





An Eco-Epidemiological Model Involving Prey Refuge and Prey Harvesting with Beddington–DeAngelis, Crowley–Martin and Holling Type II Functional Responses [†]

Megala Thangavel ^{1,*} , Nandha Gopal Thangaraj ¹ , Siva Pradeep Manickasundaram ¹ 
and Yasotha Arunachalam ² 

¹ Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore 641 020, Tamilnadu, India; nandhu792002@yahoo.co.in (N.G.T.); sivapradeep@rmv.ac.in (S.P.M.)

² Department of Mathematics, United Institute of Technology, Coimbatore 641 020, Tamilnadu, India; yasotha@uit.ac.in

* Correspondence: megala369@rmv.ac.in

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Abstract: This paper represents a three-species food web model based on the connections between susceptible prey, infected prey, and predator species. It is considered that in the absence of predator species, prey species grow logistically. Predators consume susceptible and infected prey in the form of Crowley–Martin and Beddington–DeAngelis functional responses. Also, infected prey consumes susceptible prey in the form of Holling type II interactions. Here, prey refuge and harvesting in prey with disease in a prey population are taken into consideration. Positiveness, boundedness, and positive invariance are examined. All biologically feasible equilibrium points are investigated. The local stability of positive equilibria and their global stability are analyzed by the suitable Lyapunov functions. Finally, numerical solutions are analyzed according to our findings.

Keywords: eco-epidemiological model; Crowley–Martin functional response; Beddington–DeAngelis form; stability; equilibrium



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1. Introduction

In mathematical ecology, at the beginning of twentieth century, many plans were made to statistically predict the existence and species of evolution. Certainly, the well-known classical Lotka–Volterra model was the first major effort in this area in 1927.

A certain percentage of prey populations can receive some degree of protection from nature by using its refuges. By lowering the risk of extinction from predation [1] and dampening prey–predator oscillations [2], such refuges can aid in extending predator–prey interactions. Studies in the literature reveal that refuges have both stabilizing [3] and destabilizing effects [4] in the environment. The fundamental model of Kermack–McKendric [5] on SIRS systems describes the diseases spread through contact. Mathematical modeling of epidemics has emerged as an important area of study. A significant amount of study has been conducted in this field [6–8].

One of the essential elements of predator–prey population modeling is a “Functional Response”. Most functional responses, such as the Holling kinds, are labeled “prey-dependent” since they are dependent on either the prey or the predator. In Crowley–Martin functional responses, both the prey and the predator are considered. Prey handling along with prey hunting are seen as two distinct and independent acts in Beddington–DeAngelis functional responses. In this study, the functional responses of Crowley–Martin, Holling type II, and Beddington–DeAngelis types are taken into account. The analysis of the consequences of disease on prey refuge, and harvesting in prey in the predator–prey

system is the primary focus of this study. Here, the boundedness and positive invariance of the equilibrium points of this system, together with their local and global stabilities, have all been investigated.

To the best of our knowledge, no researchers have investigated a three-species food web eco-epidemiological model with multi-functional response. So, we explore the three-species food web eco-epidemiological model with the Holling type II functional response $(\frac{\mathcal{Z}(1-\psi)AB}{p_1+(1-\psi)A})$, Beddington–DeAngelis functional response $(\frac{f_1BC}{(1+\beta B+\gamma C)})$, and Crowley–Martin functional response $\frac{\alpha_1 AC}{(1+\eta A)(1+\mu C)}$, as well as Hopf bifurcation as a result of this fact.

The main purpose is to investigate how disease affects a predator–prey relationship. Section 2 deals with the mathematical formulation. Section 3 contains some of the preliminary findings. Section 4 depicts the boundary equilibrium points and their stability. Section 5 investigates the local stability of the interior equilibrium point $\mathcal{E}^*(a^*, b^*, c^*)$ and determines its coexistence state. Section 6 shows the global stability for \mathcal{E}^* . In addition, in Section 7, we looked into Hopf bifurcation based on refuge ψ . In Section 8, all significant conclusions are mathematically validated using MATLAB software. This research’s conclusion and the biological implications of our findings are found in Section 9, which concludes the work.

2. Mathematical Formulation of the Model

Models are developed for a predator–prey system with prey’s refuge and harvesting.

$$\left. \begin{aligned} \frac{dA}{dT} &= r_1 A \left(1 - \frac{A+B}{K}\right) - \frac{\mathcal{Z}(1-\psi)AB}{p_1+(1-\psi)A} - \frac{\alpha_1 AC}{(1+\eta A)(1+\mu C)} - \mathcal{H}_1 \mathcal{E}_1 A, \\ \frac{dB}{dT} &= \frac{\mathcal{Z}(1-\psi)AB}{p_1+(1-\psi)A} - d_1 B - \frac{f_1 BC}{(1+\beta B+\gamma C)} - \mathcal{H}_2 \mathcal{E}_2 B, \\ \frac{dC}{dT} &= -d_2 C + \frac{n\alpha_1 AC}{(1+\eta A)(1+\mu C)} + \frac{nf_1 BC}{(1+\beta B+\gamma C)}, \end{aligned} \right\} \quad (1)$$

by the conditions of non-negative terms $A(0) = A_0 \geq 0, B(0) = B_0 \geq 0$, and $C(0) = C_0 \geq 0$. The detailed environmental illustrations of the parameters are given in Table 1.

Table 1. Environmental illustration of the system.

Parameters	Environmental Illustration
A, B, C	Susceptible prey, Infected prey, Predator
\mathcal{Z}, r_1, ψ	Infection rate, Growth rate of prey, refuge of prey
K, η, \mathcal{E}	Carrying capacity, Predator’s handling time, harvesting effort
p_1 and β	Half-saturation constant among infected prey and predators
α_1, n	Rate of predation on susceptible prey, Conversion of prey to predators
γ, μ	Magnitude of interference among predators by Crowley and Beddington
f_1	Capture rate by predator on susceptible prey
d_1 and d_2	Death rate of infected prey and predators
$\mathcal{H}_1, \mathcal{H}_2$	Catchability coefficients of susceptible and infected prey

To reduce the system (1) parameters, it is appropriate to change the variables as $a = \frac{A}{K}, b = \frac{B}{K}, c = \frac{C}{K}$, and the dimensionless time $t = \mathcal{Z}KT$.

In non-dimensional form,

$$\left. \begin{aligned} \frac{da}{dt} &= ra(1-a-b) - \frac{ab(1-\psi)}{p+(1-\psi)a} - \frac{\alpha ac}{(1+\eta a)(1+\mu c)} - h_1 a, \\ \frac{db}{dt} &= \frac{(1-\psi)ab}{p+(1-\psi)a} - db - \frac{\theta bc}{(1+\beta b+\gamma c)} - h_2 b, \\ \frac{dc}{dt} &= -\delta c + \frac{n\alpha ac}{(1+\eta a)(1+\mu c)} + \frac{n\theta bc}{(1+\beta b+\gamma c)} \end{aligned} \right\} \quad (2)$$

where $r = \frac{r_1}{\mathcal{Z}K}, \alpha = \frac{\alpha_1}{\mathcal{Z}K}, p = \frac{p_1}{K}, h_1 = \frac{\mathcal{H}_1 \mathcal{E}_1}{\mathcal{Z}K}, \theta = \frac{f_1}{\mathcal{Z}K}, h_2 = \frac{h_2 \mathcal{E}_2}{\mathcal{Z}K}, d = \frac{d_1}{\mathcal{Z}K}, \delta = \frac{d_2}{\mathcal{Z}K}$. The system’s initial conditions are $a(0) = a_0 \geq 0, b(0) = b_0 \geq 0$, and $c(0) = c_0 \geq 0$.

3. Positive Invariance and Boundedness

Let $X \equiv (a(t), b(t), c(t))^T$ and $\mathcal{S}(X) = (\mathcal{S}_1(X), \mathcal{S}_2(X), \mathcal{S}_3(X))^T$, where

$$\begin{aligned}\mathcal{S}_1(X) &= ra(1-a-b) - \frac{ab(1-\psi)}{p+(1-\psi)a} - \frac{\alpha ac}{(1+\eta a)(1+\mu c)} - h_1 a, \\ \mathcal{S}_2(X) &= \frac{(1-\psi)ab}{p+(1-\psi)a} - db - \frac{\theta ab}{(1+\beta b+\gamma a)} - h_2 b, \\ \mathcal{S}_3(X) &= -\delta c + \frac{n\alpha ac}{(1+\eta a)(1+\mu c)} + \frac{n\theta bc}{(1+\beta b+\gamma c)}.\end{aligned}$$

The model (2) can be stated as $\frac{dX}{dt} = \mathcal{S}(X)$ where $\mathcal{S} : C_+ \rightarrow \mathcal{R}_+^3$ with $X(0) = X_0 \in \mathcal{R}_+^3$, thus, $\mathcal{S}_k \in \mathcal{C}^\infty(\mathcal{R})$ for $k = 1, 2, 3$. The Lipschitzian function is \mathcal{S} and continuous on \mathcal{R}_+^3 . The model (2) contains positive initial conditions. Hence, (2) the region \mathcal{R}_+^3 is an invariant.

Theorem 1. All the system's solutions (2) are bounded in \mathcal{R}_+^3 .

Proof. Let $(a(t), b(t), c(t))$ be the solutions of the system (2) with non-negative conditions.

$$\frac{da}{dt} \leq a(1-a)$$

$\limsup_{t \rightarrow \infty} a(t) \leq 1$ (by the above inequality).

Let $\rho = a + b + c$.

$$\begin{aligned}\frac{d\rho}{dt} &= ra(1-a) - ab\left(r + \frac{(1-\psi)}{p+(1-\psi)a}\right) - \frac{\alpha ac(1-n)}{(1+\eta a)(1+\mu c)} - h_1 a + \frac{(1-\psi)ab}{p+(1-\psi)a} - db \\ &\quad - \frac{\theta bc}{(1+\beta b+\gamma c)}(1-n) - h_2 b - \delta c, \\ &\leq ra(1-a) - h_1 a - abr - b(d+h_2) - \delta c, \text{ (since } n < 1\text{)}, \\ &\leq \frac{r}{4} - h_1 a - b(d+h_2) - \delta c \text{ (since } \max\{ra(1-a)\} = \frac{r}{4}\text{)}, \\ &\leq \frac{r}{4} - \beta\rho. \text{ where, } \beta = \min\{h_1, d+h_2, \delta\}.\end{aligned}$$

Thus, we have $\frac{d\rho}{dt} + \beta\rho \leq \frac{r}{4}$. By differential inequality theory, we have $0 < \rho \leq \frac{r}{4\beta}(1 - \exp^{-\beta t}) + \rho(a_0, b_0, c_0)\exp^{-\beta t}$. For $t \rightarrow \infty$, since $0 < \rho \leq \frac{r}{4\beta}$. So, every solution of model (2) is confined to non-negative initial conditions around Ω

$$\Omega = \{(a, b, c) \in \mathcal{R}_+^3 : a + b + c \leq \frac{r}{4\beta} + \epsilon\}.$$

Hence, the result. \square

4. Boundary Equilibrium Points

- \mathcal{E}_0 is the point of trivial Equilibrium. Here, $\mathcal{E}_0(0, 0, 0)$ exists.
- \mathcal{E}_1 , diseased prey and no predator Equilibria, $\mathcal{E}_1(\frac{r-h_1}{r}, 0, 0)$ exists for $h_1 < r$.
- \mathcal{E}_2 is the equilibria with no predator, $\mathcal{E}_2(\bar{a}, \bar{b}, 0)$ where $\bar{a} = \frac{p(d+h_2)}{(1-d-h_2)(1-\psi)}$ and $\bar{b} = \frac{(r(1-a)-h_1)(p+(1-\psi)a)}{r+(1-\psi)}$. \mathcal{E}_2 exists for $p(d+h_2) < (1-d-h_2)$ and $h_1 < r(1-a)$.
- \mathcal{E}_3 is the equilibria with no disease, $\mathcal{E}_3(\bar{a}, 0, \bar{c})$ where $\bar{a} = \frac{\delta(1+\mu c)}{n\alpha - \eta\delta(1+\mu c)}$ and $\bar{c} = \frac{(r(1-a)-h_1)(1+\eta a)(1+\mu)}{\alpha}$. \mathcal{E}_3 exists for $\eta\delta(1+\mu c) < n\alpha$ and $h_1 < r(1-a)$.
- \mathcal{E}^* is the equilibria of interior which is positive, by system (2) $\mathcal{E}^*(a^*, b^*, c^*)$. It exists for $\delta > n\alpha$, $(1+\beta b^* + \gamma c^*) > 0$, $r(1-a^* - b^*) > h_1 - b^*$, $\alpha p > 0$. Where, $a^* = \frac{(p+(1-\psi))((d+h_2)+(1+\beta b^* + \gamma c^*)+\theta c^*)}{(1-\psi)(1+\beta b^* + \gamma c^*)}$,

$$b^* = \frac{(1+\beta+\gamma c^*)(\delta(1+\eta a^*)(1+\mu c^*)-n\alpha a^*)}{(1+\eta a^*)(1+\mu c^*)n\theta},$$

$$c^* = \frac{((1+\eta a^*)(1+\mu))(p+(1-\psi)a^*(r(1-a^*-b^*)-h_1)-b^*(1-\psi))}{\alpha(p+(1-\psi)a^*)}.$$

5. Local Stability

The Jacobian matrix for local stability analysis at an arbitrary point (a, b, c) is

$$n_{11} = r(1-2a-b) - \frac{pb(1-\psi)}{(p+a-\psi a)^2} - \frac{\alpha c}{(1+\eta a)^2(1+\mu c)} - h_1, n_{12} = -ar - \frac{a(1-\psi)}{a-a\psi+p},$$

$$n_{13} = -\frac{\alpha a}{(1+\mu c)^2(1+\eta a)^2}, n_{21} = \frac{pb(1-\psi)}{(p+a-\psi a)^2}, n_{31} = \frac{\alpha cn}{(1+\eta a)^2(1+\mu c)},$$

$$n_{22} = \frac{a(1-\psi)}{a-a\psi+p} - d - \frac{c\theta(1+\gamma c)}{(1+\beta b+\gamma c)^2} - h_2, n_{23} = -\frac{b\theta(1+\beta b)}{(1+\beta b+\gamma c)^2},$$

$$n_{32} = \frac{\theta cn(1+\gamma c)}{(1+\beta b+\gamma c)^2}, n_{33} = -\delta + \frac{\alpha an}{(1+\mu c)^2(1+\eta a)} + \frac{\theta bn(1+\beta b)}{(1+\beta b+\gamma c)^2}.$$

Theorem 2. In the system (2), we have

1. The equilibria of trivial point $\mathcal{E}_0(0,0,0)$ is locally stable if $r < h_1$; otherwise, it is unstable.
2. The equilibria without infection and predator $\mathcal{E}_1(\frac{r-h_1}{r}, 0, 0)$ is locally asymptotically stable if $r < h_1$, $-d-h_2 > \frac{(r-h_1)(1-\psi)}{(r-h_1)(1-\psi)+p}$, $\delta < \frac{(r-h_1)n\alpha}{1+\eta(r-h_1)}$.
3. The equilibria with no predator $(\bar{a}, \bar{b}, 0)$ is locally asymptotically stable if $\mathcal{Y}_{11} > 0$, $\mathcal{Y}_{12} > 0$, and $\delta > \frac{n\alpha\bar{a}}{1+\eta\bar{a}} + \frac{bn\theta(1+\beta\bar{b})}{(\beta\bar{b}+1)^2}$.

Proof.

1. The eigenvalues of $\mathcal{E}_0(0,0,0)$ are $r-h_1$, $-d-h_2$, $-\delta$. Hence, it is locally asymptotically stable when $r < h_1$. If not, it is unstable.
2. The eigen values of $\mathcal{E}_1(\frac{r-h_1}{r}, 0, 0)$ are h_1-r , $\frac{(r-h_1)(1-\psi)}{(r-h_1)(1-\psi)+p} - d - h_2$, $\delta + \frac{(r-h_1)n\alpha}{(r+\eta(r-h_1))}$. Hence, it is locally asymptotically stable if $r < h_1$, $-d-h_2 > \frac{(r-h_1)(1-\psi)}{(r-h_1)(1-\psi)+p}$, $\delta < \frac{(r-h_1)n\alpha}{1+\eta(r-h_1)}$. If not, it is unstable.
3. The Jacobian matrix is

$$n_{11} = r(1-2\bar{a}-\bar{b}) - \frac{p\bar{b}(1-\psi)}{(p+\bar{a}-\psi\bar{a})^2} - h_1, n_{12} = -\bar{a}r - \frac{\bar{a}(1-\psi)}{\bar{a}+\bar{a}\psi+p}, n_{13} = -\frac{\alpha\bar{a}}{1+\eta\bar{a}},$$

$$n_{21} = \frac{p\bar{b}(1-\psi)}{(p+\bar{a}-\psi\bar{a})^2}, n_{22} = \frac{\bar{a}(1-\psi)}{\bar{a}-\bar{a}\psi+p} - d - h_2, n_{23} = -\frac{\bar{b}\theta(1+\beta\bar{b})}{(\beta\bar{b}+1)^2}, n_{31} = 0, n_{32} = 0,$$

$$n_{33} = -\delta + \frac{\bar{a}n\alpha}{1+\eta\bar{a}} + \frac{bn\theta(1+\beta\bar{b})}{(\beta\bar{b}+1)^2}.$$

The characteristic form of $J(\mathcal{E}_2)$ is $(n_{33}-\lambda)(\lambda^2+\mathcal{Y}_{11}\lambda+\mathcal{Y}_{12})=0$, where $\mathcal{Y}_{11}=-(n_{11}+n_{22})$ and $\mathcal{Y}_{12}=n_{11}n_{22}-n_{12}n_{21}$. Hence, the one of the eigenvalues of the above characteristic equation is n_{33} , which is negative, and the other two eigenvalues also must be negative. Hence, \mathcal{E}_2 is locally asymptotically stable if $\mathcal{Y}_{11} > 0$, $\mathcal{Y}_{12} > 0$ and $\delta > \frac{\bar{a}n\alpha}{1+\eta\bar{a}} + \frac{bn\theta(1+\beta\bar{b})}{(\beta\bar{b}+1)^2}$. \square

Theorem 3. The equilibria with no infection $(\bar{a}, 0, \bar{c})$ is locally asymptotically stable if $\mathcal{Y}_{11} > 0$, $\mathcal{Y}_{12} > 0$ and $-(d + \frac{c\theta(1+\gamma c)}{(\beta b+\gamma c+1)^2} + h_2) > \frac{a(1-\psi)}{a-a\psi+p}$. (This proof is similar to Theorem 2.(3))

Theorem 4. The point of equilibria \mathcal{E}^* is locally asymptotically stable if $\mathcal{Y}_1 > 0$, $\mathcal{Y}_3 > 0$, and $\mathcal{Y}_1\mathcal{Y}_2 - \mathcal{Y}_3 > 0$.

Proof. The Jacobian form of system (2) at $\mathcal{E}^*(a^*, b^*, c^*)$ where,

$$\begin{aligned} g_{11} &= r(1 - 2a^* - b^*) - \frac{pb^*(1 - \psi)}{(p + a^* - \psi a^*)^2} - \frac{\alpha c^*}{(1 + \eta a^*)^2(1 + \mu c^*)} - h_1, \\ g_{12} &= -a^*r - \frac{a^*(1 - \psi)}{a^* - a^*\psi + p}, g_{13} = -\frac{\alpha a^*}{(1 + \mu c^*)^2(1 + \eta a^*)^2}, g_{21} = \frac{pb^*(1 - \psi)}{(p + a^* - \psi a^*)^2}, \\ g_{31} &= \frac{\alpha c^*n}{(1 + \eta a^*)^2(1 + \mu c^*)}, g_{22} = \frac{a^*(1 - \psi)}{a^* - a^*\psi + p} - d - \frac{c^*\theta(1 + \gamma c^*)}{(1 + \beta b^* + \gamma c^*)^2} - h_2, \\ g_{23} &= -\frac{b^*\theta(1 + \beta b^*)}{(1 + \beta b^* + \gamma c^*)^2}, g_{32} = \frac{\theta c^*n(1 + \gamma c^*)}{(1 + \beta b^* + \gamma c^*)^2}, \\ g_{33} &= -\delta + \frac{\alpha a^*n}{(1 + \mu c^*)^2(1 + \eta a^*)} + \frac{\theta b^*n(1 + \beta b^*)}{(1 + \beta b^* + \gamma c^*)^2}. \end{aligned}$$

The equation for the cubic characteristic of $J(\mathcal{E}^*)$ is

$$\lambda^3 + \mathcal{Y}_1\lambda^2 + \mathcal{Y}_2\lambda + \mathcal{Y}_3 = 0. \quad (3)$$

$\mathcal{Y}_1 = -(g_{11} + g_{22} + g_{33})$, $\mathcal{Y}_2 = -(g_{12}g_{21} + g_{13}g_{31} + g_{23}g_{32} - g_{11}g_{22} - g_{11}g_{33} - g_{22}g_{33})$, $\mathcal{Y}_3 = -(g_{11}g_{22}g_{33} + g_{12}g_{23}g_{31} + g_{13}g_{21}g_{32} - g_{13}g_{31}g_{22} - g_{12}g_{21}g_{33} - g_{11}g_{23}g_{32})$. If $\mathcal{Y}_1 > 0$, $\mathcal{Y}_3 > 0$, and $\mathcal{Y}_1\mathcal{Y}_2 - \mathcal{Y}_3 > 0$. Negative real parts are the root of characteristic equation if and only if \mathcal{Y}_1 , \mathcal{Y}_3 , and $\mathcal{Y}_1\mathcal{Y}_2 - \mathcal{Y}_3 > 0$. By Routh–Hurwitz, \mathcal{E}^* is locally asymptotically stable. \square

6. Global Stability

Theorem 5. If \mathcal{E}^* is globally asymptotically stable in $H = \{(a, b, c) : a > a^*, b > b^* \text{ and } c > c^* \text{ or } a < a^*, b < b^* \text{ and } c < c^*\}$.

Proof. A Lyapunov function is in the form of

$$\mathcal{L}_1(a, b, c) = \mathcal{L}_2(a - a^* - a^*\ln \frac{a}{a^*}) + (b - b^* - b^*\ln \frac{b}{b^*}) + \mathcal{L}_3(c - c^* - c^*\ln \frac{c}{c^*}),$$

where $\mathcal{L}_2, \mathcal{L}_3$ are positive constants.

Differentiating \mathcal{L}_1 with respect to t along with the solution of (2),

$$\begin{aligned} \frac{d\mathcal{L}_1}{dt} &= \left(\frac{a - a^*}{a}\right) \frac{da}{dt} + \mathcal{L}_2 \left(\frac{b - b^*}{b}\right) \frac{db}{dt} + \mathcal{L}_3 \left(\frac{c - c^*}{c}\right) \frac{dc}{dt} \\ &= [r(1 - a - b) - \frac{(1 - \psi)b}{p + (1 - \psi)a} - \frac{\alpha c}{(1 + \eta a)(1 + \mu c)} - h_1](a - a^*) \\ &\quad + \mathcal{L}_2 \left[\frac{(1 - \psi)a}{p + (1 - \psi)a} - d - \frac{\theta c}{(1 + \beta b + \gamma c)} - h_2\right](b - b^*) \\ &\quad + \mathcal{L}_3 \left[-\delta + \frac{n\alpha a}{(1 + \eta a)(1 + \mu c)} + \frac{n\theta b}{(1 + \beta b + \gamma c)}\right](c - c^*). \text{ (After simplifications we get),} \\ \frac{d\mathcal{L}_1}{dt} &= -(a - a^*)[r(a + b) - (a^* + b^*)] + (1 - \psi) \left(\frac{b}{p + (1 - \psi)a} - \frac{b^*}{p + (1 - \psi)a^*}\right) \\ &\quad + \alpha \left(\frac{c}{(1 + \eta a)(1 + \mu c)} - \frac{c^*}{(1 + \eta a^*)(1 + \mu c^*)}\right) \\ &\quad - \mathcal{L}_2(b - b^*) \left[(1 - \psi) \left(\frac{a}{p + (1 - \psi)a} - \frac{a^*}{p + (1 - \psi)a^*}\right) \right. \\ &\quad \left. - \theta \left(\frac{c}{a + (1 + \beta b + \gamma c)} - \frac{c^*}{1 + \beta b^* + \gamma c^*}\right) \right] \\ &\quad - \mathcal{L}_3(c - c^*)n \left[\left(\frac{\alpha(a - a^*) + c\mu^*(a - a^*)}{(1 + \eta a)(1 + \mu c)(1 + \eta a^*)(1 + \mu c^*)}\right) + \theta \left(\frac{(b - b^*) + \gamma(bc^* - b^*c)}{(1 + \beta b + \gamma c)(1 + \beta b^* + \gamma c^*)}\right)\right]. \end{aligned}$$

We see that for $\frac{d\mathcal{L}_1}{dt}$, the region is negative:

$H = \{(a, b, c) : a > a^*, b > b^* \text{ and } c > c^* \text{ or } a < a^*, b < b^* \text{ and } c < c^*\}$ and as a result, \mathcal{L}_1 is a Lyapunov function for all solutions in H . \square

7. Hopf Bifurcation Analysis

Theorem 6. For Hopf bifurcation occurring in the model (2), suppose that the bifurcation parameter ψ exceeds a substantial value. The occurrence of the following Hopf bifurcation criteria, $\psi = \psi^*$

1. $\mathcal{H}(\psi^*)\mathcal{I}(\psi^*) - \mathcal{J}(\psi^*) = 0$;
2. $\frac{d}{d\psi}(\text{Re}(\rho(\psi)))|_{\psi=\psi^*} \neq 0$, where ρ is the zero of the characteristic equation, which equates to the equilibrium point's positive value.

Proof. For $\psi = \psi^*$, the characteristic Equation (3) implies that $(\rho^2(\psi^*) + \mathcal{I}(\psi^*))(\rho(\psi^*) + \mathcal{H}(\psi^*)) = 0$. (i.e., $\pm i\sqrt{\mathcal{I}(\psi^*)}$ and $-\mathcal{H}(\psi^*)$ are the roots of the Equation (7). To establish that Hopf bifurcation exists at the point, we must fulfill the transversality requirement. $\psi^* = \psi$. $\frac{d}{d\psi}(\text{Re}(\rho(\psi)))|_{\psi=\psi^*} \neq 0$. For all ψ , the general roots of the form $\rho_{1,2}(\psi) = r(\psi) \pm is(\psi)$, and $\rho_3(\psi) = -\mathcal{H}(\psi)$. Now, we check the condition $\frac{d}{d\psi}(\text{Re}(\rho_j(\psi)))|_{\psi=\psi^*} \neq 0, j = 1, 2$. If $\rho_1(\psi) = r(\psi) + is(\psi)$ in (7), we obtain $\gamma_1(\psi) + i\gamma_2(\psi) = 0$, where, $\gamma_1(\psi) = r^3(\psi) + r^2(\psi)\mathcal{H}(\psi) - 3r(\psi)s^2(\psi) - s^2(\psi)\mathcal{H}(\psi) + r(\psi)\mathcal{I}(\psi) + \mathcal{H}(\psi)\mathcal{I}(\psi)$, $\gamma_2(\psi) = 3r^2(\psi)s(\psi) + 2r(\psi)s(\psi)\mathcal{H}(\psi) - s^3(\psi) + s(\psi)\mathcal{I}(\psi)$.

In order to complete Equation (7), we must have $\gamma_1(\psi) = 0$ and $\gamma_2(\psi) = 0$. Then, differentiating γ_1 and γ_2 with respect to ψ , we have

$$\frac{d\gamma_1}{d\psi} = \mathcal{T}_1(\psi)r'(\psi) - \mathcal{T}_2(\psi)s'(\psi) + \mathcal{T}_3(\psi) = 0, \quad (4)$$

$$\frac{d\gamma_2}{d\psi} = \mathcal{T}_2(\psi)r'(\psi) + \mathcal{T}_1(\psi)s'(\psi) + \mathcal{T}_4(\psi) = 0, \quad (5)$$

$$\mathcal{T}_1(\psi) = 3r^2(\psi) + 2r(\psi)\mathcal{H}(\psi) - 3s^2(\psi) + \mathcal{I}(\psi),$$

$$\mathcal{T}_2(\psi) = 6r(\psi)s(\psi) + 2s(\psi)\mathcal{H}(\psi),$$

$$\mathcal{T}_3(\psi) = r^2(\psi)\mathcal{H}'(\psi) - s^2(\psi)\mathcal{H}'(\psi) + \mathcal{J}'(\psi) + \mathcal{I}'(\psi)r(\psi),$$

$$\mathcal{T}_4(\psi) = 2r(\psi)s(\psi)\mathcal{H}'(\psi) + s(\psi)\mathcal{I}'(\psi).$$

$$r'(\psi) = -\frac{\mathcal{T}_1(\psi)\mathcal{T}_3(\psi) + \mathcal{T}_2(\psi)\mathcal{T}_4(\psi)}{\mathcal{T}_1^2(\psi) + \mathcal{T}_2^2(\psi)}. \quad (6)$$

$r(\psi) = 0$ and $s(\psi) = \sqrt{\mathcal{I}(\psi)}$ at $\psi = \psi^*$ on $\mathcal{T}_1(\psi), \mathcal{T}_2(\psi), \mathcal{T}_3(\psi)$ and $\mathcal{T}_4(\psi)$.

So, $\mathcal{T}_1(\psi^*) = -2\mathcal{I}(\psi^*)$, $\mathcal{T}_2(\psi^*) = 2\sqrt{\mathcal{I}(\psi^*)}\mathcal{H}(\psi^*)$, $\mathcal{T}_3(\psi^*) = -\mathcal{I}(\psi^*)\mathcal{H}'(\psi^*) + \mathcal{J}'(\psi^*)$, $\mathcal{T}_4(\psi^*) = \sqrt{\mathcal{I}(\psi^*)}\mathcal{I}'(\psi^*)$.

$$r'(\psi^*) = \frac{\mathcal{J}'(\psi^*) - (\mathcal{H}(\psi^*)\mathcal{I}'(\psi^*) + \mathcal{I}(\psi^*)\mathcal{H}'(\psi^*))}{2(\mathcal{I}^2(\psi^*) + \mathcal{H}^2(\psi^*))}, \quad (7)$$

If $\mathcal{J}'(\psi^*) - (\mathcal{H}(\psi^*)\mathcal{I}'(\psi^*) + \mathcal{I}(\psi^*)\mathcal{H}'(\psi^*)) \neq 0$,

(i.e., $\frac{d}{d\psi}(\text{Re}(\rho_j(\psi)))|_{\psi=\psi^*} = r'(\psi^*) \neq 0$. $j = 1, 2$, and $\rho_3(\psi^*) = -\mathcal{H}(\psi^*) \neq 0$.

Thus, the condition $\mathcal{J}'(\psi^*) - (\mathcal{H}(\psi^*)\mathcal{I}'(\psi^*) + \mathcal{I}(\psi^*)\mathcal{H}'(\psi^*)) \neq 0$, the transversality criteria are confirmed, and the system (2) experiences Hopf bifurcation at $\psi = \psi^*$. \square

8. Numerical Simulation

In this section, a few simulations on the system (2) are performed to support the theoretical conclusions. The refuge ψ is used as a control parameters. For the fixed parameter, MATLAB and MATHEMATICA software tools are used to carry out the simulation. Here, $r = 0.1, d = 0.1, \delta = 0.2, \theta = 0.23, \mu = 0.15, \eta = 0.14, \alpha = 0.4$, and $\psi = \text{variable}$. For Bifurcation of refuge ψ , if $\psi = 0.2$, the model (2) of positive equilibria is asymptotically stable $\mathcal{E}^*(0.52754, 0.0916718, 0.204662)$ and the other parameter values are the identical. As a result of increasing the bifurcation parameter value, $\psi = 0.5$, the model (2) lost its stability, meaning it is asymptotically unstable at $\mathcal{E}^*(0.53814, 0.0917798, 0.320138)$. Then, the model (2) passes the transversality conditions for $(\text{Re}(\rho(\psi)))|_{\psi=\psi^*} = 0.002105 \neq 0$. The graph Figure 1 depicts the system's (2) behavioral alterations at refuge, $\psi = 0.5$.

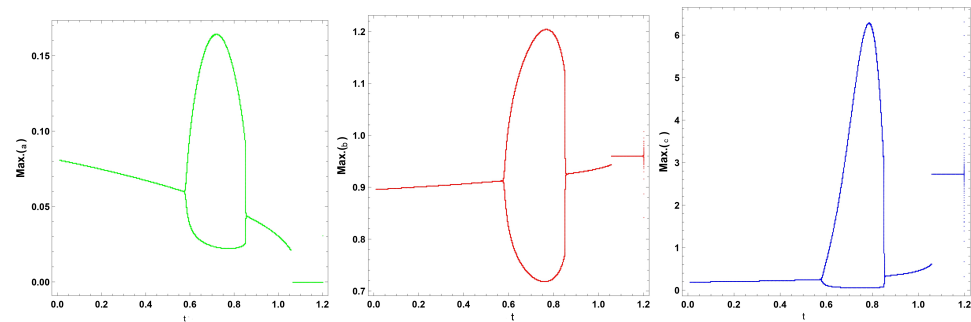


Figure 1. Dynamical changes of the system (2) at refuge $\psi = 0.5$.

9. Conclusions and Discussion

We explored an eco-epidemiological model involving refuge of prey and harvesting in prey with illness in the prey population, in which the predator predate both the sick and susceptible prey. The results of the boundedness and positivity indicate that the constructed system (2) is well behaved biologically. When the intrinsic growth rate of the susceptible prey is smaller than the harvesting rate of the susceptible prey, the population will go extinct. The system's local stability at each biologically feasible equilibrium point and the equilibria for interior (2) has been established. The analytic and numerical results for hopf bifurcation are observed above. This study demonstrates complex behavior, such as infectious prey refuge and prey harvesting, which provides rich dynamics.

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