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Dynamical Invariant for Dissipative Systems via Complex Quantum Hydrodynamics

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Abstract: For Hamiltonian systems with time-dependent potential, the Hamiltonian, and thus the energy, is no longer a constant of motion. However, for such systems as the parametric oscillator, i.e., an oscillator with time-dependent frequency $\omega(t)$, still, a dynamical invariant can be found that now has the dimension of action. The question, if such an invariant still exists after the addition of a dissipative friction force is analyzed for the classical as well as for the quantum mechanical case from different perspectives, particularly from that of a complex hydrodynamic formulation of quantum mechanics.

1. Introduction

As classical Hamiltonian mechanics in terms of Newton's equation of motion is not invariant under coordinate transformations, a more general formulation is required, leading finally to Hamiltonian mechanics, which is even invariant under so-called canonical transformations. This form is also the basis for a transition to quantum mechanics via canonical quantization. The Hamiltonian function not only generates the dynamics of the system, but it also represents its energy; the same also applies after quantization. Usually, systems with time-independent potential energy are considered, and hence the Hamiltonian is a constant of motion and the energy is conserved.

A different situation arises for systems with time-dependent potential, such as the parametric oscillator, i.e., an oscillator whose frequency of oscillation, ω, is no longer constant, like it is for the harmonic oscillator ($\omega = \omega_0 = $ const.), but explicitly time-dependent ($\omega = \omega(t)$). There exist only few frequencies $\omega(t)$ for which still exact analytic solutions exist, but in any case, the Hamiltonian is no longer a constant of motion. However, already in 1880, Ermakov was able to show [1] that for those systems, still a dynamical invariant can be constructed. For this purpose, he used the Newtonian equation of motion for the system plus another auxiliary Newtonian equation that contained the same frequency $\omega(t)$. By eliminating the frequency between the two equations, he arrived at the invariant that is named after him and has essentially the dimension of action (not energy, like the Hamiltonian).

The same invariant was also obtained by Lewis [2] in a quantum mechanical context by solving the corresponding time-dependent Schrödinger equation. Since an oscillator potential is quadratic in position coordinate, also for $\omega(t)$, there exist exact analytic solutions in the form of Gaussian wave packets for this system. These Gaussians are uniquely determined by their maximum and width, and it was shown that the equation of motion for the maximum was just the classical Newtonian equation of motion as expected according to Ehrenfest, but the equation for the dynamics of the width could be brought in a form that was exactly that of Ermakov's auxiliary equation. Therefore, now, a physical meaning

could be associated with the quantity obeying this equation, namely that of the wave packet width and thus the position uncertainty. Although the two equations were obtained from a Hamiltonian, the operator in the Schrödinger equation, the construction of the invariant followed exactly Ermakov's procedure, using only Newtonian equations. A quantum version of the invariant is obtained simply by canonical quantization.

Recently, in [3] it was shown that the same invariant can also be obtained in a different way without Newtonian equations but from a Hamiltonian formalism that is based on a complex version of Madelung's hydrodynamic formulation of quantum mechanics in connection with a complex nonlinear Riccati equation that contains the information about the dynamics of the wave packet width. The question to be answered in this paper is, if there are other systems where the energy is not conserved, but still a dynamical invariant can be obtained via our complex hydrodynamic approach and the corresponding Hamiltonian formalism. Our formalism shall also be compared with other approaches trying to describe the same systems, and differences or similarities shall be specified, giving a hint as to which types of problems which approach could be most favorable.

In this category obviously fall open systems, i.e., systems that interact with some kind of environment. In those systems, such as Brownian motion, energy is dissipated due to friction forces, and the evolution is irreversible, reflected by the broken time-reversal symmetry of the equations of motion.

This situation actually represents a many-body problem, but one is essentially interested in the dynamics and energetics of the observable Brownian particle, not in the degrees of freedom of the surrounding environment. This can be taken into account by an effective description of this situation.

In the trajectory picture, taking into account the particle aspect, this can be achieved by the phenomenological Newton-type Langevin equation that contains the environmental effect in the form of a (deterministic) friction force that is linearly proportional to velocity (or momentum) and a (statistical) stochastic fluctuating force that vanishes on average.

Looking at the same situation from the viewpoint of distribution functions, an equivalent description can be given by so-called Fokker–Planck equations, i.e., continuity equations that take into account not only reversible convection currents, but also irreversible diffusion currents.

As this description is, in the trajectory picture, in terms of a Newton-type equation of motion, the question arises if a more general or invariant formulation in terms of an efficient Hamiltonian for the system of interest is possible. If this is possible, one could try to quantize this approach in order to obtain the corresponding equations of motion for position and position uncertainty that hopefully could again be combined to find a dynamical invariant for this system.

Trying to find a corresponding Hamiltonian function by simply adding a (real) "friction potential" to the classical Hamiltonian function fails to provide the desired equations of motion. Another type of approach tries to reach this goal by a non-canonical transformation of the position and momentum variables leading to (eventually explicitly time-dependent) Hamiltonians that supply the correct equations of motion, including the friction force; see, for example, [4,5]. This approach has the disadvantage that after canonical quantization, the uncertainty principle seems to be violated (a solution of this problem is shown in Section 4).

Other approaches try to circumvent the classical part and start already on the quantum mechanical level by adding a "friction potential" to the Hamiltonian operator, leading to nonlinear modifications of the Schrödinger equation [6–10]. Although, by construction, according to the Ehrenfest theorem, the correct equation of motion, including the friction force is obtained for the average values, there are other problems with the physical results due to ambiguities in the definition of the friction term. In particular, for real "friction potentials" [6,7], the equation for the probability density $\rho = \psi^* \psi$ remains the reversible continuity equation, contradicting the irreversible character of the process. Irreversibility can only be achieved by adding a term to the continuity equation, which causes an additional imaginary contribution to the Hamiltonian operator, turning into a non-Hermitian

one. An attempt like that will be used by us in Section 3. In this context, an approach by Dekker [11,12] shall only be mentioned that also uses non-Hermitian Hamiltonians obtained by introducing complex variables as linear combinations of position and momentum, used to define complex Lagrange and Hamilton functions that finally allow canonical quantization.

In our recent complex hydrodynamic description of quantum mechanics [13,14], we started with the two Madelung equations [15] obtained using a polar ansatz for the wave function, $\psi = \sqrt{\rho} \exp\left(\frac{i}{\hbar} S\right)$, a continuity equation for ρ, and a modified Hamilton–Jacobi equation for the phase S. Introducing our complex quantities, we were able to bring the dynamics in a Hamiltonian form with complex Hamiltonian. From one of the corresponding Hamiltonian equations of motion, we were able to obtain directly the Ermakov invariant.

For the above-mentioned open system, we want to follow the same scheme, however, guaranteeing the irreversibility of the dynamics by introducing a diffusion term into the continuity equation, changing it (like in the theory of Brownian motion) into a Fokker–Planck-type equation in position space, called the Smoluchowski equation. To complete our hydrodynamic description, the modified Hamilton–Jacobi equation for the phase, including the environmental effect, is missing. This can be obtained by separation of the equation for $\rho = \psi^* \psi$ into two equations containing only ψ or ψ^*, following a method proposed by Madelung [16] and Mrowka [17]. Due to the diffusion term, this is not possible in general, but for special separation conditions that still allow Gaussian solutions for the resulting nonlinear Schrödinger equation with non-Hermitian Hamiltonian. This equation provides the missing Hamilton–Jacobi equation for the phase S as well as the modified Riccati equation that enters our derivation of the Ermakov invariant.

The paper is organized as follows: In Section 2, the situation without an environment is summarized to an extent that is necessary to keep this paper self-contained. The equations of motion for the wave packet maximum and width are given explicitly as well as the resulting Ermakov invariant, based on the Newtonian form of the equations. Then, Madelung's hydrodynamic formulation [15], that is also the formal basis for Bohmian mechanics, is presented and extended to a complex version. It should be stressed that this is different from approaches that write the wave function in terms of a complex action function [18–26], from which a complex momentum field arises, similar to our formulation. However, these approaches then draw the incorrect conclusion that also the (independent) position variable must be complex. For further details, see [13] and references cited therein. In our version, a complex Hamiltonian equation of motion can be used to find an alternative way to obtain the invariant.

In Section 3, it is shown how the introduction of a diffusion term into the Madelung formalism can provide the above-mentioned modification leading to a Schrödinger equation with complex nonlinearity. Solving this equation provides two modified Newtonian equations for the maximum and width of the Gaussian solution that can again be combined to a dynamical invariant, following Ermakov's procedure. It is shown that, using our complex hydrodynamic formalism, it is also possible to obtain the same invariant via a Hamiltonian equation of motion, where the Hamiltonian is now derived from the nonlinear Schrödinger equation.

The derivation of this invariant shows similarities to the approaches using non-canonical transformations [5,6]. A way to avoid nonphysical violations of the uncertainty principle is shown in Section 4.

Section 5 shows the connection between our complex Hamiltonian approach and generalized creation and annihilation operators, including the open system. This allows also comparison with the complex formalism of Dekker.

So, a complete picture of our complex hydrodynamic approach and its connections with formally different effective approaches for the description of the Brownian system is given.

In the conclusions, the advantages of the hydrodynamic approach for certain problems, such as time-dependent tunneling, are specified, and certain fields of possible application are mentioned.

2. Time-Dependent Schrödinger Equation with Gaussian Solutions, Hydrodynamic Formulation and Dynamical Invariant

In order to obtain exact analytical expressions, the following discussion is restricted to generalized coherent states, i.e., to Gaussian wave packets with time-dependent width in position space. These functions are solutions to the time-dependent Schrödinger equation with potentials that are, at most, quadratic in the position variable. Without loss of generality, we consider the one-dimensional case, i.e.,

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left\{ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t) \right\}\psi(x,t), \tag{1}$$

where $\psi(x,t)$ is the wave packet in position representation.

As we restrict our discussion to the quadratic potentials and assume a constant mass m, the time-dependence of the potential $V(x,t)$ originates from the time dependence of the oscillator frequency $\omega(t)$ and has the form $V(x,t) = \frac{1}{2}m\omega^2(t)x^2$, describing the so-called parametric oscillator.

In the following, we use a generalized Gaussian wave packet ansatz for the solution of Equation (1) that provides the classical Newtonian equation of motion and the auxiliary Newton-type Ermakov equation that was used by Ermakov to eliminate the frequency $\omega(t)$ and to obtain the dynamical invariant.

The Gaussian solution can be written in the form

$$\psi(x,t) = \langle x|\psi(t)\rangle = N(t)\exp\left[\frac{i}{\hbar}\left(\frac{m}{2}\mathcal{C}\widetilde{x}^2 + \langle p\rangle\widetilde{x} + f(t)\right)\right], \tag{2}$$

where $\widetilde{x} \doteq x - \langle x\rangle = x - \eta(t)$, i.e., the maximum of the wave function is at $\langle x\rangle = \int_{-\infty}^{+\infty}\psi^*(x,t)x\psi(x,t)\mathrm{d}x$. This is in agreement with Ehrenfest's theorem that the mean value $\langle x\rangle$ follows the classical trajectory, here denoted as $\eta(t)$. The coefficient of the quadratic term, $\mathcal{C} = \mathcal{C}(t)$, is a complex function of time and $\langle p\rangle = m\dot\eta$, where overdots denote time derivatives. The possibly time-dependent normalization factor $N(t)$ and the purely time-dependent function $f(t)$ are not relevant for the following discussion, as they do not contribute to the equations of motion of $\eta(t)$ and $\mathcal{C}(t)$. (If the ground state energy is included in a complex $N(t)$, the function $f(t)$ is given by $\frac{1}{2}\langle x\rangle\langle p\rangle$; see also [27].) Inserting (2) into (1) provides the two equations of motion for $\eta(t)$ and $\mathcal{C}(t)$,

$$\ddot\eta + \omega^2(t)\eta = 0 \tag{3}$$

and

$$\dot{\mathcal{C}} + \mathcal{C}^2 + \omega^2(t) = 0. \tag{4}$$

Equation (3) is Newton's equation of motion for the trajectory. Equation (4) is a complex nonlinear Riccati equation that is connected with the time-dependence of the wave packet width. This connection becomes clearer by introducing a new variable $\alpha(t)$ according to $\mathcal{C}_I = \frac{1}{\alpha^2} = \frac{\hbar}{2m}\frac{1}{\sigma_x^2}$ with $\sigma_x^2 = \langle\widetilde{x}^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ being the mean square deviation of position, so $\alpha(t)$ represents, up to a constant factor, the position uncertainty. With this definition of $\mathcal{C}_I$, it follows from the imaginary part of Equation (4) for the real part

$$\mathcal{C}_R = \frac{\dot\alpha}{\alpha}. \tag{5}$$

Inserting $\mathcal{C}_I$ and $\mathcal{C}_R$ in this form into the real part of (4) turns this into

$$\ddot{\alpha} + \omega^2(t)\alpha = \frac{1}{\alpha^3},$$ (6)

a real nonlinear differential equation known as the Ermakov equation.

Ermakov used this auxiliary equation to obtain a dynamical invariant by eliminating the frequency $\omega(t)$ with the help of Equation (3) (see, for example, [1,28]), leading to

$$I_L = \frac{1}{2}\left[(\dot{\eta}\alpha - \dot{\alpha}\eta)^2 + \left(\frac{\eta}{\alpha}\right)^2\right] = \text{const.}$$ (7)

So, Ermakov used two Newton-type equations, Equations (3) and (6), to derive this invariant. In the following, we show how this is also possible using a (complex) Hamiltonian equation of motion and the complex nonlinear Riccati Equation (4). For this purpose, we consider next Madelung's hydrodynamic version of quantum mechanics.

Shortly after Schrödinger published his wave equation, Madelung [15] achieved a decomposition of this linear complex equation into two real equations that have similarity with hydrodynamic equations, namely the continuity equation

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\left(\rho\frac{1}{m}\frac{\partial}{\partial x}S\right) = 0$$ (8)

and a modified Hamilton–Jacobi equation

$$\frac{\partial}{\partial t}S + \frac{1}{2m}\left(\frac{\partial}{\partial x}S\right)^2 + V + V_{qu} = 0$$ (9)

by using a polar ansatz for the wave function, i.e., $\psi(x,t) = \sqrt{\rho}\exp\left(\frac{i}{\hbar}S\right)$, where $\rho = \psi^*\psi$. The appearance of the "quantum potential"

$$V_{qu} = -\frac{\hbar^2}{2m}\frac{\frac{\partial^2}{\partial x^2}\sqrt{\rho}}{\sqrt{\rho}}$$ (10)

distinguishes the modified Hamilton–Jacobi Equation (9) from the classical counterpart.

This formulation was adopted 25 years later by Bohm [29,30] in his interpretation of quantum mechanics in terms of "hidden variables". We have argued recently [31] that these variables cannot be considered as real trajectories of really existing "quantum particles", but have to be seen in a probabilistic context, however, different from the conventional probabilistic viewpoint usually taken in quantum mechanics. The experiments of this year's Nobel laureates, particularly those of Aspect [32–34], indicate that hidden variables do not exist and support our result. In the following, we do not explore ontological interpretations of quantum mechanics but stick to the formal structure that was established by Madelung.

In [13], it was shown that this hydrodynamic version of quantum mechanics can be formulated in terms of complex quantities that are obtained by applying a quantum mechanical operator F_{op} onto a quantum state or wave function $\langle a|\psi(t)\rangle$ in the $\{|a\rangle\}$ representation (in this paper "a" is position x and $\langle x|\psi(t)\rangle = \psi(x,t)$) and dividing the result by $\langle a|\psi(t)\rangle$, in our case here,

$$F = \frac{F_{op}\psi(x,t)}{\psi(x,t)} = F_R + iF_I.$$ (11)

The result is, in general, complex; however, the mean value of the imaginary part always vanishes ($\langle F_I\rangle = 0$), i.e., it cannot be observed directly (for further details, see [13,14]).

Taking the gradient of Equation (9) and using that $\frac{\partial}{\partial x}S = P_R$ is the real part of the complex momentum according to (11), the result can be written in the form of an Euler-type equation:

$$\frac{\partial}{\partial t}P_R + \frac{1}{m}P_R\frac{\partial}{\partial x}P_R = \frac{D}{Dt}P_R = -\frac{\partial}{\partial x}\left(V + V_{qu}\right) \tag{12}$$

with the substantial or convective time-derivative $\frac{D}{Dt}$ for a co-moving observer.

For the Gaussian wave packets (2), the relevant complex quantities are

$$\begin{align}
X &= x \tag{13}\\
P &= m\dot{\eta} + m\mathcal{C}\tilde{x} = P_R + iP_I \tag{14}\\
V &= \frac{m}{2}\omega^2(t)x^2 \tag{15}\\
T &= \frac{1}{2m}\left[(P)^2 + \frac{\hbar}{i}\frac{\partial}{\partial x}P\right] = \frac{1}{2m}\left[(P)^2 - i\hbar m\mathcal{C}\right] = T_R + iT_I \tag{16}\\
T_R &= \frac{1}{2m}\left[P_R^2 - P_I^2 + \hbar\frac{\partial}{\partial x}P_I\right] = \frac{1}{2m}P_R^2 + V_{qu} \tag{17}\\
T_I &= -\frac{1}{2m}\left[2P_RP_I - \hbar\frac{\partial}{\partial x}P_R\right] \tag{18}\\
H_L &= T + V. \tag{19}
\end{align}$$

As shown in [13], in position space, the Hamiltonian equation for the complex momentum P is given by

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial x}H_L. \tag{20}$$

With the expression for P as given for the Gaussians in (14), one obtains

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial x}\left(\frac{1}{2m}(P)^2 - \frac{i\hbar}{2}\mathcal{C} + \frac{m}{2}\omega^2(t)x^2\right). \tag{21}$$

Using Riccati Equation (4) to replace $\omega^2(t)$ and $\frac{\partial}{\partial x}P = m\mathcal{C}$ leads to

$$\frac{\partial}{\partial t}P = -P\mathcal{C} + m\dot{\mathcal{C}}x + m\mathcal{C}^2x \tag{22}$$

which can be rearranged to yield

$$\frac{\partial}{\partial t}\ln\left(P - m\mathcal{C}x\right) = -\mathcal{C} = -\frac{\partial}{\partial t}\ln\alpha - i\frac{1}{\alpha^2}. \tag{23}$$

As the logarithm of a complex quantity $w = |w|\exp(i\phi)$ is given by $\ln w = \ln|w| + i\phi$, Equation (23) can be rewritten as

$$\frac{\partial}{\partial t}\ln|P - m\mathcal{C}x| + i\frac{\partial}{\partial t}\tan^{-1}\left(\frac{P_I - m\mathcal{C}_Ix}{P_R - m\mathcal{C}_Rx}\right) = -\frac{\partial}{\partial t}\ln\alpha - i\frac{1}{\alpha^2}, \tag{24}$$

where $\frac{1}{\alpha^2} = \dot{\phi}$, i.e., the time derivative of the angle determined by the expression in the round brackets. The meaning of this angle has been discussed and will not be affected by the inclusion of the environment.

For the absolute value, this finally leads to

$$\alpha^2|P - m\mathcal{C}x|^2 = \alpha_0^2|P_0 - m\mathcal{C}_0x|^2. \tag{25}$$

Using $P - m\mathcal{C}x = -m\mathcal{C}\eta + m\dot{\eta}$ yields

$$\alpha^2\left[\left(-m\mathcal{C}_R\eta + m\dot{\eta}\right)^2 + \left(-m\mathcal{C}_I\eta\right)^2\right] = \text{const.} \tag{26}$$

In the cases under consideration, the initial classical position η_0 and the initial spreading $\dot{\alpha}_0$ can be taken equal to zero. With the initial classical momentum $p_0 = m\dot{\eta}_0$, Equations (25) and (26) can be simplified to

$$(\dot{\eta}\alpha - \dot{\alpha}\eta)^2 + \left(\frac{\eta}{\alpha}\right)^2 = \left(\frac{\alpha_0 p_0}{m}\right)^2 \tag{27}$$

which is, up to a factor $\frac{1}{2}$, the Ermakov invariant.

3. Open Systems with Irreversibility and Dissipation

As mentioned in the Introduction, one characteristic feature of the open systems that can be found in Brownian motion is the irreversibility of the dynamics, expressed in the broken time symmetry of the corresponding equations of motion. Considering Madelung's hydodynamic equations, this can be achieved by adding a corresponding diffusion term to the continuity equation, transforming it into a Fokker–Planck-type equation, in position space specifically called the Smoluchowski equation:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\left(\rho\frac{1}{m}\frac{\partial}{\partial x}S\right) - D\frac{\partial^2}{\partial x^2}\rho = \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\left[\rho\left(\frac{1}{m}\frac{\partial}{\partial x}S - D\frac{\frac{\partial\rho}{\partial x}}{\rho}\right)\right] = 0, \tag{28}$$

with the convection velocity field $v_{\mathrm{con}} = \frac{1}{m}\frac{\partial}{\partial x}S = \frac{1}{m}P_{\mathrm{R}} = \dot{\eta} + \mathcal{C}_{\mathrm{R}}\tilde{x} = \dot{\eta} + \left(\frac{\dot{\alpha}}{\alpha} - \frac{\gamma}{2}\right)\tilde{x}$ and the diffusion velocity $v_{\mathrm{dif}} = -D(\frac{\partial}{\partial x}\rho)/\rho = D\frac{2}{\hbar}P_{\mathrm{I}} = D\frac{2m}{\hbar}\mathcal{C}_{\mathrm{I}}\tilde{x} = \frac{\gamma}{2}\tilde{x}$.

In conventional quantum mechanics, the continuity equation for $\rho = \psi^*\psi$ is obtained by combining the Schrödinger equation for ψ with its complex conjugate equation for ψ^*. So, the question arises of whether this procedure can be reversed to obtain the complex modified Schrödinger equation corresponding to (28) by the separation of this equation. A method to achieve this was found by Madelung [16] and Mrowka [17] in their attempt to derive the Schrödinger equation without making use of Hamiltonian mechanics. Unfortunately, due to the diffusion term, ψ and ψ^* are coupled, and this method cannot be applied in general. However, there might be special conditions that can be fulfilled by the diffusion term to still allow separation.

One such separation condition is given by

$$-D\frac{\frac{\partial^2}{\partial x^2}\rho}{\rho} = \gamma\left(\ln\rho - \langle\ln\rho\rangle\right) \tag{29}$$

that particularly is also fulfilled for Gaussian functions, the ones we consider in our analysis. The subtraction of the mean value of $\ln\rho$ is necessary to guarantee normalizability of the solutions of the corresponding Schrödinger equation, as the diffusion term will lead to an imaginary contribution in the Hamiltonian, thus turning it into a non-Hermitian one.

The separation of (29) leads to a nonlinear Schrödinger equation with complex logarithmic nonlinearity:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V + \gamma\frac{\hbar}{i}\left(\ln\psi - \langle\ln\psi\rangle\right)\right)\psi(x,t) = H_{\mathrm{NL,\,op}}\psi(x,t). \tag{30}$$

Further details concerning the derivative of this equation, its properties and connection with similar nonlinear approaches can be found in [27].

The additional nonlinear term has also a real contribution that is not arbitrary but fixed by the separation condition, provides the missing second hydrodynamic equation for our approach, and has a clear physical meaning. A first idea about this meaning can

be obtained by looking at the modified Hamilton–Jacobi equation. Due to the additional nonlinear term and the polar form of the wave function, it turns into

$$\frac{\partial}{\partial t}S + \frac{1}{2m}\left(\frac{\partial}{\partial x}S\right)^2 + V + V_{qu} + \gamma\left(S - \langle S \rangle\right) = 0. \tag{31}$$

Taking the gradient of this equation now provides a kind of Euler equation with a substantial time derivative and an additional friction term that is linearly proportional to momentum, as it is known from the Langevin equation, but without putting it in by the assumption of a corresponding "friction potential", as in other similar approaches [6–10],

$$\left(\frac{\partial}{\partial t} + \frac{1}{m}P_R\frac{\partial}{\partial x}\right)P_R = \frac{D}{Dt}P_R = -\frac{\partial}{\partial x}\left(V + V_{qu}\right) - \gamma P_R. \tag{32}$$

Additionally, the nonlinear Schrödinger Equation (30) has Gaussian-shaped solutions, as the separation condition (29) that determines the additional nonlinear term is also compatible with Gaussian functions. Inserting the Gaussian ansatz (2) into (30) provides the equations of motion for the maximum and width,

$$\ddot{\eta} + \gamma\dot{\eta} + \omega^2\eta = 0 \tag{33}$$
$$\dot{\mathcal{C}} + \gamma\mathcal{C} + \mathcal{C}^2 + \omega^2 = 0. \tag{34}$$

The one for the maximum, and thus the classical trajectory, contains the linear friction term, the complex Riccati equation, an additional term linear in $\mathcal{C}$.

Also in this case, an Ermakov equation can be obtained from the Riccati equation, using the same definition of $\mathcal{C}_I$ in terms of $\alpha(t)$, i.e., $\mathcal{C}_I = \frac{1}{\alpha^2}$. Inserting this into the imaginary part of (34) now yields

$$\mathcal{C}_R = \frac{\dot{\alpha}}{\alpha} - \frac{\gamma}{2}. \tag{35}$$

With these expressions for $\mathcal{C}_R$ and $\mathcal{C}_I$, the real part of Equation (34) turns into the modified Ermakov equation:

$$\ddot{\alpha} + \left(\omega^2 - \frac{\gamma^2}{4}\right)\alpha = \frac{1}{\alpha^3}. \tag{36}$$

In a procedure similar to Ermakov's in the case without dissipation, from Equations (33) and (36), also in this case a dynamical invariant can be obtained that can be written as

$$\begin{aligned}
I_{NL} &= \frac{1}{2}\exp(\gamma t)\left[\left(\dot{\eta}\alpha - (\dot{\alpha} - \frac{\gamma}{2})\eta\right)^2 + \left(\frac{\eta}{\alpha}\right)^2\right] \\
&= \frac{1}{2}\exp(\gamma t)\alpha^2\left[(\dot{\eta} - \mathcal{C}_R\eta)^2 + (\mathcal{C}_I\eta)^2\right] = \text{const.}
\end{aligned} \tag{37}$$

Next we want to show how the same invariant can also be obtained via the complex hydrodynamic formulation, using again a (complex) Hamiltonian equation of motion with the Hamiltonian $H_{NL} = H_L + W_{NL}$ where W_{NL} is given by

$$W_{NL} = \gamma\frac{\hbar}{i}\left(\ln\psi - \langle\ln\psi\rangle\right) = \gamma\left(\frac{m}{2}\mathcal{C}\tilde{x}^2 + m\dot{\eta}\tilde{x} - \frac{m}{2}\mathcal{C}\langle\tilde{x}^2\rangle\right) \tag{38}$$

and the complex Riccati Equation (34).

From the Hamiltonian equation of motion follows

$$\frac{\partial}{\partial t}P = -\frac{\partial}{\partial x}H_{NL} = -P\mathcal{C} + m\dot{\mathcal{C}}x + m\mathcal{C}^2x + m\gamma\mathcal{C}x - \gamma P \tag{39}$$

where the last two terms on the rhs originate from W_{NL} and the additional linear term in the Riccati Equation (34).

Equation (39) can be rewritten using the relations

$$\frac{\partial}{\partial t} P + \gamma P = \exp(-\gamma t) \frac{\partial}{\partial t} \left(P \exp(\gamma t) \right) = \exp(-\gamma t) \frac{\partial}{\partial t} \widehat{P} \tag{40}$$

$$\dot{\mathcal{C}} + \gamma \mathcal{C} = \exp(-\gamma t) \frac{\partial}{\partial t} \left(\mathcal{C} \exp(\gamma t) \right) = \exp(-\gamma t) \frac{\partial}{\partial t} \widehat{\mathcal{C}} \tag{41}$$

$$\mathcal{C}_R = \frac{\dot{\alpha}}{\alpha} - \frac{\gamma}{2} = \frac{\frac{\partial}{\partial t} \alpha \exp(-\frac{\gamma}{2} t)}{\alpha \exp(-\frac{\gamma}{2} t)} = \frac{\frac{\partial}{\partial t} \widetilde{\alpha}}{\widetilde{\alpha}} = \frac{\partial}{\partial t} \ln \widetilde{\alpha} \tag{42}$$

with $\widehat{P} = P \exp(\gamma t)$, $\widehat{C} = C \exp(\gamma t)$ and $\widetilde{\alpha} = \alpha \exp(-\frac{\gamma}{2} t)$ to yield

$$\frac{\partial}{\partial t} \ln \left(\widehat{P} - m \widehat{C} x \right) = -\frac{\partial}{\partial t} \ln \widetilde{\alpha} - i \frac{1}{\alpha^2}. \tag{43}$$

This equation is identical to Equation (23), with only P and $\mathcal{C}$ replaced by $\widehat{P}$ and $\widehat{\mathcal{C}}$, as well as $\ln \alpha$ being replaced by $\ln \widetilde{\alpha}$.

With the same assumptions about the initial conditions as in the case without dissipation, the equation for the absolute value can be written as

$$\widetilde{\alpha}^2 \left| \widehat{P} - m \widehat{C} x \right|^2 = \alpha^2 \exp(\gamma t) \left| P - m \mathcal{C} x \right|^2 = m^2 \left(\frac{\alpha_0 p_0}{m} \right)^2. \tag{44}$$

Again using $P - m \mathcal{C} x = -m \mathcal{C} \eta + m \dot{\eta}$, but now with $\mathcal{C}_R = \frac{\dot{\alpha}}{\alpha} - \frac{\gamma}{2}$ finally leads to

$$\exp(\gamma t) \alpha^2 \left[\left(\dot{\eta} - \mathcal{C}_R \eta \right)^2 + (\mathcal{C}_I \eta)^2 \right] = \exp(\gamma t) \left[\left(\dot{\eta} \alpha - (\dot{\alpha} - \frac{\gamma}{2} \alpha) \eta \right)^2 + \left(\frac{\eta}{\alpha} \right)^2 \right] = \left(\frac{\alpha_0 p_0}{m} \right)^2, \tag{45}$$

which is, apart from a factor $1/2$, identical to the invariant I_{NL} in (37).

Since the common factor $\exp(\gamma t)$ in $\widehat{P}$ and $\widehat{\mathcal{C}}$ does not change the ratio of their real and imaginary parts, the phase of the complex quantity remains unchanged.

Concerning the variables involved, the major difference between the systems without and with dissipation is the replacement of P by $\widehat{P} = P \exp(\gamma t)$, which corresponds, according to our definition (11), to the replacement of $P_{op} = \frac{\hbar}{i} \frac{\partial}{\partial x}$ by $\widehat{P}_{op} = \exp(\gamma t) P_{op} = \frac{\hbar}{i} \exp(\gamma t) \frac{\partial}{\partial x}$.

This definition of "canonical" momentum is, however, identical to the one used by Caldirola [4] and Kanai [5] in their approach to describe the dissipative system within a canonical formalism that is connected with the physical position and momentum via a non-canonical transformation. Using canonical quantization, they obtain the corresponding operators and modified Schrödinger equation. As already mentioned, the quantized version of their approach seems to violate the uncertainty principle, which has been criticized by several authors [35–37], and attempts to justify or modify the Caldirola–Kanai approach [38–40] in order to avoid this nonphysical behavior are not convincing. In the next section, we will show how our nonlinear approach can be linked to the quantum version of Caldirola and Kanai, and how this can avoid the above-mentioned violation.

4. Connection with Canonical Approaches for the Description of Dissipative Systems

To show the connection with the canonical approaches, we recall Schrödinger's original definition of the wave function ψ via the action S in his first communication in 1926 [41];

$$S = K \ln \psi. \tag{46}$$

In the beginning, Schrödinger assumed K to be real and equal to $K = \hbar$ as well as ψ to be a real function of position and time. Later in the same year [42], he accepted that ψ must be complex and K has to include the imaginary unit $i = \sqrt{-1}$, i.e.,

$$S_{\mathrm{c}} = \frac{\hbar}{i} \ln \psi \tag{47}$$

with complex ψ, and therefore the subscript "c".

Using (47), the friction term (38) in our logarithmic nonlinear Schrödinger Equation (30) can be written as

$$W_{\mathrm{NL}} = \gamma \left(S_{\mathrm{c}} - \langle S_{\mathrm{c}} \rangle \right). \tag{48}$$

With the definition of our complex quantum hydrodynamic quantities according to (11), and dividing Equation (30) by ψ, allows to write this equation as

$$-\frac{\partial}{\partial t} S_{\mathrm{c}} = \mathrm{H_L} + \gamma \left(S_{\mathrm{c}} - \langle S_{\mathrm{c}} \rangle \right). \tag{49}$$

Ignoring the purely time-dependent term $\langle S_{\mathrm{c}} \rangle$ at the moment (we will see that this term is essentially necessary for normalization purposes) and using

$$\frac{\partial}{\partial t} S_{\mathrm{c}} + \gamma S_{\mathrm{c}} = \exp(-\gamma t) \frac{\partial}{\partial t} \left(S_{\mathrm{c}} \exp(\gamma t) \right) = \exp(-\gamma t) \frac{\partial}{\partial t} \widehat{S}_{\mathrm{c}} \tag{50}$$

with

$$\widehat{S}_{\mathrm{c}} = \exp(\gamma t) S_{\mathrm{c}}, \tag{51}$$

Equation (49) can be written as the transformed Hamilton–Jacobi equation,

$$\frac{\partial}{\partial t} \widehat{S}_{\mathrm{c}} + \widehat{\mathrm{H}} = 0 \tag{52}$$

with $\widehat{\mathrm{H}} = \exp(\gamma t) \mathrm{H_L}$.

A crucial point that is usually ignored by the canonical approaches is the change of the wave function, taking into account Schrödinger's definition (47) and Equation (51), according to

$$\ln \widehat{\psi} = \exp(\gamma t) \ln \psi, \tag{53}$$

i.e., a non-unitary transformation between the wave function $\psi(x, t)$, depending on the physical variables, and $\widehat{\psi}$ depending on the canonical variables, requesting a corresponding "renormalization" via $\langle S_{\mathrm{c}} \rangle$. In the case of the Caldirola–Kanai approach, only the momentum variable is non-canonically transformed, $\widehat{p} = \exp(\gamma t) p$, the position variable remains unchanged, $\widehat{x} = x$. This leads, unfortunately, to the wrong assumption that $\widehat{\psi}(\widehat{x}, t) = \widehat{\psi}(x, t)$ is identical to $\psi(x, t)$, ignoring (53) and thus causing an apparent violation of the uncertainty principle that can be avoided when operators and wave functions are transformed simultaneously [43].

Further details as well as an additional canonical/unitary transformation on the canonical level that leads to a time-independent Hamiltonian can be found in [27].

5. Connection with Generalized Creation-/Annihilation-Operators

The dynamical invariant derived in the cases without and with dissipation is the sum of two quadratic expressions that can also be expressed as the product of a linear expression and its complex conjugate. It will now be shown how such linear expressions are connected with our complex hydrodynamic approach and the creation/annihilation operators and their generalizations for coherent states with time-dependent width as well as those for systems with dissipation.

Keeping in mind the derivation of the dynamical invariant from our complex Hamiltonian Equations (20) and (39), the key points are the expressions (25) and (44), where the absolute values can also be considered as the product of two complex conjugate quantities.

Recalling the definition of our complex hydrodynamic quantities according to Equation (11), $P - m\mathcal{C}x$ can be considered as the application of an operator to the wave function ψ and then divided by ψ. The corresponding operator would be $P_{\text{op}} - m\mathcal{C}x = \frac{\hbar}{i}\frac{\partial}{\partial x} - m\mathcal{C}x$. Comparing this to the definition of the dynamical invariant as a product of two complex conjugate operators $a(t)$ and $a^{\dagger}(t)$, a corresponding operator would in our case be

$$a(t) \sim m\left(\frac{P_{\text{op}}}{m} - \mathcal{C}x\right) \tag{54}$$

or, taking into account the factor α in (25) and the constant factors,

$$a(t) = i\sqrt{\frac{m}{2\hbar}}\alpha\left(\frac{P_{\text{op}}}{m} - \mathcal{C}x\right). \tag{55}$$

This operator is a generalization of the conventional annihilation operator of the harmonic oscillator, that can be rewritten for $\dot{\alpha} = 0$ and $\frac{1}{\alpha^2} = \omega_0$ as

$$a_0 = i\sqrt{\frac{m}{2\hbar}}\left(\frac{P_{\text{op}}}{m} - i\omega_0 x\right). \tag{56}$$

Whereas the application of (56) can provide a Gaussian coherent state with constant width (although the general solution provides Gaussians with oscillating width), the application of (55) provides always generalized coherent states with a time-dependent width.

A further generalization of the annihilation (and corresponding creation) operator can be found in agreement with our results in Section 3 for dissipative systems, particularly with Equation (44) as

$$a_{\text{NL}}(t) = i\sqrt{\frac{m}{2\hbar}}\alpha\exp(\frac{\gamma}{2}t)\left(\frac{P_{\text{op}}}{m} - \mathcal{C}_{\text{NL}}x\right), \tag{57}$$

where $\mathcal{C}_{\text{NL}}$ is a solution of (34).

This operator also provides the Gaussian wave packet solution of the nonlinear Schrödinger Equation (30) (see also [27]).

For a time-independent solution of Riccati Equation (34), i.e., a particular solution $\widetilde{\mathcal{C}}$ for the damped harmonic oscillator, $\widetilde{\mathcal{C}} = -\frac{\gamma}{2} \pm i\Omega$ with $\Omega = \sqrt{\omega_0 - \frac{\gamma^2}{4}}$, the classical expression corresponding to (57) would be identical to Dekker's definition of his complex variable q in [11] (where his $\omega \doteq$ our Ω and his $\lambda \doteq$ our $\frac{\gamma}{2}$), apart from the missing exponential factor in his case. So, also a link to this approach and generalization to time-dependent uncertainties is possible.

This shows clearly how our complex hydrodynamic Hamiltonian approach is not only connected with Dekker's approach, but also with the powerful tool of creation and annihilation operators, even if time-dependent uncertainties and environmental effects are included.

6. Conclusions

Considering Hamiltonian systems where the Hamiltonian function, and consequently also the corresponding quantum mechanical operator, is no longer a constant of motion shows that there are systems where still another constant of motion exists. This is the case for the so-called parametric oscillator, i.e., a harmonic oscillator with time-dependent frequency $\omega = \omega(t)$. In this case, Ermakov showed [1] that from the classical Newtonian equation of the system with the help of an auxiliary, also a Newtonian-type equation, a dynamical invariant with the dimension of an action, can be obtained. In an earlier work, we showed that the same invariant can also be obtained via the more general Hamiltonian

formalism, if a complex Hamiltonian function is applied that originates from a complex hydrodynamic version of quantum mechanics, following Madelung's idea.

The questions we wanted to answer in this paper are whether it is also possible to extend our formalism to other systems where the Hamiltonian is no longer a constant of motion, particularly to open systems with dissipative forces and irreversible dynamics, and how our approach is related to other attempts trying to describe these systems and aiming to find corresponding invariants.

To be comparable with the situation describing the parametric oscillator and the other approaches mentioned above, we chose an effective one for the description of the open system, i.e., one where the degrees of freedom of the environment do not enter explicitly, only the effect of the environment on the system of interest.

To take into account the irreversibility of the dynamics, we added to the continuity equation for the probability density a time-symmetry breaking diffusion term, leading to a Fokker–Planck-type Smoluchowski equation. In order to find the corresponding changes in the modified Hamilton–Jacobi equation, the separation of the Smoluchowski equation into complex conjugate equations for ψ and ψ^* was necessary, which was possible for a well-defined separation condition, leading to a nonlinear Schrödinger equation with complex logarithmic nonlinearity. From the real part of this nonlinearity, an additional term to the modified Hamilton–Jacobi equation was obtained that, when taking the gradient of this equation, provided the desired friction force proportional to momentum or velocity.

Using a Gaussian wave packet ansatz for the solution of the nonlinear Schrödinger equation, two equations of motion for the time-dependence of the wave packet maximum and width were obtained that were Newtonian equations, similar to the ones used by Ermakov, but including the friction effects. These two equations could be combined to provide the dynamical invariant, but again only in a Newtonian formalism. However, within our hydrodynamic formulation, it was possible, with a new complex Hamiltonian, derived from the Hamiltonian operator of the nonlinear Schrödinger equation, to derive the same invariant via a Hamiltonian formalism. In this context, it turned out that the quantity (44) obtained from our complex Hamiltonian equation of motion and providing the invariant is closely related to non-canonical variables used in an approach by Caldirola [4] and Kanai [5]. Since our approach uses only physical position and momentum variables, no problems with apparent violations of the uncertainty principle occur.

Furthermore, this quantity is also, apart from constant factors, identical to generalized creation and annihilation operators for this open system. This could provide closer insight into the similarities between approaches using these operators and our complex hydrodynamic approach.

Additionally, a formal similarity to a complex Hamiltonian approach of Dekker [11,12] could be established, where our approach seems to be a generalization, including the time dependence of the quantum degrees of freedom.

Therefore, our complex hydrodynamic approach with complex Hamilton structures does not only supply an alternative to other effective descriptions of open systems, but can also be directly translated into the relevant variables and parameters of these approaches and vice versa.

These connections in themselves would already be interesting, but what could be the advantages and applications of our approach? A major motivation of our investigation is the treatment of time-dependent tunneling, also including environmental effects. Whereas time-independent tunneling is well understood, knowing that the boundary conditions at the borders to the barriers have to be fixed by the assumption of appropriate behavior of the wave function and its first derivative at these borders, the situation is more involved for time-dependent systems, as these boundary conditions also change in time. There are many different approaches to find analytical solutions to this problem using the time-dependent Schrödinger equation, but none of them is really completely satisfactory.

Alternative approaches are based on the hydrodynamic formulation of Madelung, but mostly used in the Bohmian interpretation [44,45]. In particular, an attempt by Nassar [46]

tries to describe this problem involving more hydrodynamic quantities, such as momentum and energy densities. In [14], we showed how to generalize this approach. A problem with these approaches based on Bohmian mechanics is that they consider real so-called Bohmian trajectories as hidden variables. The experiments of this year's Nobel laureates, particularly those of Aspect [32–34], however, indicate that this kind of variable does not exist. Does this rule out these approaches? Not necessarily, particularly not their numerical results, as in [31] we showed that a different, probabilistic interpretation of these trajectories can be given. Furthermore, Brandt et al. [47] have shown that an interpretation like ours can also be used to describe the tunneling process of interest. Therefore, our approach would also be applicable, but now even including dissipative effects, such as those in inelastic scattering. As nuclear fusion very recently has gained renewed remarkable interest and tunneling processes are involved, this might also be an interesting field of application for our approach.

Another field of problems that is gaining importance is the effect of tunneling in nano-structures, as our electronic devices become more and more miniaturized. When do tunneling effects play a crucial role for the functionality of these devices, and what is the influence of the environment, e.g., its temperature? To find at least qualitative answers would be very useful, and we think our hydrodynamic approach could be helpful in this direction.

Not to forget, our formalism can also be connected via the creation/annihilation operators with other approaches using these operators. So, for example, two-level systems as they are considered for quantum computing, can be described using these operators, e.g., in the Dicke model [48] or Jaynes–Cumming model [49,50]. Including environmental effects would be quite useful to estimate the influence on the performance of these devices.

We hope this shows that our complex hydrodynamic formulation of quantum mechanics can provide an alternative method to tackle important problems in quantum mechanics, such as time-dependent tunneling, with wide fields of application and that it is possible to link our results with other approaches, using formally different methods for the same purpose. In particular, the relation of our complex Hamiltonian approach to derive the dynamical invariant for the open system with generalized creation/annihilation operators appears promising.

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