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Random Solutions for Generalized Caputo Periodic and Non-Local Boundary Value Problems

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Abstract: In this article, we present some results on the existence and uniqueness of random solutions to a non-linear implicit fractional differential equation involving the generalized Caputo fractional derivative operator and supplemented with non-local and periodic boundary conditions. We make use of the fixed point theorems due to Banach and Krasnoselskii to derive the desired results. Examples illustrating the obtained results are also presented.

Keywords: generalized Caputo fractional derivative; fractional integral; existence; random solution; non-local condition; periodic condition; fixed point

MSC: 26A33; 34A08; 34B10; 34K05

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1. Introduction

Fractional differential equations are found to be of great interest in view of their utility in modeling and explaining natural phenomena occurring in biophysics, quantum mechanics, wave theory, polymers, continuum mechanics, etc. [1–3]. In fact, fractional order derivative operators have been successfully applied to generalize fundamental laws of nature, especially in the transport phenomena. For more details, we refer the reader to the works [4–13], and the references cited therein.

In [14], a non-linear coupled system involving both Caputo and Riemann–Liouville generalized fractional derivatives equipped with coupled integral boundary conditions was studied. One can find some existence results for the generalized Caputo fractional differential equations and inclusions with Steiltjes-type fractional integral boundary conditions in [15].

In [16], some properties of Caputo-type modification of the Erdélyi–Kober fractional derivative are provided by the authors. More information are available in [12,17]. In [18], the authors have presented several properties related to the generalized Caputo fractional differential equations involving retardation and anticipation. For integer-order differential equations with retardation and anticipation, for instance, see [19].

The values of the coefficients, parameters, and initial conditions in a differential equation are often expressed by the mean of the values acquired as a consequence of certain experimental determinations. As a result, physical constants and parameters may be thought of as random variables whose values are determined by a probability distribution or law. The same may be stated for coefficients and forcing functions, which can be random variables or random functions. We refer to publications [17,20,21] for results and further references on differential equations with random parameters.

In [22], Abd El-Salam studied the existence of at least one solution to the second-order boundary value problem of the form

$$\begin{cases} x''(\tau) = f(\tau, x(\tau), x'(\tau)), \text{ for } \tau \in (0, 2\pi), \\ x(0) = x(2\pi) \;\; and \;\; \sum_{j=1}^{m} \lambda_j x(\tau_j) = x_0, \end{cases}$$

where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = 2\pi$.

Inspired by the above-mentioned papers, and with the goal of extending previous results in mind, in this paper, we investigate the existence and uniqueness of random solutions for the following fractional boundary value problem

$${}^{C}D_{0^+}^{\alpha,\rho}(x(\tau,\delta) - \psi(\tau, x(\tau,\delta), \delta)) = f(\tau, x(\tau,\delta), {}^{C}D_{0^+}^{\alpha,\rho}x(\tau,\delta), \delta), \; \tau \in J := [0, 2\pi], \quad (1)$$

$$x(0,\delta) = x(2\pi,\delta) \;\; and \;\; \sum_{j=1}^{m} \lambda_j x(\tau_j, \delta) = d(\delta), \quad (2)$$

where $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_m < \tau_{m+1} = 2\pi$, $1 < \alpha \leq 2$, ${}^{C}D_{0^+}^{\alpha,\rho}$ is the generalized Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \times \Psi \to \mathbb{R}$ and $\psi : J \times \mathbb{R} \times \Psi \to \mathbb{R}$, are given functions, λ_j are real constants such that $\sum_{j=1}^{m} \lambda_j \neq 0$ and Ψ is the sample space in a probability space and δ is a random variable. For the sake of simplicity, we assume that $\psi(\tau_j, x(\tau_j, \delta), \delta) = 0$; $j = 0, 1, \ldots, m+1$.

The structure of this paper is as follows. Section 2 presents certain notations and preliminaries about generalized fractional derivatives used throughout this manuscript. In Section 3, we present two existence and uniqueness results for the problem (1) and (2) which rely on the Banach contraction mapping principle and Krasnoselskii's fixed point theorem. In Section 4, two examples are presented in support of the results obtained.

2. Preliminaries

First, we give the definitions and notations used in this paper. We denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions from J into $\mathbb{R}$ with the following norm

$$\|\chi\|_J = \sup\{|\chi(s)| : 0 \leq s \leq 2\pi\}.$$

By $B_{\mathbb{R}}$, we denote the σ-algebra of Borel subsets of $\mathbb{R}$. A mapping $\delta : \Psi \to \mathbb{R}$ is said to be measurable if $\delta^{-1}(\mathcal{G}) = \{\xi \in \Psi : \delta(\xi) \in \mathcal{G}\} \subset \mathcal{A}$ for any $\mathcal{G} \in B_{\mathbb{R}}$, where $\mathcal{A}$ is a σ-algebra defined in Ψ.

Consider the space $X_b^p(0, 2\pi)$, $(1 \leq p, b \in \mathbb{R})$ of those complex-valued Lebesgue measurable functions χ on J for which $\|\chi\|_{X_b^p} < \infty$, with the norm:

$$\|\chi\|_{X_b^p} = \left(\int_0^{2\pi} |\tau^b \chi(\tau)|^p \frac{d\tau}{\tau} \right)^{\frac{1}{p}}, \; (1 \leq p < \infty, b \in \mathbb{R}).$$

Definition 1 (Generalized Riemann–Liouville integral [23]). *Let $v \in \mathbb{R}, b \in \mathbb{R}$ and $\tilde{f} \in X_b^p(0, 2\pi)$, the generalized RL fractional integral of order v is given by*

$$({}^{\rho}I_{0^+}^{v}\tilde{f})(\tau) = \frac{\rho^{1-v}}{\Gamma(v)} \int_0^{\tau} s^{\rho-1}(\tau^{\rho} - s^{\rho})^{v-1}\tilde{f}(s)ds, \qquad \tau > 0, \rho > 0 \quad (3)$$

where the Euler gamma function $\Gamma(\cdot)$ is given by

$$\Gamma(v) = \int_0^{\infty} s^{v-1}e^{-s}ds, \; v > 0.$$

Definition 2 ([24])**.** *Let $\tau > 0$. The generalized fractional derivative is given by*

$$\begin{aligned} {}^{\rho}D^{v}_{0^+}\psi(\tau) &= \frac{1}{\Gamma(n-v)}\left(\tau^{1-\rho}\frac{d}{d\tau}\right)^{n}\int_0^{\tau}\frac{s^{\rho-1}\rho^{1-n+v}}{(\tau^{\rho}-s^{\rho})^{1-n+v}}\psi(s)ds \\ &= \tilde{\delta}^{n}_{\rho}({}^{\rho}I^{n-v}_{0^+}\psi)(\tau), \end{aligned} \tag{4}$$

where $\tilde{\delta}^{n}_{\rho} = \left(\tau^{1-\rho}\frac{d}{d\tau}\right)^{n}$.

Definition 3 ([16,24])**.** *The Caputo-type generalized fractional derivative ${}^{\rho}_{c}D^{v}_{0^+}$ is defined by*

$$({}^{\rho}_{c}D^{v}_{0^+}\tilde{f})(\tau) = \left({}^{\rho}D^{v}_{0^+}\left[\tilde{f}(\tau) - \sum_{j=0}^{n-1}\frac{\tilde{f}^{(j)}(a)}{j!}s^{j}\right]\right). \tag{5}$$

Lemma 1 ([24])**.** *Let $v, \rho \in \mathbb{R}^{+}$, then*

$$({}^{\rho}I^{v}_{0^+}\, {}^{C}D^{v,\rho}_{0^+}\tilde{f})(\tau) = \tilde{f}(\tau) - \sum_{j=0}^{n-1}\iota_j\left(\frac{\tau^{\rho}}{\rho}\right)^{j}, \tag{6}$$

for some $\iota_j \in \mathbb{R}$, $n = [v] + 1$.

Lemma 2 ([25])**.** *If $x > n$, then we have*

$$\left[{}^{\rho}I^{v}_{0^+}\left(\frac{\tau^{\rho}}{\rho}\right)^{\epsilon-1}\right](x) = \frac{\Gamma(\epsilon)}{\Gamma(\epsilon+v)}\left(\frac{x^{\rho}}{\rho}\right)^{v+\epsilon-1}. \tag{7}$$

Definition 4. *A mapping $N : \Psi \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if for any $\mathcal{G} \in B_{\mathbb{R}}$, one has*

$$N^{-1}(\mathcal{G}) = \{(\xi, x) \in \Psi \times \mathbb{R} : N(\xi, x) \in \mathcal{G}\} \subset \mathcal{A} \times B_{\mathbb{R}},$$

where $\mathcal{A} \times B_{\mathbb{R}}$ is the product of the σ-algebras $\mathcal{A}$ defined in Ψ and $B_{\mathbb{R}}$.

Definition 5. *A function $N : \Psi \times \mathbb{R} \to \mathbb{R}$ is called jointly measurable if $N(\cdot, x)$ is measurable for all $x \in \mathbb{R}$ and $N(\xi, \cdot)$ is continuous for all $\xi \in \Psi$.*

Then, the map $\tilde{\kappa} : \Psi \times \mathbb{R} \to \mathbb{R}$ is called a random operator if $\tilde{\kappa}(\delta, x)$ is measurable in δ for all $x \in \mathbb{R}$ and it is written as $\tilde{\kappa}(\delta)x = \tilde{\kappa}(\delta, x)$. In this situation, $\tilde{\kappa}(\delta)$ is a random operator on $\mathbb{R}$. This operator is called continuous (resp. compact, totally bounded and completely continuous) if $\tilde{\kappa}(\delta, x)$ is continuous (resp. compact, totally bounded and completely continuous) in x for all $\delta \in \Psi$; (see [26] for more details).

Definition 6 ([27])**.** *Let $\mathcal{D}(X)$ be the family of all non-empty subsets of X and F be a mapping from Ψ into $\mathcal{D}(X)$. A mapping $\tilde{\kappa} : \{(\delta, y) : \delta \in \Psi,\ y \in F(\delta)\} \to X$ is called random operator with stochastic domain F if F is measurable (i.e., for all closed $\mathcal{T}_1 \subset X$, $\{\delta \in \Psi, F(\delta) \cap \mathcal{T}_1 \neq \emptyset\}$ is measurable) and for all open $D \subset X$ and all $y \in X$, $\{\delta \in \Psi : y \in F(\delta), \tilde{\kappa}(\delta, y) \in D\}$ is measurable. $\tilde{\kappa}$ will be called continuous if every $\tilde{\kappa}(\delta)$ is continuous. For a random operator $\tilde{\kappa}$, a mapping $y : \Psi \to X$ is called random (stochastic) fixed point of $\tilde{\kappa}$ if for almost all $\delta \in \Psi$, $y(\delta) \in F(\delta)$ and $\tilde{\kappa}(\delta)y(\delta) = y(\delta)$ and for all open $D \subset X$, $\{\delta \in \Psi : y(\delta) \in D\}$ is measurable.*

Definition 7. *A function $\gamma : J \times \mathbb{R} \times \Psi \to \mathbb{R}$ is called random Carathéodory if*

(i) *The function $(\chi, \xi) \to \gamma(\chi, x, \xi)$ is jointly measurable for each $x \in \mathbb{R}$,*
(ii) *The function $x \to \gamma(\chi, x, \xi)$ is continuous for almost each $\chi \in J$ and $\xi \in \Psi$.*

3. Existence of Solutions

Let us begin by defining what we mean by a random solution of the problem (1) and (2).

Definition 8. *A random solution of problem (1) and (2) is a measurable function $x(\cdot,\delta) \in C(J,\mathbb{R})$ which satisfies the Equation (1) and the conditions (2).*

Lemma 3. *Let $1 < \alpha \le 2$ and $\kappa, \xi : J \times \Psi \to \mathbb{R}_+$ be measurable functions, such that $\xi(\tau_j,\delta) = 0$; $j = 0, 1, \dots, m+1$. Then, the linear problem*

$$ {}^{C}D^{\alpha,\rho}_{0^+}(x(\tau,\delta) - \xi(\tau,\delta)) = \kappa(\tau,\delta), \textit{ for a.e. } \tau \in J,\ \delta \in \Psi, \tag{8} $$

$$ x(0,\delta) = x(2\pi,\delta) \ \textit{ and } \ \sum_{j=1}^{m} \lambda_j x(\tau_j,\delta) = d(\delta) \tag{9} $$

has a random solution given by

$$ \begin{aligned} x(\tau,\delta) &= \xi(\tau,\delta) + \frac{d(\delta)}{\sum_{j=1}^{m}\lambda_j} \\ &+ \left[\frac{\sum_{j=1}^{m}\lambda_j\tau_j^{\rho}}{\sum_{j=1}^{m}\lambda_j} - \tau^{\rho}\right]\frac{1}{(2\pi)^{\rho}\Gamma(\alpha)}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds \\ &- \frac{1}{\Gamma(\alpha)\sum_{j=1}^{m}\lambda_j}\sum_{j=1}^{m}\lambda_j\int_0^{\tau_j}\left(\frac{\tau_j^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^{\tau}\left(\frac{\tau^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds. \end{aligned} \tag{10} $$

Proof. Applying Lemma 1 to Equation (8), we obtain

$$ x(\tau,\delta) - \xi(\tau,\delta) = {}^{\rho}I^{\alpha}_{0^+}\kappa(\tau,\delta) + \iota_0 + \iota_1\left(\frac{\tau^{\rho}}{\rho}\right), \tag{11} $$

where ι_1 and $\iota_2 \in \mathbb{R}$. Since $\xi(\tau_j,\delta) = 0$; $j = 0, 1, \dots, m+1$, then

$$ \iota_0 = x(0,\delta) = x(2\pi,\delta) = \iota_0 + \iota_1\frac{(2\pi)^{\rho}}{\rho} + \frac{1}{\Gamma(\alpha)}\int_a^{2\pi}\left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds $$

and

$$ d(\delta) = \sum_{j=1}^{m}\lambda_j x(\tau_j,\delta) = \iota_0\sum_{j=1}^{m}\lambda_j + \iota_1\sum_{j=1}^{m}\lambda_j\frac{\tau_j^{\rho}}{\rho} + \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{m}\int_a^{\tau_j}\left(\frac{\tau_j^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds. $$

Therefore, we have

$$ \iota_1 = \frac{-\rho}{(2\pi)^{\rho}\Gamma(\alpha)}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds, $$

$$ \begin{aligned} \iota_0 &= \frac{d(\delta)}{\sum_{j=1}^{m}\lambda_j} + \frac{\sum_{j=1}^{m}\lambda_j\tau_j^{\rho}}{(2\pi)^{\rho}\Gamma(\alpha)\sum_{j=1}^{m}\lambda_j}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds \\ &- \frac{1}{\Gamma(\alpha)\sum_{j=1}^{m}\lambda_j}\sum_{j=1}^{m}\lambda_j\int_0^{\tau_j}\left(\frac{\tau_j^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds. \end{aligned} $$

Substituting the values of ι_0 and ι_1 in (11) leads to the Equation (10). $\square$

Lemma 4. *Let $f : J \times \mathbb{R} \times \mathbb{R} \times \Psi \longrightarrow \mathbb{R}$ be a random Carathéodory function. A function $x(\cdot,\delta) \in C(J,\mathbb{R})$ is a random solution of the non-local and periodic problems (1) and (2) if, and only if, x satisfies the integral equation*

$$\begin{aligned} x(\tau,\delta) &= \psi(\tau,x(\tau,\delta),\delta) + \frac{d(\delta)}{\sum_{j=1}^m \lambda_j} \\ &+ \left[\frac{\sum_{j=1}^m \lambda_j \tau_j^\rho}{\sum_{j=1}^m \lambda_j} - \tau^\rho\right] \frac{1}{(2\pi)^\rho \Gamma(\alpha)} \int_0^{2\pi} \left(\frac{(2\pi)^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds \\ &- \frac{1}{\Gamma(\alpha)\sum_{j=1}^m \lambda_j} \sum_{j=1}^m \lambda_j \int_0^{\tau_j} \left(\frac{\tau_j^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^\tau \left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds, \end{aligned}$$

where $\kappa \in C(J,\mathbb{R})$ satisfies the functional equation

$$\kappa(\tau,\delta) = f(\tau,x(\tau,\delta),\kappa(\tau,\delta),\delta). \tag{12}$$

The hypotheses

Hypothesis 1. *The functions $f : J \times \mathbb{R} \times \mathbb{R} \times \Psi \longrightarrow \mathbb{R}$ and $\psi : J \times \mathbb{R} \times \Psi \longrightarrow \mathbb{R}$ are random Carathéodory.*

Hypothesis 2. *There exist measurable and essentially bounded functions $p,q,b : J \to L^\infty(\Psi,\mathbb{R}_+)$, such that*

$$|f(\tau,y_1,v_1,\delta) - f(\tau,y_2,v_2,\delta)| \le p(\tau,\delta)|y_1 - y_2| + q(\tau,\delta)|v_1 - v_2|,$$

and

$$|\psi(\tau,y_1,\delta) - \psi(\tau,y_2,\delta)| \le b(\tau,\delta)|y_1 - y_2|,$$

for $\tau \in J$, $\delta \in \Psi$ and each $y_i, v_i \in \mathbb{R}$; $i = 1,2$, with

$$p(\delta) = ess\sup_{\tau\in J}|p(\tau,\delta)|, \;\; q(\delta) = ess\sup_{\tau\in J}|q(\tau,\delta)| < 1,$$

and

$$b(\delta) = ess\sup_{\tau\in J}|b(\tau,\delta)|.$$

Set

$$d^* = ess\sup_{\delta\in\Psi}|d(\delta)|.$$

Remark 1. *For the definition of essential supremum (ess sup), see Definition 15.23 in the book [28].*

Now we state and prove our existence result for problem (1) and (2) by applying the Banach contraction mapping principle [29].

Theorem 1. *Assume that $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ hold. If*

$$b(\delta)+\left(\frac{\left|\sum_{j=1}^{m}\lambda_j\tau_j^{\rho}\right|}{\left|\sum_{j=1}^{m}\lambda_j\right|}+2(2\pi)^{\rho}\right)\frac{p(\delta)(2\pi)^{\rho(\alpha-1)}}{(1-q(\delta))\rho^{\alpha}\Gamma(\alpha+1)}+\frac{p(\delta)\sum_{j=1}^{m}\left|\lambda_j\left(\frac{\tau_j^{\rho}}{\rho}\right)^{\alpha}\right|}{(1-q(\delta))\Gamma(\alpha+1)\left|\sum_{j=1}^{m}\lambda_j\right|}<1, \tag{13}$$

then the problem (1) and (2) have a unique solution.

Proof. Let the operator $\mathcal{S}: C(J,\mathbb{R})\times\Psi\longmapsto C(J,\mathbb{R})$ be defined by

$$\begin{aligned}(\mathcal{S}x)(\tau,\delta)&=\frac{d(\delta)}{\sum_{j=1}^{m}\lambda_j}+\left[\frac{\sum_{j=1}^{m}\lambda_j\tau_j^{\rho}}{\sum_{j=1}^{m}\lambda_j}-\tau^{\rho}\right]\frac{1}{(2\pi)^{\rho}\Gamma(\alpha)}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\kappa(s,\delta)ds\\&+\psi(\tau,x(\tau,\delta),\delta)-\frac{1}{\Gamma(\alpha)\sum_{j=1}^{m}\lambda_j}\sum_{j=1}^{m}\lambda_j\int_0^{\tau_j}\left(\frac{\tau_j^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\kappa(s,\delta)ds\\&+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\kappa(s,\delta)ds,\end{aligned} \tag{14}$$

where κ satisfies (12).

According to Lemma 4, the fixed points of $\mathcal{S}$ are random solutions to problem (1) and (2).

Let $x_1(\cdot,\delta)$ and $x_2(\cdot,\delta)\in\Psi$. Then, for $\tau\in J$, we have

$$\begin{aligned}&|(\mathcal{S}x_1)(\tau,\delta)-(\mathcal{S}x_2)(\tau,\delta)|\le|\psi(\tau,x_1(\tau,\delta),\delta)-\psi(\tau,x_2(\tau,\delta),\delta)|\\&+\left(\frac{\left|\sum_{j=1}^{m}\lambda_j\tau_j^{\rho}\right|}{\left|\sum_{j=1}^{m}\lambda_j\right|}+\tau^{\rho}\right)\frac{1}{(2\pi)^{\rho}\Gamma(\alpha)}\int_0^{2\pi}\left(\frac{(2\pi)^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}|\kappa_{x_1}(s,\delta)-\kappa_{x_2}(s,\delta)|ds\\&+\frac{1}{\Gamma(\alpha)\left|\sum_{j=1}^{m}\lambda_j\right|}\sum_{j=1}^{m}|\lambda_j|\int_0^{\tau_j}\left(\frac{\tau_j^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}|\kappa_{x_1}(s,\delta)-\kappa_{x_2}(s,\delta)|ds\\&+\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\left(\frac{\tau^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}|\kappa_{x_1}(s,\delta)-\kappa_{x_2}(s,\delta)|ds.\end{aligned} \tag{15}$$

By $(\mathcal{H}_2)$, we have

$$\begin{aligned}|\kappa_{x_1}(\tau,\delta)-\kappa_{x_2}(\tau,\delta)|&=|f(\tau,x_1(\tau,\delta),\kappa_{x_1}(\tau,\delta),\delta)-f(\tau,x_2(\tau,\delta),\kappa_{x_2}(\tau,\delta),\delta)|\\&\le p(\tau,\delta)|x_1(\tau,\delta)-x_2(\tau,\delta)|+q(\tau,\delta)|\kappa_{x_1}(\tau,\delta)-\kappa_{x_2}(\tau,\delta)|\\&\le p(\delta)|x_1(\tau,\delta)-x_2(\tau,\delta)|+q(\delta)|\kappa_{x_1}(\tau,\delta)-\kappa_{x_2}(\tau,\delta)|.\end{aligned}$$

Then

$$|\kappa_{x_1}(\tau,\delta)-\kappa_{x_2}(\tau,\delta)|\le\frac{p(\delta)}{1-q(\delta)}|x_1(\tau,\delta)-x_2(\tau,\delta)|.$$

Therefore, for each $\tau\in J$, we have

$$\begin{aligned}&|(\mathcal{S}x_1)(\tau,\delta)-(\mathcal{S}x_2)(\tau,\delta)|\\&\le b(\tau,\delta)|x_1(\tau,\delta)-x_2(\tau,\delta)|+\left(\frac{\left|\sum_{j=1}^{m}\lambda_j\tau_j^{\rho}\right|}{\left|\sum_{j=1}^{m}\lambda_j\right|}+\tau^{\rho}\right)\end{aligned}$$

$$
\begin{aligned}
&\times \frac{p(\delta)}{(1-q(\delta))(2\pi)^\rho\Gamma(\alpha)}\int_0^{2\pi}\left(\frac{(2\pi)^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}|x_1(s,\delta)-x_2(s,\delta)|ds \\
&+ \frac{p(\delta)}{(1-q(\delta))\Gamma(\alpha)\left|\sum_{j=1}^m \lambda_j\right|}\sum_{j=1}^m |\lambda_j| \int_0^{\tau_j}\left(\frac{\tau_j^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}|x_1(s,\delta)-x_2(s,\delta)|ds \\
&+ \frac{p(\delta)}{(1-q(\delta))\Gamma(\alpha)}\int_0^{\tau}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}|x_1(s,\delta)-x_2(s,\delta)|ds \\
\le\ & b(\tau)\|x_1(\cdot,\delta)-x_2(\cdot,\delta)\|_J + \|x_1(\cdot,\delta)-x_2(\cdot,\delta)\|_J \left(\frac{\left|\sum_{j=1}^m \lambda_j \tau_j^\rho\right|}{\left|\sum_{j=1}^m \lambda_j\right|} + \tau^\rho\right) \\
&\times \frac{p(\delta)}{(1-q(\delta))(2\pi)^\rho\Gamma(\alpha)}\int_0^{2\pi}\left(\frac{(2\pi)^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}ds \\
&+ \|x_1(\cdot,\delta)-x_2(\cdot,\delta)\|_J \frac{p(\delta)}{(1-q(\delta))\Gamma(\alpha)\left|\sum_{j=1}^m \lambda_j\right|}\sum_{j=1}^m |\lambda_j| \int_0^{\tau_j}\left(\frac{\tau_j^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}ds \\
&+ \|x_1(\cdot,\delta)-x_2(\cdot,\delta)\|_J \frac{p(\delta)}{(1-q(\delta))\Gamma(\alpha)}\int_0^{\tau}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}ds.
\end{aligned}
$$

Since

$$
\begin{aligned}
\frac{1}{\Gamma(\alpha)}\int_0^{\tau}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}ds &= \frac{1}{\alpha\Gamma(\alpha)}\left(\frac{\tau^\rho - s^\rho}{\rho}\right)^{\alpha}\Bigg|_{s=0}^{s=\tau} \\
&= \frac{1}{\Gamma(\alpha+1)}\left(\frac{\tau^\rho}{\rho}\right)^{\alpha},
\end{aligned}
$$

and $\tau \le 2\pi$, then we obtain

$$
\begin{aligned}
&|(\mathcal{S}x_1)(\tau,\delta) - (\mathcal{S}x_2)(\tau,\delta)| \\
&\le \left[b(\delta) + \left(\frac{\left|\sum_{j=1}^m \lambda_j \tau_j^\rho\right|}{\left|\sum_{j=1}^m \lambda_j\right|} + 2(2\pi)^\rho\right)\frac{p(\delta)(2\pi)^{\rho(\alpha-1)}}{(1-q(\delta))\rho^\alpha\Gamma(\alpha+1)} \right.\\
&\quad \left. + \frac{p(\delta)\sum_{j=1}^m \left|\lambda_j \left(\frac{\tau_j^\rho}{\rho}\right)^\alpha\right|}{(1-q(\delta))\Gamma(\alpha+1)\left|\sum_{j=1}^m \lambda_j\right|}\right] \|x_1(\cdot,\delta)-x_2(\cdot,\delta)\|_J.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|(\mathcal{S}x_1)(\cdot,\delta) - (\mathcal{S}x_2)(\cdot,\delta)\|_J \\
\le\ & \left[b(\delta) + \left(\frac{\left|\sum_{j=1}^m \lambda_j \tau_j^\rho\right|}{\left|\sum_{j=1}^m \lambda_j\right|} + 2(2\pi)^\rho\right)\frac{p(\delta)(2\pi)^{\rho(\alpha-1)}}{(1-q(\delta))\rho^\alpha\Gamma(\alpha+1)} \right.\\
+\ & \left. \frac{p(\delta)\sum_{j=1}^m \left|\lambda_j \left(\frac{\tau_j^\rho}{\rho}\right)^\alpha\right|}{(1-q(\delta))\Gamma(\alpha+1)\left|\sum_{j=1}^m \lambda_j\right|}\right] \|x_1(\cdot,\delta)-x_2(\cdot,\delta)\|_J.
\end{aligned}
$$

Hence, by the Banach contraction principle, $\mathcal{S}$ has a unique fixed point which is a unique random solution of the problem (1) and (2). □

For the following existence result, we set our terminology as follows. Let $p(\tau,\delta)=\beta_1(\tau,\delta)$, $q(\tau,\delta)=\beta_2(\tau,\delta)$, $b(\tau,\delta)=\beta_4(\tau,\delta)$, $\beta_0(\tau,\delta)=|f(\tau,0,0,\delta)|$ and $\beta_3(\tau,\delta)=|\psi(\tau,0,\delta)|$. Then, it follows by the hypothesis $(\mathcal{H}_2)$ that

$$|f(\tau,y_1,y_2,\delta)|\leq\beta_0(\tau,\delta)+\beta_1(\tau,\delta)|y_1|+\beta_2(\tau,\delta)|y_2|,$$

and

$$|\psi(\tau,y_1,\delta)|\leq\beta_3(\tau,\delta)+\beta_4(\tau,\delta)|y_1|,$$

for $\tau\in J$, $\delta\in\Psi$ and each $y_1,y_2\in\mathbb{R}$, where $\beta_i:J\to L^\infty(\Psi,\mathbb{R}_+)$; $i=0,1,\ldots,4$ are measurable functions, with

$$\beta_i(\delta)=ess\sup_{\tau\in J}|\beta_i(\tau,\delta)| \text{ and } \beta_2(\delta)<1. \tag{16}$$

Theorem 2. *Suppose that $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and (16) hold. If*

$$\ell=\beta_4(\delta)+\left[\left(\frac{\left|\sum_{j=1}^m\lambda_j\tau_j^\rho\right|}{\left|\sum_{j=1}^m\lambda_j\right|}+2(2\pi)^\rho\right)\frac{(2\pi)^{\rho(\alpha-1)}}{\rho^\alpha\Gamma(\alpha+1)}+\frac{\sum_{j=1}^m\left|\lambda_j\left(\frac{\tau_j^\rho}{\rho}\right)^\alpha\right|}{\Gamma(\alpha+1)\left|\sum_{j=1}^m\lambda_j\right|}\right]\frac{\beta_1(\delta)}{1-\beta_2(\delta)}<1, \tag{17}$$

then problem (1) and (2) has at least one random solution defined on J.

Proof. Consider the set

$$\mathcal{G}_{\eta^*(\delta)}=\{\xi\in\Psi:||\xi(\cdot,\delta)||_J\leq\eta^*(\delta)\},$$

where

$$\begin{aligned}\eta^*(\delta)\geq{}&\frac{\beta_3(\delta)}{1-\ell}+\frac{d^*}{\left|\sum_{j=1}^m\lambda_j\right|(1-\ell)}+\left[\left(\frac{\left|\sum_{j=1}^m\lambda_j\tau_j^\rho\right|}{\left|\sum_{j=1}^m\lambda_j\right|}+2(2\pi)^\rho\right)\frac{(2\pi)^{\rho(\alpha-1)}}{\rho^\alpha\Gamma(\alpha+1)}\right.\\&\left.+\frac{\sum_{j=1}^m\left|\lambda_j\left(\frac{\tau_j^\rho}{\rho}\right)^\alpha\right|}{\Gamma(\alpha+1)\left|\sum_{j=1}^m\lambda_j\right|}\right]\frac{\beta_0(\delta)}{(1-\ell)(1-\beta_2(\delta))}.\end{aligned} \tag{18}$$

We define the operators $\mathcal{S}_1$ and $\mathcal{S}_2$ on $\mathcal{G}_{\eta^*(\delta)}$ by

$$\begin{aligned}(\mathcal{S}_1x)(\tau,\delta)={}&\frac{d(\delta)}{\sum_{j=1}^m\lambda_j}+\left[\frac{\sum_{j=1}^m\lambda_j\tau_j^\rho}{\sum_{j=1}^m\lambda_j}-\tau^\rho\right]\frac{1}{(2\pi)^\rho\Gamma(\alpha)}\int_0^{2\pi}\left(\frac{(2\pi)^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\kappa(s,\delta)ds\\&+\psi(\tau,x(\tau,\delta),\delta)-\frac{1}{\Gamma(\alpha)\sum_{j=1}^m\lambda_j}\sum_{j=1}^m\lambda_j\int_0^{\tau_j}\left(\frac{\tau_j^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\kappa(s,\delta)ds,\end{aligned} \tag{19}$$

$$(\mathcal{S}_2x)(\tau,\delta)=\frac{1}{\Gamma(\alpha)}\int_0^\tau\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\kappa(s,\delta)ds, \tag{20}$$

where κ satisfies (12). Then the fractional integral Equation (14) can be written as the operational equation

$$(\mathcal{S}x)(\tau,\delta)=(\mathcal{S}_1x)(\tau,\delta)+(\mathcal{S}_2x)(\tau,\delta),\ \ x(\cdot,\delta)\in\Psi.$$

The proof will be given in several steps.

Step 1: We prove that $\mathcal{S}_1 x_1(\cdot,\delta) + \mathcal{S}_2 x_2(\cdot,\delta) \in \mathcal{G}_{\eta^*(\delta)}$ for any $x_1(\cdot,\delta), x_2(\cdot,\delta) \in \mathcal{G}_{\eta^*(\delta)}$. For $\tau \in J$, we have

$$\begin{aligned}
&|(\mathcal{S}_1 x_1)(\tau,\delta)| \\
&\leq \left(\frac{\left|\sum_{j=1}^m \lambda_j \tau_j^\rho\right|}{\left|\sum_{j=1}^m \lambda_j\right|} + \tau^\rho \right) \frac{1}{(2\pi)^\rho \Gamma(\alpha)} \int_0^{2\pi} \left(\frac{(2\pi)^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} |\kappa_{x_1}(s,\delta)| ds \\
&+ \frac{d^*}{\left|\sum_{j=1}^m \lambda_j\right|} + |\psi(\tau, x_1(\tau,\delta))| + \frac{\sum_{j=1}^m |\lambda_j| \int_0^{\tau_j} \left(\frac{\tau_j^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} |\kappa_{x_1}(s,\delta)| ds}{\Gamma(\alpha) \left|\sum_{j=1}^m \lambda_j\right|}.
\end{aligned}$$

In view of $(\mathcal{H}_2)$ and (16), we have

$$\begin{aligned}
|\kappa_{x_1}(\tau,\delta)| &= |f(\tau, x_1(\tau,\delta), \kappa_{x_1}(\tau,\delta),\delta)| \\
&\leq \beta_1(\tau,\delta)|x_1(\tau,\delta)| + \beta_2(\tau,\delta)|\kappa_{x_1}(\tau,\delta)| + \beta_0(\tau,\delta) \\
&\leq \beta_1(\delta)|x_1(\tau,\delta)| + \beta_2(\delta)|\kappa_{x_1}(\tau,\delta)| + \beta_0(\delta).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
|\kappa_{x_1}(\tau,\delta)| &\leq \frac{\beta_1(\delta)|x_1(\tau,\delta)| + \beta_0(\delta)}{1-\beta_2(\delta)} \\
&\leq \frac{\beta_1(\delta)\|x_1(\tau,\delta)\|_J + \beta_0(\delta)}{1-\beta_2(\delta)}.
\end{aligned}$$

Since $x_1(\cdot,\delta) \in \mathcal{G}_{\eta^*(\delta)}$, then

$$|\kappa_{x_1}(\tau,\delta)| \leq \frac{\beta_1(\delta)\eta^*(\delta) + \beta_0(\delta)}{1-\beta_2(\delta)}. \tag{21}$$

For each $\tau \in J$, we have

$$\begin{aligned}
|\psi(\tau, x_1(\tau,\delta),\delta)| &\leq \beta_4(\tau,\delta)|x_1(\tau,\delta)| + \beta_3(\tau,\delta) \\
&\leq \beta_4(\delta)|x_1(\tau,\delta)| + \beta_3(\delta).
\end{aligned}$$

Then, for each $\tau \in J$, we obtain

$$|\psi(\tau, x_1(\tau,\delta),\delta)| \leq \beta_4(\delta)\eta^*(\delta) + \beta_3(\delta). \tag{22}$$

Thus, by (21) and (22), and since $\tau \leq 2\pi$, we obtain

$$\begin{aligned}
|(\mathcal{S}_1 x_1)(\tau,\delta)| \leq\ & \beta_4(\delta)\eta^*(\delta) + \beta_3(\delta) + \left[\left(\frac{\left|\sum_{j=1}^m \lambda_j \tau_j^\rho\right|}{\left|\sum_{j=1}^m \lambda_j\right|} + (2\pi)^\rho \right) \frac{(2\pi)^{\rho(\alpha-1)}}{\rho^\alpha \Gamma(\alpha+1)} \right. \\
&+ \left. \frac{\sum_{j=1}^m \left|\lambda_j \left(\frac{\tau_j^\rho}{\rho}\right)^\alpha\right|}{\Gamma(\alpha+1)\left|\sum_{j=1}^m \lambda_j\right|} \right] \frac{\beta_1(\delta)\eta^*(\delta) + \beta_0(\delta)}{1-\beta_2(\delta)} + \frac{d^*}{\left|\sum_{j=1}^m \lambda_j\right|}.
\end{aligned}$$

Consequently, we obtain

$$\|(\mathcal{S}_1x_1)(\cdot,\delta)\|_J \leq \beta_4(\delta)\eta^*(\delta)+\beta_3(\delta)+\left[\left(\frac{\left|\sum_{j=1}^m \lambda_j\tau_j^\rho\right|}{\left|\sum_{j=1}^m \lambda_j\right|}+(2\pi)^\rho\right)\frac{(2\pi)^{\rho(\alpha-1)}}{\rho^\alpha\Gamma(\alpha+1)} + \frac{\sum_{j=1}^m\left|\lambda_j\left(\frac{\tau_j^\rho}{\rho}\right)^\alpha\right|}{\Gamma(\alpha+1)\left|\sum_{j=1}^m \lambda_j\right|}\right]\frac{\beta_1(\delta)\eta^*(\delta)+\beta_0(\delta)}{1-\beta_2(\delta)}+\frac{d^*}{\left|\sum_{j=1}^m \lambda_j\right|}. \tag{23}$$

Now, for operator $\mathcal{S}_2$ and $\tau\in J$, we have

$$|(\mathcal{S}_2x_2)(\tau,\delta)| \leq \frac{1}{\Gamma(\alpha)}\int_0^\tau\left(\frac{\tau^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|\kappa_{x_2}(s,\delta)|ds.$$

Therefore,

$$|(\mathcal{S}_2x_2)(\tau,\delta)| \leq \left[\frac{(2\pi)^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\right]\frac{\beta_1(\delta)\eta^*(\delta)+\beta_0(\delta)}{1-\beta_2(\delta)}.$$

Hence,

$$\|(\mathcal{S}_2x_2)(\cdot,\delta)\|_J \leq \left[\frac{(2\pi)^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)}\right]\frac{\beta_1(\delta)\eta^*(\delta)+\beta_0(\delta)}{1-\beta_2(\delta)}. \tag{24}$$

Linking (23) and (24) for every $x_1(\cdot,\delta),x_2(\cdot,\delta)\in\mathcal{G}_{\eta^*(\delta)}$, we obtain

$$\begin{aligned}&\|(\mathcal{S}_1x_1)(\cdot,\delta)+(\mathcal{S}_2x_2)(\cdot,\delta)\|_J\\ &\leq \beta_4(\delta)\eta^*(\delta)+\beta_3(\delta)+\left[\left(\frac{\left|\sum_{j=1}^m \lambda_j\tau_j^\rho\right|}{\left|\sum_{j=1}^m \lambda_j\right|}+2(2\pi)^\rho\right)\frac{(2\pi)^{\rho(\alpha-1)}}{\rho^\alpha\Gamma(\alpha+1)}\right.\\ &\left.+\frac{\sum_{j=1}^m\left|\lambda_j\left(\frac{\tau_j^\rho}{\rho}\right)^\alpha\right|}{\Gamma(\alpha+1)\left|\sum_{j=1}^m \lambda_j\right|}\right]\frac{\beta_1(\delta)\eta^*(\delta)+\beta_0(\delta)}{1-\beta_2(\delta)}+\frac{d^*}{\left|\sum_{j=1}^m \lambda_j\right|}.\end{aligned}$$

By (18), we have

$$\|(\mathcal{S}_1x_1)(\cdot,\delta)+(\mathcal{S}_2x_2)(\cdot,\delta)\|_J\leq\eta^*(\delta),$$

which implies that $\mathcal{S}_1x_1(\cdot,\delta)+\mathcal{S}_2x_2(\cdot,\delta)\in\mathcal{G}_{\eta^*(\delta)}$.

Step 2: $\mathcal{S}_1$ is a contraction.

Let $x_1(\cdot,\delta),x_2(\cdot,\delta)\in\Psi$. Then, for $\tau\in J$, we have

$$\begin{aligned}&|(\mathcal{S}_1x_1)(\tau,\delta)-(\mathcal{S}_1x_2)(\tau,\delta)|\leq|\psi(\tau,x_1(\tau,\delta),\delta)-\psi(\tau,x_2(\tau,\delta),\delta)|\\ &+\left(\frac{\left|\sum_{j=1}^m \lambda_j\tau_j^\rho\right|}{\left|\sum_{j=1}^m \lambda_j\right|}+\tau^\rho\right)\frac{1}{(2\pi)^\rho\Gamma(\alpha)}\int_0^{2\pi}\left(\frac{(2\pi)^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|\kappa_{x_1}(s,\delta)-\kappa_{x_2}(s,\delta)|ds\\ &+\frac{1}{\Gamma(\alpha)\left|\sum_{j=1}^m \lambda_j\right|}\sum_{j=1}^m|\lambda_j|\int_0^{\tau_j}\left(\frac{\tau_j^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|\kappa_{x_1}(s,\delta)-\kappa_{x_2}(s,\delta)|ds.\end{aligned}$$

Therefore, for each $\tau\in J$, we have

$$\|(\mathcal{S}_1x_1)(\cdot,\delta)-(\mathcal{S}_1x_2)(\cdot,\delta)\|_J$$

$$\leq \left[b(\delta) + \left(\frac{\left|\sum_{j=1}^{m} \lambda_j \tau_j^{\rho}\right|}{\left|\sum_{j=1}^{m} \lambda_j\right|} + (2\pi)^{\rho} \right) \frac{p(\delta)(2\pi)^{\rho(\alpha-1)}}{(1-q(\delta))\rho^{\alpha}\Gamma(\alpha+1)} + \frac{p(\delta)\sum_{j=1}^{m}\left|\lambda_j \left(\frac{\tau_j^{\rho}}{\rho}\right)^{\alpha}\right|}{(1-q(\delta))\Gamma(\alpha+1)\left|\sum_{j=1}^{m} \lambda_j\right|} \right] \|x_1(\cdot,\delta) - x_2(\cdot,\delta)\|_J.$$

By (17), the operator $\mathcal{S}_1$ is a contraction.

Step 3: $\mathcal{S}_2$ is compact and continuous.

Observe that continuity of $\mathcal{S}_2$ follows from that of f. Next, we prove that $\mathcal{S}_2$ is uniformly bounded on $\mathcal{G}_{\eta^*(\delta)}$. Let $x_2(\cdot,\delta) \in \mathcal{G}_{\eta^*(\delta)}$. Then, by (24), we have

$$\|(\mathcal{S}_2 x_2)(\cdot,\delta)\|_J \leq \left[\frac{(2\pi)^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right] \frac{\beta_1(\delta)\eta^*(\delta) + \beta_0(\delta)}{1-\beta_2(\delta)}.$$

This means that $\mathcal{S}_2$ is uniformly bounded on $\mathcal{G}_{\eta^*(\delta)}$. Next, we show that $\mathcal{S}_2\left(\mathcal{G}_{\eta^*(\delta)}\right)$ is equicontinuous. Let $x(\cdot,\delta) \in \mathcal{G}_{\eta^*(\delta)}$ and $0 < \tau_1 < \tau_2 \leq 2\pi$. Then

$$\begin{aligned}
&|(\mathcal{S}_2 x)(\tau_2,\delta) - (\mathcal{S}_2 x)(\tau_1,\delta)| \\
&\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{\tau_2} \left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds - \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left(\frac{\tau_1^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\kappa(s,\delta)ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{\tau_1}^{\tau_2} \left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}|\kappa(s,\delta)|ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{\tau_1} \left| \left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} - \left(\frac{\tau_1^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} \right| |\kappa(s,\delta)|ds \\
&\leq \left[\frac{(\tau_2^{\rho} - \tau_1^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right] \frac{\beta_1(\delta)\eta^*(\delta) + \beta_0(\delta)}{1-\beta_2(\delta)} \\
&+ \frac{\beta_1(\delta)\eta^*(\delta) + \beta_0(\delta)}{\Gamma(\alpha)(1-\beta_2(\delta))} \int_0^{\tau_1} \left| \left(\frac{\tau_2^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} - \left(\frac{\tau_1^{\rho} - s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1} \right| ds,
\end{aligned}$$

which tends to zero as $\tau_2 \to \tau_1$. This shows that $\mathcal{S}_2(\mathcal{G}_{\eta^*(\delta)})$ is equicontinuous on J. Therefore, $\mathcal{S}_2(\mathcal{G}_{\eta^*(\delta)})$ is relatively compact on $\mathcal{G}_{\eta^*(\delta)}$. By the Arzela–Ascoli Theorem, we deduce that $\mathcal{S}_2$ is compact on $\mathcal{G}_{\eta^*(\delta)}$.

As a consequence of Krasnoselskii's fixed point theorem, the operator $\mathcal{S}$ has at least one fixed point, which is a solution of the problem (1) and (2). □

Remark 2. *It is noteworthy to observe that Banach's contraction principle is more advantageous, as it establishes the existence, as well as uniqueness of a solution to the problem at hand. On the other hand, Krasnoselskii's fixed point theorem solely ensures the existence of a solution to the problem at hand. Obviously, the contractive condition for the operator $\mathcal{S}_1$ used in Theorem 2 is different from the one used in Theorem 1. Moreover, we require that $\beta_0(\delta) = ess\sup_{\tau\in J}|\beta_0(\tau,\delta)| = ess\sup_{\tau\in J}|f(\tau,0,0,\delta)|$, and $\beta_3(\delta) = ess\sup_{\tau\in J}|\beta_3(\tau,\delta)| = ess\sup_{\tau\in J}|\psi(\tau,0,\delta)|$ in Theorem 2. In case we interchange the role of operators $\mathcal{S}_1$ and $\mathcal{S}_2$ in the proof of Theorem 2, the contractive condition also changes.*

4. Examples

Example 1. *Let the space $\mathbb{R}_{-}^{*} := (-\infty, 0)$ be equipped with the usual σ-algebra consisting of Lebesgue measurable subsets of $\mathbb{R}_{-}^{*}$. Consider the boundary value problem involving a generalized Caputo fractional differential equation given by*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{3}{2},\rho}(x(\tau,\delta) - \psi(\tau, x(\tau,\delta),\delta)) = \frac{\sin(\tau)(x(\tau,\delta)+1)}{100(\delta^2+1)}, \tau \in J, \\ x(0,\delta) = x(2\pi,\delta), \ \sum_{j=1}^{2} \frac{j}{3} x(\frac{j\pi}{3}) = d(\delta). \end{cases} \tag{25}$$

Set

$$f(\tau, x(\tau,\delta), ({}^{C}D_{0^+}^{\frac{3}{2},\rho}x)(\tau,\delta),\delta) = \frac{\sin(\tau)(x(\tau,\delta)+1)}{100(\delta^2+1)}, \ \ \tau \in J, x \in \mathbb{R},$$

and

$$\psi(\tau, x(\tau,\delta),\delta) = \frac{(\sin^2(\tau) - \frac{\sqrt{3}}{2}\sin(\tau))x(\tau,\delta)}{1000(\delta^2+1)}, \ \ \tau \in J, x \in \mathbb{R},$$

with $\psi(2\pi, x(2\pi,\delta),\delta) = \psi(0, x(0,\delta),\delta) = \psi(\tau_j, x(\tau_j,\delta),\delta) = 0$, $j = 1,2$, $\alpha = \frac{3}{2}$, $\rho = \frac{1}{5}$, $\tau_j = \frac{j\pi}{3}$.

For each $x_1, x_2, v_1, v_2 \in \mathbb{R}$ and $\tau \in J$, we have

$$\begin{aligned} |f(\tau, x_1, v_1, \delta) - f(\tau, x_2, v_2, \delta)| &\leq \left| \frac{\sin(\tau)(x_1+1)}{100(\delta^2+1)} - \frac{\sin(\tau)(x_2+1)}{100(\delta^2+1)} \right| \\ &\leq \frac{|\sin(\tau)|}{100(\delta^2+1)} |x_1 - x_2|, \end{aligned}$$

and

$$|\psi(\tau, x_1, \delta) - \psi(\tau, x_2, \delta)| \ \leq \ \frac{\left|\sin^2(\tau) - \frac{\sqrt{3}}{2}\sin(\tau)\right|}{1000(\delta^2+1)} |x_1 - x_2|.$$

Therefore, $(\mathcal{H}_2)$ is verified with

$$p(\tau,\delta) = \frac{|\sin(\tau)|}{100(\delta^2+1)}, \ b(\tau,\delta) = \frac{\left|\sin^2(\tau) - \frac{\sqrt{3}}{2}\sin(\tau)\right|}{1000(\delta^2+1)}, \ \ q(\tau,\delta) = 0.$$

The condition

$$\begin{aligned} & b(\delta) + \left(\frac{\left|\sum_{j=1}^{m} \lambda_j \tau_j^{\rho}\right|}{\left|\sum_{j=1}^{m} \lambda_j\right|} + 2(2\pi)^{\rho} \right) \frac{p(\delta)(2\pi)^{\rho(\alpha-1)}}{(1-q(\delta))\rho^{\alpha}\Gamma(\alpha+1)} + \frac{p(\delta)\sum_{j=1}^{m}\left|\lambda_j \left(\frac{\tau_j^{\rho}}{\rho}\right)^{\alpha}\right|}{(1-q(\delta))\Gamma(\alpha+1)\left|\sum_{j=1}^{m}\lambda_j\right|} \\ = \ & \frac{2+\sqrt{3}}{2000(\delta^2+1)} + \left(\frac{\sum_{j=1}^{2} \frac{j}{3} (\frac{j\pi}{3})^{\frac{1}{5}}}{\sum_{j=1}^{2} \frac{j}{3}} + 2(2\pi)^{\frac{1}{5}} \right) \frac{(2\pi)^{\frac{1}{5}(\frac{3}{2}-1)}}{100(\delta^2+1)(\frac{1}{5})^{\frac{3}{2}}\Gamma(\frac{3}{2}+1)} \\ + \ & \frac{\sum_{j=1}^{2} \frac{j}{3} \left(\frac{(\frac{j\pi}{3})^{\frac{1}{5}}}{\frac{1}{5}} \right)^{\frac{3}{2}}}{100(\delta^2+1)\Gamma(\frac{3}{2}+1)\sum_{j=1}^{2}\frac{j}{3}} \approx \frac{0.43478131}{\delta^2+1} < 1, \end{aligned}$$

is satisfied with $\alpha = \frac{3}{2}$. Thus, all the conditions of Theorem 1 hold true, so the problem (25) admits a unique random solution.

Example 2. *Consider the following problem,*

$$\begin{cases} {}^{C}D_{0^+}^{\frac{4}{3},\rho}(x(\tau,\delta)-\psi(\tau,x(\tau,\delta),\delta)) = f(\tau,x(\tau,\delta),({}^{C}D_{0^+}^{\frac{4}{3},\rho}x)(\tau,\delta),\delta), \ \ \tau\in J, \\ x(0,\delta)=x(2\pi,\delta), \ \ \sum_{j=1}^{2} 2jx(\tau_j) = d(\delta), \end{cases} \tag{26}$$

where

$$f(\tau,x_1,x_2,\delta) = \frac{|x_1|+|x_2|+3}{411e^{\tau}(1+|x_1|+|x_2|)(|\delta|+2)}, \ \ \tau\in J, \ x_1,x_2\in\mathbb{R},$$

and

$$\psi(\tau,x,\delta) = \frac{\sin(\tau)(\cos^3(\tau)-\frac{\cos(\tau)}{2})|x|}{300(|\delta|+2)}, \ \ \tau\in J, x\in\mathbb{R}, j=1,2.$$

Notice that

$$\psi(2\pi,x(2\pi,\delta),\delta) = \psi(0,x(0,\delta),\delta) = \psi(\tau_j,x(\tau_j,\delta),\delta) = 0, \ \ j=1,2, \ \alpha=\frac{4}{3},$$

$$\rho=1, \ \tau_1=\frac{\pi}{4}, \ \tau_2=\frac{7\pi}{4}.$$

All conditions of Theorem 2 are satisfied with

$$\beta_0(\delta)=\frac{3}{411(|\delta|+2)}, \beta_1(\delta)=\beta_2(\delta)=p(\delta)=q(\delta)=\frac{1}{411(|\delta|+2)},$$

$$\beta_3(\delta)=0, \beta_4(\delta)=b(\delta)=\frac{1}{200(|\delta|+2)},$$

and

$$\begin{aligned} \ell &= \frac{1}{200(|\delta|+2)} + \left[\left(\frac{\frac{\pi}{2}+7\pi}{6}+4\pi\right)\frac{(2\pi)^{\frac{1}{3}}}{\Gamma(\frac{7}{3})} + \frac{2(\frac{\pi}{4})^{\frac{4}{3}}+4(\frac{7\pi}{4})^{\frac{4}{3}}}{6\Gamma(\frac{7}{3})}\right]\frac{1}{411|\delta|+821} \\ &\approx \frac{1}{200(|\delta|+2)} + \frac{31.1975967219243}{411|\delta|+821} \\ &< 1. \end{aligned}$$

Hence, by the conclusion of Theorem 2, the problem (26) admits at least one random solution.

5. Conclusions

In this paper, we have obtained the existence and uniqueness results concerning the random solutions of a non-local and periodic boundary value problem of non-linear generalized Caputo type implicit fractional differential equations by applying the standard fixed point theorems. For the applicability of the main results, illustrative examples are presented. Our results are new and enrich the related literature.

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