty)$ or $g \in L_\infty(0,\infty)$. Roughly speaking, if $g \in L_1(0,\infty)$, then $z(x)$ tends to a specific finite limit as $x \to \infty$. Moreover, under appropriate conditions, a submultiplicative rate of convergence was given in the form $o(1/\varphi(x))$. If $g \in L_\infty(0,\infty)$, then $z(x) = O(x)$ or even $z(x) = f(\infty)x/\mu$ as $x \to \infty$, provided $f(\infty) := \lim_{x\to\infty} f(x)$ exists.

The existence of the solution to Equation (1) and its explicit form (5) were established in [4] for $g \in L_\infty(0,\infty)$ and arbitrary probability distributions $F$, regardless of whether $F$ is of oscillating or drifting type. If $\mu = 0$ and if some other hypotheses are fulfilled, then $z(x)$ tends to a specific finite limit as $x \to \infty$ (Theorem 4 of [4]).

The stability of an integro-differential equation with a convolution type kernel was studied in [5,6].

## 2. Preliminaries

Consider the collection $S(\varphi)$ of all complex-valued measures $\varkappa$, such that

$$\|\varkappa\|_\varphi := \int_{\mathbb{R}} \varphi(x)\, |\varkappa|(dx) < \infty;$$

here, $|\varkappa|$ stands for the total variation of $\varkappa$. The collection $S(\varphi)$ is a Banach algebra with norm $\|\cdot\|_\varphi$ by the usual operations of addition and scalar multiplication of measures; the product of two elements $\nu$ and $\varkappa$ of $S(\varphi)$ is defined as their convolution $\nu * \varkappa$ (Section 4.16) of [2]. The unit element of $S(\varphi)$ is the measure $\delta_0$ of unit mass concentrated at zero. Define the Laplace transform of a measure $\varkappa$ as $\widehat{\varkappa}(s) := \int_{\mathbb{R}} \exp(sx)\, \varkappa(dx)$. It follows from (2) that the Laplace transform of any $\varkappa \in S(\varphi)$ converges absolutely with respect to $|\varkappa|$ for all $s$ in the strip $\Pi(r_-, r_+) := \{s \in \mathbb{C} : r_- \le \Re s \le r_+\}$. Let $\nu$ and $\varkappa$ be two complex-valued measures on the $\sigma$-algebra $\mathscr{B}$ of Borel sets in $\mathbb{R}$. Their *convolution* is the measure

$$\nu * \varkappa(A) := \iint_{\{x+y\in A\}} \nu(dx)\, \varkappa(dy) = \int_{\mathbb{R}} \nu(A-x)\, \varkappa(dx), \qquad A \in \mathscr{B},$$

provided the integrals make sense; here, $A - x := \{y \in \mathbb{R} : x + y \in A\}$. Denote by $F^{n*}$ the $n$-th convolution power of $F$:

$$F^{0*} := \delta_0, \quad F^{1*} := F, \quad F^{(n+1)*} := F^{n*} * F, \quad n \ge 1.$$

Let $U$ be the renewal measure generated by $F$: $U := \sum_{n=0}^{\infty} F^{n*}$.

Let $X_k$, $k \ge 1$, be independent random variables with the same distribution $F$ not concentrated at zero. These variables generate the random walk $S_0 = 0$, $S_n = X_1 + \ldots + X_n$, $n \ge 1$. Put $\overline{\mathscr{T}}_+ := \min\{n \ge 1 : S_n \ge 0\}$. The random variable $\overline{\mathscr{H}}_+ := S_{\overline{\mathscr{T}}_+}$ is called the *first weak ascending ladder height*. Similarly, $\overline{\mathscr{T}}_- := \min\{n \ge 1 : S_n < 0\}$ and $\overline{\mathscr{H}}_- := S_{\overline{\mathscr{T}}_-}$

is the *first strong descending ladder height*. We have the factorization identity (the symbol $\mathsf{E}$ stands for "expectation").

$$1 - \xi \mathsf{E}(e^{sX_1}) = \Big(1 - \mathsf{E}\big(\xi^{\overline{\mathscr{T}}_-} e^{s\overline{\mathscr{H}}_-}\big)\Big)\Big(1 - \mathsf{E}\big(\xi^{\overline{\mathscr{T}}_+} e^{s\overline{\mathscr{H}}_+}\big)\Big), \quad |\xi| \le 1, \quad \Re s = 0. \tag{3}$$

This can easily be deduced from an analogous identity in Section XVIII.3 of [1] for another collection of ladder variables. Denote by $F_\pm$ the distributions of the random variables $\overline{\mathscr{H}}_\pm$, respectively. It follows from the identity (3) that

$$\delta_0 - F = (\delta_0 - F_-) * (\delta_0 - F_+). \tag{4}$$

Let $U_\pm := \sum_{k=0}^\infty F_\pm^{k*}$ be the renewal measures generated by the distributions $F_\pm$, respectively. Denote by $\mathbf{1}_{\mathbb{R}_+}$ the indicator of the subset $\mathbb{R}_+$ in $\mathbb{R}$: $\mathbf{1}_{\mathbb{R}_+}(x) = 1$ for $x \in \mathbb{R}_+$ and $\mathbf{1}_{\mathbb{R}_+}(x) = 0$ for $x \in \mathbb{R}_-$. Extend the function $g$ onto the whole line: $g(x) := 0$, $x < 0$. This convention will be valid throughout. Let $\nu$ be a measure defined on $\mathscr{B}$, and $a(x)$, $x \in \mathbb{R}$, a function. Define the convolution $\nu * a(x)$ as the function $\int_{\mathbb{R}} a(x-y)\,\nu(dy)$, $x \in \mathbb{R}$. The following theorem has been proven in [4].

**Theorem 1.** *Let $F$ be a probability distribution and $g \in L_1(\mathbb{R}_+)$. Then, the function*

$$z(x) = U_+ * \big((U_- * g)\mathbf{1}_{\mathbb{R}_+}\big)(x), \qquad x \in \mathbb{R}_+, \tag{5}$$

*is the solution to Equation (1), which coincides with the solution obtained by successive approximations.*

If $\mu$ is finite and positive, then $\mu_+ := \int_{\mathbb{R}} x\, F_+(dx)$ is also finite and positive (Section XII.2, Theorem 2 of [1]). We have

$$\mu = \mu_+(1 - F_-(\mathbb{R}_-)), \qquad U_-(\mathbb{R}_- \cup \{0\}) = \frac{1}{1 - F_-(\mathbb{R}_-)}. \tag{6}$$

In fact, pass in (4) to Laplace transforms and divide both sides by $s$. We get

$$\frac{1 - \widehat{F}(s)}{s} = \big(1 - \widehat{F}_-(s)\big)\frac{1 - \widehat{F}_+(s)}{s}, \qquad s \ne 0, \quad \Re s = 0.$$

Let $s$ tend to zero. Then, the fractions on both sides will tend to $\mu$ and $\mu_+$, respectively. The second equality in (6) is a consequence of the fact that the distribution $F_-$ is defective, i.e., $F_-(\mathbb{R}_-) < 1$.

**Lemma 1.** *Let $F$ be a nonarithmetic probability distribution, such that*

$$\mu = \int_{\mathbb{R}} x\, F(dx) \in (0, \infty)$$

*and let $\varphi(x)$, $x \in \mathbb{R}$, be a submultiplicative function with $r_- \le 0 \le r_+$. Assume that*

$$\int_{-\infty}^0 \varphi(x) F((-\infty, x])\, dx < \infty.$$

*Suppose additionally that $\widehat{F}(r_-) < 1$ if $r_- < 0$. Then $U_- \in S(\varphi)$.*

**Proof.** By Theorem 4 in [7] with $n = 1$ and Remark 5 therein, we have

$$\int_{-\infty}^0 \varphi(x)\, F_-(dx) < \infty,$$

i.e., $F_- \in S(\varphi)$. Let us prove that the element $\nu := \delta_0 - F_-$ is invertible in $S(\varphi)$. Let $\nu = \nu_{\mathfrak{c}} + \nu_{\mathfrak{d}} + \nu_{\mathfrak{s}}$ be the decomposition of $\nu$ into absolutely continuous, discrete, and singular components. By Theorem 1 of [8], the element $\nu \in S(\varphi)$ has an inverse if $\widehat{\nu}(s) \neq 0$ for all $s \in \Pi(r_1, r_2)$, and if

$$\inf_{s\in\Pi(r_-,r_+)} \left|\widehat{\nu_{\mathfrak{d}}}(s)\right| > \max\left\{\widehat{|\nu_{\mathfrak{s}}|}(r_-),\, \widehat{|\nu_{\mathfrak{s}}|}(r_+)\right\}. \tag{7}$$

Let $F_- = F_-^{\mathfrak{c}} + F_-^{\mathfrak{d}} + F_-^{\mathfrak{s}}$ be the decomposition of $F_- \in S(\varphi)$ into absolutely continuous, discrete, and singular components. Then, $\nu^{\mathfrak{d}} = \delta_0 - F_-^{\mathfrak{d}}$ and $\nu^{\mathfrak{s}} = -F_-^{\mathfrak{s}}$. We have

$$\inf_{s\in\Pi(r_-,r_+)} \left|\widehat{\nu_{\mathfrak{d}}}(s)\right| \geq 1 - \sup_{s\in\Pi(r_-,r_+)} \left|\widehat{F_-^{\mathfrak{d}}}(s)\right| = 1 - \widehat{F_-^{\mathfrak{d}}}(r_-).$$

On the other hand, $\max\left\{\widehat{|\nu_{\mathfrak{s}}|}(r_-),\, \widehat{|\nu_{\mathfrak{s}}|}(r_+)\right\} = \widehat{F_-^{\mathfrak{s}}}(r_-)$. Hence, in order to prove (7), it suffices to show that

$$1 - \widehat{F_-^{\mathfrak{d}}}(r_-) - \widehat{F_-^{\mathfrak{s}}}(r_-) \geq 1 - \widehat{F_-}(r_-) > 0.$$

If $r_- = 0$, this follows from the fact that the distribution $F_-$ is defective. Let $r_- < 0$. By assumption, $\widehat{F}(r_-) < 1$ and, obviously, $\widehat{F}_+(r_-) < 1$. Relation (4) implies

$$1 - \widehat{F}(s) = \left(1 - \widehat{F_-}(s)\right)\left(1 - \widehat{F_+}(s)\right), \qquad s \in \Pi(r_-, r_+), \tag{8}$$

whence $1 - \widehat{F_-}(r_-) > 0$ and (7) follows. Finally,

$$|\widehat{\nu}(s)| \geq 1 - |\widehat{F_-}(s)| \geq 1 - \widehat{F_-}(|s|) \geq 1 - \widehat{F_-}(r_-) > 0, \qquad s \in \Pi(r_-, r_+).$$

Therefore, by Theorem 1 in [8], the measure $\delta_0 - F_-$ is invertible in the Banach algebra $S(\varphi)$ and $U_- = (\delta_0 - F_-)^{-1} \in S(\varphi)$. The proof of the lemma is complete. □

**Lemma 2.** *Let $a(x)$, $x \in \mathbb{R}_+$, be a monotone nondecreasing positive function. Suppose that* $\lim_{x\to\infty} a(x+y)/a(x) = 1$ *for each $y \in \mathbb{R}$. Then,*

$$a(x) = o\left(\int_0^x a(y)\, dy\right) \qquad \text{as } x \to \infty.$$

**Proof.** Let $M > 0$ be arbitrary. We have

$$\int_0^x \frac{a(y)}{a(x)}\, dy \geq \int_{x-M}^x \frac{a(y)}{a(x)}\, dy \geq \int_{x-M}^x \frac{a(x-M)}{a(x)}\, dy = M\frac{a(x-M)}{a(x)}.$$

It follows that $\liminf_{x\to\infty} \int_0^x a(y)\, dy / a(x) = \infty$. The proof of the lemma is complete. □

**Lemma 3.** *Let $G$ be a nonarithmetic probability distribution on $\mathbb{R}_+$, such that*

$$\mu_G := \int_{\mathbb{R}} x\, G(dx) \in (0, \infty)$$

*and let $U_G$ be the corresponding renewal measure: $U_G := \sum_{n=0}^{\infty} G^{n*}$. Suppose that $a(x)$ and $b(x)$, $x \in \mathbb{R}_+$, are nonnegative functions such that $a(x) \sim b(x)$ as $x \to \infty$.Then,*

$$I(x) := U_G * a(x) \sim U_G * b(x) =: J(x) \qquad \text{as } x \to \infty.$$

**Proof.** Given $\varepsilon > 0$, choose $A > 0$, such that

$$(1 - \varepsilon)b(x) \leq a(x) \leq (1 + \varepsilon)b(x), \qquad x \geq A.$$

Let
$$I(x) = \left(\int_0^{x-A} + \int_{x-A}^{x}\right) a(x-y)\, U_G(dy) =: I_1(x) + I_2(x).$$
Similarly, let $J(x) = J_1(x) + J_2(x)$. Obviously,
$$1-\varepsilon \le \liminf_{x\to\infty} \frac{I_1(x)}{J_1(x)} \le \limsup_{x\to\infty} \frac{I_1(x)}{J_1(x)} \le 1+\varepsilon.$$

Since $\varepsilon$ is arbitrary, $\lim_{x\to\infty} I_1(x)/J_1(x) = 1$, i.e., $I_1(x) \sim J_1(x)$ as $x \to \infty$. Moreover, $I_1(x) \ge a(x-A)U_G([0, x-A]) \to \infty$ as $x \to \infty$ by the elementary renewal theorem for the measure $U_G$: $U_G([0,x]) \sim x/\mu_G$ as $x \to \infty$ (see Section 1.2 of [9]). According to Blackwell's theorem (Section XI.1, Theorem 1 of [1]),
$$I_2(x) \le a(A)U_G((x-A, x]) \to a(A)A/\mu_G \qquad \text{as } x \to \infty.$$

Hence, $I(x) \sim I_1(x)$ as $x \to \infty$. A similar relation also holds for $J(x)$, which completes the proof of the lemma. □

**Lemma 4.** *Let $\varphi(x)$, $x \in \mathbb{R}_+$, be a submultiplicative function, such that there exists $\nu(y) := \lim_{x\to\infty} \varphi(x+y)/\varphi(x)$ for each $y \in \mathbb{R}$. Then $\nu(y) = \exp(r_+ y)$, $y \in \mathbb{R}$.*

**Proof.** By the Corollary of Theorem 4.17.3 in Section 4.17 of [2], $\nu(y) = \exp(\alpha y)$ for some $\alpha \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$, such that $\log \dfrac{\varphi(n+1)}{\varphi(n)} \le \alpha + \varepsilon$ for $n \ge n_0$.

Hence, $\varphi(n_0 + m) \le \varphi(n_0)e^{m(\alpha+\varepsilon)}$ and
$$r_+ = \lim_{m\to\infty} \frac{\log \varphi(n_0+m)}{m} \le \lim_{m\to\infty} \frac{\log \varphi(n_0)}{m} + \lim_{m\to\infty} \frac{m(\alpha+\varepsilon)}{m} = \alpha + \varepsilon.$$

Similarly, $r_+ \ge \alpha - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\alpha = r_+$. The proof of the lemma is complete. □

## 3. Main Results

**Theorem 2.** *Let $F$ be a nonarithmetic probability distribution, such that*
$$\mu = \int_{\mathbb{R}} x\, F(dx) \in (0, \infty)$$
*and let $\varphi(x)$, $x \in \mathbb{R}_+$, be a nondecreasing continuous submultiplicative function tending to infinity as $x \to \infty$, such that $r_+ = 0$ and there exists $\lim_{x\to\infty} \varphi(x+y)/\varphi(x)$ for each $y \in \mathbb{R}$. Suppose that the inhomogeneous term $g(x)$, $x \in \mathbb{R}_+$, is bounded on finite intervals and satisfies the relation $g(x) \sim c\varphi(x)$ as $x \to \infty$, where $c \in \mathbb{C}$. Assume that*
$$\int_{-\infty}^{0} \varphi(|x|)F((-\infty, x])\, dx < \infty.$$
*Then, the function $z(x)$, $x \in \mathbb{R}_+$, defined by (5) is a solution to Equation (1) and satisfies the asymptotic relation*
$$z(x) \sim \frac{c}{\mu} \int_0^x \varphi(y)\, dy \qquad \text{as } x \to \infty.$$

**Proof.** Put $M(x) = \int_0^x \varphi(y)\, dy$. By Lemma 4, $\lim_{x\to\infty} \varphi(x+y)/\varphi(x) = 1$ for each $y \in \mathbb{R}$. Extend the function $\varphi(x)$ onto the whole line $\mathbb{R}$ by setting $\varphi(x) = \varphi(|x|)$ for $x \in \mathbb{R}_-$. The extended function retains the submultiplicative property and $r_\pm = 0$. To prove the first statement of the theorem, it suffices to assume $g \geq 0$. Choose $C > 0$, such that $g(x) \leq C\varphi(x)$, $x \in \mathbb{R}_+$. The function $z(x)$ defined by (5) is finite, since

$$U_- * g(x) \leq CU_- * \varphi(x) = C\int_{-\infty}^{0+} \varphi(x-y)\,U_-(dy) \leq C\varphi(x)\|U_-\|_\varphi,$$
$$z(x) \leq C\|U_-\|_\varphi \int_0^x \varphi(x-y)\,U_+(dy) \leq C\|U_-\|_\varphi \varphi(x)U_+([0,x]) < \infty$$

for all $x \in \mathbb{R}_+$. Let $n$ be a natural number. Denote by $\mathbf{1}_{[0,n]}$ the indicator of $[0,n]$. Consider Equation (1) with the inhomogeneous term $g_n(x) = g(x)\mathbf{1}_{[0,n]}(x)$. Let $z_n$ be the solution to the equation

$$z_n(x) = \int_{-\infty}^{x} z_n(x-y)\,F(dy) + g_n(x), \qquad x \in \mathbb{R}_+, \tag{9}$$

defined by formula (5):

$$z_n(x) = U_+ * \big((U_- * g_n)\mathbf{1}_{\mathbb{R}_+}\big)(x), \qquad x \in \mathbb{R}_+. \tag{10}$$

The integral in (9) can be written as

$$\int_{\mathbb{R}} z_n(x-y)\mathbf{1}_{[0,x]}(y)\,F(dy) \leq z_n(x) \leq z(x) < \infty.$$

The last two inequalities are consequences of (5). Obviously, $z_n(x) \uparrow$ as $n \uparrow$. By Section 27, Theorem B of [10], the integral tends to $\int_{-\infty}^{x} z(x-y)\,F(dy)$ as $n \uparrow \infty$. Letting $n \uparrow \infty$ in (9) and (10), we get that $z$ is a solution to (1). Let us prove the assertion of the theorem for the solution $z_\varphi$ to (1) for $g = \varphi$. Let us show that

$$\frac{U_- * \varphi(x)}{\varphi(x)} \to U_-(\mathbb{R}_- \cup \{0\}) \qquad \text{as } x \to \infty. \tag{11}$$

We have

$$\frac{U_- * \varphi(x)}{\varphi(x)} = \int_{-\infty}^{0} \frac{\varphi(x-y)}{\varphi(x)}\,U_-(dy). \tag{12}$$

By Lemma 4, the integrand tends to 1 as $x \to \infty$ and it is majorized by the $U_-$-integrable function $\varphi(y)$, since

$$\frac{\varphi(x-y)}{\varphi(x)} \leq \varphi(-y) = \varphi(y)$$

and $U_- \in S(\varphi)$ by Lemma 1. Applying Lebesgue's bounded convergence theorem (Section 26, Theorem D of [10]), we can pass to the limit under the integral sign in (12), which proves (11). Apply Lemma 3 with the following choice of $G$, $a(x)$ and $b(x)$:

$$G := F_+, \qquad a(x) := \mathbf{1}_{\mathbb{R}_+}(x)U_- * \varphi(x), \qquad b(x) := U_-(\mathbb{R}_- \cup \{0\})\mathbf{1}_{\mathbb{R}_+}(x)\varphi(x).$$

We get

$$z_\varphi(x) = \int_0^x U_- * \varphi(x-y)\,U_+(dy) \sim U_-(\mathbb{R}_- \cup \{0\})\int_0^x \varphi(x-y)\,U_+(dy) \qquad \text{as } x \to \infty.$$

Recalling (6), we see that in order to prove the theorem for $z_\varphi$, it suffices to establish

$$U_+ * (\mathbf{1}_{\mathbb{R}_+}\varphi)(x) = \int_0^x \varphi(x-y)\,U_+(dy) \sim \frac{1}{\mu_+}\int_0^x \varphi(y)\,dy = \frac{1}{\mu_+}M(x) \qquad \text{as } x \to \infty. \tag{13}$$

Integrating by parts, we get

$$\begin{aligned}\int_0^x \varphi(x-y)\,U_+(dy) &= \varphi(x-y)U_+([0,y])\big|_{y=0}^x - \int_0^x U_+([0,y])\,d_y\varphi(x-y)\\ &= U_+([0,x]) - \varphi(x) - \int_0^x U_+([0,y])\,d_y\varphi(x-y).\end{aligned} \tag{14}$$

The following three estimates hold:

$$\varphi(x),\ x,\ U_+([0,x]) = o(M(x)) \qquad \text{as } x\to\infty. \tag{15}$$

The first estimate follows from Lemma 2 with $a(x)=\varphi(x)$. The second one follows from the assumption $\varphi(y)\to\infty$ as $y\to\infty$. The third estimate follows from the second one and the elementary renewal theorem for the measure $U_+$: $U_+([0,x])\sim x/\mu_+$ as $x\to\infty$.

Show that

$$-\int_0^x U_+([0,y])\,d_y\varphi(x-y) \sim -\frac{1}{\mu_+}\int_0^x y\,d_y\varphi(x-y) \qquad \text{as } x\to\infty, \tag{16}$$

$$-\frac{1}{\mu_+}\int_0^x y\,d_y\varphi(x-y) \sim \frac{1}{\mu_+}M(x) \qquad \text{as } x\to\infty. \tag{17}$$

We prove first (17). This follows from the second estimate in (15) and the equality

$$-\int_0^x y\,d_y\varphi(x-y) = -y\varphi(x-y)\big|_{y=0}^x + \int_0^x \varphi(x-y)\,dy = -x + M(x).$$

Let $\varepsilon>0$ be arbitrary. Use the elementary renewal theorem and choose $y_0 = y_0(\varepsilon)$, such that

$$(1-\varepsilon)U_+([0,y]) \le \frac{y}{\mu_+} \le (1+\varepsilon)U_+([0,y]), \qquad y\ge y_0.$$

Write the left-hand side of (16) in the form

$$-\left(\int_0^{y_0} + \int_{y_0}^x\right) U_+([0,y])\,d_y\varphi(x-y) =: K_1(x)+K_2(x),$$

and let $M_1(x)+M_2(x)$ be a similar decomposition for the right-hand side. Obviously,

$$(1-\varepsilon)M_2(x) \le K_2(x) \le (1+\varepsilon)M_2(x). \tag{18}$$

Let us prove that, as $x\to\infty$, both sides in (16) are asymptotically equivalent to $K_2(x)$ and $M_2(x)$, respectively. We have

$$\begin{aligned}M_2(x) &= -\frac{1}{\mu_+}\int_{y_0}^x y\,d_y\varphi(x-y) = -\frac{y}{\mu_+}\varphi(x-y)\Big|_{y=y_0}^x + \frac{1}{\mu_+}\int_{y_0}^x \varphi(x-y)\,dy\\ &= -\frac{x}{\mu_+} + \frac{y_0}{\mu_+}\varphi(x-y_0) + \frac{1}{\mu_+}\int_0^{x-y_0}\varphi(y)\,dy.\end{aligned}$$

Let us show that

$$M_3(x) := \int_0^{x-y_0}\varphi(y)\,dy \sim M(x) \qquad \text{as } x\to\infty.$$

Using the first estimate in (15), we get

$$\begin{aligned}\int_{x-y_0}^x \varphi(y)\,dy &\le \varphi(x)\int_{x-y_0}^x \varphi(y-x)\,dy\\ &= \varphi(x)\int_0^{y_0}\varphi(y)\,dy \le \varphi(x)\varphi(y_0)y_0 = o(M(x)) \qquad \text{as } x\to\infty.\end{aligned}$$

Finally,

$$\frac{M_3(x)}{M(X)} = \frac{1}{M(x)}\left(\int_0^x \varphi(y)\,dy - \int_{x-y_0}^x \varphi(y)\,dy\right)$$
$$= 1 - \frac{1}{M(x)}\int_{x-y_0}^x \varphi(y)\,dy = 1 - o(1) \to 1 \qquad \text{as } x \to \infty,$$

which establishes the desired equivalence $M_3(x) \sim M(x)$ as $x \to \infty$. Taking into account the estimates in (15), we see that $M_2(x) \sim M(x)/\mu_+$ as $x \to \infty$. Moreover,

$$M_1(x) = -\frac{-y_0\varphi(x-y_0)}{\mu_+} + \frac{1}{\mu_+}\int_{x-y_0}^x \varphi(u)\,du.$$

The integral is estimated by $y_0\varphi(x)/\mu_+$. Thus, $M_1(x) = o(M(x))$ as $x \to \infty$ (see (15)). Relation (17) is proven. Now, divide all parts of (18) by $M_2(x)$ and let $x$ tend to infinity. We obtain

$$1 - \varepsilon \le \liminf_{x\to\infty} \frac{K_2(x)}{M_2(x)} \le \limsup_{x\to\infty} \frac{K_2(x)}{M_2(x)} \le 1 + \varepsilon.$$

Hence, $K_2(x) \sim M_2(x) \sim M(x)$ as $x \to \infty$. Relation (16) is proven, since, as $x \to \infty$,

$$K_1(x) \le -U_+([0,y_0])\int_0^{y_0} d_y\varphi(x-y)$$
$$= U_+([0,y_0])[\varphi(x) - \varphi(x-y_0)] \le U_+([0,y_0])\varphi(x) = o(M(x)).$$

The equivalence (13) now follows from (14)–(17), which proves the theorem in the particular case $g = \varphi$. Let $g$ satisfy the hypotheses of the theorem. If, for some $C > 0$, $|g(x)| \le C\varphi(x)$, $x \in \mathbb{R}_+$, then

$$\limsup_{x\to\infty} |z(x)| \Big/ \int_0^x \varphi(y)\,dy \le \frac{C}{\mu}.$$

It follows that if $c = 0$, then $z(x) = o(z_\varphi(x))$ as $x \to \infty$. To see this, choose a small $\varepsilon > 0$ and a natural number $n$, such that $|g(x)| \le \varepsilon\varphi(x)$, $x \ge n$. Write

$$g = \mathbf{1}_{[0,n]}g + (g - \mathbf{1}_{[0,n]}g) =: g_1 + g_2.$$

Let $z_1$ and $z_2$ be the solutions to (1) corresponding to $g_1$ and $g_2$, respectively. Then, $z = z_1 + z_2$ and $|z_2(x)| \le \varepsilon z_\varphi(x)$, $x \in \mathbb{R}_+$. By Theorem 6.2 in [3], $z_1(x) = o(x)$ as $x \to \infty$. Since $\varphi(x) \ge 1$, $x \in \mathbb{R}_+$, it follows that $z_1(x) = o\big(\int_0^x \varphi(y)\,dy\big)$ as $x \to \infty$. Therefore,

$$\limsup_{x\to\infty} |z(x)| \Big/ \int_0^x \varphi(y)\,dy \le \frac{\varepsilon}{\mu}.$$

Since $\varepsilon > 0$ is arbitrary, the assertion of the theorem is true for $c = 0$. Let $c \ne 0$. Write $g$ in the form $g = c\varphi + g_1$. Then, $g_1(x) = o(\varphi(x))$ as $x \to \infty$, and we have $z = cz_\varphi + z_1$, where $z_1$ is the solution to Equation (1) with the inhomogeneous term $g_1$. The proof of the theorem is complete. □

**Theorem 3.** *Let $F$ be a nonarithmetic probability distribution, such that*

$$\mu = \int_{\mathbb{R}} x\,F(dx) \in (0,\infty),$$

*and let $\varphi(x)$, $x \in \mathbb{R}_+$, be a nondecreasing submultiplicative function, such that $r_+ > 0$, and there exists $\lim_{x\to\infty} \varphi(x+y)/\varphi(x)$ for each $y \in \mathbb{R}$. Suppose that the inhomogeneous term $g(x)$,*

*$x \in \mathbb{R}_+$, is bounded on finite intervals and satisfies the relation $g(x) \sim c\varphi(x)$ as $x \to \infty$, where $c \in \mathbb{C}$. Assume that*

$$\int_{-\infty}^{0} \varphi(|x|)F((-\infty, x])\,dx < \infty$$

*and $\widehat{F}(-r_+) < 1$. Then, the function $z(x)$, $x \in \mathbb{R}_+$, defined by (5) is a solution to Equation (1) and satisfies the asymptotic relation*

$$z(x) \sim \frac{c}{1-\widehat{F}(-r_+)}\varphi(x) \qquad as\ x \to \infty.$$

**Proof.** As in the proof of the preceding theorem, we verify that $z(x)$ is a solution to (1). First, let us prove the assertion of the theorem for the solution $z_\varphi$ to (1) corresponding to $g = \varphi$, i.e., let us prove that, as $x \to \infty$,

$$\frac{z_\varphi(x)}{\varphi(x)} = \int_0^x \frac{U_- * \varphi(x-y)}{\varphi(x)}\,U_+(dy) \to \widehat{U}_-(-r_+)\widehat{U}_+(-r_+) = \frac{1}{1-\widehat{F}(-r_+)}. \tag{19}$$

Write the integrand in the form

$$I(x,y) := \mathbf{1}_{[0,x]}(y)\frac{U_- * \varphi(x-y)}{\varphi(x-y)}\frac{\varphi(x-y)}{\varphi(x)}, \qquad y \in \mathbb{R}_+.$$

Notice that

$$\frac{U_- * \varphi(x)}{\varphi(x)} = \int_{-\infty}^{0} \frac{\varphi(x-y)}{\varphi(x)}\,U_-(dy) \to \widehat{U}_-(-r_+) \qquad \text{as } x \to \infty. \tag{20}$$

In fact, $\varphi(x-y)/\varphi(x) \to e^{-r_+y}$ as $x \to \infty$ by Lemma 4 and, according to Lemma 1, this ratio is majorized by the $U_-$-integrable function $\varphi(y)$, $y \in \mathbb{R}_-$:

$$\frac{U_- * \varphi(x)}{\varphi(x)} = \int_{-\infty}^{0} \frac{\varphi(x-y)}{\varphi(x)}\,U_-(dy) \le \int_{-\infty}^{0} \varphi(|y|)\,U_-(dy) = \|U_-\|_\varphi < \infty.$$

Relation (20) now follows from Lebesgue's bounded convergence theorem. Our further actions are as follows. We will pick out a majorant for the function $I(x,y)$, $y \in \mathbb{R}_+$, in the form $Me^{\beta y}$ with $\beta \in (-r_+, 0)$. Then, by Lebesgue's theorem, we pass to the limit under the integral sign in the left-side integral in (19) as $x \to \infty$, and thus prove relation (19). Put $f(x) = \log\varphi(x) - r_+x$. By hypothesis, we have

$$f(x-y) - f(x) = \log\varphi(x-y) - \log\varphi(x) + r_+x \to 0 \qquad \text{as } x \to \infty \tag{21}$$

for each $y \in \mathbb{R}$. According to Lemma 1.1 in [11], relation (21) is fulfilled uniformly in $y \in [0,1]$. Hence,

$$\frac{\varphi(x-y)\exp(r_+y)}{\varphi(x)} \to 1 \qquad \text{as } x \to \infty$$

uniformly in $y \in [0,1]$. Choose a small $\varepsilon > 0$ such that $\beta := \log(1+\varepsilon) - r_+ < 0$. Let $N = N(\varepsilon) > 0$ be an integer such that

$$\frac{\varphi(x-y)\exp(r_+y)}{\varphi(x)} \le 1+\varepsilon, \qquad x \ge N, \quad y \in [0,1].$$

Denote by $[x]$ the integral part of a real number $x$; i.e., $[x]$ is the maximal integer not exceeding $x$: $x = [x] + \vartheta$, $\vartheta \in [0, 1)$. For $y \in [l, l+1]$, $l = 0, \dots, [x] - N - 1$, we have

$$\frac{\varphi(x-y)}{\varphi(x)} = \frac{\varphi(x-l-(y-l))}{\varphi(x-l)} \frac{\varphi(x-l)}{\varphi(x)},$$
$$\frac{\varphi(x-l-(y-l))}{\varphi(x-l)} \le (1+\varepsilon)\exp(-r_+(y-l)),$$
$$\frac{\varphi(x-l)}{\varphi(x)} = \frac{\varphi(x-l)}{\varphi(x-l+1)} \frac{\varphi(x-l+1)}{\varphi(x-l+2)} \cdots \frac{\varphi(x-1)}{\varphi(x)} \le (1+\varepsilon)^l \exp(-lr_+).$$

Ultimately,

$$\frac{\varphi(x-y)}{\varphi(x)} \le (1+\varepsilon)^{l+1} \exp(-r_+(y-l)) \exp(-lr_+) = (1+\varepsilon)^{l+1} \exp(-r_+ y)$$
$$\le (1+\varepsilon)\exp(\beta y), \qquad y \in [l, l+1], \quad l = 0, \dots, [x] - N - 1.$$

Now, let $y \in ([x] - N - 1, x]$. We have

$$\frac{\varphi(x-y)}{\varphi(x)} \le \frac{\varphi(N+2)}{\varphi(x)} \le \frac{\varphi(N+2)}{\exp(r_+ x)} \le \frac{\varphi(N+2)}{\exp(r_+ y)} \le \varphi(N+2)\exp(\beta y).$$

Thus, the $U_+$-integrable majorant sought for the function $I(x, y)$, $y \in \mathbb{R}_+$, which does not depend on $x$, is of the form

$$\|U_-\|_\varphi \max\{(1+\varepsilon), \varphi(N+2)\} \exp(\beta y), \qquad y \in \mathbb{R}_+.$$

Now, in order to prove relation (19), it suffices, by Lebesgue's theorem, to pass to the limit under the integral sign in (19). The last equality in (19) is a consequence of (8) for $\Re s = -r_+$:

$$\widehat{U}(s) = \frac{1}{1-\widehat{F}(s)} = \frac{1}{1-\widehat{F}_-(s)} \frac{1}{1-\widehat{F}_+(s)} = \widehat{U}_-(s)\widehat{U}_+(s),$$

which is admissible, since

$$|\widehat{F}(s)| \le \widehat{F}(-r_+) < 1, \qquad |\widehat{F}_\pm(s)| \le \widehat{F}_\pm(-r_+) < 1, \qquad \Re s = -r_+.$$

In the general case, it suffices to repeat the concluding reasoning of the previous proof using the estimate

$$\limsup_{x\to\infty} \frac{|z(x)|}{\varphi(x)} \le \frac{C}{1-\widehat{F}(-r_+)}$$

for $|g(x)| \le C\varphi(x)$, $x \in \mathbb{R}_+$, and, considering the case $c = 0$, take into account the relation $z_1(x) = o(x)$ as $x \to \infty$ and all the more $z_1(x) = o(\varphi(x))$ as $x \to \infty$, since $x \le e^{r_+ x} \le \varphi(x)$, $x \in \mathbb{R}_+$. $\square$

## 4. Conclusions

We have established the asymptotic behavior of the solution $z$ of the generalized Wiener–Hopf Equation (1), where the inhomogeneous term $g$ behaves like an unbounded submultiplicative function, up to a constant factor, i.e., $g(x) \sim c\varphi(x)$ as $x \to \infty$. Depending on whether $r_+ = 0$ or $r_+ > 0$, there are two different types of asymptotics for $z$ (Theorems 2 and 3): either $z(x) \sim c_1 \int_0^x \varphi(y)\,dy$ or $z(x) \sim c_2\varphi(x)$ as $x \to \infty$, where $c_1$ and $c_2$ are specific constants. Here are two simple examples ($c = 1$):

(i) If $\varphi(x) = (x+1)^r$, $r > 0$, then

$$z(x) \sim \frac{x^{r+1}}{\mu(r+1)} \qquad \text{as } x \to \infty;$$

(ii) If $\varphi(x) = \exp(\gamma x)$, $\gamma > 0$, then

$$z(x) \sim \frac{e^{\gamma x}}{1 - \widehat{F}(-r_+)} \qquad \text{as } x \to \infty.$$

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