AppliedMath

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Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Terms of the Green's Function, in Nonstandard Analysis

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Abstract: Discussions are presented by Morita and Sato on the problem of obtaining the particular solution of an inhomogeneous differential equation with polynomial coefficients in terms of the Green's function. In a paper, the problem is treated in distribution theory, and in another paper, the formulation is given on the basis of nonstandard analysis, where fractional derivative of degree, which is a complex number added by an infinitesimal number, is used. In the present paper, a simple recipe based on nonstandard analysis, which is closely related with distribution theory, is presented, where in place of Heaviside's step function $H(t)$ and Dirac's delta function $\delta(t)$ in distribution theory, functions $H_\epsilon(t) := \frac{1}{\Gamma(1+\epsilon)} t^\epsilon H(t)$ and $\delta_\epsilon(t) := \frac{d}{dt} H_\epsilon(t) = \frac{1}{\Gamma(\epsilon)} t^{\epsilon-1} H(t)$ for a positive infinitesimal number ϵ, are used. As an example, it is applied to Kummer's differential equation.

Keywords: Green's function; differential equations with polynomial coefficients; nonstandard analysis; distribution theory

Citation: Morita, T. Solution of Inhomogeneous Differential Equations with Polynomial Coefficients in Terms of the Green's Function, in Nonstandard Analysis. *AppliedMath* **2022**, *2*, 379–392. https://doi.org/10.3390/appliedmath2030022

Academic Editors : Valery Karachik and Leonid Shaikhet

Received: 15 April 2022
Accepted: 28 June 2022
Published: 15 July 2022

1. Introduction

In the present paper, we treat the problem of obtaining the particular solutions of a differential equation with polynomial coefficients in terms of the Green's function.

In a preceding paper [1], this problem is studied in the framework of distribution theory, where the method is applied to Kummer's and the hypergeometric differential equation. In another paper [2], this problem is studied in the framework of nonstandard analysis, where a recipe of solution of the present problem is presented, and it is applied to a simple fractional and a first-order ordinary differential equation.

In the present paper, we present a compact recipe based on nonstandard analysis, which is obtained by revising the one given in [2]. As an example, it is applied to Kummer's differential equation.

The presentation in this paper follows those in [1,2], in Introduction and in many descriptions in the following sections.

We consider a fractional differential equation, which takes the form:

$$p_n(t, {}_RD_t)u(t) := \sum_{l=0}^{n} a_l(t)\, {}_RD_t^{\rho_l} u(t) = f(t), \tag{1}$$

where $n \in \mathbb{Z}_{>-1}$, $t \in \mathbb{R}$, $a_l(t)$ for $l \in \mathbb{Z}_{[0,n]}$ are polynomials of t, $\rho_l \in \mathbb{C}$ for $l \in \mathbb{Z}_{[0,n]}$ satisfy $\mathrm{Re}\,\rho_0 > \mathrm{Re}\,\rho_1 \geq \cdots \geq \mathrm{Re}\,\rho_n$ and $\mathrm{Re}\,\rho_0 > 0$. We use Heaviside's step function $H(t)$, which is equal to 1 if $t > 0$, and to 0 if $t \leq 0$. Here ${}_RD_t^{\rho_l}$ are the Riemann–Liouville fractional integrals and derivatives defined by the following definition; see [3].

Definition 1. *Let $t \in \mathbb{R}$, $\tau \in \mathbb{R}$, $u_0(t)$ be locally integrable on $\mathbb{R}_{>\tau}$, $u(t) = u_0(t)H(t-\tau)$, $\lambda \in \mathbb{C}_+$, $n \in \mathbb{Z}_{>-1}$ and $\rho = n - \lambda$. Then ${}_R D_t^{-\lambda} u(t)$ is the Riemann–Liouville fractional integral defined by*

$$
\begin{aligned}
{}_R D_t^{-\lambda} u(t) &= \frac{1}{\Gamma(\lambda)} \int_{-\infty}^{t} (t-x)^{\lambda-1} u_0(x) H(x-\tau) dx \\
&= \frac{1}{\Gamma(\lambda)} \int_{\tau}^{t} (t-x)^{\lambda-1} u_0(x) dx \cdot H(t-\tau),
\end{aligned} \tag{2}
$$

and ${}_R D_t^{-\lambda} u(t) = 0$ for $t \leq \tau$, where $\Gamma(\lambda)$ is the gamma function, ${}_R D_t^{\rho} u(t) = {}_R D_t^{n-\lambda} u(t)$ is the Riemann–Liouville fractional derivative defined by

$$
{}_R D_t^{\rho} u(t) = {}_R D_t^{n-\lambda} u(t) = \frac{d^n}{dt^n} [{}_R D_t^{-\lambda} u_0(t)] \cdot H(t-\tau), \tag{3}
$$

when $n \geq \mathrm{Re}\, \lambda$, and ${}_R D_t^{n} u(t) = \frac{d^n}{dt^n} u_0(t) \cdot H(t-\tau)$ when $\rho = n \in \mathbb{Z}_{>-1}$.

Here $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all integers, all real numbers and all complex numbers, respectively, and $\mathbb{Z}_{>a} = \{n \in \mathbb{Z} \mid n > a\}$, $\mathbb{Z}_{<b} = \{n \in \mathbb{Z} \mid n < b\}$ and $\mathbb{Z}_{[a,b]} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ for $a, b \in \mathbb{Z}$ satisfying $a < b$. We also use $\mathbb{R}_{>a} = \{x \in \mathbb{R} \mid x > a\}$ for $a \in \mathbb{R}$, and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 0\}$.

In accordance with Definition 1, when $u_0(t) = \frac{1}{\Gamma(\nu)}(t-\tau)^{\nu-1}$, we adopt

$$
{}_R D_t^{\rho} \frac{(t-\tau)^{\nu-1}}{\Gamma(\nu)} H(t-\tau) = \begin{cases} \frac{(t-\tau)^{\nu-\rho-1}}{\Gamma(\nu-\rho)} H(t-\tau), & \nu - \rho \in \mathbb{C} \backslash \mathbb{Z}_{<1}, \\ 0, & \nu - \rho \in \mathbb{Z}_{<1}, \end{cases} \tag{4}
$$

for $\nu \in \mathbb{C} \backslash \mathbb{Z}_{<1}$ and $\tau \in \mathbb{R}$. Here ${}_R D_t$ is used in place of usually used notation ${}_\tau D_R$, in order to show that the variable is t.

Remark 1. *Let $g_\nu(t) := \frac{1}{\Gamma(\nu)} t^{\nu-1} H(t)$ for $\nu \in \mathbb{C}$. Then $g_\nu(t) = 0$ if $\nu \in \mathbb{Z}_{<1}$, and Equation (4) shows that if $\nu \notin \mathbb{Z}_{<1}$, ${}_R D_t^{\rho} g_\nu(t) = g_{\nu-\rho}(t)$. As a consequence, we have ${}_R D_t^{\nu+n} g_\nu(t) = g_{-n}(t) = 0$ for $n \in \mathbb{Z}_{>-1}$.*

Remark 2. *Let $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$, $\nu \in \mathbb{C} \backslash \mathbb{Z}_{<1}$ and $g_\nu(t) := \frac{1}{\Gamma(\nu)} t^{\nu-1} H(t)$. Then, the index law: ${}_R D_t^{\rho_1} {}_R D_t^{\rho_2} g_\nu(t) = {}_R D_t^{\rho_2} {}_R D_t^{\rho_1} g_\nu(t)$ does not always hold. An example is given in the book [4] (p. 108); see also [5] (p. 48).*

In [1,6], discussions are made of an ordinary differential equation, which is expressed by (1) for $\rho_l = n - l$, in terms of distribution theory, and with the aid of the analytic continuation of Laplace transform, respectively. In those papers, solutions are given of differential equations with an inhomogeneous term $f(t)$, which satisfies one of the following three conditions.

Condition 1. (i) $f(t) = f_0(t)H(t)$, *where $f_0(t)$ is locally integrable on $\mathbb{R}_{>0}$.*

(ii) $f(t) = {}_R D_t^{\beta}[f_\beta(t)H(t)]$, *where $\beta \in \mathbb{C} \backslash \mathbb{Z}_{>-1}$, and $f_\beta(t)$ is locally integrable on $\mathbb{R}_{>0}$.*

(iii) $f(t) = {}_R D_t^{\beta+1} H(t) = \frac{1}{\Gamma(-\beta)} t^{-\beta-1} H(t)$, *where $\beta \in \mathbb{C} \backslash \mathbb{Z}_{>-1}$.*

1.1. Green's Function in Distribution Theory

In a recent paper [5], the solution of Euler's differential equation in distribution theory is compared with the solution in nonstandard analysis. In distribution theory [1,7–9], we use distribution $\tilde{H}(t)$, which corresponds to function $H(t)$, differential operator D and distribution $\delta(t) = D\tilde{H}(t)$, which is called Dirac's delta function.

When $\nu \in \mathbb{C}_+$ and $n \in \mathbb{Z}_{>0}$, $\tilde{g}_\nu(t) := \frac{1}{\Gamma(\nu)} t^{\nu-1}\tilde{H}(t) = D^{-\nu+1}\tilde{H}(t) = D^{-\nu}\delta(t)$ is a regular distribution, and $D^n\tilde{g}_\nu(t) = D^{n-\nu+1}\tilde{H}(t) = D^{n-\nu}\delta(t)$ is a distribution but is not a regular one, if $\nu - n \in \mathbb{C}\backslash\mathbb{C}_+$.

As a consequence, when $\rho = -\nu \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$, we have

$$D^{\rho+1}\tilde{H}(t) = D^{\rho}\delta(t) = D^{-\nu}\delta(t) = \begin{cases} \tilde{g}_\nu(t), & \nu = -\rho \in \mathbb{C}_+, \\ D^n\tilde{g}_{-\rho+n}(t), & -\rho + n \in \mathbb{C}_+. \end{cases} \tag{5}$$

In place of (4), for $\rho_1 \in \mathbb{C}$ and $\rho \in \mathbb{C}$, we now have $D^{\rho_1}D^{\rho}\delta(t) = D^{\rho_1+\rho}\delta(t)$.

Remark 3. *Let $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$ and $\rho \in \mathbb{C}$. Then, the index law:*

$$D^{\rho_1}D^{\rho_2}D^{\rho}\delta(t) = D^{\rho_1+\rho_2}D^{\rho}\delta(t) = D^{\rho_1+\rho_2+\rho}\delta(t), \tag{6}$$

always holds.

Remark 4. *In solving (1) in [1], the Green's function $\tilde{G}(t,\tau)$ in distribution theory is introduced by*

$$p_n(t, D)\tilde{G}(t,\tau) = \delta(t-\tau). \tag{7}$$

Lemma 1. *Let $u_c(t,\tau)$ be a complementary solution of Equation (1) for $t > \tau$, and $G_0(t,\tau)$, which is given by*

$$G_0(t,\tau) = u_c(t,\tau)H(t-\tau), \tag{8}$$

satisfy

$$\int_{-\infty}^{t} [p_n(x, {}_RD_x)G_0(x,\tau)]dx = H(t-\tau) = \begin{cases} 1, & t > \tau, \\ 0, & t \le \tau. \end{cases} \tag{9}$$

Then $\tilde{G}(t,\tau) = G_0(t,\tau)\tilde{H}(t-\tau)$ is the Green's function defined in Remark 4.

In [1], the following theorem is given.

Theorem 1. *Let $f(t)$ satisfy Condition 1 (i) and $G_0(t,\tau)$ be the one given in Lemma 1. Then $u_f(t)$ given by*

$$u_f(t) = \int_{-\infty}^{t} G_0(t,\tau)f(\tau)d\tau = \int_{-\infty}^{\infty} G_0(t,\tau)f(\tau)d\tau, \tag{10}$$

is a particular solution of Equation (1).

Proof. By using Equations and (9), we have

$$\begin{aligned} \int_0^t p_n(x, {}_RD_x)u_f(x)dx &= \int_0^t dx[p_n(x, {}_RD_x)\int_0^x G_0(x,\tau)f(\tau)H(\tau)d\tau] \\ &= \int_0^t [\int_\tau^t p_n(x, {}_RD_x)G_0(x,\tau)dx]f(\tau)H(\tau)d\tau \\ &= \int_0^t H(t-\tau)f(\tau)H(\tau)d\tau = \int_0^t f(\tau)H(\tau)d\tau. \end{aligned} \tag{11}$$

By taking the derivative of the first and the last member in this equation with respect to t, we confirm that Equation (1) is satisfied by $u(t) = u_f(t)$. □

1.2. Preliminaries on Nonstandard Analysis

In the present paper, we use nonstandard analysis [10], where infinitesimal numbers are used. We denote the set of all infinitesimal real numbers by $\mathbb{R}^0$. We also use

$\mathbb{R}^0_{>0} = \{\epsilon \in \mathbb{R}^0 \mid \epsilon > 0\}$, which is such that if $\epsilon \in \mathbb{R}^0_{>0}$ and $N \in \mathbb{Z}_{>0}$, then $\epsilon < \frac{1}{N}$. We use $\mathbb{R}^{ns}$, which has subsets $\mathbb{R}$ and $\mathbb{R}^0$. If $x \in \mathbb{R}^{ns}$ and $x \notin \mathbb{R}$, x is expressed as $x_1 + \epsilon$ by $x_1 \in \mathbb{R}$ and $\epsilon \in \mathbb{R}^0$, where x_1 may be $0 \in \mathbb{R}$. Equation $x \simeq y$ for $x \in \mathbb{R}^{ns}$ and $y \in \mathbb{R}^{ns}$, is used, when $x - y \in \mathbb{R}^0$. We denote the set of all infinitesimal complex numbers by $\mathbb{C}^0$, which is the set of complex numbers z which satisfy $|\mathrm{Re}\, z| + |\mathrm{Im}\, z| \in \mathbb{R}^0$. We use $\mathbb{C}^{ns}$, which has subsets $\mathbb{C}$ and $\mathbb{C}^0$. If $z \in \mathbb{C}^{ns}$ and $z \notin \mathbb{C}$, z is expressed as $z_1 + \epsilon$ by $z_1 \in \mathbb{C}$ and $\epsilon \in \mathbb{C}^0$, where z_1 may be $0 \in \mathbb{C}$.

Remark 5. *In nonstandard analysis [10], in addition to infinitesimal numbers, we use unlimited numbers, which are often called infinite numbers. In the present paper, we do not use them, but if we use them, we have to consider sets $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$ such that if $\omega \in \mathbb{R}^\infty$, there exists $\epsilon \in \mathbb{R}^0$ satisfying $\omega = \frac{1}{\epsilon}$, and if $\omega \in \mathbb{C}^\infty$, there exists $\epsilon \in \mathbb{C}^0$ satisfying $\omega = \frac{1}{\epsilon}$, and then $\mathbb{R}^{ns} = \mathbb{R} \cup \mathbb{R}^0 \cup \mathbb{R}^\infty$ and $\mathbb{C}^{ns} = \mathbb{C} \cup \mathbb{C}^0 \cup \mathbb{C}^\infty$.*

In place of (4), we now use

$$ {}_R D_t^\rho \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu-1+\epsilon} H(t) = \frac{1}{\Gamma(\nu-\rho+\epsilon)} t^{\nu-\rho-1+\epsilon} H(t), \tag{12}$$

for all $\rho \in \mathbb{C}$ and $\nu \in \mathbb{C}$, where $\epsilon \in \mathbb{R}^0_{>0}$.

Lemma 2. *Let $\rho_1 \in \mathbb{C}$, $\rho_2 \in \mathbb{C}$, $\nu \in \mathbb{C}$, $\epsilon \in \mathbb{R}^0_{>0}$ and $g_{\nu+\epsilon}(t) := \frac{1}{\Gamma(\nu+\epsilon)} t^{\nu+\epsilon-1} H(t)$. Then, the index law:*

$$ {}_R D_t^{\rho_1}\, {}_R D_t^{\rho_2} g_{\nu+\epsilon}(t) = {}_R D_t^{\rho_1+\rho_2} g_{\nu+\epsilon}(t) = g_{\nu-\rho_1-\rho_2+\epsilon}(t), \tag{13}$$

always holds.

Remark 6. *When $\epsilon \in \mathbb{R}^0$ or $\epsilon \in \mathbb{C}^0$, we often ignore terms of $O(\epsilon)$ compared with a term of $O(\epsilon^0)$. For instance, when $\nu \in \mathbb{R}_{>0}$ and $\nu - \rho \in \mathbb{R}_{>0}$, we adopt $\frac{1}{\Gamma(\nu+\epsilon)} t^{\nu-1+\epsilon} H(t) \simeq \frac{1}{\Gamma(\nu)} t^{\nu-1+\epsilon} H(t)$, and also*

$$ {}_R D_t^\rho \frac{1}{\Gamma(\nu)} t^{\nu-1+\epsilon} H(t) \simeq \frac{1}{\Gamma(\nu-\rho)} t^{\nu-\rho-1+\epsilon} H(t), \tag{14}$$

in place of (12). In the following, we often use "=" in place of "$\simeq$".

In the present study in nonstandard analysis, $\epsilon \in \mathbb{R}^0_{>0}$ is used, and $H(t)$ and $\delta(t) = D\tilde{H}(t)$, respectively, are replaced by

$$ H_\epsilon(t) := {}_R D_t^{-\epsilon} H(t) = \frac{1}{\Gamma(\epsilon+1)} t^\epsilon H(t) \simeq t^\epsilon H(t), \tag{15}$$

which tends to $H(t)$ in the limit $\epsilon \to 0$, and by

$$ \delta_\epsilon(t) := \frac{d}{dt} H_\epsilon(t) = \frac{d}{dt} \frac{1}{\Gamma(\epsilon+1)} t^\epsilon H(t) = \frac{1}{\Gamma(\epsilon)} t^{\epsilon-1} H(t) \simeq \epsilon t^{\epsilon-1} H(t). \tag{16}$$

Lemma 3. *In the notation in Remark 1, $H_\epsilon(t) = g_{1+\epsilon}(t)$, $\delta_\epsilon(t) = g_\epsilon(t)$, and we have*

$$ {}_R D_t^\epsilon H_\epsilon(t) = {}_R D_t^\epsilon g_{1+\epsilon}(t) = g_1(t) = H(t), \quad {}_R D_t^\epsilon \delta_\epsilon(t) = {}_R D_t^\epsilon g_\epsilon(t) = g_0(t) = 0. \tag{17}$$

Lemma 4. *Let $\epsilon \in \mathbb{R}^0_{>0}$, $\tau \in \mathbb{R}$, and $f(t)$ be locally integrable on $\mathbb{R}_{>\tau}$. Then*

$$ \int_{-\infty}^{\infty} \delta_\epsilon(t-x) f(x) H(x-\tau) dx = {}_R D_t^{-\epsilon} [f(t) H(t-\tau)]. \tag{18}$$

Proof. Since $\delta_\epsilon(t-x) = {}_R D_t^{-\epsilon+1} H(t-x)$, we have

$$\int_{-\infty}^{\infty} \delta_\epsilon(t-x) f(x) H(x-\tau) dx = {}_R D_t^{-\epsilon+1} \int_{-\infty}^{\infty} H(t-x) f(x) H(x-\tau) dx$$
$$= {}_R D_t^{-\epsilon+1} \int_{\tau}^{t} f(x) H(x-\tau) dx = {}_R D_t^{-\epsilon} [f(t) H(t-\tau)]. \tag{19}$$

□

1.3. Summary of the Following Sections

In Section 2, a recipe of solution of Equation (1), in nonstandard analysis, is presented. We there consider the solution of the following equation for $\tilde{u}(t)$:

$$\tilde{p}_{n,\epsilon}(t, {}_R D_t) \tilde{u}(t) = \tilde{f}(t), \tag{20}$$

where $\epsilon \in \mathbb{R}^0_{>0}$ and

$$\tilde{p}_{n,\epsilon}(t, {}_R D_t) := {}_R D_t^{-\epsilon} p_n(t, {}_R D_t) {}_R D_t^{\epsilon}. \tag{21}$$

Here, the inhomogeneous terms $f(t)$ and $\tilde{f}(t)$ are assumed to satisfy one of the following four conditions.

Condition 2. *Let $\epsilon \in \mathbb{R}^0_{>0}$ and $\beta \in \mathbb{C}$.*

(i) *$f(t) = f_0(t) H(t)$ and $\tilde{f}(t) = {}_R D_t^{-\epsilon} f(t) + c_\epsilon \delta_\epsilon(t)$, where $f_0(t)$ is locally integrable on $\mathbb{R}_{>0}$ and c_ϵ is a constant.*

(ii) *$f(t) = {}_R D_t^{\beta} f_\beta(t)$ and $\tilde{f}(t) = {}_R D_t^{\beta} \tilde{f}_\beta(t)$, where*

$$\tilde{f}_\beta(t) = {}_R D_t^{-\epsilon} f_\beta(t) + c_{\beta,\epsilon} \delta_\epsilon(t), \quad f_\beta(t) = f_{\beta,0}(t) H(t), \tag{22}$$

$f_{\beta,0}(t)$ is locally integrable on $\mathbb{R}_{>0}$, and $c_{\beta,\epsilon}$ is a constant.

(iii) *$\tilde{f}(t) = {}_R D_t^{\beta} \tilde{f}_\beta(t)$, where $\tilde{f}_\beta(t) = {}_R D_t H_\epsilon(t) = \delta_\epsilon(t)$. When $\beta \in \mathbb{Z}_{>-1}$, $f(t) = 0$, and when $\beta \notin \mathbb{Z}_{>-1}$, $f(t) = {}_R D_t^{\beta+1} H(t)$.*

(iv) *$\tilde{f}(t)$ and $f(t)$ are expressed as follows:*

$$\tilde{f}(t) = \sum_{l=1}^{\infty} c_l \cdot {}_R D_t^{\beta_l} \delta_\epsilon(t) = \sum_{l=1}^{\infty} c_l \cdot \frac{t^{\epsilon-1-\beta_l}}{\Gamma(\epsilon-\beta_l)} H(t), \quad f(t) = \sum_{l=1}^{\infty} d_l \cdot {}_R D_t^{\beta_l+1} H(t), \tag{23}$$

respectively, where $c_l \in \mathbb{C}$ are constants, $\beta_l \in \mathbb{C}$ satisfy $-\mathrm{Re}\,\beta_l \geq -\mathrm{Re}\,\beta_1 \in \mathbb{R}$, for all $l \in \mathbb{Z}_{>0}$, and $d_l = c_l$ if $\beta_l \notin \mathbb{Z}_{>-1}$, and $d_l = 0$ if $\beta_l \in \mathbb{Z}_{>-1}$.

Remark 7. *Lemma 3 shows that when Condition 2 (i) is satisfied, ${}_R D_t^{\epsilon} \tilde{f}(t) = f(t)$, and $\tilde{f}(t) = {}_R D_t^{-\epsilon} f(t)$ does not always hold, and when Condition 2 (iii) is satisfied, ${}_R D_t^{\epsilon} \tilde{f}_\beta(t) = 0$.*

In Sections 3 and 4, full expressions of the Green's functions and the solutions, are derived along the recipe given in Section 2, for Kummer's differential equation:

$$p_K(t, {}_R D_t) u(t) := [t \frac{d^2}{dt^2} + (c - bt) \frac{d}{dt} - ab] u(t) = f(t), \tag{24}$$

where a, b and c are constants satisfying $a \neq 0$ and $b \neq 0$.

Section 5 is for Conclusion. In Section 6, a concluding remark is given.

2. Recipe of Solution of Differential Equation, in Nonstandard Analysis

In obtaining a particular solution of Equation (1) for $\tilde{f}(t)$ satisfying Condition 2 (i), in place of the Green's function defined in Remark 4, we use it defined in the following definition.

Definition 2. *Let $\tilde{p}_{n,\epsilon}(t, {}_R D_t)$ be given by Equation (21). Then for $\epsilon \in \mathbb{R}^0_{>0}$ and $\tau \in \mathbb{R}$, the Green's function $G_\epsilon(t,\tau)$ for Equation (1) satisfies*

$$\tilde{p}_{n,\epsilon}(t, {}_R D_t) G_\epsilon(t,\tau) = \delta_\epsilon(t-\tau). \tag{25}$$

Lemma 5. *Let $G_\epsilon(t,\tau)$ be defined as in Definition 2, and $G_0(t,\tau) := {}_R D_t^\epsilon G_\epsilon(t,\tau)$. Then $G_0(t,\tau)$ is a complementary solution of Equation (1) on $\mathbb{R}_{>\tau}$, and ${}_R D_t^{-1} p_n(t, {}_R D_t) G_0(t,\tau) = 1$ at any value of t satisfying $t > \tau$.*

Proof. These are confirmed by applying ${}_R D_t^\epsilon$ and ${}_R D_t^{-1+\epsilon}$ to Equation (25), by noting Lemma 3. □

Lemma 6. *Let $\tilde{u}_c(t)$ be a complementary solution of Equation (20) on $\mathbb{R}_{>0}$, and $u_c(t) := {}_R D_t^\epsilon \tilde{u}_c(t)$. Then $u_c(t)$ is a complementary solution of Equation (1) on $\mathbb{R}_{>0}$.*

Proof. This is confirmed by replacing $\tilde{u}(t)$ and $\tilde{f}(t)$ by $\tilde{u}_c(t)$ and 0 in Equation (20), and then applying ${}_R D_t^\epsilon$ to the equation. □

Theorem 2. *Let Condition 2 (i) be satisfied, $G_\epsilon(t,\tau)$ and $G_0(t,\tau)$ be given as in Lemma 5. Then $\tilde{u}_f(t)$ given by*

$$\tilde{u}_f(t) = \int_{-\infty}^{\infty} G_\epsilon(t,\tau) f(\tau) d\tau + c_\epsilon G_\epsilon(t,0), \tag{26}$$

is the particular solution of Equation (20) for the term $\tilde{f}(t)$, and $u_f(t)$ given by

$$u_f(t) = {}_R D_t^\epsilon \tilde{u}_f(t) = \int_{-\infty}^{\infty} G_0(t,\tau) f(\tau) d\tau + c_\epsilon G_0(t,0), \tag{27}$$

consists of the particular solution for the term $f(t)$ and a complementary solution of Equation (1).

Proof. By using Equations (27), (25) and (18), we obtain

$$\begin{aligned} \tilde{p}_{n,\epsilon}(t, {}_R D_t)\tilde{u}_f(t) &= \tilde{p}_{n,\epsilon}(t, {}_R D_t)[\int_{-\infty}^{\infty} G_\epsilon(t,\tau) f(\tau) d\tau + c_\epsilon G_\epsilon(t,0)] \\ &= \int_{-\infty}^{\infty} \delta_\epsilon(t-\tau) f(\tau) d\tau + c_\epsilon \delta_\epsilon(t) = {}_R D_t^{-\epsilon} f(t) + c_\epsilon \delta_\epsilon(t) = \tilde{f}(t), \end{aligned} \tag{28}$$

which is a proof for $\tilde{u}_f(t)$. □

When Condition 2 (ii) is satisfied, we introduce the transformed differential equations for $w(t) = {}_R D_t^{-\beta} u(t)$ and $\tilde{w}(t) = {}_R D_t^{-\epsilon} w(t)$ from Equations (1) and (20), respectively, by

$$\tilde{p}_{n,\beta}(t, {}_R D_t) w(t) = f_\beta(t), \tag{29}$$

$$\tilde{p}_{n,\beta+\epsilon}(t, {}_R D_t) \tilde{w}(t) = \tilde{f}_\beta(t), \tag{30}$$

where

$$\tilde{p}_{n,\beta}(t, {}_R D_t) := {}_R D_t^{-\beta} p_n(t, {}_R D_t) {}_R D_t^\beta, \tag{31}$$

$$\tilde{p}_{n,\beta+\epsilon}(t, {}_R D_t) := {}_R D_t^{-\beta-\epsilon} p_n(t, {}_R D_t) {}_R D_t^{\beta+\epsilon}. \tag{32}$$

Lemma 7. *Let Equation (30) and $\tilde{f}(t) = {}_R D_t^\beta \tilde{f}_\beta(t)$ hold. Then by using (32), we confirm that Equation (20) for $\tilde{u}(t) = {}_R D_t^\beta \tilde{w}(t)$ holds.*

Remark 8. *Let $\tilde{u}_c(t)$ and $\tilde{w}_c(t)$ be complementary solutions of Equation (20) and (30), respectively, on $\mathbb{R}_{>0}$. Then by using (32), we confirm that they are related by $\tilde{u}_c(t) = {}_R D_t^\beta \tilde{w}_c(t)$.*

Definition 3. *For $\epsilon \in \mathbb{R}^0_{>0}$ and $\tau \in \mathbb{R}$, the Green's function $G_{\beta,\epsilon}(t,\tau)$ for Equation (29) satisfies*

$$\tilde{p}_{n,\beta+\epsilon}(t, {}_R D_t) G_{\beta,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau). \tag{33}$$

Lemma 8. *Let $G_{\beta,\epsilon}(t,\tau)$ be defined as in Definition 3, and $G_{\beta,0}(t,\tau) := {}_R D_t^\epsilon G_{\beta,\epsilon}(t,\tau)$. Then $G_{\beta,0}(t,\tau)$ is a complementary solution of Equation (29) on $\mathbb{R}_{>\tau}$.*

Proof. A proof of this lemma is obtained from that of Lemma 5, by replacing (25) by (33), $\tilde{p}_{n,\epsilon}$ by $\tilde{p}_{n,\beta+\epsilon}$, G_ϵ by $G_{\beta,\epsilon}$, p_n by $\tilde{p}_{n,\beta}$, G_0 by $G_{\beta,0}$, and (1) by (29). □

Theorem 3. *Let Condition 2 (ii) be satisfied, and $G_{\beta,\epsilon}(t,\tau)$ satisfy Equation (33). Then $\tilde{w}_f(t)$ and $\tilde{u}_f(t)$ given by*

$$\tilde{w}_f(t) := \int_{-\infty}^{\infty} G_{\beta,\epsilon}(t,\tau) f_\beta(\tau) d\tau + c_{\beta.\epsilon} G_{\beta,\epsilon}(t,0), \quad \tilde{u}_f(t) := {}_R D_t^\beta \tilde{w}_f(t), \tag{34}$$

are particular solutions of Equations (30) and (20), respectively.

Proof. Theorem 2 states that when $\tilde{f}(t)$ satisfies Condition 2 (i) and $G_\epsilon(t,\tau)$ satisfies (30), the solution $\tilde{u}_f(t)$ of (20) is expressed as (26). This shows that when $\tilde{f}_\beta(t)$ satisfies Condition 2 (ii) and $G_{\beta,\epsilon}(t,\tau)$ satisfies (33), the solution $\tilde{w}_f(t)$ of (30) is given by the first equation in (34). The second equation in it is due to Lemma 7. □

When Condition 2 (iii) is satisfied, Equation (20) is expressed as

$$\tilde{p}_{n,\epsilon}(t, {}_R D_t)\tilde{u}(t) = {}_R D_t^\beta \delta_\epsilon(t) = \frac{1}{\Gamma(\epsilon-\beta)} t^{\epsilon-\beta-1} H(t). \tag{35}$$

Since Condition 2 (iii) is a special case of Condition 2 (ii) in which $f_\beta(t) = 0$ and $c_{\beta,\epsilon} = 1$, we obtain the following theorem from Theorem 3.

Theorem 4. *Let Condition 2 (iii) be satisfied, and $G_{\beta,\epsilon}(t,0)$ satisfy Equation (33) for $\tau = 0$. Then $\tilde{w}_f(t)$ and $\tilde{u}_f(t)$ given by*

$$\tilde{w}_f(t) = G_{\beta,\epsilon}(t,0), \quad \tilde{u}_f(t) = {}_R D_t^\beta G_{\beta,\epsilon}(t,0), \tag{36}$$

are particular solutions of Equations (30) and (20), respectively.

Theorem 4 shows that if $\tilde{f}(t) = {}_R D_t^\beta \delta_\epsilon(t)$, the particular solution of (20) is given by $\tilde{u}_f(t) = {}_R D_t^\beta G_{\beta,\epsilon}(t,0)$. As a consequence, we have

Theorem 5. *Let $\tilde{f}(t)$ satisfy Condition 2 (iv), so that it is given by Equation (23). Then the particular solution of Equation (20) is given by*

$$\tilde{u}_f(t) = \sum_{l=1}^{\infty} c_l \cdot {}_R D_t^{\beta_l} G_{\beta_l,\epsilon}(t,0). \tag{37}$$

3. Solution of Kummer's Differential Equation, I

We construct the transformed differential equation of Equation (24), which corresponds to Equation (20). For this purpose, we use the following lemma.

Lemma 9. *Let $\lambda \in \mathbb{C}_+$, $m \in \mathbb{Z}_{>-1}$ and $\rho = m - \lambda$. Then*

$${}_R D_t^\rho [t u(t)] = t \cdot {}_R D_t^\rho u(t) + \rho \cdot {}_R D_t^{\rho-1} u(t). \tag{38}$$

Proof. When $m=0$ and $\rho=-\lambda$, this is confirmed with the aid of Formula (2), as follows:

$$ {}_RD_t^{-\lambda}[tu(t)] = \frac{1}{\Gamma(\lambda)}\int_{-\infty}^t (t-\xi)^{\lambda-1}\xi u(\xi)d\xi $$
$$ = \frac{1}{\Gamma(\lambda)}\int_{-\infty}^t (t-\xi)^{\lambda-1}(t-(t-\xi))u(\xi)d\xi = t\cdot {}_RD_t^{-\lambda}u(t) - \lambda\cdot {}_RD_t^{-\lambda-1}u(t). $$

We prove (38) by mathematical induction. In fact, when (38) holds for a value $n\in\mathbb{Z}_{>-1}$ of m, we confirm it to hold even for $m=n+1$, by applying $\frac{d}{dt}$ to (38). □

Remark 9. *When $u(t)=\frac{t^{\nu+\epsilon}}{\Gamma(\nu+\epsilon+1)}H(t)$, by using (12), we confirm (38) as follows:*

$$ {}_RD_t^{\rho}[tu(t)] = {}_RD_t^{\rho}[t\frac{t^{\nu+\epsilon}}{\Gamma(\nu+\epsilon+1)}H(t)] = (\nu+\epsilon+1){}_RD_t^{\rho}[\frac{t^{\nu+\epsilon+1}}{\Gamma(\nu+\epsilon+2)}H(t)] $$
$$ = ((\nu+\epsilon+1-\rho)+\rho)\frac{t^{\nu+\epsilon-\rho+1}}{\Gamma(\nu+\epsilon-\rho+2)}H(t) = t\cdot {}_RD_t^{\rho}u(t) + \rho\cdot {}_RD_t^{\rho-1}u(t). $$

With the aid of Formula (38) for $\rho=-\epsilon$, we construct the following transformation of Equation (24) for $\tilde{u}(t)={}_RD_t^{-\epsilon}u(t)$, which corresponds to Equation (20):

$$ \tilde{p}_{K,\epsilon}(t,{}_RD_t)\tilde{u}(t) := {}_RD_t^{-\epsilon}p_K(t,{}_RD_t){}_RD_t^{\epsilon}\tilde{u}(t) = {}_RD_t^{-\epsilon}[t\frac{d^2}{dt^2}+(c-bt)\frac{d}{dt}-ab]{}_RD_t^{\epsilon}\tilde{u}(t) $$
$$ = [t\frac{d^2}{dt^2}+(c-\epsilon-bt)\frac{d}{dt}-(a-\epsilon)b]\tilde{u}(t) = \tilde{f}(t). \tag{39} $$

When Condition 2 (i) is satisfied, in accordance with Definition 2, we define the Green's function $G_{K,\epsilon}(t,\tau)$, which satisfies

$$ \tilde{p}_{K,\epsilon}(t,{}_RD_t)G_{K,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau), \tag{40} $$

for $\tau\in\mathbb{R}$. The solutions of Equations (39) and (24) are then given with the aid of Theorem 2 and the following lemma.

Lemma 10. *Let $c\notin\mathbb{Z}_{<1}$. Then there exist two complementary solutions of Equation (24), which are given by*

$$ K_1(t) = {}_1F_1(a;c;bt) := \sum_{k=0}^{\infty}\frac{(a)_kb^k}{k!(c)_k}t^k, \quad t>0, \tag{41} $$
$$ K_2(t) = \frac{1}{\Gamma(2-c)}t^{1-c}\cdot {}_1F_1(a-c+1;2-c;bt) = \sum_{k=0}^{\infty}\frac{(a-c+1)_kb^k}{k!\Gamma(2-c+k)}t^{1-c+k}, \quad t>0, \tag{42} $$

where $(a)_k$ for $k\in\mathbb{Z}_{>0}$ and $k=0$, denote $(a)_k=\prod_{l=0}^{k-1}(a+l)=\frac{\Gamma(a+k)}{\Gamma(a)}$ and $(a)_0=1$, respectively.

In the present paper, these equations are proved in Lemmas 11 and 12 given below.

Lemma 11. *Let $K_1(t)$ be given by (41). Then $G_{K,\epsilon}(t,0)$ and $G_{K,0}(t,0)$, given by*

$$ G_{K,\epsilon}(t,0) = \frac{1}{-1+c}\sum_{k=0}^{\infty}\frac{(a)_kb^k}{(c)_k\Gamma(k+\epsilon+1)}t^{k+\epsilon}H(t), \tag{43} $$
$$ G_{K,0}(t,0) = {}_RD_t^{\epsilon}G_{K,\epsilon}(t,0) = \frac{1}{-1+c}K_1(t)H(t), \tag{44} $$

are a particular solution of Equation (40) for $\tau = 0$, and a complementary solution of Equation (24), respectively.

A proof of the statement for $G_{K,\epsilon}(t,0)$ is given in Section 3.1, and the statement for $G_{K,0}(t,0)$ is due to Lemma 5.

Lemma 12. *Let $K_2(t)$ be given by (42). Then $\tilde{u}_c(t)$ and $u_c(t)$, given by*

$$\tilde{u}_c(t) = \sum_{k=0}^{\infty} \frac{(a-c+1)_k b^k}{k!\Gamma(2-c+k+\epsilon)} t^{1-c+\epsilon+k} H(t), \tag{45}$$

$$u_c(t) = {}_R D_t^{\epsilon} \tilde{u}_c(t) = K_2(t)H(t), \tag{46}$$

are complementary solutions of Equations (39) and (24), respectively.

A proof of the statement for $\tilde{u}_c(t)$ is given in Section 3.1, and the statement for $u_c(t)$ is due to Lemma 6.

The differential equation satisfied by the Green's function $G_{K,\epsilon}(t,\tau)$ for Equation (24) is given by Equation (40).

Lemma 13. *Let $0 < \tau < t$, $K_1(t)$ and $K_2(t)$ be those in Lemma 10, and $G_{K,0}(t,\tau)$ be given by*

$$G_{K,0}(t,\tau) = \frac{1}{\tau_K \psi'_\tau(\tau)} \psi_\tau(t) H(t-\tau) = \frac{1}{\tau_K \psi'_\tau(\tau)} \sum_{k=1}^{\infty} \frac{1}{k!} \psi_\tau^{(k)}(\tau)(t-\tau)^k H(t-\tau), \tag{47}$$

where $\tau_K = \tau$ and $\psi_\tau(t) = K_1(\tau)K_2(t) - K_2(\tau)K_1(t)$.

Then $G_{K,\epsilon}(t,\tau)$, given by $G_{K,\epsilon}(t,\tau) = {}_R D_t^{-\epsilon} G_{K,0}(t,\tau)$, satisfies Equation (40).

Proof. Taking account of Lemma 5, we choose the complementary solution of Equation (24) on $\mathbb{R}_{>\tau}$, given by $\tilde{G}_{K,0}(t,\tau) = C_1 \cdot G_{K,0}(t,\tau)$, where C_1 is a constant, and then confirm that $\tilde{G}_{K,\epsilon}(t,\tau) = C_1 \cdot {}_R D_t^{-\epsilon} G_{K,0}(t,\tau)$ satisfies (40), when $C_1 = 1$, as follows.

We put $x = t - \tau$, and we express $\tilde{G}_{K,\epsilon}(t,\tau)$ by

$$\tilde{v}(x) := \tilde{G}_{K,\epsilon}(\tau + x, \tau) = \sum_{k=1}^{\infty} a_k \frac{x^{k+\epsilon}}{\Gamma(k+\epsilon+1)} H(x), \tag{48}$$

where a_k are constants, and $a_1 \neq 0$. Then (40) is expressed as

$$\tilde{p}_{K,\epsilon}(\tau + x, {}_R D_x)\tilde{v}(x) = [\tau_K a_1 \frac{x^{\epsilon-1}}{\Gamma(\epsilon)} + O(x^\epsilon)]H(x) = \frac{x^{\epsilon-1}}{\Gamma(\epsilon)} H(x). \tag{49}$$

This is satisfied when $a_1 = \frac{1}{\tau_K}$. □

Theorem 6. *Let $\tilde{f}(t)$ satisfy Condition 2 (i), $G_{K,\epsilon}(t,\tau)$ satisfy Equation (40), $G_{K,\epsilon}(t,\tau)$ and $G_{K,0}(t,\tau)$ for $\tau > 0$ be given in Lemma 13, and $G_{K,\epsilon}(t,0)$ and $G_{K,0}(t,0)$ be given in Lemma 11. Then Theorem 2 shows that we have the solutions $\tilde{u}_f(t)$ and $u_f(t)$ of Equations (39) and (24), respectively, which are given by*

$$\tilde{u}_f(t) := \int_{-\infty}^{\infty} G_{K,\epsilon}(t,\tau) f(\tau) d\tau + c_\epsilon G_{K,\epsilon}(t,0), \tag{50}$$

$$u_f(t) := {}_R D_t^{\epsilon} \tilde{u}_f(t) = \int_{-\infty}^{\infty} G_{K,0}(t,\tau) f(\tau) d\tau + c_\epsilon G_{K,0}(t,0). \tag{51}$$

See Lemma 12 for the complementary solutions $\tilde{u}_c(t)$ and $u_c(t)$.

This result is derived with the aid of the complementary solutions given by Equations (41) and (42), and hence by assuming $c \notin \mathbb{Z}_{<1}$.

3.1. Derivations of Equations for $G_{K,\epsilon}(t,0)$ and $\tilde{u}_c(t)$ by Using Frobenius' Method

Equation (40) shows that $\tilde{u}(t) = G_{K,\epsilon}(t,0)$, given by Equation (43), is the particular solution of Equation (39) in which $\tilde{f}(t) = \delta_\epsilon(t)$, and $\tilde{u}(t) = \tilde{u}_c(t)$, given by Equation (45), is the complementary solution of Equation (39) in which $\tilde{f}(t) = 0$.

We assume that the solution $\tilde{u}(t)$ of Equation (39) is expressed by

$$\tilde{u}(t) = \sum_{k=0}^{\infty} p_k \frac{1}{\Gamma(\alpha+k+1)} t^{\alpha+k} H(t), \tag{52}$$

where α and p_k are constants, and $p_0 \neq 0$. Then Equation (39) is expressed as

$$\begin{aligned}
\tilde{p}_{K,\epsilon}(t, {}_RD_t)\tilde{u}(t) = &\sum_{k=0}^{\infty} p_k[(\alpha+k-1+c-\epsilon)\frac{t^{\alpha+k-1}}{\Gamma(\alpha+k)} \\
&-b(\alpha+k+a-\epsilon)\frac{t^{\alpha+k}}{\Gamma(\alpha+k+1)}]H(t) \\
= &\, p_0(\alpha-1+c-\epsilon)\frac{t^{\alpha-1}}{\Gamma(\alpha)}H(t) + \sum_{k=1}^{\infty}[p_k(\alpha+k-1+c-\epsilon) \\
&-bp_{k-1}(\alpha+k-1+a-\epsilon)]\frac{t^{\alpha+k-1}}{\Gamma(\alpha+k)}H(t) = \tilde{f}(t).
\end{aligned} \tag{53}$$

When $\tilde{f}(t) = \delta_\epsilon(t) = \frac{t^{\epsilon-1}}{\Gamma(\epsilon)}H(t)$, Equation (53) is satisfied, if

$$\alpha = \epsilon, \quad p_0 = \frac{1}{-1+c}; \quad p_k = bp_{k-1}\frac{k-1+a}{k-1+c} = b^k p_0 \frac{(a)_k}{(c)_k}, \quad k \in \mathbb{Z}_{>0}. \tag{54}$$

By using these in Equation (52) and putting $\tilde{u}(t) = G_{K,\epsilon}(t,0)$, we obtain Equation (43).
When $\tilde{f}(t) = 0$, Equation (53) is satisfied, if

$$\alpha = 1-c+\epsilon; \quad p_k = bp_{k-1}\frac{k+a-c}{k} = b^k p_0 \frac{(a-c+1)_k}{k!}, \quad k \in \mathbb{Z}_{>0}. \tag{55}$$

By using these in Equation (52) and putting $\tilde{u}(t) = p_0\tilde{u}_c(t)$, we obtain Equation (45).

4. Solution of Kummer's Differential Equation, II

We construct the transformed differential equations of Equation (24), which appear in Theorems 3–5. Corresponding to Equations (29) and (30), we have the following equations for $w(t) = {}_RD_t^{-\beta}u(t)$ and $\tilde{w}(t) = {}_RD_t^{-\epsilon}w(t)$ from Equation (24) satisfying Condition 2 (ii), as follows:

$$\begin{aligned}
&\tilde{p}_{K,\beta}(t, {}_RD_t)w(t) := {}_RD_t^{-\beta}p_K(t, {}_RD_t){}_RD_t^{\beta}w(t) \\
&= [t\frac{d^2}{dt^2} + (c-\beta-bt)\frac{d}{dt} - (a-\beta)b]w(t) = f_\beta(t),
\end{aligned} \tag{56}$$

$$\begin{aligned}
&\tilde{p}_{K,\beta+\epsilon}(t, {}_RD_t)\tilde{w}(t) := {}_RD_t^{-\beta-\epsilon}p_K(t, {}_RD_t){}_RD_t^{\beta+\epsilon}\tilde{w}(t) \\
&= [t\frac{d^2}{dt^2} + (c-\beta-\epsilon-bt)\frac{d}{dt} - (a-\beta-\epsilon)b]\tilde{w}(t) = \tilde{f}_\beta(t).
\end{aligned} \tag{57}$$

Remark 10. *In this section, we consider Equations (56) and (57) in place of Equations (24) and (39), respectively, and hence the equations in this section are obtained from the corresponding equations in Section 3, by replacing c by $c-\beta$, a by $a-\beta$, f by f_β, $\tilde{f}$ by $\tilde{f}_\beta$, u by w, and $\tilde{u}$ by $\tilde{w}$. They will be given without derivation.*

Lemma 14. *Lemma 10 and Remark 10 show that if $c-\beta \notin \mathbb{Z}_{<1}$, there exist two complementary solutions of Equation (56), which are given by*

$$K_{\beta,1}(t) = {}_1F_1(a-\beta; c-\beta; bt) = \sum_{k=0}^{\infty} \frac{(a-\beta)_k b^k}{k!(c-\beta)_k} t^k, \quad t>0, \tag{58}$$

$$\begin{aligned} K_{\beta,2}(t) &= \frac{1}{\Gamma(2-c+\beta)} t^{1-c+\beta} \cdot {}_1F_1(a-c+1; 2-c+\beta; bt) \\ &= \sum_{k=0}^{\infty} \frac{(a-c+1)_k b^k}{k!\Gamma(2-c+\beta+k)} t^{1-c+\beta+k} = {}_R D_t^{-\beta} K_2(t) H(t), \quad t>0. \end{aligned} \tag{59}$$

In accordance with Definition 3, we define the Green's function $G_{K,\beta,\epsilon}(t,\tau)$, which satisfies

$$\tilde{p}_{K,\beta+\epsilon}(t, {}_R D_t) G_{K,\beta,\epsilon}(t,\tau) = \delta_\epsilon(t-\tau), \tag{60}$$

for $\tau \in \mathbb{R}$. The solutions of Equations (57), (56), (39) and (24) are then given with the aid of Theorems 3, 4 and 5, and Lemma 14.

Remark 11. *Equation (60) is obtained from Equation (40), by replacing c by $c-\beta$, a by $a-\beta$, and $G_{K,\epsilon}$ by $G_{K,\beta,\epsilon}$.*

In Section 4, formulas are derived with the aid of two complementary solutions given by (58) and (59), and hence they hold when $c-\beta \notin \mathbb{Z}_{<1}$.

Lemma 15. *Let $K_{\beta,1}(t)$ be given by Equation (58). Then Lemma 11, Remark 10 and Lemmas 14 and 5 show that $G_{K,\beta,\epsilon}(t,0)$ and $G_{K,\beta,0}(t,0)$, given by*

$$G_{K,\beta,\epsilon}(t,0) = {}_R D_t^{-\epsilon} G_{K,\beta,0}(t,0), \quad G_{K,\beta,0}(t,0) = \frac{1}{-1+c-\beta} K_{\beta,1}(t) H(t), \tag{61}$$

are a particular solution of Equation (60) for $\tau = 0$, and a complementary solution of Equation (56), respectively.

With the aid of Remark 11, we have the following lemma for $G_{K,\beta,\epsilon}(t,\tau)$ for $\tau > 0$.

Lemma 16. *The lemma, which is obtained from Lemma 13 by replacing K_1 by $K_{\beta,1}$, Lemma 10 by Lemma 14, K_2 by $K_{\beta,2}$, $G_{K,\epsilon}$ by $G_{K,\beta,\epsilon}$, and $G_{K,0}$ by $G_{K,\beta,0}$, holds.*

Theorem 7. *Let Condition 2 (iii) be satisfied, and $G_{K,\beta,\epsilon}(t,0)$ be given in Equation (61). Then, Theorem 4 shows that $\tilde{w}_f(t) := G_{K,\beta,\epsilon}(t,0)$ and $\tilde{u}_f(t)$, given by*

$$\tilde{u}_f(t) := {}_R D_t^{\beta} \tilde{w}_f(t) = \frac{1}{-1+c-\beta} \sum_{k=0}^{\infty} \frac{(a-\beta)_k b^k}{(c-\beta)_k \Gamma(k-\beta+1+\epsilon)} t^{k-\beta+\epsilon} H(t), \tag{62}$$

are particular solutions of Equations (57) and (39), respectively.

Corollary 1. *Let $\beta = n \in \mathbb{Z}_{>-1}$, and $\tilde{u}_f(t)$ be the solution of (39), given by Equation (62). Then $u_f(t) = {}_R D_t^{\epsilon} \tilde{u}_f(t)$ and $\tilde{u}_f(t)$ are expressed by*

$$u_f(t) = \sum_{k=n}^{\infty} \frac{(a-n)_k b^k}{(-1+c-n)_{k+1}} \frac{1}{(k-n)!} t^{k-n} H(t), \tag{63}$$

$$\tilde{u}_f(t) \simeq {}_R D_t^{-\epsilon} u_f(t) + \epsilon \sum_{k=0}^{n-1} \frac{(a-n)_k b^k (-1)^{n-k-1}}{(-1+c-n)_{k+1}} (n-k-1)! t^{k-n+\epsilon} H(t), \tag{64}$$

where $u_f(t)$ is a complementary solution of Equation (24), for $n \in \mathbb{Z}_{>-1}$.

In obtaining the last term in Equation (64), we use the following formulas:

$$\frac{1}{\Gamma(z)} = \frac{\sin(\pi z)\Gamma(1-z)}{\pi}; \quad \frac{1}{\Gamma(-m+\epsilon)} \simeq (-1)^m \epsilon \cdot m!, \quad m \in \mathbb{Z}_{>-1}. \tag{65}$$

Theorem 7 shows that if $\tilde{f}(t) = {}_R D_t^\beta \delta_\epsilon(t)$, the particular solution of Equation (39) is given by Equation (62). As a consequence, we have the following theorem.

Theorem 8. *Let $\tilde{f}(t)$ satisfy Condition 2 (iv), so that it is given by Equation (23). Then the particular solution of Equation (39) is given by*

$$\tilde{u}_f(t) = \sum_{l=1}^{\infty} c_l \cdot \frac{1}{-1+c-\beta_l} \sum_{k=0}^{\infty} \frac{(a-\beta_l)_k b^k}{(c-\beta_l)_k \Gamma(k-\beta_l+1+\epsilon)} t^{k-\beta_l+\epsilon} H(t). \tag{66}$$

Condition $c - \beta \notin \mathbb{Z}_{<1}$ in Lemma 14 requires the condition $c - \beta_l \notin \mathbb{Z}_{<1}$ for all $l \in \mathbb{Z}_{>0}$, in the present case.

Lemma 17. *Lemma 12, Remark 10 and Lemma 6 show that $\tilde{w}_c(t)$ and $w_c(t)$, given by*

$$\tilde{w}_c(t) = {}_R D_t^{-\epsilon} w_c(t), \quad w_c(t) := K_{\beta,2}(t)H(t) = {}_R D_t^{-\beta} K_2(t) H(t), \tag{67}$$

are complementary solutions of Equations (57) and (56), respectively, and then Remark 8 shows that $\tilde{u}_c(t)$ and $u_c(t)$, given by $\tilde{u}_c(t) = {}_R D_t^\beta \tilde{w}_c(t)$ and $u_c(t) = {}_R D_t^\epsilon \tilde{u}_c(t)$, respectively, are the complementary solutions of Equations (39) and (24), which are given in Lemma 12.

Theorem 9. *Let $f_\beta(t)$ satisfy Condition 2 (ii), $G_{K,\beta,\epsilon}(t,\tau)$ for $\tau > 0$, satisfy Equation (60), and be determined by Lemma 16, and $G_{K,\beta,\epsilon}(t,0)$ be given in Equation (61). Then Theorem 3 shows that the particular solutions of Equations (57) and (39), respectively, are given by*

$$\tilde{w}_f(t) = \int_{-\infty}^{\infty} G_{K,\beta,\epsilon}(t,\tau) f_\beta(\tau) d\tau + c_{\beta,\epsilon} G_{K,\beta,\epsilon}(t,0), \quad \tilde{u}_f(t) = {}_R D_t^\beta \tilde{w}_f(t). \tag{68}$$

Their complementary solutions $\tilde{w}_c(t)$ and $\tilde{u}_c(t)$ are given in Lemma 17.

5. Conclusions

In [1], the problem of obtaining the particular solution of an inhomogeneous ordinary differential equation with polynomial coefficients is discussed in terms of the Green's function, in the framework of distribution theory. It is applied to Kummer's and the hypergeometric differential equation.

In [2], a compact recipe is presented, which is applicable to the case of an inhomogeneous fractional differential equation, which is expressed by Equation (1). In the recipe, the particular solution is given by Theorems 2, 3 or 4, according as the inhomogeneous part satisfies Condition 2 (i), (ii) or (iii), in the framework of nonstandard analysis. It is applied to a simple fractional and an ordinary differential equation.

In Section 2, in the present paper, a compact revised recipe in nonstandard analysis is presented, which is more closely related with distribution theory. In this case, the particular solution is given by Theorems 2, 3, 4 or 5, according as the inhomogeneous part satisfies Condition 2 (i), (ii), (iii) or (iv). In Sections 3 and 4, it is applied to inhomogeneous Kummer's differential Equation (24). In solving Equation (24) in nonstandard analysis, we construct transformed Equation (39) from it. In Section 3, we obtain the solution of Equation (39) by using the Green's function, and obtain the solution of Equation (24) from it. In Section 4, we construct further transformed Equation (57) from Equation (39), obtain the solutions of Equation (57) by using the Green's function, and then obtain the solutions of

Equations (39) and (24) from them. In Corollary 1, a nonstandard solution, which involves infinitestimal terms, is presented.

In [11], an ordinary differential equation is expressed in terms of blocks of classified terms. When the equation is expressed by two blocks of classified terms, the complementary solutions are obtained by using Frobenius' method. In Section 3.1, the Green's function and a complementary solution for Equation (39) are presented by using Frobenius' method.

One of reviewers of this paper asked the author to cite papers [12–14], which discuss the solutions of fractional differential equations. When the solutions of the differential equations, which are obtained with the aid of distribution theory, are of interest, the solution by using nonstandard analysis will be useful.

6. Concluding Remark

In the book of [9], Dirac's delta function $\delta(t)$ is introduced as a limit of zero width, of a function which has a single peak at $t = 0$ and unit area, and is defined as a functional. In the present paper, we study problems in nonstandard analysis, by using a function $\delta_\epsilon(t)$ which has an infinitesimal width ϵ and unit area.

In a preceding paper [1], the problem of obtaining the particular solution of an inhomogeneous ordinary differential equation, is discussed in terms of distribution theory. In another paper [2], we discussed solution of a fractional and a simple ordinary differential equation, in terms of nonstandard analysis by using two functions $\delta_{\epsilon_1}(t)$ and $\delta_\epsilon(t)$ expressed by two infinitesimal numbers ϵ_1 and ϵ. In the present paper, we proposed a revised recipe in terms of nonstandard analysis, by using the function $\delta_\epsilon(t)$ in place of distribution $\delta(t)$ in distribution theory. In the present paper, the recipe is applied only to Kummer's differential equation. The application of the present recipe to other differential equations studied in [1,2], will be given in a separate paper in preparation.

The author desires to have a day when we discuss the merit of using two functions $\delta_{\epsilon_1}(t)$ and $\delta_\epsilon(t)$.

Funding: This research received no external funding.

Acknowledgments: The author is indebted to Ken-ichi Sato, who collaborated in writing preceding papers and an early stage of the present paper, but left from mathematics because of his illness.

Conflicts of Interest: The author declares no conflict of interest.

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