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Review

A Comprehensive Review of the Hermite–Hadamard Inequality Pertaining to Quantum Calculus

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Abstract: A review of results on Hermite–Hadamard (H-H) type inequalities in quantum calculus, associated with a variety of classes of convexities, is presented. In the various classes of convexities this includes classical convex functions, quasi-convex functions, p-convex functions, (p,s)-convex functions, modified (p,s)-convex functions, (p,h)-convex functions, tgs-convex functions, η-quasi-convex functions, ϕ-convex functions, (α,m)-convex functions, ϕ-quasi-convex functions, and coordinated convex functions. Quantum H-H type inequalities via preinvex functions and Green functions are also presented. Finally, H-H type inequalities for (p,q)-calculus, h-calculus, and $(q-h)$-calculus are also included.

Keywords: H-H inequality; quantum calculus; post-quantum calculus; convex function

1. Introduction

The term Quantum Calculus in mathematics describes a form of calculus that proceeds without the concept of a limit. It is also referred to as q-calculus and is essentially constructed around the concept of finite difference re-scaling. It was first addressed at the beginning of the 18th century. Euler first developed the q-calculus and Jackson first presented the q-integral and q-derivative in Ref. [1] (see also Ref. [2]). There are many implementations of q-calculus in physics and mathematics, including orthogonal polynomials, quantum theory, mechanics, number theory, combinatorics, fundamental hypergeometric functions, and theory of relativity. For examples, see Refs. [3–7]. The basic understanding in addition to the underlying ideas of quantum theory are discussed and explored in the famous book by Cheung and Kac [8].

Tariboon and Ntouyas [9] created an entirely novel field of study, acquired numerous q-analogues of classical mathematical objects, and presented the notions of quantum calculus on finite intervals. They have, for example, extended some prominent integral inequalities to the q-calculus. This stimulated other investigators and, as a result, multiple innovative results via quantum equivalents of classical mathematical results were put forward in the literature.

In the domain of applied mathematics, fractional calculus encompasses the investigation and implementation of arbitrary order integrals and derivatives. Tariboon et al. [10] investigated a new operator, namely q-shifting operator ${}_{x_1}\Phi_q(m) = qm + (1-q)x_1$ for analyzing new ideas related to fractional q-calculus. In addition, since numerous inequalities are crucial for mathematical analysis, which depends on inequalities, Tariboon et al. examined and discussed some q-integral inequalities in the frame of fractional calculus such as the q-Korkine equality, the q-Grüss, the q-Hölder, the fractional q-H-H, the q-Polya–Szeqö, and the q-Grüss–Chebyshev integral inequality on finite intervals. For details, see the monograph [11].

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The (p,q)-calculus is an extended form of q-calculus. This calculus has a lot of importance and plays remarkable roles in physics and applied mathematics such as dynamical systems, mechanics, special functions, combinatorics, fractals, and number theory. In case of $p=1$, this calculus collapses to the q-calculus.

Due to its numerous implementations in physics and mathematics, mathematical inequalities have significant implications in both of these fields. Convex functions are among the most important functions that have been utilized to analyze numerous intriguing inequalities, which is defined as:

Definition 1. *A function $\pi : I \to \mathbb{R}$ is called convex, if*

$$\pi(\theta x_1 + (1-\theta)x_2) \le \theta\pi(x_1) + (1-\theta)\pi(x_2)$$

for all $x_1, x_2 \in I$ and $\theta \in [0,1]$.

Hermite [12] and Hadamard [13] first introduced and investigated the H-H inequality. This inequality is considered one of the most important concepts in applied and pure mathematics, with diverse and significant applications. It holds a prominent place in the study of convexity and is widely recognized for its geometrical interpretation. The H-H inequality has been extensively explored in the literature, highlighting its importance and unique properties. Numerous scholars have contributed various ideas in the field of inequalities. Due to its widespread perspective in the field of science, this inequality has become a dynamic and highly intriguing topic, and it has been extensively discussed in the context of convex functions. This inequality states that if real valued function π is convex on I and $x_1, x_2 \in I$ with $x_1 < x_2$, then

$$\pi\left(\frac{x_1+x_2}{2}\right) \le \frac{1}{x_2-x_1}\int_{x_1}^{x_2} \pi(x)dx \le \frac{\pi(x_1)+\pi(x_2)}{2}. \tag{1}$$

Dragomir and Aqarwal presented the following inequality associated with the right part of (1).

Theorem 1 ([14])**.** *Let $\pi : I^\circ \subset \mathbb{R}$ be a differentiable mapping on I°, (the interior of an interval I), $x_1, x_2 \in I^\circ$ with $x_1 < x_2$. If $|\pi'|$ is convex on $[x_1, x_2]$, then*

$$\left|\frac{\pi(x_1)+\pi(x_2)}{2} - \frac{1}{x_2-x_1}\int_{x_1}^{x_2} \pi(x)dx\right| \le \frac{(x_2-x_1)}{8}(|\pi'(x_1)| + |\pi'(x_2)|). \tag{2}$$

The goal of this study is to provide an in-depth and current overview of H-H-type inequalities for various types of convexities in the frame of quantum calculus. In each part and subsection, we initially describe the fundamental definitions of various types of convexities and quantum calculus, followed by the results on H-H inequalities. We anticipate that compiling practically all current H-H-type inequalities in one file will assist new researchers in the field in learning about prior work on the topic before creating new findings.

This survey is devoted to reviewing the results on H-H type inequalities in quantum calculus, which is associated with a variety of classes of convexities. This review article is constructed in the following manner. In Section 2 we introduce the reader to the basic concepts of q-calculus and summarize quantum H-H inequalities for many classes of convexities, including classical convex functions, quasi-convex functions, p-convex functions, (p,s)-convex functions, modified (p,s)-convex functions, (p,h)-convex functions, tgs-convex functions, η-quasi-convex functions, ϕ-convex functions, (α,m)-convex functions, ϕ-quasi-convex functions, and coordinated convex functions. Quantum H-H type inequalities via preinvex functions and Green functions are also presented. In Section 3, we present H-H inequalities via fractional quantum calculus, while in Section 4 we consider H-H inequalities regarding (p,q)-calculus. In Section 5, we include results for h-calculus and finally, in Section 6, we present the results on $q-h$-calculus.

It is crucial to consider that the primary goal of this review paper is to provide insight into the current state of the field, address evident gaps, highlight essential research, and potentially establish consensus in areas where it has not yet been achieved. The goal of this review paper is to provide a concise overview of the most recent advances in a specific field, namely convex analysis in the context of quantum calculus. Overall, the present level of knowledge on convexity is summarized in this review paper. It helps the reader comprehend the topic by discussing the findings reported in current research documents. The incorporation of relevant results is essential to demonstrate the progress in the field, as our goal is to present a more comprehensive and accurate review. However, lengthy proofs are excluded from this paper, and readers are instead directed to the respective article for more detailed information.

2. H-H Type Inequalities via Quantum Calculus

Here, we add some fundamental concepts of q-calculus.

Definition 2 ([1])**.** *Assume that $0 < q < 1$ and π is a function defined on a q-geometric set I, i.e., $qt \in I$, $\forall t \in I$. Then the q-derivative is defined as*

$$D_q\pi(t) = \frac{\pi(t) - \pi(qt)}{(1-q)t},\ t \in I \setminus \{0\}, \quad D_q\pi(0) = \lim_{t\to 0} D_q\pi(t).$$

For $t \geq 0$ we set $J_t = \{tq^n : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and define the definite q-integral of a function $\pi : J_t \to \mathbb{R}$ by

$$I_q\pi(t) = \int_0^t \pi(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n \pi(tq^n)$$

provided that the series converges.

Now we extend the notions of the q-integral and q-derivative on finite intervals.

Definition 3 ([15])**.** *Assume that function $\pi : [x_1, x_2] \to \mathbb{R}$ is continuous. Then*

$${}_{x_1}D_q\pi(t) = \frac{\pi(t) - \pi(qt + (1-q)x_1)}{(1-q)(t-x_1)}, t \neq x_1, \quad {}_{x_1}D_q\pi(x_1) = \lim_{t\to a} {}_{x_1}D_q\pi(t),$$

is called the q_{x_1}-derivative of π at $t \in [x_1, x_2]$.

Definition 4 ([15])**.** *The q-integral states that*

$$\int_{x_1}^t \pi(s) d_q s = (1-q)(t - x_1) \sum_{n=0}^{\infty} q^n \pi(q^n t + (1-q^n)x_1)$$

for $t \in I$, where π is a real-valued continuous function.

Quantum H-H type inequalities for q_{x_1}-integral and a variety of convex functions

In the following theorems we present quantum H-H type inequalities for many kinds of convex function. We start with results on classical convex and quasi-convex functions.

Definition 5 ([16])**.** *A real-valued function π is called quasi-convex, if*

$$\pi(\theta x_1 + (1-\theta)x_2) \leq \max\{\pi(x_1), \pi(x_2)\},$$

for all $x_1, x_2 \in I$ and $\theta \in [0, 1]$.

Theorem 2 ([17]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be differentiable and convex on (x_1, x_2) and $0 < q < 1$. Then*

$$\pi\Big(\frac{qx_1 + x_2}{1+q}\Big) \leq \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(t)\ {}_{x_1}d_q t \leq \frac{q\pi(x_1) + \pi(x_2)}{1+q}.$$

Note that when $q \to 1-$ the above inequality is reduced to classical H-H inequality (1).

Theorem 3 ([17]). *Assume that π is as in Theorem 2. Then we have*

$$\max\{I_1, I_2\} \leq \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(t)\ {}_{x_1}d_q t \leq \frac{q\pi(x_1) + \pi(x_2)}{1+q},$$

where

$$\begin{aligned} I_1 &= \pi\Big(\frac{x_1 + qx_2}{1+q}\Big) + \frac{(1-q)(x_2 - x_1)}{1+q}\pi'\Big(\frac{x_1 + qx_2}{1+q}\Big) \\ I_2 &= \pi\Big(\frac{x_1 + x_2}{2}\Big) + \frac{(1-q)(x_2 - x_1)}{2(1+q)}\pi'\Big(\frac{x_1 + x_2}{2}\Big). \end{aligned}$$

Theorem 4 ([17]). *Assume that function $\pi : [x_1, x_2] \to \mathbb{R}$ is q-differentiable on (x_1, x_2), ${}_{x_1}D_q\pi$ is integrable and continuous on $[x_1, x_2]$ and $0 < q < 1$. If $|{}_{x_1}D_q\pi|$ is convex on $[x_1, x_2]$, then*

$$\begin{aligned} &\left|\pi\Big(\frac{qx_1 + x_2}{1+q}\Big) - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(t)\ {}_{x_1}d_q t\right| \\ \leq\ & q(x_2 - x_1)\Big[|{}_{x_1}D_q\pi(x_2)|\frac{3}{(1+q)^3(1+q+q^2)} + |{}_{x_1}D_q\pi(x_1)|\frac{-1+2q+2q^2}{(1+q)^3(1+q+q^2)}\Big]. \end{aligned}$$

Theorem 5 ([17]). *Assume that π is as in Theorem 4. If $|{}_{x_1}D_q\pi|^r$ is convex on $[x_1, x_2]$ for $r \geq 1$, then*

$$\begin{aligned} &\left|\pi\Big(\frac{qx_1 + x_2}{1+q}\Big) - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(t)\ {}_{x_1}d_q t\right| \\ \leq\ & q(x_2 - x_1)\frac{1}{(1+q)^{3-\frac{3}{r}}}\Big[\Big(|{}_{x_1}D_q\pi(x_2)|^r\frac{1}{(1+q)^3(1+q+q^2)} \\ &+ |{}_{x_1}D_q\pi(x_1)|^r\frac{q}{(1+q)^2(1+q+q^2)}\Big)^{\frac{1}{r}}\Big(|{}_{x_1}D_q\pi(x_2)|^r\frac{2}{(1+q)^3(1+q+q^2)} \\ &+ |{}_{x_1}D_q\pi(x_1)|^r\frac{-1+q+q^2}{(1+q)^3(1+q+q^2)}\Big)^{\frac{1}{r}}\Big]. \end{aligned}$$

Theorem 6 ([17]). *Assume that π is as in Theorem 4. If $|{}_{x_1}D_q\pi|^r$ is quasi-convex on $[x_1, x_2]$ for $r > 1$, then*

$$\begin{aligned} &\left|\pi\Big(\frac{qx_1 + x_2}{1+q}\Big) - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(t)\ {}_{x_1}d_q t\right| \\ \leq\ & q(x_2 - x_1)\sup\{|{}_{x_1}D_q\pi(x_1)|, |{}_{x_1}D_q\pi(x_2)|\}\Big[\Big(\frac{1}{(1+q)^{s+1}}\frac{1-q}{1-q^{s+1}}\Big)^{\frac{1}{s}}\Big(\frac{1}{1+q}\Big)^{\frac{1}{r}} \\ &+ \Big(\int_{\frac{1}{1+q}}^{1}\Big(\frac{1}{q} - t\Big)^s\ {}_0d_q t\Big)^{\frac{1}{s}}\Big(\frac{q}{1+q}\Big)^{\frac{1}{r}}\Big], \end{aligned}$$

where $\dfrac{1}{r} + \dfrac{1}{s} = 1$.

Theorem 7 ([18])**.** *Let function $\pi : I = [x_1, x_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q-differentiable on I° with ${}_{x_1}D_q^2\pi$ be integrable and continuous on I where $0 < q < 1$. If $|{}_{x_1}D_q^2\pi|^r$ is convex on $[x_1, x_2]$ where $r \geq 1$, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1) + \pi(x_2)}{1+q} - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q x\right| \\
\leq\ & \frac{q^2(x_2 - x_1)^2}{(1+q)^{2-\frac{1}{r}}}\left((1-q)\sum_{n=0}^{\infty}(q^{2n} - q^{3n})(1 - q^{n+1})^r |{}_{x_1}D_q^2\pi(x_1)|^r\right. \\
&\left. +(1-q)\sum_{n=0}^{\infty}(q^{3n}(1-q^{n+1})^r |{}_{x_1}D_q^2\pi(x_2)|^r\right)^{\frac{1}{r}}.
\end{aligned}$$

Theorem 8 ([18])**.** *Assume that π is as in Theorem 7. If $|{}_{x_1}D_q^2\pi|^r$ is convex on $[x_1, x_2]$, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1) + \pi(x_2)}{1+q} - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q x\right| \\
\leq\ & \frac{q^2(x_2 - x_1)^2}{1+q}\left[(1-q)\sum_{n=0}^{\infty} q^{2n}(1-q^{n+1})^p\right]^{\frac{1}{p}}\left(\frac{q^2|{}_{x_1}D_q^2\pi(x_1)|^r + (1+q)|{}_{x_1}D_q^2\pi(x_2)|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}},
\end{aligned}$$

where $p, r > 1$, $\dfrac{1}{p} + \dfrac{1}{r} = 1$.

Theorem 9 ([19])**.** *Assume that π is as in Theorem 7. If $|{}_{x_1}D_q^2\pi|^r$ is quasi-convex on $[x_1, x_2]$, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1) + \pi(x_2)}{1+q} - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q x\right| \\
\leq\ & \frac{q^2(x_2 - x_1)^2}{1+q}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left((1-q)\sum_{n=0}^{\infty} q^{2n}(1-q^{n+1})^r \sup\left\{|{}_{x_1}D_q^2\pi(x_1)|^r, |{}_{x_1}D_q^2\pi(x_2)|^r\right\}\right)^{\frac{1}{r}},
\end{aligned}$$

for $r \geq 1$.

Theorem 10 ([19])**.** *Assume that π is as in Theorem 7. If $|{}_{x_1}D_q^2\pi|^r$ is quasi-convex on $[x_1, x_2]$, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1) + \pi(x_2)}{1+q} - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q x\right| \\
\leq\ & \frac{q^2(x_2 - x_1)^2}{1+q}\left((1-q)\sum_{n=0}^{\infty} q^{2n}(1-q^{n+1})^p\right)^{\frac{1}{p}}\left(\frac{\sup\left\{|{}_{x_1}D_q^2\pi(x_1)|^r, |{}_{x_1}D_q^2\pi(x_2)|^r\right\}}{1+q}\right)^{\frac{1}{r}},
\end{aligned}$$

where $p, r > 1$, $\dfrac{1}{p} + \dfrac{1}{r} = 1$.

Theorem 11 ([20]). *Let function $\pi : I \subset \mathbb{R} \to \mathbb{R}$ be q-differentiable on the interior I° with ${}_{x_1}D_q$ be integrable and continuous on I. If $|{}_{x_1}D_q|^r$ is convex, then*

$$\begin{aligned}
&\left|\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)_{x_1}d_qx-\frac{q\pi(x_1)+\pi(x_2)}{1+q}\right| \\
\leq\ & \frac{q(x_2-x_1)}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{1-\frac{1}{r}}\left[\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}|{}_{x_1}D_q\pi(x_1)|^r\right. \\
&\left.+\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}|{}_{x_1}D_q\pi(x_2)|^r\right]^{\frac{1}{r}},
\end{aligned}$$

where $0 < q < 1$ and $r \geq 1$.

Theorem 12 ([20]). *Assume that π is as in Theorem 11. If function $|{}_{x_1}D_q|^r$, $r \geq 1$ is convex, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1)+\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)_{x_1}d_qx\right| \\
\leq\ & \frac{q(x_2-x_1)}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{\frac{1}{p}}\left[\frac{q(1+4q+q^2)}{(1+q+q^2)(1+q)^3}|{}_{x_1}D_q\pi(x_1)|^r\right. \\
&\left.+\frac{q(1+3q^2+2q^3)}{(1+q+q^2)(1+q)^3}|{}_{x_1}D_q\pi(x_2)|^r\right]^{\frac{1}{r}},
\end{aligned}$$

where $p, r > 1$, $\dfrac{1}{p}+\dfrac{1}{r}=1$.

Theorem 13 ([20]). *Assume that π is as in Theorem 11. If function $|{}_{x_1}D_q|^r$, $r \geq 1$ is quasi-convex, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1)+\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)_{x_1}d_qx\right| \\
\leq\ & \frac{q(x_2-x_1)}{1+q}\left(\frac{q(2+q+q^3)}{(1+q)^3}\right)^{\frac{1}{p}}\left[\frac{q(2+q+q^3)}{(1+q)^3}[\sup\{|{}_{x_1}D_q\pi(x_1)|,|{}_{x_1}D_q\pi(x_2)|\}]\right]^{\frac{1}{r}},
\end{aligned}$$

where $p, r > 1$, $\dfrac{1}{p}+\dfrac{1}{r}=1$.

Theorem 14 ([20]). *Assume that π is as in Theorem 11. If function $|{}_{x_1}D_q|^r$, is quasi-convex, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1)+\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)_{x_1}d_qx\right| \\
\leq\ & \frac{q^2(x_2-x_1)(2+q+q^3)}{(1+q)^4}\left[\sup\{|{}_{x_1}D_q\pi(x_1)|,|{}_{x_1}D_q\pi(x_2)|\}\right]^{\frac{1}{r}},
\end{aligned}$$

where $r > 1$.

Theorem 15 ([21]). *Assume that real-valued function π is continuous on I. If $|{}_{x_1}D_q\pi|$ is convex and integrable on I°, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1)+\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx\right| \\
\leq\ & \frac{q^2(x_2-x_1)}{(1+q+q^2)(1+q)^4}\left[(1+4q+q^2)|{}_{x_1}D_q\pi(x_2)|+q(1+3q^2+2q^3)|{}_{x_1}D_q\pi(x_1)|\right].
\end{aligned}$$

Theorem 16 ([21]). *Assume that $\pi : I = [x_1, x_2] \to \mathbb{R}$ is continuous. If $|{}_{x_1}D_q\pi|^r$ is convex and integrable on I°, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1)+\pi(x_2)}{1+q} - \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx\right| \\
\leq\ & \frac{q^2(2+q+q^2)(x_2-x_1)}{(1+q)^4}\left[\frac{(1+4q+q^2)|{}_{x_1}D_q\pi(x_2)|^r+(1+3q^2+2q^3)|{}_{x_1}D_q\pi(x_1)|^r}{(1+q+q^2)(2+q+q^3)}\right]^{\frac{1}{r}},
\end{aligned}$$

where $r \geq 1$.

We continue with quantum H-H type inequalities for s-convex functions in the second sense.

Definition 6 ([22]). *A real-valued function π on $[0,\infty)$ is s-convex in the second sense if*

$$\pi(\lambda x+(1-\lambda)y) \leq \lambda^s\pi(x)+(1-\lambda)^s\pi(y),$$

for all $x, y \in [0,\infty)$, $\lambda \in [0,1]$ and $s \in (0,1]$.

Theorem 17 ([23]). *Let $\pi : [0,\infty) \to [0,\infty)$ be a continuous function which is s-convex in the second sense, $s > 0$, $0 < q < 1$. If $x_1, x_2 \in [0,\infty)$, $x_1 < x_2$ and π is q-differentiable on $[x_1, x_2]$, then*

$$\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx \leq \left[(1-q)\sum_{n=0}^{\infty}q^n(1-q^n)^s\right]\pi(x_1)+\left[(1-q)\sum_{n=0}^{\infty}(q^{s+1})^n\right]\pi(x_2).$$

Theorem 18 ([23]). *Suppose that for $\pi : [x_1, x_2] \to \mathbb{R}$ the q-derivative exists on (x_1, x_2) with $[0,\infty) \subset (x_1, x_2)$. If ${}_{x_1}D_q\pi$ is continuous and q-integrable on $[x_1, x_2]$, $x_1, x_2 \in [0,\infty)$, $x_1 < x_2$, $0 < q < 1$ and $|{}_{x_1}D_q\pi|^r$ is s-convex in the second sense, with $r \geq 1$, then*

$$\begin{aligned}
&\left|\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)_{x_1}d_qx - \frac{q\pi(x_1)+\pi(x_2)}{1+q}\right| \\
\leq\ & \frac{q(x_2-x_1)}{1+q}\left[\frac{2q}{(1+q)^2}\right]^{1-\frac{1}{r}}\left[I_1(q,s)|{}_{x_1}D_q\pi(x_1)|^r+I_2(q,s)|{}_{x_1}D_q\pi(x_2)|^r\right]^{\frac{1}{r}},
\end{aligned}$$

where

$$\begin{aligned}
I_1(q,s) &= 2\frac{1-q}{1+q}\sum_{n=0}^{\infty}q^n(1-q^n)\left(1-\frac{q^n}{1+q}\right)^s+(1-q)\sum_{n=0}^{\infty}q^n((1+q)q^n-1)(1-q^n)^s, \\
I_2(q,s) &= 2\frac{1-q}{1+q}\sum_{n=0}^{\infty}q^n(1-q^n)\left(\frac{q^n}{1+q}\right)^s+(1-q)\sum_{n=0}^{\infty}q^n[(1+q)q^n-1)q^{ns}].
\end{aligned}$$

Results of the q-H–H type inequalities for differentiable convex functions with a critical point are included in the next theorems.

Theorem 19 ([24]). *Assume that function $\pi : [x_1, x_2] \to \mathbb{R}$ is differentiable convex on (x_1, x_2) and $\pi'(c) = 0$, for $0 < q < 1$ and $c \in (x_1, x_2)$. Then*

$$\max\{I_1, I_2,\} \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx \leq \frac{q\pi(x_1)+\pi(x_2)}{1+q},$$

where

$$
\begin{aligned}
I_1 &= \pi\left(\frac{q(x_1+c)+(1-q)x_2}{1+q}\right)+\pi'\left(\frac{q(x_1+c)+(1-q)x_2}{1+q}\right)\left(\frac{q(x_2-c)}{1+q}\right),\\
I_2 &= \pi\left(\frac{(1-q)x_1+q(c+x_2)}{1+q}\right)+\pi'\left(\frac{(1-q)x_1+q(c+x_2)}{1+q}\right)\left(\frac{q(2x_1-x_2-c)+x_2-x_1}{1+q}\right).
\end{aligned}
$$

In the following we state results on H-H type quantum inequalities based on *tgs*-convex functions.

Definition 7 ([25]). *The non-negative real-valued function π is tgs-convex function on I, if*

$$\pi((1-\theta)x_1+\theta x_2)\leq\theta(1-\theta)[\pi(x_1)+\pi(x_2)],\quad \forall x_1,x_2\in I,\ \theta\in[0,1].$$

Theorem 20 ([26]). *Let function $\pi:[x_1,x_2]\to\mathbb{R}$ be tgs-convex. Then*

$$2\pi\left(\frac{x_1+x_2}{2}\right)\leq\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx\leq\left(\frac{1}{1+q}-\frac{1}{1+q+q^2}\right)[\pi(x_1)+\pi(x_2)].$$

Theorem 21 ([26]). *Let function $\pi:[x_1,x_2]\to\mathbb{R}$ be q-differentiable on (x_1,x_2) with ${}_{x_1}D_q$ be integrable and continuous on $[x_1,x_2]$. If function $|{}_{x_1}D_q|$ is tgs-convex, then*

$$
\begin{aligned}
&\left|\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx-\frac{q\pi(x_1)+\pi(x_2)}{1+q}\right|\\
\leq\ &\frac{q^4(1-q)}{(1+q)^4(1+q+q^2)(1+q+q^2+q^3)}(x_2-x_1)\Big[|{}_{x_1}D_q\pi(x_1)|+|{}_{x_1}D_q\pi(x_2)|\Big],
\end{aligned}
$$

where $0<q<1$.

Theorem 22 ([26]). *Assume that π is as in Theorem 21. If function $|{}_{x_1}D_q|^r$, $r\geq1$, is tgs-convex, then*

$$
\begin{aligned}
&\left|\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx-\frac{q\pi(x_1)+\pi(x_2)}{1+q}\right|\\
\leq\ &\frac{q(x_2-x_1)}{1+q}\left(\frac{2q}{(1+q)^2}\right)^{1-\frac{1}{r}}\frac{q^3(1-q)}{(1+q)^3(1+q+q^2)(1+q+q^2+q^3)}\\
&\times\Big[|{}_{x_1}D_q\pi(x_1)|^r+|{}_{x_1}D_q\pi(x_2)|^r\Big]^{\frac{1}{r}}.
\end{aligned}
$$

The q-H-H type inequalities for double integrals are given next.

Theorem 23 ([27]). *Assume that $0<q<1$ and real-valued function π is convex on $[x_1,x_2]$. Then*

$$
\begin{aligned}
\pi\left(\frac{qx_1+x_2}{1+q}\right) &\leq \frac{1}{(x_2-x_1)^2}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\pi\left(\frac{z+y}{2}\right)\ {}_{x_1}d_qz\ {}_{x_1}d_qy\\
&\leq \frac{1}{(x_2-x_1)^2}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\frac{1}{2}\left[\pi\left(\frac{\alpha z+\beta y}{\alpha+\beta}\right)+\pi\left(\frac{\beta z+\alpha y}{\alpha+\beta}\right)\right]\ {}_{x_1}d_qz\ {}_{x_1}d_qy\\
&\leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(z)\ {}_{x_1}d_qz,
\end{aligned}
$$

hold for all $\alpha,\beta\geq0$ with $\alpha+\beta>0$.

Theorem 24 ([28]). *Let π be as in Theorem 23. Then we have*

$$\begin{aligned}\pi\Big(\frac{qx_1+x_2}{1+q}\Big) &\leq \frac{1}{(x_2-x_1)^2}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\pi(\theta x_1+(1-\theta)x_2)\ {}_{x_1}d_qx\ {}_{x_1}d_qy\\ &\leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx\\ &\leq \frac{q\pi(x_1)+\pi(x_2)}{1+q},\end{aligned}$$

for all $\theta\in[0,1]$.

Theorem 25 ([28]). *Assume that $0<q<1$ and real-valued function π is q-differentiable function on $[x_1,x_2]$. Then*

$$\begin{aligned}0 &\leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx-\frac{1}{(x_2-x_1)^2}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\pi(\theta x_1+(1-\theta)x_2)\ {}_{x_1}d_qx\ {}_{x_1}d_qy\\ &\leq \theta\Big[\frac{\pi(x_1)+q\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(qx+(1-q)x_1)\ {}_{x_1}d_qx\Big],\end{aligned}$$

for all $\quad\theta\in[0,1]$.

Theorem 26 ([28]). *Let real-valued function π be a q-differentiable convex continuous, which is defined at the point $\frac{qx_1+x_2}{1+q}\in(x_1,x_2)$ and $0<q<1$. Then*

$$\begin{aligned}0 &\leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi\Big(\theta x+(1-\theta)\frac{qx_1+x_2}{1+q}\Big)\ {}_{x_1}d_qx\\ &\leq (1-\theta)\Big[\frac{\pi(x_1)+q\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(qx+(1-q)x_1)\ {}_{x_1}d_qx\Big],\end{aligned}$$

for all $\theta\in[0,1]$.

Theorem 27 ([28]). *Assume that π is as in Theorem 26. Then*

$$\begin{aligned}&(1-\theta)\frac{(1-q)(x_2-x_1)}{1+q}\pi'\Big(\frac{x_1+qx_2}{1+q}\Big)\\ \leq\ & \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi\Big(\theta x+(1-\theta)\frac{qx_1+x_2}{1+q}\Big)\ {}_{x_1}d_qx\\ \leq\ & (1-\theta)\Big[\frac{\pi(x_1)+q\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(qx+(1-q)x_1)\ {}_{x_1}d_qx\Big],\end{aligned}$$

for all $\theta\in[0,1]$.

We will now give some results on quantum H-H type inequalities via η-quasiconvex functions.

Definition 8 ([29]). *A real-valued function π is called η-quasiconvex on $[x_1,x_2]$ with respect to $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ if*

$$\pi(\theta x+(1-\theta)y)\leq\max\{\pi(y),\pi(y)+\eta(\pi(x),\pi(y))\}$$

for all $x,y\in[x_1,x_2]$ and $\theta\in[0,1]$.

Theorem 28 ([30]). *Assume that $0 < q < 1$ and a real-valued function π is q-differentiable on (x_1, x_2), with ${}_{x_1}D_q\pi$ continuous on $[x_1, x_2]$. If $|{}_{x_1}D_q\pi|^r$, $r \geq 1$, is η-quasiconvex on $[x_1, x_2]$, then*

$$\begin{aligned} &\left|\mu\pi(x_2) + (1-\mu)\pi(x_1) - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q x\right| \\ \leq\ &\begin{cases} \dfrac{(1-\mu-\mu q)(x_2 - x_1)}{1+q}\left[\mathcal{D}_{x_1}^{x_2}\left(|{}_{x_1}D_q\pi|^r;\eta\right)\right]^{\frac{1}{r}}, & 0 \leq \mu \leq 1-q, \\ \dfrac{(2\mu^2 + \mu(q-3)+1)(x_2 - x_1)}{1+q}\left[\mathcal{D}_{x_1}^{x_2}\left(|{}_{x_1}D_q\pi|^r;\eta\right)\right]^{\frac{1}{r}}, & 1-q < \mu \leq 1, \end{cases} \end{aligned}$$

where $\mathcal{D}_{x_1}^{x_2}(f;\eta) = \max\{\pi(x_1), \pi(x_1) + \eta(\pi(x_2), \pi(x_1))\}$.

Theorem 29 ([30]). *Assume that π is as in Theorem 28. If $|{}_{x_1}D_q\pi|^r$, is η-quasiconvex on $[x_1, x_2]$, for $r > 1$ with $\frac{1}{s} + \frac{1}{r} = 1$, then for all $\mu \in [0, 1]$, we have*

$$\begin{aligned} &\left|\mu\pi(x_2) + (1-\mu)\pi(x_1) - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q x\right| \\ \leq\ &(x_2 - x_1)[\Omega_q(1,\mu,\nu)]^{\frac{1}{r}}(\mathcal{D}_{x_1}^{x_2}|{}_{x_1}D_q\pi|^s, \eta))^{\frac{1}{s}}, \end{aligned}$$

where $\mathcal{D}_{x_1}^{x_2}(f;\eta) = \max\{\pi(x_1), \pi(x_1) + \eta(\pi(x_2), \pi(x_1))\}$ and $\Omega_q(\lambda,\mu,\theta) = \int_0^1 |q\tau - (1 - \lambda\mu)|^\theta\ {}_0d_q\tau$.

Following are the results on quantum H-H type inequalities concerning φ-convex functions.

Definition 9 ([31]). *A real-valued function π is convex with respect to φ (or φ-convex) on I, if*

$$\pi(\theta x_1 + (1-\theta)x_2) \leq \pi(x_2) + \theta\varphi(\pi(x_1), \pi(x_2)),$$

for all $x_1, x_2 \in I$ and $\theta \in [0, 1]$.

Definition 10 ([31]). *A real-valued function π is φ-quasiconvex on I, if*

$$\pi(\theta x_1 + (1-\theta)x_2) \leq \max\{\pi(x_2), \pi(x_2) + \varphi(\pi(x_1), \pi(x_2))\},$$

for all $x_1, x_2 \in I$ and $\theta \in [0, 1]$.

Theorem 30 ([32]). *Let $0 < q < 1$ and function $\pi : [x_1, x_2] \to \mathbb{R}$ be a twice q-differentiable on (x_1, x_2) with ${}_{x_1}D_q^2\pi$ integrable and continuous on $[x_1, x_2]$. If $|{}_{x_1}D_q^2\pi|^m$ is φ-convex on $[x_1, x_2]$ for $m \geq 1$, then*

$$\begin{aligned} &\left|\frac{q\pi(x_1) + \pi(x_2)}{1+q} - \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q x\right| \\ \leq\ &\frac{q^2(x_2 - x_1)^2}{(1+q)^{2-\frac{1}{m}}}\Big((1-q)\sum_{n=0}^{\infty} q^{2n}(1-q^{n+1})^m|{}_{x_1}D_q^2\pi(x_1)|^m \\ &+(1-q)\sum_{n=0}^{\infty} q^{3n}(1-q^{n+1})^m\varphi\Big(|{}_{x_1}D_q^2\pi(x_2)|^m, |{}_{x_1}D_q^2\pi(x_1)|^m\Big)\Big)^{\frac{1}{m}}. \end{aligned}$$

Theorem 31 ([32]). *Assume that π is as in Theorem 30. If $|_{x_1}D_q^2\pi|^m$ is φ-convex on $[x_1,x_2]$ where $m,n\geq 1$, $\frac{1}{n}+\frac{1}{m}=1$, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1)+\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx\right| \\
\leq\ & \frac{q^2(x_2-x_1)^2}{1+q}\left((1-q)\sum_{i=0}^{\infty}q^{2i}(1-q^{i+1})^n\right)^{\frac{1}{n}} \\
&\times\left(\frac{|_{x_1}D_q^2\pi(x_2)|^m}{1+q}+\frac{\varphi\left(|_{x_1}D_q^2\pi(x_2)|^m,|_{x_1}D_q^2\pi(x_1)|^m\right)}{1+q+q^2}\right)^{\frac{1}{m}}.
\end{aligned}$$

We will provide in the next results on q-Hermite–Hadamard inequalities based on (α,m)-convex functions.

Definition 11 ([33]). *A function $\pi:[0,x_2]\to\mathbb{R}$ is called (α,m)-convex, if for every $x,y\in[0,x_2]$ and $\theta\in[0,1]$ one has*

$$\pi(\theta x+m(1-\theta)y)\leq\theta^{\alpha}\pi(x)+m(1-\theta^{\alpha})\pi(y),$$

where $(\alpha,m)\in[0,1]^2$.

Theorem 32 ([34]). *Let function $\pi:[x_1,x_2]\subset\mathbb{R}\to\mathbb{R}$ be a twice q-differentiable on (x_1,x_2), and ${}_{x_1}D_q^2\pi$ be integrable and continuous on $[x_1,x_2]$. If $|_{x_1}D_q^2\pi|$ is (α,m)-convex on $[x_1,x_2]$, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1)+\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx\right| \\
\leq\ & \frac{q^2(x_2-x_1)^2(1-q)^2}{(1+q)(1-q^{\alpha+2})(1-q^{\alpha+3})}\left[\left(\frac{(1-q^{\alpha+2})(1-q^{\alpha+3})}{(1-q^2)(1-q^3)}-1\right)|_{x_1}D_q^2\pi(x_1)|+m\left|_{x_1}D_q^2\pi\left(\frac{x_2}{m}\right)\right|\right].
\end{aligned}$$

Theorem 33 ([34]). *Let π be as in Theorem 32. If $|_{x_1}D_q^2\pi|^r$, $r>1$, is (α,m)-convex on $[x_1,x_2]$, then*

$$\begin{aligned}
&\left|\frac{q\pi(x_1)+\pi(x_2)}{1+q}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx\right| \\
\leq\ & \frac{q^2(x_2-x_1)^2(1-q)^2}{(1+q)^{2-\frac{1}{r}}(1+q+q^2)}\left[\frac{(1-q^3)(1-q)}{(1-q^{\alpha+2})(1-q^{\alpha+3})}\right]^{\frac{1}{r}} \\
&\times\left[\left(\frac{(1-q^{\alpha+2})(1-q^{\alpha+3})}{(1-q^2)(1-q^3)}-1\right)|_{x_1}D_q^2\pi(x_1)|+m\left|_{x_1}D_q^2\pi\left(\frac{x_2}{m}\right)\right|\right]^{\frac{1}{r}}.
\end{aligned}$$

Now we define the q^{x_2}-derivative and q^{x_2}-integral of a function $\pi:[x_1,x_2]\to\mathbb{R}$, and present the corresponding H-H type quantum inequalities.

Definition 12 ([35]). *Assume $\pi:[x_1,x_2]\to\mathbb{R}$ is a continuous function. Then the expression*

$$^{x_2}D_q\pi(t)=\frac{\pi(t)-\pi(qt+(1-q)x_2)}{(1-q)(t-x_2)},t\neq x_2,\quad {}^{x_2}D_q\pi(x_1)=\lim_{t\to b}{}^{x_2}D_q\pi(t),$$

is called q^{x_2}-derivative of π at $t\in[x_1,x_2]$.

Definition 13 ([35])**.** *The right q-integral of $\pi : [x_1, x_2] \to \mathbb{R}$ is given by*

$$\int_{x}^{x_2} \pi(t)\ {}^{x_2}d_q t = (1-q)(x_2 - x) \sum_{n=0}^{\infty} q^n \pi(q^n x + (1-q^n)x_2).$$

Quantum H-H type inequalities for q^{x_2}-integral.

Theorem 34 ([35])**.** *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be convex on (x_1, x_2) and $0 < q < 1$. Then*

$$\pi\left(\frac{x_1 + qx_2}{1+q}\right) \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi(t)\ {}^{x_2}d_q t \leq \frac{\pi(x_1) + q\pi(x_2)}{1+q}.$$

Theorem 35 ([36])**.** *Let function $\pi : [x_1, x_2] \subset \mathbb{R} \to \mathbb{R}$ be twice q^{x_2}-differentiable on (x_1, x_2) such that ${}^{x_2}D_q^2 \pi$ is integrable and continuous on $[x_1, x_2]$. If $|{}^{x_2}D_q^2 \pi|$ is convex on $[x_1, x_2]$, then*

$$\begin{aligned} &\left| \frac{\pi(x_1) + q\pi(x_2)}{1+q} - \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi(x)\ {}^{x_2}d_q x \right| \\ \leq\ & \frac{q^2(x_2 - x_1)^2}{(1+q)(q^2+q+1)(q^3+q^2+q+1)} \left[|{}^{x_2}D_q^2 \pi(x_1)| + q^2 |{}^{x_2}D_q^2 \pi(x_2)| \right],\ \ 0 < q < 1. \end{aligned}$$

Theorem 36 ([36])**.** *Assume that π is as in Theorem 35. If $|{}^{x_2}D_q^2 \pi|^p$, $p > 1$, is convex on $[x_1, x_2]$, then*

$$\begin{aligned} &\left| \frac{\pi(x_1) + q\pi(x_2)}{1+q} - \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi(x)\ {}^{x_2}d_q x \right| \\ \leq\ & \frac{q^2(x_2 - x_1)^2}{(1+q)^{2-\frac{1}{p}}(q^2+q+1)} \left(\frac{1}{(q^3+q^2+q+1)} \right)^{\frac{1}{p}} \left[|{}^{x_2}D_q^2 \pi(x_1)|^p + q^2 |{}^{x_2}D_q^2 \pi(x_2)|^p \right]^{\frac{1}{p}}, \end{aligned}$$

where $0 < q < 1$.

Theorem 37 ([36])**.** *Assume that π is as in Theorem 35. If $|{}^{x_2}D_q^2 \pi|^p$, $p > 1$, is convex on $[x_1, x_2]$, and $\dfrac{1}{r} + \dfrac{1}{p} = 1$, then*

$$\begin{aligned} &\left| \frac{\pi(x_1) + q\pi(x_2)}{1+q} - \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi(x)\ {}^{x_2}d_q x \right| \\ \leq\ & \frac{q^2(x_2 - x_1)^2}{1+q} \left[(1-q) \sum_{n=0}^{\infty} (q^n)^{r+1} (1 - q^{n+1})^r \right]^{\frac{1}{r}} \left(\frac{|{}^{x_2}D_q^2 \pi(x_1)|^p + |{}^{x_2}D_q^2 \pi(x_2)|^p}{q+1} \right)^{\frac{1}{p}}. \end{aligned}$$

Theorem 38 ([37])**.** *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be q-differentiable on (x_1, x_2) and $q \in (0, 1)$. If ${}^{x_2}D_q^2 \pi$ is integrable and continuous on $[x_1, x_2]$, and $|{}^{x_2}D_q^2 \pi|$ is convex on $[x_1, x_2]$, then*

$$\begin{aligned} &\left| \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi(s)\ {}^{x_2}d_q s - \pi\left(\frac{x_1 + qx_2}{[2]_q} \right) \right| \\ \leq\ & \frac{(x_2 - x_1)^2}{[2]_q^5 [3]_q [4]_q} \left[(2q + 4q^2 + 2q^3) |{}^{x_2}D_q^2 \pi(x_1)| + (-q - q^2 + 2q^3 + 4q^4 + 3q^5 + q^6) |{}^{x_2}D_q^2 \pi(x_2)| \right]. \end{aligned}$$

Theorem 39 ([37]). *Assume that π is as in Theorem 38. Then*

$$
\begin{aligned}
&\left|\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(s)\,{}^{x_2}d_qs-\pi\left(\frac{x_1+qx_2}{[2]_q}\right)\right| \\
\leq\ & \frac{q^3(x_2-x_1)^2}{[2]_q^{1+\frac{3}{r}+2+\frac{1}{s}}[2s+1]_q^{\frac{1}{s}}}\left(|{}^{x_2}D_q^2\pi(x_1)|^r+(2q+q^2)|{}^{x_2}D_q^2\pi(x_2)|^r\right)^{\frac{1}{r}} \\
&+\frac{q^{\frac{1}{r}}(x_2-x_1)^2\left(1-\frac{1}{[2]_q}\right)_q^{2+\frac{1}{s}}}{[2]_q^{1+\frac{3}{r}}[2s+1]_q^{\frac{1}{s}}}\left((2+q)|{}^{x_2}D_q^2\pi(x_1)|^r+(q+q^2-1)|{}^{x_2}D_q^2\pi(x_2)|^r\right)^{\frac{1}{r}},
\end{aligned}
$$

where $\frac{1}{r}+\frac{1}{s}=1$.

Theorem 40 ([38]). *Let function $\pi : [x_1,x_2]\to\mathbb{R}$ be convex differentiable on $[x_1,x_2]$ and $0<q<1$. Then we have:*

$$
\begin{aligned}
\pi\left(\frac{qx_1+x_2}{[2]_q}\right)-\frac{(1-q)(x_2-x_1)}{[2]_q}\pi'\left(\frac{qx_1+x_2}{[2]_q}\right) &\leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\,{}^{x_2}d_qx \\
&\leq \frac{\pi(x_1)+q\pi(x_2)}{[2]_q}.
\end{aligned}
$$

Theorem 41 ([38]). *Let π be as in Theorem 40. Then we have:*

$$
\begin{aligned}
\pi\left(\frac{x_1+x_2}{2}\right)-\frac{(1-q)(x_2-x_1)}{2[2]_q}\pi'\left(\frac{x_1+x_2}{2}\right) &\leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\,{}^{x_2}d_qx \\
&\leq \frac{\pi(x_1)+q\pi(x_2)}{[2]_q}.
\end{aligned}
$$

Theorem 42 ([38]). *Let $\pi : [x_1,x_2]\to\mathbb{R}$ be a convex function on $[x_1,x_2]$ and $0<q<1$. Then*

$$
\begin{aligned}
\pi\left(\frac{x_1+qx_2}{[2]_q}\right) &\leq \frac{1}{(x_2-x_1)^2}\int_{x_1}^{x_2}\int_{x_1}^{x_2}\pi(\theta x_1+(1-\theta)x_2)\,{}^{x_2}d_qx\,{}^{x_2}d_qy \\
&\leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(x)\,{}^{x_2}d_qx \\
&\leq \frac{\pi(x_1)+q\pi(x_2)}{[2]_q},
\end{aligned}
$$

for all $\theta\in[0,1]$.

Theorem 43 ([39]). *Let function $\pi : [x_1,x_2]\to\mathbb{R}$ be differentiable and convex on (x_1,x_2), $c\in(x_1,x_2)$ and $0<q<1$. Then*

$$
\begin{aligned}
&\pi\left(\frac{q(x_1+c)+(1-q)x_2}{1+q}\right)+\pi'\left(\frac{q(x_1+c)+(1-q)x_2}{1+q}\right)\left(\frac{q(2x_2-x_1-c)+x_1-x_2}{1+q}\right) \\
\leq\ & \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi(t)\,{}^{x_2}d_qt\leq\frac{\pi(x_1)+q\pi(x_2)}{1+q}.
\end{aligned}
$$

Theorem 44 ([39]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be differentiable and convex on (x_1, x_2), such that $\pi'(c) = 0$ for $c \in (x_1, x_2)$ and $0 < q < 1$. Then*

$$\begin{aligned}
&\pi\Big(\frac{(1-q)x_1 + q(c+x_2)}{1+q}\Big) + \pi'\Big(\frac{(1-q)x_1 + q(c+x_2)}{1+q}\Big)\Big(\frac{q(x_1-c)}{1+q}\Big) \\
\leq\ & \frac{1}{x_2 - x_1}\int_{x_1}^{x_2} \pi(t)\ {}^{x_2}d_q t \leq \frac{\pi(x_1) + q\pi(x_2)}{1+q}.
\end{aligned}$$

The next results concern quantum q^{x_2}-H-H type inequalities for (α, m)-convex functions.

Theorem 45 ([40]). *Let function $\pi : I = [x_1, x_2] \subset \mathbb{R} \to \mathbb{R}$ be a twice q^{x_2}-differentiable on (x_1, x_2) such that ${}^{x_2}D_q^2\pi$ is integrable and continuous on $[x_1, x_2]$. If $|{}^{x_2}D_q^2\pi|$ is (α, m)-convex on $[x_1, x_2]$, then*

$$\begin{aligned}
&\left|\frac{\pi(x_1) + q\pi(mx_2)}{1+q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2} \pi(x)\ {}^{mx_2}d_q x\right| \\
\leq\ & \frac{q^2(mx_2 - x_1)^2}{1+q}\Big[\frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\left|{}^{x_2}D_q^2\pi(x_1)\right| \\
&+ m\Big(\frac{1}{[3]_q[2]_q} - \frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\Big)\left|{}^{x_2}D_q^2\pi(x_2)\right|\Big],
\end{aligned}$$

where $[n]_q = \dfrac{1-q^n}{1-q}$.

Theorem 46 ([40]). *Assume that π is as in Theorem 45. If $|{}^{x_2}D_q^2\pi|^p$, $p \geq 1$, is (α, m)-convex on $[x_1, x_2]$, then*

$$\begin{aligned}
&\left|\frac{\pi(x_1) + q\pi(mx_2)}{1+q} - \frac{1}{mx_2 - x_1}\int_{x_1}^{mx_2} \pi(x)\ {}^{mx_2}d_q x\right| \\
\leq\ & \frac{q^2(mx_2 - x_1)^2}{([2]_q)^{2-\frac{1}{p}}(]3]_q)^{\frac{1}{p}}}\Big(\frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\left|{}^{x_2}D_q^2\pi(x_1)\right|^p \\
&+ m\Big(\frac{1}{[3]_q[2]_q} - \frac{[\alpha+3]_q - q[\alpha+2]_q}{[\alpha+3]_q[\alpha+2]_q}\Big)\left|{}^{x_2}D_q^2\pi(x_2)\right|^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

Quantum H-H type inequalities for q_{x_1} and q^{x_2}-integrals.

Now, we give H-H inequalities involving left and right quantum integrals.

Theorem 47 ([41]). *Let function $\pi : [x_1, x_2] \subset \mathbb{R} \to \mathbb{R}$ be q-differentiable such that ${}_{x_1}D_q\pi$ and ${}^{x_2}D_q\pi$ are integrable and continuous on $[x_1, x_2]$. If ${}_{x_1}D_q\pi$ and ${}^{x_2}D_q\pi$ are convex, then:*

$$\begin{aligned}
&\left|\frac{\pi(x_1) + \pi(x_2)}{2} - \frac{1}{2(x_2 - x_1)}\Big[\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q x + \int_{x_1}^{x_2} \pi(x)\ {}^{x_2}d_q x\Big]\right| \\
\leq\ & \frac{q(x_2 - x_1)}{2[3]_q}\Big[|{}_{x_1}D_q\pi(x_2)| + |{}^{x_2}D_q\pi(x_1)| + \frac{q^2(|{}_{x_1}D_q\pi(x_1)| + |{}^{x_2}D_q\pi(x_2)|)}{[2]_q}\Big].
\end{aligned}$$

Theorem 48 ([41]). *Let π be as in Theorem 47. If $|_{x_1}D_q\pi|^{p_1}|$ and $|^{x_2}D_q\pi|^{p_1}$, $p_1 \geq 1$, are convex, then:*

$$\begin{aligned}
&\left|\frac{\pi(x_1)+\pi(x_2)}{2} - \frac{1}{x_2-x_1}\Big[\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx + \int_{x_1}^{x_2}\pi(x)\ {}^{x_2}d_qx\Big]\right| \\
\leq\ & \frac{q(x_2-x_1)}{2[2]_q}\Big[\Big(\frac{[2]_q|_{x_1}D_q\pi(x_2)|^{p_1} + q^2|_{x_1}D_q\pi(x_1)|^{p_1}}{[3]_q}\Big)^{\frac{1}{p_1}} \\
&+\Big(\frac{[2]_q|^{x_2}D_q\pi(x_1)|^{p_1} + q^2|^{x_2}D_q\pi(x_2)|^{p_1}}{[3]_q}\Big)^{\frac{1}{p_1}}\Big].
\end{aligned}$$

Theorem 49 ([41]). *Assume that π is as in Theorem 47. If $|^{x_2}D_q\pi|$ and $|_{x_1}D_q\pi|$ are convex, then:*

$$\begin{aligned}
&\left|\pi\Big(\frac{x_1+x_2}{2}\Big) - \frac{1}{2(x_2-x_1)}\Big[\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx + \int_{x_1}^{x_2}\pi(x)\ {}^{x_2}d_qx\Big]\right| \\
\leq\ & \frac{x_2-x_1}{2}\Big[|_{x_1}D_q\pi(x_2)|\frac{3}{4([4]_q+q[2]_q)} + |_{x_1}D_q\pi(x_1)|\frac{5q^2+4q-2q^3-1}{4([4]_q+q[2]_q)} \\
&+|^{x_2}D_q\pi(x_1)|\frac{3}{4([4]_q+q[2]_q)} + |^{x_2}D_q\pi(x_2)|\frac{5q^2+4q-2q^3-1}{4([4]_q+q[2]_q)}\Big].
\end{aligned}$$

Theorem 50 ([42]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be convex. Then*

$$\pi\Big(\frac{x_1+x_2}{2}\Big) \leq \frac{1}{x_2-x_1}\Big[\int_{x_1}^{\frac{x_1+x_2}{2}}\pi(x)\ {}_{x_1}d_qx + \int_{\frac{x_1+x_2}{2}}^{x_2}\pi(x)\ {}^{x_2}d_qx\Big] \leq \frac{\pi(x_1)+\pi(x_2)}{2}.$$

Theorem 51 ([42]). *Let $\pi : [x_1, x_2] \to \mathbb{R}$ and ${}_{x_1}D_q\pi$ and ${}^{x_2}D_q\pi$ be continuous and integrable on $[x_1, x_2]$. If $|_{x_1}D_q\pi|$ and $|^{x_2}D_q\pi|$ are convex, then*

$$\begin{aligned}
&\left|\frac{1}{x_2-x_1}\Big[\int_{x_1}^{\frac{x_1+x_2}{2}}\pi(x)\ {}_{x_1}d_qx + \int_{\frac{x_1+x_2}{2}}^{x_2}\pi(x)\ {}^{x_2}d_qx\Big] - \pi\Big(\frac{x_1+x_2}{2}\Big)\right| \\
\leq\ & \frac{x_2-x_1}{8[2]_q[3]_q}\Big[q[2]_q|^{x_2}D_q\pi(x_1)| + q([3]_q+q^2)|^{x_2}D_q\pi(x_2)|\Big] \\
&+\frac{x_2-x_1}{8[2]_q[3]_q}\Big[q([3]_q+q^2)|_{x_1}D_q\pi(x_1)| + q[2]_q|_{x_1}D_q\pi(x_2)|\Big].
\end{aligned}$$

Theorem 52 ([42]). *Assume that π is as in Theorem 51. If $|_{x_1}D_q\pi|^s$ and $|^{x_2}D_q\pi|^s$, $s \geq 1$ are convex, then*

$$\begin{aligned}
&\left|\frac{1}{x_2-x_1}\Big[\int_{x_1}^{\frac{x_1+x_2}{2}}\pi(x)\ {}_{x_1}d_qx + \int_{\frac{x_1+x_2}{2}}^{x_2}\pi(x)\ {}^{x_2}d_qx\Big] - \pi\Big(\frac{x_1+x_2}{2}\Big)\right| \\
\leq\ & \frac{q(x_2-x_1)}{4[2]_q}\Big[\Big(\frac{[2]_q|^{x_2}D_q\pi(x_1)|^s + ([3]_q+q^2)|^{x_2}D_q\pi(x_2)|^s}{2[3]_q}\Big)^{\frac{1}{s}} \\
&+\Big(\frac{([3]_q+q^2)|_{x_1}D_q\pi(x_1)|^s + [2]_q|_{x_1}D_q\pi(x_2)|^s}{2[3]_q}\Big)^{\frac{1}{s}}\Big].
\end{aligned}$$

Theorem 53 ([42]). *Let π be as in Theorem 51. If $|_{x_1}D_q\pi|$ and $|^{x_2}D_q\pi|$ are convex, then*

$$\begin{aligned}
&\left|\frac{\pi(x_1)+\pi(x_2)}{2}-\frac{1}{x_2-x_1}\left[\int_{x_1}^{\frac{x_1+x_2}{2}}\pi(x)\ {}_{x_1}d_qx+\int_{\frac{x_1+x_2}{2}}^{x_2}\pi(x)\ {}^{x_2}d_qx\right]\right| \\
\leq\ & \frac{x_2-x_1}{8[2]_q[3]_q}\left[|^{x_2}D_q\pi(x_1)|+([3]_q+q+q^2)|^{x_2}D_q\pi(x_2)|\right] \\
&+\frac{x_2-x_1}{8[2]_q[3]_q}\left[([3]_q+q+q^2)|_{x_1}D_q\pi(x_1)|+|_{x_1}D_q\pi(x_2)|\right].
\end{aligned}$$

Theorem 54 ([42]). *Assume that π is as in Theorem 51. If $|_{x_1}D_q\pi|^s$ and $|^{x_2}D_q\pi|^s$, $s>1$ are convex, then*

$$\begin{aligned}
&\left|\frac{\pi(x_1)+\pi(x_2)}{2}-\frac{1}{x_2-x_1}\left[\int_{x_1}^{\frac{x_1+x_2}{2}}\pi(x)\ {}_{x_1}d_qx+\int_{\frac{x_1+x_2}{2}}^{x_2}\pi(x)\ {}^{x_2}d_qx\right]\right| \\
\leq\ & \frac{x_2-x_1}{4}\left(\int_0^1(1-qt)^r\ d_qt\right)^{\frac{1}{r}}\left[\left(\frac{|^{x_2}D_q\pi(x_1)|^s+([2]_q+q)|^{x_2}D_q\pi(x_2)|^s}{2[2]_q}\right)^{\frac{1}{s}}\right. \\
&\left.+\left(\frac{([2]_q+q)|_{x_1}D_q\pi(x_1)|^s+|_{x_1}D_q\pi(x_2)|^s}{2[3]_q}\right)^{\frac{1}{s}}\right],
\end{aligned}$$

where $\dfrac{1}{s}+\dfrac{1}{r}=1$.

Theorem 55 ([43]). *Let the real valued function π be s-convex in the second sense and $x_1,x_2\in\mathbb{R}^+$ with $x_1<x_2$. If $f\in L_1[x_1,x_2]$, then for $s\in(0,1]$:*

$$2^{s-1}\pi\left(\frac{x_1+x_2}{2}\right)\leq\frac{1}{2(x_2-x_1)}\left[\int_{x_1}^{x_2}\pi(x)\ {}_{x_1}d_qx+\int_{x_1}^{x_2}\pi(x)\ {}^{x_2}d_qx\right]\leq\frac{\pi(x_1)+\pi(x_2)}{[s+1]_q}.$$

The following H-H type inequalities depend on a parameter.

Theorem 56 ([44]). *Let function $\pi:\mathbb{R}^+\to\mathbb{R}$ be convex on $[x_1,x_2]$ with $x_1<x_2$. Then*

$$\begin{aligned}
\pi\left(\frac{x_1+x_2}{2}\right)\ &\leq\ \frac{1}{2\lambda(x_2-x_1)}\left[\int_{x_1}^{\lambda x_2+(1-\lambda)x_1}\pi(x)\ {}_{x_1}d_qx+\int_{\lambda x_1+(1-\lambda)x_2}^{x_2}\pi(x)\ {}^{x_2}d_qx\right] \\
&\leq\ \frac{\pi(x_1)+\pi(x_2)}{2},
\end{aligned}$$

for all $\lambda\in(0,1]$.

Theorem 57 ([44]). *Let function $\pi:\mathbb{R}^+\to\mathbb{R}$ be a convex on $[x_1,x_2]$ with $x_1<x_2$. Then*

$$\begin{aligned}
\pi\left(\frac{x_1+x_2}{2}\right)\ &\leq\ \frac{1}{2}\left[\pi\left(\frac{x_1+qx_2}{[2]_q}\right)+\pi\left(\frac{qx_1+x_2}{[2]_q}\right)\right] \\
&\leq\ \frac{1+q^2}{2q\lambda[2]_q(x_2-x_1)}\left(\int_{x_1}^{\lambda x_2+(1-\lambda)x_1}\pi(x)\ {}_{x_1}d_qx+\int_{\lambda x_1+(1-\lambda)x_2}^{x_2}\pi(x)\ {}^{x_2}d_qx\right) \\
&\quad-\frac{1-q}{q[2]_q}[\pi(\lambda x_1+(1-\lambda)x_2)+\pi((1-\lambda x_1)+\lambda x_2] \\
&\leq\ \frac{\pi(x_1)+\pi(x_2)}{2},
\end{aligned}$$

for all $\lambda\in(0,1]$.

Theorem 58 ([44]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be q differentiable, with ${}_{x_1}D_q\pi$ and ${}^{x_2}D_q\pi$ be q-integrable and continuous over $[x_1, x_2]$. If $|{}_{x_1}D_q\pi|$ and $|{}^{x_2}D_q\pi|$ are convex on $[x_1, x_2]$, then*

$$\begin{aligned}
&\Bigg|\frac{1}{2\lambda(x_2-x_1)}\Big(\int_{x_1}^{\lambda x_2+(1-\lambda)x_1}\pi(x)\ {}_{x_1}d_qx+\int_{\lambda x_1+(1-\lambda)x_2}^{x_2}\pi(x)\ {}^{x_2}d_qx\Big)\\
&-\frac{1}{2}\big[\pi(\lambda x_2+(1-\lambda)x_1)+\pi(\lambda x_1+(1-\lambda)x_2)\big]\Bigg|\\
\leq\ &\frac{\lambda q(x_2-x_1)}{2[2]_q[3]_q}\Big[([3]_q-\lambda[2]_q)\big[|{}^{x_2}D_q\pi(x_2)|+|{}_{x_1}D_q\pi(x_1)|\big]\Big]\\
&+\lambda[2]_q\big[|{}^{x_2}D_q\pi(x_1)|+|{}_{x_1}D_q\pi(x_2)|\big],
\end{aligned}$$

for all $\lambda \in (0, 1]$.

Now we present q-H-H integral inequalities regarding the family of p, (p, s) and extended sort of (p, s)-convex functions.

Definition 14 ([45]). *A function $\pi : I \subset (0, \infty) \to \mathbb{R}$ is called p-convex, if*

$$\pi\Big([\theta x_1^p+(1-\theta)x_2^p]^{\frac{1}{p}}\Big)\leq\theta\pi(x_1)+(1-\theta)\pi(x_2),$$

for all $x_1, x_2 \in I$ and $\theta \in [0, 1]$.

Theorem 59 ([46]). *Let $\pi : I \subset (0, \infty) \to \mathbb{R}$ be p-convex on I°, $x_1^p, x_2^p \in I$ with with $x_1^p < x_2^p$ and $p > 0$, $q \in (0, 1)$. Then*

$$\begin{aligned}
\pi\Big(\Big(\frac{x_1^p+x_2^p}{2}\Big)^{\frac{1}{p}}\Big)\ &\leq\ \frac{1}{2(x_2^p-x_1^p)}\Big[\int_{x_1^p}^{x_2^p}\pi(x_2^{\frac{1}{p}})\ {}_{x_1^p}d_qx_2+\int_{x_1^p}^{x_2^p}\pi(x_2^{\frac{1}{p}})\ {}^{x_2^p}d_qx_2\Big]\\
&\leq\ \frac{\pi(x_1)+\pi(x_2)}{2}.
\end{aligned}$$

Definition 15 ([46]). *A real-valued function π is (p, s)-convex function, if*

$$\pi\Big((\theta x_1^p+(1-\theta)x_2^p)^{\frac{1}{p}}\Big)\leq\theta^s\pi(x_1)+(1-\theta)^s\pi(x_2),$$

for all $x_1, x_2 \in I$, $p > 0$, $s \in (0, 1]$ and $\theta \in [0, 1]$.

Theorem 60 ([46]). *Let the real-valued function π be (p, s)-convex on I°, $x_1^p, x_2^p \in I$ with $x_1^p < x_2^p$. Then*

$$\begin{aligned}
2^s\pi\Big(\Big(\frac{x_1^p+x_2^p}{2}\Big)^{\frac{1}{p}}\Big)\ &\leq\ \frac{1}{(x_2^p-x_1^p)}\Big[\int_{x_1^p}^{x_2^p}\pi(x_2^{\frac{1}{p}})\ {}_{x_1^p}d_qx_2+\int_{x_1^p}^{x_2^p}\pi(x_2^{\frac{1}{p}})\ {}^{x_2^p}d_qx_2\Big]\\
&\leq\ [\pi(x_1)+\pi(x_2)]\Big(\frac{1}{[s+1]_q}+\int_0^1(1-\tau)^s d_q\tau\Big).
\end{aligned}$$

where $p > 0$, $q \in (0, 1)$ and $s \in (0, 1]$.

Theorem 61 ([46]). *Let the real-valued function π be modified type (p,s)-convex on I°, $x_1^p, x_2^p \in I$ with $x_1^p < x_2^p$. Then*

$$\begin{aligned} 2^s \pi\Big(\Big(\frac{x_1^p + x_2^p}{2}\Big)^{\frac{1}{p}}\Big) &\leq \frac{1}{(x_2^p - x_1^p)}\Big[\int_{x_1^p}^{x_2^p} \pi(x_2^{\frac{1}{p}})\ {}_{x_1^p}d_q x_2 + (2^s - 1)\int_{x_1^p}^{x_2^p} \pi(x_2^{\frac{1}{p}})\ {}^{x_2^p}d_q x_2\Big] \\ &\leq \Big(\frac{2}{[s+1]_q} - 1\Big)[\pi(x_2) - \pi(x_1)] + 2^s\Big(\frac{\pi(x_1) - \pi(x_2)}{[s+1]_q} + \pi(x_2)\Big), \end{aligned}$$

where $p > 0$, $q \in (0,1)$ and $s \in (0,1]$.

Next, we present q-H-H integral inequalities pertaining to (p,h)-convex functions.

Definition 16 ([47]). *A function $\pi : I \to \mathbb{R}$ is called a (p,h)-convex function, if it is non-negative and*

$$\pi\Big([\theta x_1^p + (1-\theta)x_2^p]^{\frac{1}{p}}\Big) \leq h(t)\pi(x_1) + h(1-\theta)\pi(x_2)$$

for any $x_1, x_2 \in I$ and $\theta \in [0,1]$.

Theorem 62 ([48]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be q-integrable and (p,h)-convex. Then*

$$\begin{aligned} \frac{1}{h(\frac{1}{2})}\pi\Big(\Big[\frac{x_1^p + x_2^p}{2}\Big]^{\frac{1}{p}}\Big) &\leq \frac{1}{x_2^p - x_1^p}\Big\{\int_{x_1^p}^{x_2^p} \pi(x^{\frac{1}{p}})\ {}_{x_1^p}d_q x + \int_{x_1^p}^{x_2^p} \pi(x^{\frac{1}{p}})\ {}^{x_2^p}d_q x\Big\} \\ &\leq [\pi(x_1) + \pi(x_2)]\Big(\int_0^1 h(t)d_q t + \int_0^1 h(1-t)d_q t\Big). \end{aligned}$$

Quantum H-H-Mercer type inequalities.

In the following theorems we present a quantum version of the H-H-Mercer inequalities.

Theorem 63 ([49]). *For a convex function $\pi : [x_1, x_2] \to \mathbb{R}$, then*

$$\begin{aligned} \pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) &\leq \pi(x_1) + \pi(x_2) - \frac{1}{y-x}\Big[\int_x^{\frac{x+y}{2}} \pi(u)\ {}_x d_q u + \int_{\frac{x+y}{2}}^y \pi(u)\ {}^y d_q u\Big] \\ &\leq \pi(x_1) + \pi(x_2) - \pi\Big(\frac{x+y}{2}\Big) \end{aligned}$$

and

$$\begin{aligned} &\pi\Big(x_1 + x_2 - \frac{x+y}{2}\Big) \\ \leq\ & \frac{1}{y-x}\Big[\int_{x_1+x_2-y}^{x_1+x_2-\frac{x+y}{2}} \pi(u)\ {}_{x_1+x_2-y}d_q u + \int_{\frac{x_1+x_2-x+y}{2}}^{x_1+x_2-x} \pi(u)\ {}^{x_1+x_2-x}d_q u\Big] \\ \leq\ & \frac{\pi(x_1 + x_2 - x) + \pi(x_1 + x_2 - y)}{2} \\ \leq\ & \pi(x_1) + \pi(x_2) - \frac{\pi(x) + \pi(y)}{2}, \end{aligned}$$

for all $x, y \in [x_1, x_2]$ and $x < y$.

Theorem 64 ([49]). *Assume that $\pi : [x_1, x_2] \to \mathbb{R}$ is q-differentiable and ${}_{x_1}D_q\pi$, ${}^{x_2}D_q\pi$ are q-integrable and continuous. If $|{}_{x_1}D_q\pi|$, $|{}^{x_2}D_q\pi|$ are convex, then*

$$\begin{aligned}
&\left|\frac{1}{y-x}\Big[\int_{x_1+x_2-y}^{x_1+x_2-\frac{x+y}{2}} \pi(u)\ {}_{x_1+x_2-y}d_q u + \int_{\frac{x_1+x_2-x+y}{2}}^{x_1+x_2-x} \pi(u)\ {}^{x_1+x_2-x}d_q u\Big]\right. \\
&\left.-\pi\Big(x_1+x_2-\frac{x+y}{2}\Big)\right| \\
\leq\ & \frac{y-x}{4}\Big[\frac{q}{[2]_q}\Big(|{}^{x_2}D_q\pi(x_1)| + |{}^{x_2}D_q\pi(x_2)|\Big) \\
&-\Big(\frac{q([3]_q+q^2)}{2[2]_q[3]_q}|{}^{x_2}D_q\pi(x)| + \frac{q}{2[3]_q}|{}^{x_2}D_q\pi(y)|\Big) \\
&+\frac{q}{[2]_q}\Big(|{}_{x_1}D_q\pi(x_1)| + |{}_{x_1}D_q\pi(x_2)|\Big) - \Big(\frac{q([3]_q+q^2)}{2[2]_q[3]_q}|{}_{x_1}D_q\pi(y)| + \frac{q}{2[3]_q}|{}_{x_1}D_q\pi(x)|\Big)\Big],
\end{aligned}$$

for all $x, y \in [x_1, x_2]$ and $x < y$.

Quantum H-H type inequalities for ${}_{x_1}T_q$ and ${}^{x_2}T_q$ integrals.

Here, we add some new q-H-H type inequalities pertaining to convex functions utilizing the idea of q-integral.

Definition 17 ([50]). *Let $\pi : J \to \mathbb{R}$ be continuous. For $0 < q < 1$ we define the ${}_{x_1}T_q$-integral as*

$$\int_{x_1}^{x_2} \pi(s)\ {}_{x_1}d_q^T s = \frac{(1-q)(x_2-x_1)}{2q}\Big[(1+q)\sum_{n=1}^{\infty} q^n \pi(q^n x_2 + (1-q^n)x_1) - \pi(x_2)\Big].$$

Theorem 65 ([50]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be convex and differentiable on $[x_1, x_2]$ and $0 < q < 1$. Then*

$$\max\{I_1, I_2, I_3\} \leq \frac{1}{x_2-x_1}\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_q^T x \leq \frac{\pi(x_1)+\pi(x_2)}{2},$$

where

$$\begin{aligned}
I_1 &= \pi\Big(\frac{x_1+x_2}{2}\Big), \\
I_2 &= \pi\Big(\frac{qx_1+x_2}{1+q}\Big) + \frac{(q-1)(x_2-x_1)}{2(1+q)}\pi'\Big(\frac{qx_1+x_2}{1+q}\Big), \\
I_3 &= \pi\Big(\frac{x_1+qx_2}{1+q}\Big) + \frac{(1-q)(x_2-x_1)}{2(1+q)}\pi'\Big(\frac{x_1+qx_2}{1+q}\Big).
\end{aligned}$$

Next, we examine a new idea of quantum integral, the ${}^{x_2}T_q$-integral, and we present H-H type inequalities for this new integral.

Definition 18 ([51]). *Let $\pi : J \to \mathbb{R}$ be continuous. For $0 < q < 1$ we define the ${}^{x_2}T_q$-integral as*

$$\int_{x_1}^{x_2} \pi(s)\ {}^{x_2}d_q^T s = \frac{(1-q)(x_2-x_1)}{2q}\Big[(1+q)\sum_{n=1}^{\infty} q^n \pi(q^n x_1 + (1-q^n)x_2) - \pi(x_1)\Big].$$

Theorem 66 ([51])**.** *Assume that function $\pi : [x_1, x_2] \to \mathbb{R}$ is differentiable convex on $[x_1, x_2]$ and $0 < q < 1$. Then*

$$\max\{I_1, I_2, I_3\} \le \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi(x) \, {}^{x_2}d_q^T x \le \frac{\pi(x_1) + \pi(x_2)}{2},$$

where

$$\begin{aligned} I_1 &= \pi\Big(\frac{x_1 + x_2}{2}\Big), \\ I_2 &= \pi\Big(\frac{qx_1 + x_2}{1+q}\Big) + \frac{(q-1)(x_2 - x_1)}{2(1+q)} \pi'\Big(\frac{qx_1 + x_2}{1+q}\Big), \\ I_3 &= \pi\Big(\frac{x_1 + qx_2}{1+q}\Big) + \frac{(1-q)(x_2 - x_1)}{2(1+q)} \pi'\Big(\frac{x_1 + qx_2}{1+q}\Big). \end{aligned}$$

In the next theorem we give a ${}_{x_1}T_q$ and ${}^{x_2}T_q$ H-H type integral inequality for s-convexity.

Theorem 67 ([52])**.** *Assume that a real-valued function π is s-convex in the second manner with $s \in (0, 1]$. Then*

$$\begin{aligned} 2^s \pi\Big(\frac{x_1 + x_2}{2}\Big) &\le \frac{1}{x_2 - x_1} \Big[\int_{x_1}^{x_2} \pi(x) \, {}_{x_1}d_q^T x + \int_{x_1}^{x_2} \pi(x) \, {}^{x_2}d_q^T x \Big] \\ &\le \big[\pi(x_1) + \pi(x_2)\big] \Big[\frac{1}{2q} \Big(\frac{1+q}{[s+1]_q} - 1 + q \Big) + \int_0^1 (1-\xi)^s \, {}_0d_q^T \xi \Big]. \end{aligned}$$

Quantum H-H type inequalities involving coordinated convex functions.

Definition 19 ([53])**.** *A function $\pi : \Delta = [x_1, x_2] \times [c_1, d_1] \to \mathbb{R}$ will be called coordinated convex on Δ, for all $\theta, s \in [0, 1]$ and $(x, y), (u, w) \in \Delta$, if*

$$\begin{aligned} & \pi(\theta x + (1-\theta)y, su + (1-s)w) \\ \le\; & \theta s \pi(x, u) + s(1-\theta)\pi(y, u) + \theta(1-s)\pi(x, w) + (1-\theta)(1-s)\pi(y, w). \end{aligned}$$

Quantum H-H's type inequalities pertaining to coordinated convex functions are given in the next.

Theorem 68 ([54])**.** *Let $\pi : [x_1, x_2] \times [c_1, d_1] \subset \mathbb{R}^2 \to \mathbb{R}$ be convex on coordinates on $[x_1, x_2] \times [c_1, d_1]$. Then, for all $q_1, q_2 \in (0, 1)$,, we have*

$$\begin{aligned} & \pi\Big(\frac{x_1 q_1 + x_2}{1 + q_1}, \frac{c_1 q_2 + d_1}{1 + q_2}\Big) \\ \le\; & \frac{1}{2}\Big[\frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi\Big(x, \frac{c_1 q_2 + d_1}{1 + q_2}\Big) \, {}_{x_1}d_{q_1} x + \frac{1}{d_1 - c_1} \int_{c_1}^{d_1} \pi\Big(\frac{x_1 q_1 + x_2}{1 + q_1}, y\Big) \, {}_{c_1}d_{q_2} y \Big] \\ \le\; & \frac{1}{(x_2 - x_1)(d_1 - c_1)} \int_{x_1}^{x_2} \int_{c_1}^{d_1} \pi(x, y) \, {}_{c_1}d_{q_2} y \, {}_{x_1}d_{q_1} x \\ \le\; & \frac{q_1}{2(1+q_1)(d_1 - c_1)} \int_{c_1}^{d_1} \pi(x_1, y) \, {}_{c_1}d_{q_2} y + \frac{1}{2(1+q_1)(d_1 - c_1)} \int_{c_1}^{d_1} \pi(x_2, y) \, {}_{c_1}d_{q_2} y \\ & + \frac{q_2}{2(1+q_2)(x_2 - x_1)} \int_{x_1}^{x_2} \pi(x, c_1) \, {}_{x_1}d_{q_1} x + \frac{1}{2(1+q_2)(x_2 - x_1)} \int_{x_1}^{x_2} \pi(x, d_1) \, {}_{x_1}d_{q_1} x \\ \le\; & \frac{q_1 q_2 \pi(x_1, c_1) + q_1 \pi(x_1, d_1) + q_2 \pi(x_2, c_1) + \pi(x_2, d_1)}{(1+q_1)(1+q_2)}. \end{aligned}$$

Theorem 69 ([55])**.** *Assume that π is as in Theorem 68. Then for all $q_1, q_2 \in (0,1)$, we have*

$$\begin{aligned}
&\pi\Big(\frac{x_1q_1+x_2}{1+q_1}, \frac{c_1+d_1q_2}{1+q_2}\Big) \\
\leq\ & \frac{1}{2}\Big[\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\pi\Big(x, \frac{c_1+d_1q_2}{1+q_2}\Big)\ {}_{x_1}d_{q_1}x + \frac{1}{d_1-c_1}\int_{c_1}^{d_1}\pi\Big(\frac{x_1q_1+x_2}{1+q_1}, y\Big)\ {}^{d_1}d_{q_2}y\Big] \\
\leq\ & \frac{1}{(x_2-x_1)(d_1-c_1)}\int_{x_1}^{x_2}\int_{c_1}^{d_1}\pi(x,y)\ {}^{d_1}d_{q_2}y\ {}_{x_1}d_{q_1}x \\
\leq\ & \frac{q_1}{2(1+q_1)(d_1-c_1)}\int_{c_1}^{d_1}\pi(x_1,y)\ {}^{d_1}d_{q_2}y + \frac{1}{2(1+q_1)(d_1-c_1)}\int_{c_1}^{d_1}\pi(x_2,y)\ {}^{d_1}d_{q_2}y \\
&+\frac{1}{2(1+q_2)(x_2-x_1)}\int_{x_1}^{x_2}\pi(x,c_1)\ {}_{x_1}d_{q_1}x + \frac{q_2}{2(1+q_2)(x_2-x_1)}\int_{x_1}^{x_2}\pi(x,d_1)\ {}_{x_1}d_{q_1}x \\
\leq\ & \frac{q_1\pi(x_1,c_1)+q_1q_2\pi(x_1,d_1)+\pi(x_2,c_1)+q_2\pi(x_2,d_1)}{(1+q_1)(1+q_2)}.
\end{aligned}$$

Theorem 70 ([56])**.** *Assume that function $\pi : \Delta \to \mathbb{R}$ is a coordinated convex. Then*

$$\begin{aligned}
\pi\Big(\frac{x_1+x_2}{2}, \frac{c_1+d_1}{2}\Big) \leq\ & \frac{1}{(x_2-x_1)(d_1-c_1)}\Big[\int_{x_1}^{\frac{x_1+x_2}{2}}\int_{c_1}^{\frac{c_1+d_1}{2}}\pi(x,y)\ {}^{\frac{c_1+d_1}{2}}d_{q_2}y\ {}^{\frac{x_1+x_2}{2}}d_{q_1}x \\
&+\int_{x_1}^{\frac{x_1+x_2}{2}}\int_{\frac{c_1+d_1}{2}}^{d_1}\pi(x,y)\ {}^{\frac{c_1+d_1}{2}}d_{q_2}y\ {}^{\frac{x_1+x_2}{2}}d_{q_1}x \\
&+\int_{\frac{x_1+x_2}{2}}^{x_2}\int_{c_1}^{\frac{c_1+d_1}{2}}\pi(x,y)\ {}^{\frac{c_1+d_1}{2}}d_{q_2}y\ {}_{\frac{x_1+x_2}{2}}d_{q_1}x \\
&+\int_{\frac{x_1+x_2}{2}}^{x_2}\int_{\frac{c_1+d_1}{2}}^{d_1}\pi(x,y)\ {}_{\frac{c_1+d_1}{2}}d_{q_2}y\ {}_{\frac{x_1+x_2}{2}}d_{q_1}x\Big] \\
\leq\ & \frac{1}{4}\Big[\pi(x_1,c_1)+\pi(x_1,d_1)+\pi(x_2,c_1)+\pi(x_2,d_1)\Big].
\end{aligned}$$

Theorem 71 ([56])**.** *Assume that function $\pi : \Delta \to \mathbb{R}$ is a coordinated convex. Then*

$$\begin{aligned}
\pi\Big(\frac{x_1+x_2}{2}, \frac{c_1+d_1}{2}\Big) \leq\ & \frac{1}{(x_2-x_1)(d_1-c_1)}\Big[\int_{x_1}^{\frac{x_1+x_2}{2}}\int_{c_1}^{\frac{c_1+d_1}{2}}\pi(x,y)\ {}_{c_1}d_{q_2}y\ {}_{x_1}d_{q_1}x \\
&+\int_{x_1}^{\frac{x_1+x_2}{2}}\int_{\frac{c_1+d_1}{2}}^{d_1}\pi(x,y)\ {}^{d_1}d_{q_2}y\ {}_{x_1}d_{q_1}x \\
&+\int_{\frac{x_1+x_2}{2}}^{x_2}\int_{c_1}^{\frac{c_1+d_1}{2}}\pi(x,y)\ {}_{c_1}d_{q_2}y\ {}^{x_2}d_{q_1}x \\
&+\int_{\frac{x_1+x_2}{2}}^{x_2}\int_{\frac{c_1+d_1}{2}}^{d_1}\pi(x,y)\ {}^{d_1}d_{q_2}y\ {}^{x_2}d_{q_1}x\Big] \\
\leq\ & \frac{1}{4}\Big[\pi(x_1,c_1)+\pi(x_1,d_1+\pi(x_2,c_1)+\pi(x_2,d_1)\Big].
\end{aligned}$$

In what follows we introduce q-partial derivatives and definite q-integrals for the functions of two variables.

Definition 20 ([57])**.** *Let $\pi : \Delta = [x_1,x_2]\times[c_1,d_1] \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function of two variables and $0 < q_1 < 1$, $0 < q_2 < 1$. The partial q_1-derivatives, q_2-derivatives, and q_1q_2-derivatives at $(x,y) \in [x_1,x_2]\times[c_1,d_1]$ can be defined as follows:*

$$\frac{{}_{x_1}\partial_{q_1}\pi(x,y)}{{}_{x_1}\partial_{q_1}x} = \frac{\pi(q_1x+(1-q_1)x_1,y)-\pi(x,y)}{(1-q_1)(x-x_1)},\ \ x \neq x_1,$$

$$\begin{aligned}
\frac{{}_{c_1}\partial_{q_2}\pi(x,y)}{{}_{c_1}\partial_{q_2}y} &= \frac{\pi(x,q_2y+(1-q_2)c_1)-\pi(x,y)}{(1-q_2)(y-c_1)},\ y\neq c_1,\\
\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x,y)}{{}_{x_1}\partial_{q_1}x\ {}_{c_1}\partial_{q_2}y} &= \frac{1}{(1-q_1)(1-q_2)(y-c_1)(x-x_1)}\\
&\quad\times\Big[\pi(q_1x+(1-q_1)x_1,q_2y+(1-q_2)c_1)\\
&\quad-\pi(q_1x+(1-q_1)x_1,y)-\pi(x,q_2y+(1-q_2)c_1)\\
&\quad+\pi(x,y)\Big],\ x\neq x_1,\ y\neq c_1.
\end{aligned}$$

Definition 21 ([57]). *Let $\pi:\Delta=[x_1,x_2]\times[c_1,d_1]\subset\mathbb{R}^2\to\mathbb{R}$ be a continuous function of two variables and $0<q_1<1$, $0<q_2<1$. The partial q_1-derivatives, q_2-derivatives, and q_1q_2-derivatives at $(x,y)\in[x_1,x_2]\times[c_1,d_1]$ can be defined as follows:*

$$\begin{aligned}
\int_{c_1}^{y}\int_{x_1}^{x}\pi(x,y)\ {}_{x_1}d_{q_1}\ {}_{c_1}d_{q_2}y &= (1-q_1)(1-q_2)(x-x_1)(y-c_1)\\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\pi(q_1^nx+(1-q_1^n)x_1,q_2^my+(1-q_2^m)c_1)
\end{aligned}$$

for $(x,y)\in[x_1,x_2]\times[c_1,d_1]$.

We present in the following H-H-type inequalities for functions of two variables that are convex on the coordinates.

Theorem 72 ([57]). *Let $\pi:\Delta=[x_1,x_2]\times[c_1,d_1]\subset\mathbb{R}^2\to\mathbb{R}$ be convex on the coordinates on $[x_1,x_2]\times[c_1,d_1]$. Then the following inequalities hold:*

$$\begin{aligned}
&\pi\Big(\frac{x_1+x_2}{2},\frac{c_1+d_1}{2}\Big)\\
\leq\ & \frac{1}{2(x_2-x_1)}\int_{x_1}^{x_2}\pi\Big(x,\frac{c_1+d_1}{2}\Big)\ {}_{x_1}d_{q_1}x+\frac{1}{2(d_1-c_1)}\int_{c_1}^{d_1}\Big(\frac{x_1+x_2}{2},y\Big)\ {}_{c_1}d_{q_2}y\\
\leq\ & \frac{1}{2(x_2-x_1)(d_1-c_1)}\int_{x_1}^{x_2}\int_{c_1}^{d_1}\pi(x,y)\ {}_{c_1}d_{q_2}y\ {}_{x_1}d_{q_1}x\\
\leq\ & \frac{q_1}{2(1+q_1)(d_1-c_1)}\int_{c_1}^{d_1}\pi(x_1,y)\ {}_{c_1}d_{q_2}y+\frac{1}{2(1+q_1)(d_1-c_1)}\int_{c_1}^{d_1}\pi(x_2,y)\ {}_{c_1}d_{q_2}y\\
&+\frac{1}{2(1+q_2)(x_2-x_1)}\int_{x_1}^{x_2}\pi(x,d_1)\ {}_{x_1}d_{q_1}x+\frac{q_2}{2(1+q_2)(x_2-x_1)}\int_{x_1}^{x_2}\pi(x,c_1)\ {}_{x_1}d_{q_1}x\\
\leq\ & \frac{q_1q_2\pi(x_1,c_1)+q_1\pi(x_1,d_1)+q_2\pi(x_2,c_1)+\pi(x_2,d_1)}{(1+q_1)(1+q_2)}.
\end{aligned}$$

Theorem 73 ([57]). *Let $\pi:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0<q_1<1$ and $0<q_2<1$. If partial q_1q_2-derivative $\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x,y)}{{}_{x_1}\partial_{q_1}x\ {}_{c_1}\partial_{q_2}y}$ is continuous and integrable on $[x_1,x_2]\times[c_1,d_1]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x,y)}{{}_{x_1}\partial_{q_1}x\ {}_{c_1}\partial_{q_2}y}\right|^r$ is convex on the coordinates on $[x_1,x_2]\times[c_1,d_1]$ for $r\geq1$, then the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{q_1q_2\pi(x_1,c_1)+q_1\pi(x_1,d_1)+q_2\pi(x_2,c_1)+\pi(x_2,d_1)}{(1+q_1)(1+q_2)}\\
&-\frac{q_2}{(1+q_2)(x_2-x_1)}\int_{x_1}^{x_2}\pi(x,c_1)\ {}_{x_1}d_{q_1}x-\frac{1}{(1+q_2)(x_2-x_1)}\int_{x_1}^{x_2}\pi(x,d_1)\ {}_{x_1}d_{q_1}x\\
&-\frac{q_1}{(1+q_1)(d_1-c_1)}\int_{c_1}^{d_1}\pi(x_1,y)\ {}_{c_1}d_{q_2}y-\frac{1}{(1+q_1)(d_1-c_1)}\int_{c_1}^{d_1}\pi(x_2,y)\ {}_{c_1}d_{q_2}y
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{(x_2-x_1)(d_1-c_1)}\int_{x_1}^{x_2}\int_{c_1}^{d_1}\pi(x,y)\ {}_{c_1}d_{q_2}y\ {}_{x_1}d_{q_1}x\Bigg| \\
\leq\ & \frac{q_1q_2(x_2-x_1)(d_1-c_1)}{(1+q_1)(1+q_2)}\left(\Phi_{q_1}\Phi_{q_2}\right)^{1-\frac{1}{r}} \\
&\times\left\{\Psi_{q_1}\Psi_{q_2}\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x_1,c_1)}{{}_{x_1}\partial_{q_1}t\ {}_{c_1}\partial_{q_2}s}\right|^r+Q_{q_2}\Psi_{q_1}\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x_2,c_1)}{{}_{x_1}\partial_{q_1}t\ {}_{c_1}\partial_{q_2}s}\right|^r\right. \\
&\left.+Q_{q_1}\Psi_{q_2}\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x_1,d_1)}{{}_{x_1}\partial_{q_1}t\ {}_{c_1}\partial_{q_2}s}\right|^r+Q_{q_1}Q_{q_2}\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x_2,d_1)}{{}_{x_1}\partial_{q_1}t\ {}_{c_1}\partial_{q_2}s}\right|^r\right\}^{\frac{1}{r}},
\end{aligned}
$$

where

$$
\begin{aligned}
\Phi_q &= \int_0^1 |(1-(1+q)t|\ {}_0d_qt, \\
\Psi_q &= \int_0^1 (1-t)|1-(1+q)t|\ {}_0d_qt, \\
Q_q &= \int_0^1 t|1-(1+q)t|\ {}_0d_qt.
\end{aligned}
$$

The next result is for quasi-convex functions on coordinates on $[x_1,x_2]\times[c_1,d_1]$.

Definition 22 ([57])**.** *A function $\pi:\Delta\to\mathbb{R}$ is said to be quasi-convex on the coordinates on Δ, for all $t,s\in[0,1]$ and $(x,y),(u,v)\in\Delta$, if the following inequality holds:*

$$
\pi(ty+(1-t)x, sv+(1-s)u)\leq\sup\{\pi(x,y),\pi(x,v),\pi(u,y),\pi(u,v)\}.
$$

Theorem 74 ([57])**.** *Let $\pi:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ° with $0<q_1<1$ and $0<q_2<1$. If partial q_1q_2-derivative $\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x,y)}{{}_{x_1}\partial_{q_1}x\ {}_{c_1}\partial_{q_2}y}$ is continuous and integrable on $[x_1,x_2]\times[c_1,d_1]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x,y)}{{}_{x_1}\partial_{q_1}x\ {}_{c_1}\partial_{q_2}y}\right|^r$ is quasi-convex on coordinates on $[x_1,x_2]\times[c_1,d_1]$ for $r\geq1$, then the following inequality holds:*

$$
\begin{aligned}
&\Bigg|\frac{q_1q_2\pi(x_1,c_1)+q_1\pi(x_1,d_1)+q_2\pi(x_2,c_1)+\pi(x_2,d_1)}{(1+q_1)(1+q_2)} \\
&-\frac{q_2}{(1+q_2)(x_2-x_1)}\int_{x_1}^{x_2}\pi(x,c_1)\ {}_{x_1}d_{q_1}x-\frac{1}{(1+q_2)(x_2-x_1)}\int_{x_1}^{x_2}\pi(x,d_1)\ {}_{x_1}d_{q_1}x \\
&-\frac{q_1}{(1+q_1)(d_1-c_1)}\int_{c_1}^{d_1}\pi(x_1,y)\ {}_{c_1}d_{q_2}y-\frac{1}{(1+q_1)(d_1-c_1)}\int_{c_1}^{d_1}\pi(x_2,y)\ {}_{c_1}d_{q_2}y \\
&+\frac{1}{(x_2-x_1)(d_1-c_1)}\int_{x_1}^{x_2}\int_{c_1}^{d_1}\pi(x,y)\ {}_{c_1}d_{q_2}y\ {}_{x_1}d_{q_1}x\Bigg| \\
\leq\ & \frac{q_1q_2(x_2-x_1)(d_1-c_1)}{(1+q_1)(1+q_2)}\left(\frac{4q_1q_2}{(1+q_1)^2(1+q_2)^2}\right) \\
&\times\sup\left\{\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x_1,c_1)}{{}_{x_1}\partial_{q_1}t\ {}_{c_1}\partial_{q_2}s}\right|,\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x_2,c_1)}{{}_{x_1}\partial_{q_1}t\ {}_{c_1}\partial_{q_2}s}\right|,\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x_1,d_1)}{{}_{x_1}\partial_{q_1}t\ {}_{c_1}\partial_{q_2}s}\right|,\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi(x_2,d_1)}{{}_{x_1}\partial_{q_1}t\ {}_{c_1}\partial_{q_2}s}\right|\right\}.
\end{aligned}
$$

Theorem 75 ([58]). *Let $\pi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Λ°. If $\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi}{{}_{x_1}\partial_{q_1}u\,{}_{c_1}\partial_{q_2}w}$ is continuous and integrable on $[x_1,x_2]\times[c_1,d_1]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{x_1,c_1}\partial^2_{q_1,q_2}\pi}{{}_{x_1}\partial_{q_1}u\,{}_{c_1}\partial_{q_2}w}\right|^r$ is convex on the coordinates on $[x_1,x_2]\times[c_1,d_1]$ for $r>0$, then the following inequality holds:*

$$\begin{aligned}
&\Big|\pi\Big(\frac{x_1q_1+x_2}{1+q_1},\frac{c_1q_2+d_1}{1+q_2}\Big)-\frac{1}{(x_2-x_1)}\int_{x_1}^{x_2}\pi\Big(u,\frac{c_1q_2+d_1}{1+q_2}\Big)\,{}_{x_1}d_{q_1}u\\
&-\frac{1}{(d_1-c_1)}\int_{c_1}^{d_1}\pi\Big(\frac{x_1q_1+x_2}{1+q_1},w\Big)\,{}_{c_1}d_{q_2}w\\
&+\frac{1}{(x_2-x_1)(d_1-c_1)}\int_{x_1}^{x_2}\int_{c_1}^{d_1}\pi(u,w)\,{}_{c_1}d_{q_2}w\,{}_{x_1}d_{q_1}u\\
\leq\;& q_1q_2(x_2-x_1)(d_1-c_1)\Big(\int_0^1\int_0^1|k(u,w)|^p\,{}_0d_{q_2}w\,{}_0d_{q_1}u\Big)^{\frac{1}{p}}\\
&\times\Bigg[\frac{1}{(1+q_1)(1+q_2)}\Bigg\{q_1q_2\left|\frac{{}_{x_1,c_1}\partial^2_{q_1q_2}\pi(x_1,c_1)}{{}_{x_1}\partial_{q_1}u\,{}_{c_1}\partial_{q_2}w}\right|^r+q_1\left|\frac{{}_{x_1,c_1}\partial^2_{q_1q_2}\pi(x_1,d_1)}{{}_{x_1}\partial_{q_1}u\,{}_{c_1}\partial_{q_2}w}\right|^r\\
&+q_2\left|\frac{{}_{x_1,c_1}\partial^2_{q_1q_2}\pi(x_2,c_1)}{{}_{x_1}\partial_{q_1}u\,{}_{c_1}\partial_{q_2}w}\right|^r+\left|\frac{{}_{x_1,c_1}\partial^2_{q_1q_2}\pi(x_2,d_1)}{{}_{x_1}\partial_{p_1,q_1}t\,{}_{c_1}\partial_{p_2,q_2}w}\right|^r\Bigg\}\Bigg]^{\frac{1}{r}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{r}=1$ and

$$k(u,w)=\begin{cases} uw, & (u,w)\in\left[0,\frac{1}{1+q_1}\right]\times\left[0,\frac{1}{1+q_2}\right]\\ u\Big(w-\frac{1}{q_2}\Big), & (u,w)\in\left[0,\frac{1}{1+q_1}\right]\times\left(\frac{1}{1+q_2},1\right]\\ w\Big(u-\frac{1}{q_1}\Big), & (u,w)\in\left(\frac{1}{1+q_1},1\right]\times\left[0,\frac{1}{1+q_2}\right]\\ \Big(u-\frac{1}{q_1}\Big)\Big(w-\frac{1}{q_2}\Big), & (u,w)\in\left(\frac{1}{1+q_1},1\right]\times\left(\frac{1}{1+q_2},q\right]. \end{cases}$$

Quantum H-H type inequalities in the manner of Green functions.

In the next theorems we show quantum H-H type inequalities via Green functions.

Theorem 76 ([59]). *Let $\psi\in C^2([x_1,x_2])$ be twice differentiable convex on (x_1,x_2) such that $\psi(x)=\psi(x_1)+(x-x_1)\psi'(x_2)+\int_{x_1}^{x_2}G(x,\mu)\psi''(\mu)d\mu$, $G(x,u)=\begin{cases}x_1-u, & x_1\leq u\leq x,\\ x_1-x, & x\leq u\leq x_2.\end{cases}$ If $0<q<1$, then:*

$$\psi\Big(\frac{qx_1+x_2}{q+1}\Big)\leq\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\psi(x)\,{}_{x_1}d_qx\leq\frac{q\psi(x_1)+\psi(b)}{q+1}.$$

Theorem 77 ([59]). *Assume that function $\psi\in C^2([x_1,x_2])$ is convex as in the previous Theorem 76. Then:*

(i) If $|\psi''|$ is non-decreasing, then

$$\left|\frac{q\psi(x_1)+\psi(x_2)}{q+1}-\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\psi(x)\,{}_{x_1}d_qx\right|\leq|\psi''(x_2)|\Big[\frac{q(x_2-x_1)^2}{6(1+q)}\Big].$$

(ii) *If $|\psi''|$ is non-increasing, then*

$$\left| \frac{q\psi(x_1)+\psi(x_2)}{q+1} - \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\psi(x)\ {}_{x_1}d_qx \right| \leq |\psi''(x_1)| \Big[\frac{q(x_2-x_1)^2}{6(1+q)} \Big].$$

(iii) *If $|\psi''|$ is convex, then*

$$\left| \frac{q\psi(x_1)+\psi(x_2)}{q+1} - \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\psi(x)\ {}_{x_1}d_qx \right| \leq \max\{|\psi''(x_1)|, |\psi''(x_2)|\} \Big[\frac{q(x_2-x_1)^2}{6(1+q)} \Big].$$

Theorem 78 ([59])**.** *Assume that function $\psi \in C^2([x_1,x_2])$ is convex, as in Theorem 76. Then:*

(i) *If $|\psi''|$ is non-decreasing, then*

$$\begin{aligned} &\left| \psi''\Big(\frac{qx_1+x_2}{1+q}\Big) - \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\psi(x)\ {}_{x_1}d_qx \right| \\ \leq\ & |\psi''(x_2)| \Big[\frac{(qx_1+x_2)^2}{2(1+q)^2} - x_1\frac{qx_1+x_2}{1+q} + \frac{q(x_2-x_1)^2}{(1+q)^2} - \frac{(x_2-x_1)^2}{3(q+1)} \Big] \\ & - \frac{|\psi''(x_2)|}{6(x_2-x_1)}[2x_1^3+x_2^3-3x_1x_2^2]. \end{aligned}$$

(ii) *If $|\psi''|$ is non-increasing, then*

$$\begin{aligned} &\left| \psi''\Big(\frac{qx_1+x_2}{1+q}\Big) - \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\psi(x)\ {}_{x_1}d_qx \right| \\ \leq\ & |\psi''(x_1)| \Big[\frac{(qx_1+x_2)^2}{2(1+q)^2} - x_1\frac{qx_1+x_2}{1+q} + \frac{q(x_2-x_1)^2}{(1+q)^2} - \frac{(x_2-x_1)^2}{3(q+1)} \Big] \\ & - \frac{|\psi''(x_2)|}{6(x_2-x_1)}[2x_1^3+x_2^3-3x_1x_2^2]. \end{aligned}$$

(iii) *If $|\psi''|$ is convex, then*

$$\begin{aligned} &\left| \psi''\Big(\frac{qx_1+x_2}{1+q}\Big) - \frac{1}{x_2-x_1}\int_{x_1}^{x_2}\psi(x)\ {}_{x_1}d_qx \right| \\ \leq\ & \max\{|\psi''(x_1)|, |\psi''(x_2)|\} \Big[\frac{(qx_1+x_2)^2}{2(1+q)^2} - x_1\frac{qx_1+x_2}{1+q} + \frac{q(x_2-x_1)^2}{(1+q)^2} - \frac{(x_2-x_1)^2}{3(q+1)} \Big] \\ & - \min\{|\psi''(x_1)|, |\psi''(x_2)|\} \Big[\frac{2x_1^3+x_2^3-3x_1x_2^2}{6(x_2-x_1)} \Big]. \end{aligned}$$

Quantum H-H type inequalities in the mode of preinvex functions.

The following theorems deal with preinvex functions.

Definition 23 ([60])**.** *A real-valued set K is said to be an invex with respect to $\eta : K \times K \to \mathbb{R}^n$, if for all $x, y \in K$, we have*

$$x + \theta\eta(y,x) \in K.$$

Definition 24 ([60])**.** *A real-valued function π is called preinvex with respect to η if*

$$\pi(x+\theta\eta(y,x)) \leq (1-\theta)\pi(x) + \theta\pi(y),$$

for all $x, y \in K$, and $\theta \in [0,1]$.

Condition *C*. *Assume that real-valued subset A is invex subset with respect to $\eta : A \times A \to \mathbb{R}^n$. Then η verifies the condition C if for all $x, y \in A$ and $\theta \in [0,1]$,*

$$\eta(y, y + \theta\eta(x,y)) = -\theta\eta(x,y), \ \ \eta(x, y + \theta\eta(x,y)) = (1-\theta)\eta(x,y).$$

Theorem 79 ([61])**.** *Assume that function $\pi : [x_1, x_1 + \eta(x_2, x_1)] \subset \mathbb{R} \to \mathbb{R}$ is preinvex and integrable with $\eta(x_2, x_1) > 0$. If $\eta(\cdot,\cdot)$ verifies the Condition C, then*

$$\pi\left(\frac{2x_1 + \eta(x_2,x_1)}{2}\right) \le \frac{1}{\eta(x_2,x_1)} \int_{x_1}^{x_1+\eta(x_2,x_1)} \pi(x) \ {}_{x_1}d_q x \le \frac{q\pi(x_1) + \pi(x_2)}{2}.$$

Theorem 80 ([61])**.** *Assume that function $\pi : [x_1, x_1 + \eta(x_2, x_1)] \subset \mathbb{R} \to \mathbb{R}$ is q-differentiable on $(x_1, x_1 + \eta(x_2, x_1))$ with ${}_{x_1}D_q\pi$ integrable and continuous on $[x_1, x_1 + \eta(x_2, x_1)]$ where $0 < q < 1$. If function $|{}_{x_1}D_q\pi|$ is preinvex, then*

$$\begin{aligned} &\left| \frac{1}{\eta(x_2,x_1)} \int_{x_1}^{x_1+\eta(x_2,x_1)} \pi(x) \ {}_{x_1}d_q x - \frac{q\pi(x_1) + \pi(x_1 + \eta(x_2,x_1))}{1+q} \right| \\ \le \ & \frac{q^2\eta(x_2,x_1)}{(1+q)^2(1+q+q^2)} \Big[q(1+3q^2+2q^3)|{}_{x_1}D_q\pi(x_1)| + (1+4q+q^2)|{}_{x_1}D_q\pi(x_2)| \Big]. \end{aligned}$$

3. H-H Inequalities via Fractional Quantum Calculus

The following concepts are adapted by Ref. ([10]). We state a q-shifting operator as

$${}_{x_1}\Phi_q(m) = qm + (1-q)x_1, \quad 0 < q < 1, \ \ m, x_1 \in \mathbb{R}.$$

The q-analog is stated by

$$(m;q)_0 = 1, \quad (m;q)_k = \prod_{i=1}^{k-1}(1 - q^i m), \ \ k \in \mathbb{N} \cup \{\infty\}.$$

The q number is stated by

$$[m]_q = \frac{1-q^m}{1-q}, \ \ m \in \mathbb{R}.$$

The q-Gamma function is stated by

$$\Gamma_q(t) = \frac{{}_0(1 - {}_0\Phi_q(1))_q^{(t-1)}}{(1-q)^{t-1)}}, \ \ t \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}.$$

Here, we add some definitions regarding fractional q-calculus, namely the R-L fractional q-integral.

Definition 25 ([10])**.** *Let $\alpha \ge 0$ and function π be a continuous stated on $[x_1, x_2]$. Then $({}_{x_1}I_q^0\pi)(t) = \pi(t)$ is given by*

$$\begin{aligned} ({}_{x_1}I_q^\alpha\pi)(t) &= \frac{1}{\Gamma_q(\alpha)} \int_{x_1}^{t} {}_{x_1}(t - {}_{x_1}\Phi_q(s))_q^{\alpha-1} \pi(s) \ {}_{x_1}d_q s \\ &= \frac{(1-q)(t-x_1)}{\Gamma_q(\alpha)} \sum_{i=0}^{\infty} q^i \ {}_{x_1}(t - {}_{x_1}\Phi_q^{i+1}(t))_q^{\alpha-1} \pi({}_{x_1}\Phi_q^i(t)). \end{aligned}$$

Theorem 81 ([62]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be convex, $0 < q < 1$ and $\alpha > 0$. Then*

$$\begin{aligned} &\frac{2}{\Gamma_q(\alpha+1)}\pi\Big(\frac{x_1+x_2}{2}\Big) - \frac{1}{(x_2-x_1)^\alpha}\big({}_{x_1}I_q^\alpha \pi(x_1+x_2-s)\big)(x_2) \\ \leq\ & \frac{1}{(x_2-x_1)^\alpha}\big({}_{x_1}I_q^\alpha \pi(s)\big)(x_2) \leq \frac{1}{\Gamma_q(\alpha+2)}\big(([\alpha+1]_q-1)\pi(x_1)+\pi(x_2)\big). \end{aligned}$$

Theorem 82 ([63]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be convex and $\alpha > 0$. Then*

$$f\left(\frac{([\alpha+1]_q-1)x_1+x_2}{[\alpha+1]_q}\right) \leq \frac{\Gamma_q(\alpha+1)}{(x_2-x_1)^\alpha}\big({}_{x_1}I_q^\alpha \pi\big)(x_2) \leq \frac{([\alpha+1]_q-1)\pi(x_1)+\pi(x_2)}{[\alpha+1]_q}.$$

Theorem 83 ([63]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be continuous, $\alpha > 0$ and ${}_{x_1}D_q\pi$ be q-integrable on (x_1, x_2). If ${}_{x_1}D_q\pi$ is convex on $[x_1, x_2]$, then*

$$\begin{aligned} &\left|\frac{\Gamma_q(\alpha+1)}{(x_2-x_1)^\alpha}\ {}_{x_1}I_q^\alpha \pi(x_2) - \frac{([\alpha+1]_q-1)\pi(x_1)+\pi(x_2)}{[\alpha+1]_q}\right| \\ \leq\ & \frac{x_2-x_1}{[\alpha+1]_q}\Big[|{}_{x_1}D_q\pi(x_1)|\int_0^1 |[\alpha+1]_q\ {}_0(1-\Phi_q(t))_q^{(\alpha)}-1|(1-t)\ {}_0d_qt \\ &+|{}_{x_1}D_q\pi(x_2)|\int_0^1 |[\alpha+1]_q\ {}_0(1-\Phi_q(t))_q^{(\alpha)}-1|t\ {}_0d_qt\Big]. \end{aligned}$$

4. H-H Type Inequalities for (*p*,*q*)-Calculus

Definition 26 ([64]). *If function $\pi : [x_1, x_2] \to \mathbb{R}$ is continuous, then (p, q)-derivative of π at x is stated by*

$$\begin{aligned} {}_{x_1}D_{p,q}f(x) &= \frac{\pi(px+(1-p)x_1)-\pi(qx+(1-q)x_1)}{(p-q)(x-x_1)}, \quad x \neq x_1 \qquad (3) \\ {}_{x_1}D_{p,q}f(a) &= \lim_{x\to x_1} {}_{x_1}D_{p,q}\pi(x). \end{aligned}$$

If ${}_{x_1}D_{p,q}\pi(x)$ exists for all $x \in [x_1, x_2]$, then the function π is called (p, q)-differentiable on $[x_1, x_2]$.

The ${}_{x_1}(p, q)$-integral $\int_{x_1}^x \pi(t)\ {}_{x_1}d_{p,q}t$ is defined by

$$\int_{x_1}^x \pi(t)\ {}_{x_1}d_{p,q}t = (p-q)(x-x_1)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\pi\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)x_1\right).$$

Definition 27 ([65]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be continuous. Then the ${}^{x_2}(p, q)$-derivative ${}^{x_2}D_{p,q}\pi(x)$ of π at $x \in [a, b]$ is stated by*

$${}^{x_2}D_{p,q}\pi(x) = \frac{f(px+(1-p)x_2)-f(qx+(1-q)x_2)}{(p-q)(x_2-x)}, \quad x \neq x_2.$$

The ${}^{x_2}(p, q)$-integral $\int_x^{x_2} \pi(t)^{x_2}d_{p,q}t$ is stated by

$$\int_x^{x_2} \pi(t)^{x_2}d_{p,q}t = (p-q)(x_2-x)\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\pi\left(\frac{q^n}{p^{n+1}}x+\left(1-\frac{q^n}{p^{n+1}}\right)x_2\right).$$

Theorem 84 ([66]). *Let $\pi : [x_1, x_2] \to \mathbb{R}$ be a convex differentiable function on $[x_1, x_2]$ and $0 < q < p \leq 1$. Then we have:*

$$\max\{I_1, I_2, I_3\} \leq \frac{1}{p(x_2-x_1)}\int_{x_1}^{px_2+(1-p)x_1} \pi(x)\ {}_{x_1}d_{p,q}x \leq \frac{q\pi(x_1)+p\pi(x_2)}{p+q},$$

where

$$\begin{aligned} I_1 &= \pi\Big(\frac{qx_1+px_2}{p+q}\Big), \\ I_2 &= \pi\Big(\frac{qx_1+px_2}{p+q}\Big) + \frac{(p-q)(x_2-x_1)}{p+q}\pi'\Big(\frac{px_1+qx_2}{p+q}\Big), \\ I_3 &= \pi\Big(\frac{x_1+x_2}{2}\Big) + \frac{(p-q)(x_2-x_1)}{2(p+q)}\pi'\Big(\frac{x_1+x_2}{2}\Big). \end{aligned}$$

Theorem 85 ([66]). *Assume that a real-valued function π is (p,q)-differentiable over (x_1,x_2) and $0<q<p\le 1$, ${}_{x_1}D_{p,q}\pi$ is integrable and continuous over $[x_1,x_2]$. If function $|{}_{x_1}D_{p,q}\pi|$ is convex over $[x_1,x_2]$, then*

$$\begin{aligned} &\left|\pi\Big(\frac{px_1+qx_2}{p+q}\Big) - \int_{x_1}^{px_2+(1-p)x_1}\pi(x)\ {}_{x_1}d_{p,q}x\right| \\ \le\ & q(x_2-x_1)\Big[|{}_{x_1}D_{p,q}\pi(x_2)|\frac{p^3}{(p+q)^3(p^2+pq+q^2)} \\ &+|{}_{x_1}D_{p,q}\pi(x_1)|\frac{p^2(p^2+pq+q^2)-p^3}{(p+q)^3(p^2+pq+q^2)} + |{}_{x_1}D_{p,q}\pi(x_2)|\frac{2p^3}{(p+q)^3(p^2+pq+q^2)} \\ &+|{}_{x_1}D_{p,q}\pi(x_1)|\frac{p^4+p^3q+p^2q^2-2p^3}{(p+q)^3(p^2+pq+q^2)}\Big]. \end{aligned}$$

Theorem 86 ([66]). *Assume that π is as in Theorem 85. If $|{}_{x_1}D_{p,q}\pi|^r$ is a convex function over $[x_1,x_2]$, for $r\ge 1$, then*

$$\begin{aligned} &\left|\pi\Big(\frac{px_1+qx_2}{p+q}\Big) - \int_{x_1}^{px_2+(1-p)x_1}\pi(x)\ {}_{x_1}d_{p,q}x\right| \\ \le\ & q(x_2-x_1)\Big(\frac{p^2}{(p+q)^3}\Big)^{\frac{1}{s}}\Big[|{}_{x_1}D_{p,q}\pi(x_2)|\frac{p^3}{(p+q)^3(p^2+pq+q^2)} \\ &+|{}_{x_1}D_{p,q}\pi(x_1)|\Big(\frac{p^2(p^2+pq+q^2)-p^3}{(p+q)^3(p^2+pq+q^2)}\Big)^{\frac{1}{r}} + |{}_{x_1}D_{p,q}\pi(x_2)|\frac{2p^3}{(p+q)^3(p^2+pq+q^2)} \\ &+|{}_{x_1}D_{p,q}\pi(x_1)|\Big(\frac{p^4+p^3q+p^2q^2-2p^3}{(p+q)^3(p^2+pq+q^2)}\Big)^{\frac{1}{r}}\Big]. \end{aligned}$$

Theorem 87 ([66]). *Assume that π is as in Theorem 85. If function $|{}_{x_1}D_{p,q}\pi|^r$ is quasi-convex over $[x_1,x_2]$, for $r>1$, then*

$$\begin{aligned} &\left|\pi\Big(\frac{px_1+qx_2}{p+q}\Big) - \int_{x_1}^{px_2+(1-p)x_1}\pi(x)\ {}_{x_1}d_{p,q}x\right| \\ \le\ & q(x_2-x_1)\frac{p^2}{(p+q)^3}\sup\{|{}_{x_1}D_{p,q}\pi(x_1)|, |{}_{x_1}D_{p,q}\pi(x_2)|\}. \end{aligned}$$

Theorem 88 ([67]). *Assume that the real-valued function π is differentiable and convex on $[x_1,x_2]$. Then*

$$\max\{A_1,A_2\} \le \frac{1}{p(x_2-x_1)}\int_{px_1+(1-p)x_2}^{x_2}\pi(x)\ {}^{x_2}d_{p,q}x \le \frac{p\pi(x_1)+q\pi(x_2)}{[2]_{p,q}},$$

where

$$A_1 = \pi\Big(\frac{px_1+qx_2}{[2]_{p,q}}\Big),$$

$$A_2 = \pi\Big(\frac{px_1+qx_2}{[2]_{p,q}}\Big) + \frac{(p-q)(x_2-x_1)}{[2]_{p,q}}\pi'\Big(\frac{px_1+qx_2}{[2]_{p,q}}\Big),$$

and $0 < q < p \leq 1$ *and* $[n]_{p.q} = \dfrac{p^n - q^n}{p-q}$.

Theorem 89 ([67]). *Assume that* $\pi : [x_1, x_2] \to \mathbb{R}$ *is differentiable on* (x_1, x_2). *If* ${}^{x_2}D_{p,q}\pi$ *is integrable and continuous on* $[x_1, x_2]$ *and if* $|{}^{x_2}D_{p,q}\pi|$ *is convex on* $[x_1, x_2]$, *then*

$$\begin{aligned}
&\left|\int_{px_1+(1-p)x_2}^{x_2} \pi(x)\ {}^{x_2}d_{p,q}x - \pi\Big(\frac{px_1+qx_2}{[2]_{p,q}}\Big)\right| \\
\leq\ & (x_2-x_1)\Big[|{}^{x_2}D_{p,q}\pi(x_1)|\frac{qp^3}{([2]_{p,q})^3[3]_{p,q}} + |{}^{x_2}D_{p,q}\pi(x_2)|\frac{q(p^3(p^2+q^2-p)+p^2[3]_{p,q})}{([2]_{p,q})^4[3]_{p,q}} \\
&+|{}^{x_2}D_{p,q}\pi(x_1)|\Big(\frac{q(q+2p)}{[2]_{p,q}} - \frac{q^2(q^2+3p^2+3pq)}{([2]_{p,q})^3[3]_{p,q}}\Big) \\
&+|{}^{x_2}D_{p,q}\pi(x_2)|\Big\{\Big(\frac{q}{[2]_{p,q}} - \frac{q^2(q+2p)}{(]2]_{p,a})^4}\Big) - \Big(\frac{q(q+2p)}{[2]_{p,q}} - \frac{q^2(q^2+3p^2+3pq)}{([2]_{p,q})^3[3]_{p,q}}\Big)\Big\}\Big].
\end{aligned}$$

Theorem 90 ([68]). *Let function* $\pi : I = [x_1, x_2] \to \mathbb{R}$ *be* (p,q)*-differentiable on* (x_1, x_2) *and* ${}^{x_2}D^2_{p,q}\pi$ *is integrable and continuous on* I. *If* $|{}^{x_2}D_{p,q}\pi|$ *is convex on* $[x_1, x_2]$, *then*

$$\begin{aligned}
&\left|\frac{1}{x_2-x_1}\int_{p^2x_1+(1-p^2)b}^{x_2} \pi(s)\ {}^{x_2}d_{p,q}s - \pi\Big(\frac{px_1+qx_2}{[2]_{p,q}}\Big)\right| \\
\leq\ & \frac{(x_2-x_1)^2}{[2]_{p,q}}\Big[\frac{p^3[2]_{p,q}(p^3[2]^2_{p,q} - p^2 - q^2)}{[2]^3_{p,q}[3]_{p,q}[4]_{p,q}}|{}^{x_2}D^2_{p,q}\pi(x_1)| \\
&+\Big(\frac{[3]_{p,q}[4]_{p,q}\Big((1-2p)([2]^2_{p,q}-1) + q[2]_{p,q}\Big)}{[2]^3_{p,q}[3]_{p,q}[4]_{p,q}} \\
&+\frac{(p^3q^2[2]^2_{p,q} + p^3[2]^2_{p,q} - p[2]_{p,q}[4]_{p,q} - q[4]_{p,q}\Big)}{[2]^3_{p,q}[3]_{p,q}[4]_{p,q}}\Big)|{}^{x_2}D^2_{p,q}\pi(x_2)|\Big].
\end{aligned}$$

In the next theorem we include fractional (p,q)-H-H integral inequalities on $[x_1, p^\alpha x_2 + (1-p^\alpha)x_1]$.

Theorem 91 ([69]). *If function* $\pi : [x_1, x_2] \to \mathbb{R}$ *is differentiable and convex and* $\alpha > 0$, *then*

$$\begin{aligned}
\pi\Big(\frac{([\alpha+1]_{p.q} - p^\alpha)x_1 + p^\alpha x_2}{[\alpha+1]_{p,q}}\Big) &\leq \frac{\Gamma_{p,q}(\alpha+1)}{p^{\alpha^2}(x_2-x_1)^\alpha}\Big({}_{x_1}I^\alpha_{p,q}\pi(s)\Big)(p^\alpha x_2 + (1-p^\alpha)x_1) \\
&\leq \frac{([\alpha+1]_{p,q} - p^\alpha)\pi(x_1) + p^\alpha\pi(x_2)}{[\alpha+1]_{p,q}},
\end{aligned}$$

where $\Gamma_{p,q}(t) = \dfrac{(p-q)^{(t-1)}_{p,q}}{(p-q)^{t-1}}$.

Now, by utilizing the concept of post-quantum integrals, we explore H-H inequalities in the second sense via s-convexity.

Theorem 92 ([70])**.** *Let $\pi : [0, \infty) \to \mathbb{R}$ be s-convex in the second sense and $x_1, x_2 \in [0, \infty)$ with $x_1 < x_2$. Then for $s \in (0, 1]$, we have*

$$s^{s-1}\pi\Big(\frac{x_1+x_2}{2}\Big) \le \frac{1}{2p(x_2-x_1)}\Big[\int_{x_1}^{x_2} \pi(x)\ {}_{x_1}d_{p,q}x + \int_{x_1}^{x_2} \pi(x)\ {}^{x_2}d_{p,q}x\Big] \le \frac{\pi(x_1)+\pi(x_2)}{[s+1]_{p,q}}.$$

Theorem 93 ([71])**.** *Let $\pi : [x_1, x_2] \to \mathbb{R}$ be convex. Then*

$$\begin{aligned} & \pi\Big(\frac{x_1+x_2}{2}\Big) \\ \le\ & \frac{1}{p(x_2-x_1)}\Big[\int_{px_1+(1-p)\frac{x_1+x_2}{2}}^{\frac{x_1+x_2}{2}} \pi(x)\ {}^{\frac{x_1+x_2}{2}}d_{p,q}x + \int_{\frac{x_1+x_2}{2}}^{px_2+(1-p)\frac{x_1+x_2}{2}} \pi(x)\ {}_{\frac{x_1+x_2}{2}}d_{p.q}x\Big] \\ \le\ & \frac{\pi(x_1)+\pi(x_2)}{2}. \end{aligned}$$

Theorem 94 ([71])**.** *Let $\pi : [x_1, x_2] \to \mathbb{R}$. Assume that ${}_{x_1}D_{p,q}\pi$ and ${}^{x_2}D_{p,q}\pi$ are continuous and integrable mappings over $[x_1, x_2]$. If $|{}_{x_1}D_{p,q}\pi|$ and $|{}^{x_2}D_{p,q}\pi|$ are convex, then*

$$\begin{aligned} & \Bigg|\frac{1}{p(x_2-x_1)}\Big[\int_{px_1+(1-p)\frac{x_1+x_2}{2}}^{\frac{x_1+x_2}{2}} \pi(x)\ {}^{\frac{x_1+x_2}{2}}d_{p,q}x \\ & + \int_{\frac{x_1+x_2}{2}}^{px_2+(1-p)\frac{x_1+x_2}{2}} \pi(x)\ {}_{\frac{x_1+x_2}{2}}d_{p,q}x\Big] - \pi\Big(\frac{x_1+x_2}{2}\Big)\Bigg| \\ \le\ & \frac{x_2-x_1}{8p^3}\Big[\Big(\frac{p^3+p^2q-p^2}{[2]_{p,q}} - p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,a}}\Big)|{}_{x_1}D_{p,q}\pi(x_1)| \\ & + \Big(\frac{p^3+p^2q+p^2}{[2]_{p,q}} - p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\Big)|{}_{x_1}D_{p,q}\pi(x_2)| \\ & + \Big(\frac{p^3+p^2q+p^2}{[2]_{p,q}} - p^3q\frac{[3]_{p,q}+[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\Big)|{}_{x_1}D_{p,q}\pi(x_1)| \\ & + \Big(\frac{p^3+p^2q-p^2}{[2]_{p,q}} - p^3q\frac{[3]_{p,q}-[2]_{p,q}}{[2]_{p,q}[3]_{p,q}}\Big)|{}_{x_1}D_{p,q}\pi(x_2)|\Big]. \end{aligned}$$

Now, we show (p, q)-H-H inequality in the manner of double integrals.

Theorem 95 ([72])**.** *Let $0 < q < p \le 1$ and $\pi : [x_1, x_2] \to \mathbb{R}$ be a convex function on $[x_1, x_2]$. Then*

$$\begin{aligned} & \pi\Big(\frac{qx_1+px_2}{p+q}\Big) \\ \le\ & \frac{1}{(px_2-px_1)^2}\int_{x_1}^{px_2+(1-p)x_1}\int_{x_1}^{px_2+(1-p)x_1} \pi(\theta x_1 + (1-\theta)x_2)\ {}_{x_1}d_{p,q}x\ {}_{x_1}d_{p,q}y \\ \le\ & \frac{1}{p(x_2-x_1)}\int_{x_1}^{px_2+(1-p)x_1} \pi(x)\ {}_{x_1}d_{p,q}x\ {}_{x_1}d_{p,q}y \\ \le\ & \frac{q\pi(x_1)+p\pi(x_2)}{p+q}, \end{aligned}$$

for all $\theta \in [0, 1]$.

Theorem 96 ([72])**.** *Let π be as in Theorem 95. Then we have:*

$$
\begin{aligned}
& \frac{p}{(px_2-px_1)^2} \\
\leq\ & \frac{1}{(px_2-px_1)^2}\int_{x_1}^{px_2+(1-p)x_1}\int_{x_1}^{px_2+(1-p)x_1}\pi\left(\frac{px+qy}{p+q}\right)\ {}_{x_1}d_{p,q}x\ {}_{x_1}d_{p,q}y \\
\leq\ & \frac{1}{(px_2-px_1)^2}\int_0^p\int_{x_1}^{px_2+(1-p)x_1}\int_{x_1}^{px_2+(1-p)x_1}\pi(\theta x_1+(1-\theta)x_2)\ {}_{x_1}d_{p,q}x\ {}_{x_1}d_{p,q}y\ {}_{x_1}d_{p,q}t \\
\leq\ & \frac{1}{px_2-px_1}\int_{x_1}^{px_2+(1-p)x_1}\pi(x)\ {}_{x_1}d_{p,q}x.
\end{aligned}
$$

We examine the extensions of H-H inequalities involving continuous convexity pertaining to (p,q)-calculus on $J := [x_1, px_2+(1-p)x_1]$.

Theorem 97 ([73])**.** *Assume that the real-valued π is continuous and convex. Then*

$$
\begin{aligned}
& \pi\left(\frac{qx_1+px_2}{p+q}\right) \\
\leq\ & \frac{1}{p^2(x_2-x_1)^2}\int_{x_1}^{px_2+(1-p)x_1}\int_{x_1}^{px_2+(1-p)x_1}\pi\left(\frac{x+y}{2}\right)\ {}_{x_1}d_{p,q}x_{x_1}d_{p,q}y \\
\leq\ & \frac{1}{p^2(x_2-x_1)^2}\int_{x_1}^{px_2+(1-p)x_1}\int_{x_1}^{px_2+(1-p)x_1}\frac{1}{2}\left[\pi\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)\right. \\
& \left.+\pi\left(\frac{\beta x+\alpha y}{\alpha+\beta}\right)\right]{}_{x_1}d_{p,q}x_{x_1}d_{p,q}y \\
\leq\ & \frac{1}{p(x_2-x_1)}\int_{x_1}^{px_2+(1-p)x_1}\pi(x)_{x_1}d_{p,q}x,
\end{aligned}
$$

for all $\alpha,\beta \geq 0$ with $\alpha+\beta > 0$.

We give (p,q)-H-H type inequalities for coordinated convexity.

Theorem 98 ([74])**.** *Assume that the function $\pi : [x_1,x_2]\times[c_1,d_1] \to \mathbb{R}$ is differentiable and convex. Then*

$$
\begin{aligned}
& \pi\left(\frac{q_1x_1+p_1x_2}{p_1+q_1},\frac{p_2c_1+q_2d_1}{p_2+q_2}\right) \\
\leq\ & \frac{1}{2p_1(x_2-x_1)}\int_{x_1}^{p_1x_2+(1-p_1)x_1}\pi\left(x,\frac{p_2c_1+q_2d_1}{p_2+q_2}\right)\ {}_{x_1}d_{p_1,q_1}x \\
& +\frac{1}{2p_2(x_2-x_1)}\int_{p_2c_1+(1-p_2)d_1}^{d_1}\pi\left(\frac{q_1x_1+p_1x_2}{p_1+q_1},y\right)\ {}^{d_1}d_{p_2,q_2}y \\
\leq\ & \frac{1}{p_1p_2(x_2-x_1)(d_1-c_1)}\int_{x_1}^{p_1b+(1-p_1)a}\int_{p_2c+(1-p_2)d}^{d_1}\pi(x,y)\ {}^{d_1}d_{p_2,q_2}y\ {}_{x_1}d_{p_1,q_1}x \\
\leq\ & \frac{q_1}{2p_2(p_1+q_1)(d_1-c_1)}\int_{p_2c_1+(1-p_2)d_1}^{d_1}\pi(x_1,y)\ {}^{d_1}d_{p_2,q_2}y \\
& +\frac{p_1}{2p_2(p_1+q_1)(d_1-c_1)}\int_{p_2c+(1-p_2)d}^{d_1}\pi(x_2,y)\ {}^{d_1}d_{p_2,q_2}y \\
& +\frac{p_2}{2p_1(p_2+q_2)(x_2-x_1)}\int_{x_1}^{p_1x_2+(1-p_1)x_1}\pi(x,c_1)\ {}_{x_1}d_{p_1,q_1}x \\
& +\frac{q_2}{2p_1(p_2+q_2)(x_2-x_1)}\int_{x_1}^{p_1x_2+(1-p_1)x_1}\pi(x,d_1)\ {}_{x_1}d_{p_1,q_1}x
\end{aligned}
$$

$$\leq \quad \frac{q_1 p_2 \pi(x_1, c_1) + q_1 q_2 \pi(x_1, d_1) + p_1 p_2 \pi(x_2, c_1) + p_1 q_2 \pi(x_2, d_1)}{(p_1 + q_1)(p_2 + q_2)}.$$

We define now some new concepts regarding the (p, q)-calculus of the function of two variables and present H-H-type inequalities for the functions of two variables using (p, q)-calculus.

Definition 28 ([75])**.** *Let $\pi : \Delta = [x_1, x_2] \times [c_1, d_1] \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function in each variable and $0 < q_i < p_i \leq 1, i = 1, 2$. The partial $(p_1, q_1)-$, $(p_2, q_2)-$ and $(p_1 p_2, q_1 q_2)$-derivatives at $(s, t) \in [x_1, x_2] \times [c_1, d_1]$ are, respectively, defined as:*

$$\begin{aligned}
\frac{{}_{x_1}\partial_{p_1,q_1} \pi(s,t)}{{}_{x_1}\partial_{p_1,q_1} s} &= \frac{\pi(p_1 s + (1-p_1)x_1, t) - \pi(q_1 s + (1-q_1)x_1, t)}{(p_1 - q_1)(s - x_1)}, \ s \neq x_1, \\
\frac{{}_{c_1}\partial_{p_2,q_2} \pi(s,t)}{{}_{c_1}\partial_{p_2,q_2} t} &= \frac{\pi(s, p_2 t + (1-p_2)c_1) - \pi(s, q_2 t + (1-q_2)c_1)}{(p_2 - q_2)(t - c_1)}, \ t \neq c_1, \\
\frac{{}_{x_1,c_1}\partial^2_{p_1 p_2, q_1 q_2} \pi(s,t)}{{}_{x_1}\partial_{p_1,q_1} s \; {}_{c_1}\partial_{p_2,q_2} t} &= \frac{\pi(q_1 s + (1-q_1)x_1, q_2 t + (1-q_2)c_1)}{(p_1 - q_1)(p_2 - q_2)(s - x_1)(t - c_1)} \\
&\quad - \frac{\pi(q_1 s + (1-q_1)x_1, p_2 t + (1-p_2)c_1)}{(p_1 - q_1)(p_2 - q_2)(s - x_1)(t - c_1)} \\
&\quad - \frac{\pi(p_1 s + (1-p_1)x_1, q_2 t + (1-q_2)c_1)}{(p_1 - q_1)(p_2 - q_2)(s - x_1)(t - c_1)} \\
&\quad + \frac{\pi(p_1 s + (1-p_1)x_1, p_2 t + (1-p_2)c_1)}{(p_1 - q_1)(p_2 - q_2)(s - x_1)(t - c_1)}, \ s \neq x_1, \ t \neq c_1.
\end{aligned}$$

Definition 29 ([75])**.** *Let $\pi : \Delta = [x_1, x_2] \times [c_1, d_1] \subset \mathbb{R}^2 \to \mathbb{R}$ be a continuous function in each variable and $0 < q_i < p_i \leq 1, i = 1, 2$. Then the definite $(p_1 p_2, q_1 q_2)$-integral on $[x_1, x_2] \times [c_1, d_1]$ is defined as:*

$$\begin{aligned}
&\int_{c_1}^{t} \int_{x_1}^{s} \pi(u, w) \; {}_{x_1}d_{p_1,q_1} u \; {}_{c_1}d_{p_2,q_2} w = (p_1 - q_1)(p_2 - q_2)(s - x_1)(t - c_1) \\
&\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{q_1^n}{p_1^{n+1}} \frac{q_2^m}{p_2^{m+1}} \pi\Big(\frac{q_1^n}{p_1^{n+1}} s + \Big(1 - \frac{q_1^n}{p_1^{n+1}}\Big) x_1, \frac{q_2^m}{p_2^{m+1}} t + \Big(1 - \frac{q_2^m}{p_2^{m+1}}\Big) c_1 \Big)
\end{aligned}$$

for $(s, t) \in [x_1, x_2] \times [c_1, d_1]$.

Theorem 99 ([75])**.** *Let $\pi : \Lambda \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function such that $(p_1 p_2, q_1 q_2)$-derivatives exist on Λ° with $0 < q_i < p_1 \leq 1$ $i = 1, 2$. If $\frac{{}_{x_1,c_1}\partial^2_{p_1 p_2, q_1, q_2} \pi}{{}_{x_1}\partial_{p_1,q_1} u \; {}_{c_1}\partial_{p_2,q_2} w}$ is continuous and integrable on $[x_1, x_2] \times [c_1, d_1] \subseteq \Lambda^\circ$ and $\left| \frac{{}_{x_1,c_1}\partial^2_{p_1 p_2, q_1, q_2} \pi}{{}_{x_1}\partial_{p_1,q_1} u \; {}_{c_1}\partial_{p_2,q_2} w} \right|^r$ is a convex function on coordinates on $[x_1, x_2] \times [c_1, d_1]$ for $r \geq 1$, then the following inequality holds:*

$$\begin{aligned}
&\Big| \frac{q_1 q_2 \pi(x_1, c_1) + p_2 q_1 \pi(x_1, d_1) + p_1 q_2 \pi(x_2, c_1) + p_1 p_2 \pi(x_2, d_1)}{(p_1 + q_1)(p_2 + q_2)} \\
&- \frac{q_2}{p_1 (p_2 + q_2)(x_2 - x_1)} \int_{x_1}^{p_1 x_2 + (1-p_1)x_1} \pi(u, c_1) {}_{x_1}d_{p_1,q_1} u \\
&- \frac{p_2}{p_1 (p_2 + q_2)(x_2 - x_1)} \int_{x_1}^{p_1 x_2 + (1-p_1)x_1} \pi(u, d_1) {}_{x_1}d_{p_1,q_1} u \\
&- \frac{q_1}{p_2 (p_1 + q_1)(d_1 - c_1)} \int_{c_1}^{p_2 d_1 + (1-p_2)c_1} \pi(x_1, w) {}_{c_1}d_{p_2,q_2} w \\
&- \frac{p_1}{p_2 (p_1 + q_1)(d_1 - c_1)} \int_{c_1}^{p_2 d_1 + (1-p_2)c_1} \pi(x_2, w) {}_{c_1}d_{p_2,q_2} w
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{p_1(x_2-x_1)(d_1-c_1)}\int_{x_1}^{p_1x_2+(1-p_1)x_1}\int_{c_1}^{p_2d_1+(1-p_2)c_1}\pi(u,w)\ {}_{c_1}d_{p_2,q_2}w\ {}_{x_1}d_{p_1,q_1}u\Big| \\
\leq\ &\frac{q_1q_2(x_2-x_1)(d_1-c_1)}{(p_1+q_1)(p_2+q_2)}\left(\phi_{p_1,q_1}\phi_{p_2,q_2}\right)^{1-\frac{1}{r}} \\
&\times\left(\psi_{p_1,q_1}\psi_{p_2,q_2}\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1q_2}\pi(x_1,c_1)}{{}_{x_1}\partial_{p_1,q_1}u\ {}_{c_1}\partial_{p_2,q_2}w}\right|^r+\psi_{p+1,q_1}Q_{p_2,q_2}\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1q_2}\pi(x_1,d_1)}{{}_{x_1}\partial_{p_1,q_1}u\ {}_{c_1}\partial_{p_2,q_2}w}\right|^r\right. \\
&\left.+Q_{p_1,q_1}\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1q_2}\pi(x_2,c_1)}{{}_{x_1}\partial_{p_1,q_1}u\ {}_{c_1}\partial_{p_2,q_2}w}\right|^r+Q_{p_1,q_1}Q_{p_2,q_2}\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1q_2}\pi(x_2,d_1)}{{}_{x_1}\partial_{p_1,q_1}t\ {}_{c_1}\partial_{p_2,q_2}w}\right|\right)^{\frac{1}{r}},
\end{aligned}
$$

where

$$
\begin{aligned}
\phi_{p,q} &= \frac{2(p+q-1)}{(p+q)^2}, \\
\psi_{p,q} &= \frac{q[(p^3-2+2p)+(2p^2+2)q+pq^2]+2p^2-2p}{(p+q)^3(p^2+pq+q^2)}, \\
Q_{p,q} &= \frac{q[(5p^3-4p^2-2p+2)+(6p^2-4p-2)q+(5p-2)q^2+2q^3]}{(p+q)^3(p^2+pq+q^2)} \\
&\quad+\frac{2p^4-2p^3-2p^2+2p}{(p+q)^3(p^2+pq+q^2)}.
\end{aligned}
$$

Now, we present quantum integral inequalities for functions whose partial (p_1p_2,q_1q_2)-derivative is quasi-convex on coordinates.

Theorem 100 ([75])**.** *Let $\pi:\Lambda\subseteq\mathbb{R}^2\to\mathbb{R}$ be a function such that (p_1p_2,q_1q_2)-derivatives exist on Λ° with $0<q_i<p_1\leq 1$ $i=1,2$. If $\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1,q_2}\pi}{{}_{x_1}\partial_{p_1,q_1}u\ {}_{c_1}\partial_{p_2,q_2}w}$ is continuous and integrable on $[x_1,x_2]\times[c_1,d_1]\subseteq\Lambda^\circ$ and $\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1,q_2}\pi}{{}_{x_1}\partial_{p_1,q_1}u\ {}_{c_1}\partial_{p_2,q_2}w}\right|^r$ is quasi-convex on coordinates on $[x_1,x_2]\times[c_1,d_1]$ for $r\geq 1$, then the following inequality holds:*

$$
\begin{aligned}
&\Big|\frac{q_1q_2\pi(x_1,c_1)+p_2q_1\pi(x_1,d_1)+p_1q_2\pi(x_2,c_1)+p_1p_2\pi(x_2,d_1)}{(p_1+q_1)(p_2+q_2)} \\
&-\frac{q_2}{p_1(p_2+q_2)(x_2-x_1)}\int_{x_1}^{p_1x_2+(1-p_1)x_1}\pi(u,c_1)\,{}_{x_1}d_{p_1,q_1}u \\
&-\frac{p_2}{p_1(p_2+q_2)(x_2-x_1)}\int_{x_1}^{p_1x_2+(1-p_1)x_1}\pi(u,d_1)\,{}_{x_1}d_{p_1,q_1}u \\
&-\frac{q_1}{p_2(p_1+q_1)(d_1-c_1)}\int_{c_1}^{p_2d_1+(1-p_2)c_1}\pi(x_1,w)\,{}_{c_1}d_{p_2,q_2}w \\
&-\frac{p_1}{p_2(p_1+q_1)(d_1-c_1)}\int_{c_1}^{p_2d_1+(1-p_2)c_1}\pi(x_2,w)\,{}_{c_1}d_{p_2,q_2}w \\
&+\frac{1}{p_1(x_2-x_1)(d_1-c_1)}\int_{x_1}^{p_1x_2+(1-p_1)x_1}\int_{c_1}^{p_2d_1+(1-p_2)c_1}\pi(u,w)\ {}_{c_1}d_{p_2,q_2}w\ {}_{x_1}d_{p_1,q_1}u\Big| \\
\leq\ &\frac{q_1q_2(x_2-x_1)(d_1-c_1)}{(p_1+q_1)(p_2+q_2)}\left(\frac{4(p_1+q_1-1)(p_2+q_2-1)}{(p_1+q_1)^2(p_2+q_2)^2}\right) \\
&\times\sup\left\{\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1q_2}\pi(x_1,c_1)}{{}_{x_1}\partial_{p_1,q_1}u\ {}_{c_1}\partial_{p_2,q_2}w}\right|,\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1q_2}\pi(x_1,d_1)}{{}_{x_1}\partial_{p_1,q_1}u\ {}_{c_1}\partial_{p_2,q_2}w}\right|,\right. \\
&\left.\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1q_2}\pi(x_2,c_1)}{{}_{x_1}\partial_{p_1,q_1}u\ {}_{c_1}\partial_{p_2,q_2}w}\right|,\left|\frac{{}_{x_1,c_1}\partial^2_{p_1p_2,q_1q_2}\pi(x_2,d_1)}{{}_{x_1}\partial_{p_1,q_1}t\ {}_{c_1}\partial_{p_2,q_2}w}\right|\right\}.
\end{aligned}
$$

Assume that $I_1 = [x_2 - (x_2 - x_1)/p, x_2]$ and $I_2 = [x_2 - p(x_2 - x_1), x_2]$.

Theorem 101 ([76]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be $(p,q)^{x_2}$-differentiable on I_1 such that ${}^{x_2}D_{p,q}\pi$ is integrable and continuous on I_2 with $\gamma, \nu \in [0,1]$. Then*

$$\begin{aligned}
&(x_2 - x_1)\Big[\int_0^{\nu} (qt + \gamma\nu - \gamma)\ {}^{x_2}D_{p,q}\pi(tx_1 + (1-t)x_2)\ d_{p,q}t \\
&+ \int_{\nu}^{1} (qt + \gamma\nu - 1)\ {}^{x_2}D_{p,q}\pi(tx_1 + (1-t)x_2)\ d_{p,q}t\Big] \\
= \ & \frac{1}{p(x_2 - x_1)} \int_{px_1 + (1-p)x_2}^{x_2} \pi(x)\ {}^{x_2}d_{p,q}x - \gamma\Big[\nu\pi(x_1) + (1-\nu)\pi(x_2)\Big] \\
&-(1-\gamma)\pi(\nu x_1 + (1-\nu)x_2).
\end{aligned}$$

5. H-H Type Inequalities via *h*-Calculus

In this section we give first the definitions of h-derivative and h-integral and then some H-H type inequalities for convex and twice differentiable functions via h-calculus.

Definition 30 ([8]). *For a mapping $\pi : [x_1, x_2] \to \mathbb{R}$ the h-derivative of π at x is stated as*

$$D_h(x) = \frac{\pi(x+h) - \pi(x)}{h},$$

where $h \neq 0$.

Definition 31 ([8]). *For a mapping $\pi : [x_1, x_2] \to \mathbb{R}$ the definite h-integral of π is stated as*

$$\int_{x_1}^{x_2} \pi(x) d_h x = \begin{cases} h(\pi(x_1) + \pi(x_1 + h) + \cdots + \pi(x_2 - h)), & x_1 < x_2, \\ 0, & x_1 = x_2, \\ -h(\pi(x_2) + \pi(x_2 + h) + \cdots + \pi(x_1 - h)), & x_1 > x_2, \end{cases}$$

where $h \neq 0$ and $x_2 - x_1 \in h\mathbb{Z}$.

Theorem 102 ([77]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be convex. Then*

$$\pi\Big(\frac{x_1 + x_2 - h}{2}\Big) \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi(t)\ d_h t \leq \frac{\pi(x_1) + q\pi(x_2)}{2} + \frac{h}{x_2 - x_1}\Big(\frac{\pi(x_1) - \pi(x_2)}{2}\Big).$$

Theorem 103 ([77]). *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be a continuous twice differentiable on (x_1, x_2). Then*

$$\begin{aligned}
&\pi\Big(\frac{x_1 + x_2 - h}{2}\Big) - \frac{m}{2}\Big(\frac{x_1 + x_2 - h}{2}\Big)^2 + \frac{m}{2}\Big(x_1(x_2 - h) + \frac{(x_2 - x_1 - h)(2(x_2 - x_1) - h)}{6}\Big) \\
\leq \ & \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \pi(t)\ d_h t \\
\leq \ & \frac{\pi(x_1) + \pi(x_2)}{2} + \frac{h}{2(x_2 - x_1)}(\pi(x_1) - \pi(x_2)) - m\frac{x_1^2 + x_2^2}{4} + hm\Big(\frac{x_1 + x_2}{4}\Big) \\
&+ \frac{m}{2}\Big(x_1(x_2 - h) + \frac{(x_2 - x_1 - h)(2(x_2 - x_1) - h)}{6}\Big),
\end{aligned}$$

where $m = \inf_{x \in (x_1, x_2)} \pi''(x)$.

The inequalities in the above theorem are reversed if we replace m by $M = \sup_{x \in (x_1, x_2)} \pi''(x)$.

6. H-H Type Inequalities via $q-h$-Calculus

Here, we add the definitions of the $q-h$-derivative and $q-h$-integral.

Definition 32 ([78])**.** *Assume that the real-valued function π is continuous. Then the $q-h$-derivative of π is stated by:*

$$C_hD_q\pi(x) = \frac{{}_hd_q\pi(x)}{{}_hd_qx} = \frac{\pi(q(x+h)) - \pi(x)}{(q-1)x+qh}, \quad x \neq \frac{qh}{1-q} = w,$$

where $h \in \mathbb{R}$, $0<q<1$ *and* $C_hD_q\pi(w) = \lim_{x\to w} C_hD_q\pi(x)$.

Definition 33 ([78])**.** *Assume that $0<q<1$ and function $\pi : I = [x_1, x_2] \to \mathbb{R}$ is continuous. Then $I^{x_1+}_{q-h}\pi$ and $I^{x_2-}_{q-h}\pi$ are defined as follows:*

$$I^{x_1+}_{q-h}\pi(x) = \int_{x_1}^{x} \pi(t)_hd_qt = ((1-q)(x-x_1)+qh)\sum_{n=0}^{\infty} q^n\pi(q^nx_1 + (1-q^n)x + nq^nh), \;\; x > x_1,$$

and

$$I^{x_2-}_{q-h}\pi(x) = \int_{x}^{x_2} \pi(t)_hd_qt = ((1-q)(x_2-x)+qh)\sum_{n=0}^{\infty} q^n\pi(q^nx + (1-q^n)x_2 + nq^nh), \;\; x < x_2.$$

Definition 34 ([79])**.** *Assume that a real-valued function π is continuous and $q \in (0,1)$. Then the q_{x_1-h}-derivative of π at $x \in [x_1, x_2]$ is stated by:*

$$C_hD_q^{x_1}\pi(x) = \frac{\pi(x) - \pi(qx + (1-q)x_1 + qh)}{(1-q)(x-x_1) - qh}, \quad x \neq \frac{x_1(1-q)+qh}{1-q} := x_0.$$

Analogously, let the left q_{x_2-h}-derivative of π at $x \in [x_1, x_2]$ be

$$C_hD_q^{x_2}\pi(x) = \frac{\pi(x) - \pi(qx + (1-q)x_2 + qh)}{(1-q)(x-b) - qh}, \quad x \neq \frac{x_2(1-q)+qh}{1-q} := y_0.$$

In addition $C_hD_q^{x_1}\pi(x_0) = \lim_{x\to x_0} C_hD_q^{x_1}\pi(x)$ *and* $C_hD_q^{x_2}\pi(y_0) = \lim_{x\to y_0} C_hD_q^{x_2}\pi(x)$.

In the next we present $q-h$-H-H type inequalities.

Theorem 104 ([79])**.** *Let function $\pi : [x_1, x_2] \to \mathbb{R}$ be differentiable and convex on (x_1, x_2) and $0 \le x_1 < x_2$. If $\sum_{k=0}^{\infty} kq^{2k} = S$, then*

$$\begin{aligned}
&\pi\Big(\frac{qx_1+x_2}{1+q}\Big)\frac{(1-q)(x_2-x_1)+qh}{1-q} + \pi'\Big(\frac{qx_1+x_2}{1+q}\Big)((1-q)(x_2-x_1)+qh)hS \\
\le\;& \int_{x_1}^{x_2} \pi(x)\; {}_hd_q^{x_1+}x \le ((1-q)(x_2-x_1)+qh)\Big(\frac{q\pi(x_1)+\pi(x_2)}{1-q^2} + \frac{\pi(x_2)-\pi(x_1)}{x_2-x_1}hS\Big).
\end{aligned}$$

Theorem 105 ([79])**.** *Assume that π is as in Theorem 104. If $\sum_{k=0}^{\infty} kq^{2k} = S$, then*

$$\begin{aligned}
&\pi\Big(\frac{x_1+qx_2}{1+q}\Big)\frac{(1-q)(x_2-x_1)+qh}{1-q} + \pi'\Big(\frac{x_1+qx_2}{1+q}\Big)((1-q)(x_2-x_1)+qh)hS \\
\le\;& \int_{x_1}^{x_2} \pi(x)\; {}_hd_q^{x_2-}x \le ((1-q)(x_2-x_1)+qh)\Big(\frac{\pi(x_1)+q\pi(x_2)}{1-q^2} + \frac{\pi(x_2)-\pi(x_1)}{x_2-x_1}hS\Big).
\end{aligned}$$

Theorem 106 ([79]). *Assume that π is as in Theorem 104. If $\sum_{k=0}^{\infty} kq^{2k} = S$, then*

$$\begin{aligned}
&\pi\Big(\frac{x_1+x_2}{2}\Big)\frac{(1-q)(x_2-x_1)+qh}{1-q} \\
&+\pi'\Big(\frac{x_1+x_2}{2}\Big)((1-q)(x_2-x_1)+qh)\Big(hS+\frac{x_2-x_1}{2(1+q)}\Big) \\
\leq\ & \int_{x_1}^{x_2}\pi(x)\ {}_h d_q^{x_1+}x \leq ((1-q)(x_2-x_1)+qh)\Big(\frac{q\pi(x_1)+\pi(x_2)}{1-q^2}+\frac{\pi(x_2)-\pi(x_1)}{x_2-x_1}hS\Big).
\end{aligned}$$

Theorem 107 ([79]). *Assume that π is as in Theorem 104. If $\sum_{k=0}^{\infty} kq^{2k} = S$, then*

$$\begin{aligned}
&\pi\Big(\frac{x_1+x_2}{2}\Big)\frac{(1-q)(x_2-x_1)+qh}{1-q} \\
&+\pi'\Big(\frac{x_1+x_2}{2}\Big)((1-q)(x_2-x_1)+qh)\Big(hS-\frac{x_2-x_1}{2(1+q)}\Big) \\
\leq\ & \int_{x_1}^{x_2}\pi(x)\ {}_h d_q^{x_2-}x \leq ((1-q)(x_2-x_1)+qh)\Big(\frac{\pi(x_1)+q\pi(x_2)}{1-q^2}+\frac{\pi(x_2)-\pi(x_1)}{x_2-x_1}hS\Big).
\end{aligned}$$

Definition 35 ([80]). *Let $h : I \to \mathbb{R}$ be a non-negative function. We say that $\pi : I \to \mathbb{R}$ is an h-convex function, if π is non-negative and for all $x_1, x_2 \in I, \theta \in (0,1)$ we have*

$$\pi(\theta x_1 + (1-\theta)x_2) \leq h(\theta)\pi(x_1) + h(1-\theta)\pi(x_2).$$

Definition 36 ([81]). *A function $\pi : [0, x_2] \to \mathbb{R}, x_2 > 0$, is m-convex, where $m \in [0,1]$, if*

$$\pi(\theta x + m(1-\theta)y) \leq \theta\pi(x_1) + m(1-\theta)\pi(y),$$

for all $x, y \in [0, x_2], \theta \in [0,1]$.

Theorem 108 ([82]). *Assume that $\pi : I \to \mathbb{R}$ is a convex function and $q \in (0,1)$. If $\sum_{k=0}^{\infty} kq^{2k} = S$, then for $x_1, x_2 \in I,\ x_1 < x_2$, we have*

$$\begin{aligned}
&\pi\Big(\frac{x_1+qx}{1+q}+(1-q)hS\Big)+\pi\Big(\frac{x+qx_2}{1+q}+(1-q)hS\Big) \\
\leq\ & \frac{1-q}{(1-q)(x-x_1)+qh}\int_{x_1}^{x}\pi(t)\ {}_h d_q t+\frac{1-q}{(1-q)(x_2-x)+qh}\int_{x}^{x_2}\pi(t)\ {}_h d_q t \\
\leq\ & \frac{[\pi(x_1)+q\pi(x_2)](x_2-x_1)+(1+q)[\pi(x_1)(x_2-x)+\pi(x_2)(x-x_1)]}{(1+q)(x_2-x_1)} \\
&+\frac{2[\pi(x_2)-\pi(x_1)]}{x_2-x_1}hS(1-q).
\end{aligned}$$

Theorem 109 ([82]). *Assume that the statement of this theorem is defined in Theorem 108, then*

$$\begin{aligned}
\pi\Big(\frac{x_1+qx}{1+q}+(1-q)hS\Big) &\leq \frac{1-q}{(1-q)(x_2-x_1)+qh}\int_{x_1}^{x_2}\pi(t)\ {}_h d_q t \\
&\leq \frac{\pi(x_1)+q\pi(x_2)}{(1+q)}+\frac{\pi(x_2)-\pi(x_1)}{x_2-x_1}hS(1-q).
\end{aligned}$$

Theorem 110 ([82]). *Assume that $x_1, x_2 \in I,\ x_1 < x_2$ and non-negative real-valued π is h-convex such that $h\Big(\frac{1}{2}\Big) \neq 0$ and $q \in (0,1)$.*

(i) *Assume that π is symmetric about* $\frac{x_1+x}{2}$, $x \in (x_1, x_2)$, *then*

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\pi\left(\frac{x_1+x}{2}\right) &\leq \frac{1-q}{(1-q)(x-x_1)+qh}\int_{x_1}^{x}\pi(t)\ {}_{h_1}d_qt \\
&\leq \pi(x)\int_0^1 h(t)_h d_qt + \pi(x_1)\int_0^1 h(1-t)\ {}_h d_qt,
\end{aligned}$$

where $h_1 = (x-x_1)h$.

(ii) *Assume that π is symmetric about* $\frac{x+x_2}{2}$, $x \in (x_1, x_2)$, *then*

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\pi\left(\frac{x+x_2}{2}\right) &\leq \frac{1-q}{(1-q)(x_2-x)+qh}\int_{x}^{x_2}\pi(t)\ {}_{h_2}d_qt \\
&\leq \pi(x_2)\int_0^1 h(t)_h d_qt + \pi(x)\int_0^1 h(1-t)_h d_qt,
\end{aligned}$$

where $h_2 = (x_2-x)h$.

Theorem 111 ([82]). *Assume that the function* $\pi : [0,b] \to \mathbb{R}_0^+$ *is m-convex. Additionally, suppose that* $x_1, x_2 \in [0,b]$, $x_1 < x_2$.

(i) *Assume that* $\pi\left(\frac{x_1+x-z}{m}\right) = \pi(z)$, $z \in (x_1, x)$, *then*

$$\begin{aligned}
\pi\left(\frac{x_1+x}{2}\right) &\leq \frac{(1-q)(1+m)}{2((1-q)(x-x_1)+qh_1)}\int_{x_1}^{x}\pi(t)\ {}_{h_1}d_qt \\
&\leq \frac{(1-q)+qh}{2(1-q)}\left(\pi(x)\left(\frac{q}{1+q}+(1-q)hS\right)\right. \\
&\quad \left.+m\pi\left(\frac{a}{m}\right)\left(\frac{1}{1+q}-(1-q)hS\right)\right),
\end{aligned}$$

where $q \in (0,1)$.

(ii) *Assume that* $\pi\left(\frac{x+x_2-z}{m}\right) = \pi(z)$, $z \in (x, x_2)$, *then*

$$\begin{aligned}
\pi\left(\frac{x+x_2}{2}\right) &\leq \frac{(1-q)(1+m)}{2((1-q)(x_2-x)+qh_2)}\int_{x}^{x_2}\pi(t)\ {}_{h_2}d_qt \\
&\leq \frac{(1-q)+qh}{2(1-q)}\left(\pi(x_2)\left(\frac{q}{1+q}+(1-q)hS\right)\right. \\
&\quad \left.+m\pi\left(\frac{x}{m}\right)\left(\frac{1}{1+q}-(1-q)hS\right)\right),
\end{aligned}$$

where $q \in (0,1)$.

7. Conclusions

Our objective in this paper was to provide a comprehensive and up to-date review on quantum H-H inequalities. We presented various results, including integral inequalities of the H-H type, using numerous families of convexity. Quantum H-H inequalities involving preinvex functions and Green functions were also presented. Finally, H-H type inequalities for (p,q)-calculus, h-calculus, and $q-h$-calculus were also included.

The practical as well as theoretical significance of the quantum H-H inequalities were taken into consideration when compiling this overview. We believe that the current review will provide a platform for scholars working on H-H inequalities to learn more about previous research on the subject before coming up with new findings.

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