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Anisotropy-Based Estimation for Sensor Network with Non-Centered Disturbance †

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Abstract: This paper concerns the anisotropy-based estimation design for sensor networks with coloured external disturbance. The boundedness criterion of anisotropic norm for estimation problems in network systems relies on the analysis of multiplicative noise systems in the framework of anisotropy-based theory. The solution of the considered problem is reduced to a convex optimization problem.

Keywords: anisotropy-based theory; multiplicative noise system; estimation; convex optimization

1. Introduction

The estimation problem has remained a fundamental one in control and filtering theory since the 1960s, when R. Kalman proposed an optimal filtering approach based on prediction and correction concepts [1]. The other well known $\mathcal{H}_2$ and $\mathcal{H}_\infty$ methods use different assumptions on both system and input properties in order to design controllers and/or filters in the presence of measurement noise, exogenous disturbances, system uncertainties, etc. [2,3]. Each of these theories has its own benefits as well as disadvantages. For this reason, many attempts to unify and generalize $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theories have been made over the last 30 or more years [4–6]. Initially solved in Riccati equation terms, the $\mathcal{H}_2$, $\mathcal{H}_\infty$ and mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problems were then solved via linear matrix inequalities (LMI) techniques [7–9]. This brings the study into the new era of approaches and methods applicable to practice.

In 1994, I. Vladimirov suggested an approach to generalize (in a stochastic and robust sense) $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering and control theories. This approach takes into account stochastic disturbance uncertainty described by means of an information criterion called anisotropy [10,11]. Introducing a certain performance gain, this theory succeeded in generalizing the mentioned $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theories. The problem of analysis for time-varying systems was studied in [12], and the filtering problem for this type of system was considered in [13], where an optimal solution of the filtering problem was obtained in terms of the Riccati equation solution. Widely used in control theory, the LMI approach was adapted in anisotropy-based theory later in [14]. In [15], a general case of anisotropic norm boundedness sufficient conditions for the estimation problem were derived using forward-time difference LMIs. An anisotropy-based analysis for non-zero disturbances with constraint on the first and second moments of input disturbance was considered in [16], and a design was presented in [17].

Despite the fact that many problems have successfully been solved in anisotropy-based theory, only deterministic linear systems have been the subject of study. The corresponding analysis for stochastic systems was studied in [18]. It helped to take into account multiplicative noise systems. This kind of system describes financial and population models, mechanical and hybrid systems, sensor networks, etc. The boundedness criterion for the anisotropic

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norm of a multiplicative noise system is suggested in [19]. After that, the anisotropy-based analysis yields a solution of the estimation problem for time-varying sensor networks. The case of network systems was studied in [20], and the case of dropouts with corrections was considered in [21]. Both papers contained a convex optimization approach to calculate the minimal value of xanisotropic norm upper bound for an input-to-estimation error system. A multiplicative noise system analysis for non-centered disturbance was described in [22]. A natural continuation of sensor network research is to extend anisotropy-based estimation for sensor networks with non-centered disturbance, as far as possible.

In this paper, the problem of anisotropy-based estimation for a sensor network system with non-centered disturbances is considered. The external disturbances are assumed to be sequences of random vectors with bounded anisotropy and additional constrain on two stochastic moments are given.

2. Background

The basic concepts of anisotropy-based theory are the anisotropy of a random vector and the mean anisotropy of a random vector sequence. If a time-varying system with finite time horizon is considered, then the anisotropy of an extended vector of input is used in the analysis. Likewise, the mean anisotropy of a random sequence is used when time-invariant systems are under research.

The anisotropy of a real-valued random vector W with a finite second moment is defined in [13] as $\mathbf{A}(W) = \inf_{\lambda>0} \mathbf{D}(f||p_{m,\lambda}) = \frac{m}{2}\ln\left(\frac{2\pi e}{m}\mathbf{E}|W|^2\right) - h(W)$, where $\mathbf{D}$ is the Kullback–Leibler information divergence, $\mathbf{E}|\cdot|$ is the expectation, f denotes the probability density function (pdf) of W w.r.t. Lebesque measure in $\mathbb{R}^m$, $h(W) = -\int_{\mathbb{R}^m} f(w)\ln f(w)dw$ is the differential entropy of W, and λ is considered to bea positive parameter which defines the following pdf of etalon vectors: $p_{m,\lambda}(x) = (2\pi\lambda)^{-m/2}\exp\left(-\frac{|x|^2}{2\lambda}\right)$. The function $p_{m,\lambda}(x)$ corresponds to isotropic Gaussian distribution with zero mean and scalar covariance matrix λI_m, where I_m denotes an m-dimensional identity matrix. The anisotropy of a random vector has a simple meaning: it shows the measure of the difference between the pdf of a certain vector and a set of vectors that have isotropic Gaussian pdfs parameterized by λ. It can be easily calculated that, if random vector has zero mean Gaussian distribution with covariance matrix Σ, the precise formula for its anisotropy (as given in [12]) is of the following form:

$$\mathbf{A}(W) = -\frac{1}{2}\ln\det\left(\frac{m\Sigma}{\mathrm{tr}(\Sigma)}\right).$$

Anisotropy-based analysis for time-varying systems with non-centered disturbance was studied in [16,23]. As derived in [24], the anisotropy of an m-dimensional random vector with expectation μ and covariance matrix Σ is equal to the following formula:

$$\mathbf{A}(W) = -\frac{1}{2}\ln\det\left(\frac{m\Sigma}{\mathrm{tr}(\Sigma)+|\mu|^2}\right).$$

The performance criterion in anisotropy-based theory is caused by using the information functional for input disturbance. The anisotropic norm of the linear discrete time-varying system is induced by another norm. Let us consider the matrix $F \in \mathbb{L}^{p\times m}$ of a linear operator mapping for input vectors W into output vectors Z: for random input $W \in \mathbb{L}_2^m$, the output $Z \in \mathbb{L}_2^p$ is defined as $Z = FW$. In this paper, the components of vector W and matrix F are considered to be mutually independent. The root mean square (RMS) gain of the matrix F is defined as $Q(F,W) = \sqrt{\frac{\mathbf{E}|FW|^2}{\mathbf{E}|W|^2}} = \sqrt{\frac{\mathrm{tr}(\Lambda\Sigma)}{\mathrm{tr}(\Sigma)}}$, where $\Lambda = \mathbf{E}|F^{\mathrm{T}}F|$, $\Sigma = \mathbf{E}|WW^{\mathrm{T}}|$. The supremum of RMS gain coincides with the maximum singular value of the matrix Λ, and it will further be referred to as the $\mathcal{H}_\infty$ norm of F:

$\sup_{W \in \mathbb{L}_2^m} Q(F, W) = \sqrt{\max_{i=\overline{1,m}} \lambda_i(\mathbf{E}[F^{\mathrm{T}}F])} = \|F\|_\infty$. If the input vector W has zero mean Gaussian pdf with scalar covariance matrix, the RMS gain is equal to the stochastic analogue of the Frobenius norm of matrix F: $Q(F, W) = \frac{\|F\|_2}{\sqrt{m}} = \sqrt{\frac{\mathrm{tr}\Lambda}{m}}$.

To denote the set of random vectors with anisotropy by a, we use the following notation: $\mathbf{W}_a = \{W \in \mathbb{L}_2^m : \quad \mathbf{A}(W) \leqslant a\}$. With that, the anisotropic norm of F is defined by the following formula:

$$|\!|\!|F|\!|\!|_a = \sup_{W \in \mathbf{W}_a} Q(F, W). \tag{1}$$

Now, let us briefly consider the case of non-centered disturbances, see [16,23] for more details. If a random vector W is a sum of the zero-mean vector $\widetilde{W}$ and the constant vector $\mu = \mathbf{E}[W]$, the RMS gain of the system (operator) with such an input is expressed by the following equation: $Q(F, W) = \sqrt{\frac{\mathbf{E}[|F\widetilde{W}|^2] + \mathbf{E}[|F\mu|^2]}{\mathbf{E}[|\widetilde{W}|^2] + |\mu|^2}}$. Consequently, the anisotropic norm of F in this case is defined as follows:

$$|\!|\!|F|\!|\!|_a = \sup_{W \in \mathbf{W}_a} \sqrt{\frac{\mathrm{tr}(\Lambda\Sigma) + \mu^{\mathrm{T}}\Lambda\mu}{\mathrm{tr}\Sigma + \mu^{\mathrm{T}}\mu}}. \tag{2}$$

Since both anisotropy and anisotropic norm are invariant under the scaling of input W, one can consider only the case when the two following constraints are given: $|\mu| \geqslant \tau \in [0, 1)$ and $\mathrm{tr}\Sigma \leqslant \sigma$. In [16,23], it was shown that the considered constrains for the first and second moments can be equivalently replaced by those that satisfy $\sigma + \tau^2 = 1$. This condition allows (2) to be modified in the following form:

$$|\!|\!|F|\!|\!|_{a,\tau} = \sup_{\Sigma = \Sigma^{\mathrm{T}} \succ 0} \left\{ (\mathrm{tr}(\Lambda\Sigma) + \sup_{e_0 \neq 0} \mathbf{E}[|Fe_0|^2]\tau^2)^{1/2} - \frac{1}{2}\ln\det(m\Sigma) \leqslant a,\ \mathrm{tr}(\Sigma) = 1 - \tau^2 \right\}, \tag{3}$$

where e_0 corresponds to unit vector $e_0 = \mu/|\mu|$. In (3), the second term inside the supremum over Σ takes the maximum value when unit vector e_0 corresponds to the eigenvector of the maximum eigenvalue of Λ, hence giving $\sup_{e_0 \neq 0} \mathbf{E}[|Fe_0|^2] = \|F\|_\infty^2$. The first term corresponds to the problem of local maximum finding of functional $\mathrm{tr}(\Lambda\Sigma)$ subject to one equality constraint $\mathrm{tr}\Sigma = 1 - \tau^2$ and one inequality constraint $-\frac{1}{2}\ln\det(m\Sigma) \leqslant a$. Note that $\tau \in [0; 1)$ is assumed to be given. This problem can be solved by means of the Lagrange method, and all details can be found in [16,23]. Based on [22,23], the problem of anisotropy-based estimation is studied in this paper for multiplicative noise systems.

3. Problem Statement

In this section, a model of a sensor network with random failures is considered. The description of the communication scheme for the sensors is associated with an adjacency matrix of the corresponding communication graph.

Let us consider the following linear discrete time-varying (LDTV) system:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k w_k, \\ z_k &= M_k x_k + N_k w_k, \\ y_{j,k} &= \lambda_{j,k} C_{j,k} x_k + D_{j,k} w_k, \end{aligned} \tag{4}$$

with zero initial condition $x_0 = 0$ and finite-time horizon N_h, i.e., $k = 0, 1, \ldots, N_h$. The state is denoted by $x_k \in \mathbb{R}^{n_x}$, the input disturbance $w_k \in \mathbb{R}^{m_w}$ belongs to sequence of random vectors with bounded anisotropy level a of extended input vector $W_{0:N_h} = (w_0^{\mathrm{T}}, \ldots, w_n^{\mathrm{T}})^{\mathrm{T}}$, and the non-zero expectation $|\mathbf{E}W_{0:N_h}| > \tau$ and the constrained covariance $\mathrm{tr}(\mathrm{cov}(W_{0:N_h})) < 1 - \tau^2$ are given. For each jth sensor, a measurement $y_{j,k} \in \mathbb{R}^{p_y}$ is provided. The random variable $\lambda_{j,k}$ has Bernoulli distribution with given probabilities $P(\lambda_{j,k} = 1) = p_j$ and

$P(\lambda_{j,k} = 0) = 1 - p_j = q_j$. The output to be estimated is denoted by $z_k \in \mathbb{R}^{p_z}$. All matrices $A_k, B_k, M_k, N_k, C_{j,k}, D_{j,k}$, $j = \overline{1,n}$ are known for all time instants and have appropriate dimensions. The communication scheme is defined by means of an adjacency matrix $\boldsymbol{a}$ with conditions $\boldsymbol{a}_{ji} \geqslant 0$, $\sum\limits_{i=1}^{n} \boldsymbol{a}_{ji} = 1$, $\boldsymbol{a}_{jj} = \max\limits_{i} \boldsymbol{a}_{ji}$; see [21] for more details.

To implement the results from anisotropy-based analysis to LDTV systems, we will identify the norm of such a system F with the norm of the corresponding transfer matrix $F_{0:N_h}$ associated with the mapping $W_{0:N_h} \mapsto F_{0:N_h} W_{0:N_h} = Z_{0:N_h}$ between fragments $W_{0:N_h}$ and $Z_{0:N_h}$ of the input and output sequences, respectively, where for the sake of example $W_{s:t} = (w_s^{\mathrm{T}}, w_{s+1}^{\mathrm{T}}, \ldots, w_t^{\mathrm{T}})^{\mathrm{T}}$ for any $s \leqslant t$. So, the anisotropic norm $|\!|\!| F |\!|\!|_a$ of the LDTV system F operating over a finite-time horizon $k = 0, 1, \ldots, N_h$ should be understood as an anisotropic norm $|\!|\!| F_{0:N_h} |\!|\!|_a$ of its transfer matrix $F_{0:N_h}$.

The problem studied in this paper is finding the linear estimation $\widehat{z}_k$ of output z_k for system (4) using measurements $y_{j,s}$, $s \leqslant k$, such that anisotropic norm of input-to-estimation error system is bounded by a chosen threshold $\gamma > 0$.

4. Main Result

To obtain sufficient conditions of anisotropic norm boundedness for the systems with multiplicative noises and noncentered disturbances, let us firstly discuss some results of [16,21–23].

In [16,23], the anisotropy-based analysis problem was studied for non-centered disturbance with constraints on the first and second moments of the input. The central result is provided by the following statement based on the idea that for computation of the (a, τ)-anisotropic norm for the case of noncentered disturbance one needs to shift the value of anisotropy and then calculate the anisotropic norm of the corresponding system with centered disturbance, making additional corrections in the resulting value.

Lemma 1 ([16]). *Let us consider a linear discrete time-varying system F of the form (4) but without measurements $y_{j,k}$, $j = \overline{1,n}$, and assume the parameters $\tau \in [0; 1)$, $a \geqslant -\frac{l}{2}\ln(1-\tau^2)$, $l = m_w(N_h + 1)$ to be given. The (a, τ)-anisotropic norm (3) of the system F is bounded by the given $\gamma > 0$ if there exist $\gamma_1 > 0$, $\gamma_2 > 0$, such that $|\!|\!| F |\!|\!|_b \leqslant \gamma_1$, $\| F \|_\infty \leqslant \gamma_2$, $\gamma_1^2(1-\tau^2) + \gamma_2^2\tau^2 \leqslant \gamma^2$, where $b = a + \frac{l}{2}\ln(1-\tau^2)$.*

Insofar as the input-to-estimation error system for the filtering problem is reduced to a multiplicative noise system form, the conditions of anisotropic norm boundedness are associated with an anisotropy-based analysis given in [22]. Let us consider the multiplicative noise system $\widetilde{F}$ of the form

$$
\begin{aligned}
x_{k+1} &= (A_{0,k} + \sum_{i=1}^{r} \xi_{i,k} A_{i,k}) x_k + B_k w_k, \\
z_k &= M_k x_k + N_k w_k,
\end{aligned} \tag{5}
$$

with zero initial condition $x_0 = 0$. The matrices $A_{i,k}$, $i = \overline{0,r}$, are given, and the dimensions of matrices are the same as (4). Random variables $\xi_{i,k}$, $i = \overline{1,r}$ have the same distribution with zero mean, and covariances σ_i^2, and considered to be independent of each other as well of the input w_k.

Theorem 1. *Let us consider an LDTV system $\widetilde{F}$ with multiplicative noises (5), and $\tau \in [0; 1)$ and $a \geqslant -\frac{l}{2}\ln(1-\tau^2)$, $\gamma > 0$ are given. The (a, τ)-anisotropic norm of $\widetilde{F}$ is bounded by a given threshold γ if there exist some $\gamma_1 > 0$, $\gamma_2 > 0$ and the parameter $q \in [0; ||\widetilde{F}||_\infty^{-2})$, such that the following inequalities hold true:*

$$
\begin{aligned}
& R_k \succ A_{0,k}^{\mathrm{T}} R_{k+1} A_{0,k} + \sum_{i=1}^{r} \sigma_i^2 A_{i,k}^{\mathrm{T}} R_{k+1} A_{i,k} + q M_k^{\mathrm{T}} M_k + L_k^{\mathrm{T}} S_k^{-1} L_k, \\
& S_k = (I_{m_w} - B_k^{\mathrm{T}} R_{k+1} B_k - q N_k^{\mathrm{T}} N_k)^{-1}, \\
& L_k = S_k (B_k^{\mathrm{T}} R_{k+1} A_{0,k} + q N_k^{\mathrm{T}} M_k).
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
& \widetilde{R}_k \succ A_{0,k}^{\mathrm{T}} \widetilde{R}_{k+1} A_{0,k} + \sum_{i=1}^{r} \sigma_i^2 A_{i,k}^{\mathrm{T}} \widetilde{R}_{k+1} A_{i,k} + q M_k^{\mathrm{T}} M_k + \widetilde{L}_k^{\mathrm{T}} \widetilde{S}_k^{-1} \widetilde{L}_k, \\
& \widetilde{S}_k = (I_{m_w} - B_k^{\mathrm{T}} \widetilde{R}_{k+1} B_k - q N_k^{\mathrm{T}} N_k)^{-1}, \\
& \widetilde{L}_k = \widetilde{S}_k (B_k^{\mathrm{T}} \widetilde{R}_{k+1} A_{0,k} + q N_k^{\mathrm{T}} M_k),
\end{aligned} \tag{7}
$$

$$
\sum_{k=0}^{N} \ln \det S_k^{-1} \geqslant m \ln(1-\tau^2) + 2b, \tag{8}
$$

and $\gamma_1^2(1-\tau^2) + \gamma_2^2 \tau^2 \leqslant \gamma^2$*. Here,* $b = a + \frac{l}{2}\ln(1-\tau^2)$*,* $R_{N_h+1} = 0$*,* $\widetilde{R}_{N+1} = 0$*, and matrices* $R_k, \widetilde{R}_k, S_k$ *are all positively defined for* $k = 0, \ldots, N_h$*.*

Proof. This theorem immediately follows from the results of [16,22,23]. □

Theorem 2. *The anisotropic norm of the system* $\widetilde{F}$ (5) *with non-centered disturbance* w_k *with parameter* $\tau \in [0;1)$ *and a bounded anisotropy level of extended vector* $\mathbf{A}(W_{0:N_h}) \leqslant a$ *is bounded by a given* γ*, i.e.,*

$$
|\!|\!| \widetilde{F} |\!|\!|_a \leqslant \gamma,
$$

if the following inequalities

$$
\begin{bmatrix}
R_k - M_k^{\mathrm{T}} M_k & * & * & * & \cdots & * \\
N_k^{\mathrm{T}} M_k & \eta^2 I_{m_w} - N_k^{\mathrm{T}} N_k & * & * & \cdots & * \\
R_{k+1} A_{0,k} & -R_{k+1} B_k & R_{k+1} & * & \cdots & * \\
\sigma_1 R_{k+1} A_{1,k} & 0 & 0 & R_{k+1} & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_n R_{k+1} A_{n,k} & 0 & 0 & 0 & \cdots & R_{k+1}
\end{bmatrix} \succ 0,
$$

$$
\begin{bmatrix}
\widetilde{R}_k - M_k^{\mathrm{T}} M_k & * & * & * & \cdots & * \\
N_k^{\mathrm{T}} M_k & \gamma_2^2 I_{m_w} - N_k^{\mathrm{T}} N_k & * & * & \cdots & * \\
\widetilde{R}_{k+1} A_{0,k} & -\widetilde{R}_{k+1} B_k & \widetilde{R}_{k+1} & * & \cdots & * \\
\sigma_1 \widetilde{R}_{k+1} A_{1,k} & 0 & 0 & \widetilde{R}_{k+1} & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sigma_n \widetilde{R}_{k+1} A_{n,k} & 0 & 0 & 0 & \cdots & \widetilde{R}_{k+1}
\end{bmatrix} \succ 0, \tag{9}
$$

$$
\begin{bmatrix}
\eta^2 I_{m_w} - \Psi_k - N_k^{\mathrm{T}} N_k & * \\
R_{k+1} B_k & R_{k+1}
\end{bmatrix} \succ 0. \tag{10}
$$

have positive definite solutions $R_k, \widetilde{R}_k, \Psi_k$*,* $k = \overline{0, N_h - 1}$*, with boundary constraints*

$$
\begin{bmatrix}
R_{N_h} - M_{N_h}^{\mathrm{T}} M_{N_h} & * \\
N_{N_h}^{\mathrm{T}} M_{N_h} & \eta^2 I_{m_w} - N_{N_h}^{\mathrm{T}} N_{N_h}
\end{bmatrix} \succ 0,
$$

$$
\begin{bmatrix}
\widetilde{R}_{N_h} - M_{N_h}^{\mathrm{T}} M_{N_h} & * \\
N_{N_h}^{\mathrm{T}} M_{N_h} & \gamma_2^2 I_{m_w} - N_{N_h}^{\mathrm{T}} N_{N_h}
\end{bmatrix} \succ 0, \tag{11}
$$

$$
\eta^2 I_{m_w} - \Psi_{N_h} - N_{N_h}^{\mathrm{T}} N_{N_h} \succ 0, \tag{12}
$$

and special type inequalities

$$\sum_{k=0}^{N_h} \ln\det\Psi_k \geqslant 2b + l\ln(\eta^2 - \gamma_1^2), \tag{13}$$

$$\gamma_1^2(1-\tau^2) + \gamma_2^2\tau^2 \leqslant \gamma^2 \tag{14}$$

hold true, $b = a + \frac{l}{2}\ln(1-\tau^2)$.

Proof. Inequalities (9)–(12) are derived by using Schur's complement formula. Inequality (13) follows from substituting the anisotropic norm computation for the non-centered input problem for the corresponding one with a centered input and a modified anisotropy level of the extended input vector, while (14) is related to the boundedness criteria of $|||\widetilde{F}|||_{a,\tau}$, $|||\widetilde{F}|||_b$, $\|\widetilde{F}\|_\infty$ for a given system $\widetilde{F}$. □

For implementing the results of Theorem 2, some transformation of system (4) is necessary. Let us introduce a set of virtual objects as follows:

$$\begin{aligned} &x_{j,k+1} = A_k x_{j,k} + B_k w_k, \quad x_{j,0} = 0,\\ &z_{j,k} = M_k x_{j,k} + N_k w_k,\\ &y_{j,k} = \lambda_{j,k} C_{j,k} x_{j,k} + D_{j,k} w_k, \end{aligned} \tag{15}$$

where $x_{j,k} \in \mathbb{R}^{n_x}$ denotes the virtual state, $z_{j,k} \in \mathbb{R}^{p_z}$ is the output to be estimated, $y_{j,k} \in \mathbb{R}^{p_y}$ denotes measurement of the jth sensor. In this representation, each sensor has its own virtual dynamics, and the original sensor network is virtually separated to independent objects.

Let us choose an estimation model for every virtual system (15) in the following form:

$$\begin{aligned} &\widehat{x}_{j,k+1} = \sum_{i=1}^{n} \boldsymbol{a}_{ji}(A_k\widehat{x}_{i,k} + H_{ji,k}(y_{i,k} - \widehat{y}_{i,k})), \quad \widehat{x}_{j,0} = 0,\\ &\widehat{z}_{j,k} = \sum_{i=1}^{n} \boldsymbol{a}_{ji} M_k \widehat{x}_{i,k}, \end{aligned} \tag{16}$$

where $\widehat{y}_{i,k} = p_i C_{i,k}\widehat{x}_{i,k}$. Matrices H_{ji} are considered to be found. The input-to-estimation error system, which is using estimation model (16), has to have a bounded anisotropic norm.

The state estimation error and output estimation error are depicted by the following notations: $\widetilde{x}_{j,k} = x_{j,k} - \widehat{x}_{j,k}$, $\widetilde{z}_{j,k} = z_{j,k} - \widehat{z}_{j,k}$, $j = \overline{1,n}$. The dynamics of each virtual object error are as follows:

$$\begin{aligned} &\widetilde{x}_{j,k+1} = \sum_{i=1}^{n} \boldsymbol{a}_{ji}\Big(\mathcal{A}_{ji,k} x_{i,k} + \widetilde{\mathcal{A}}_{ji,k}\widetilde{x}_{i,k} + \mathcal{B}_{ji,k} w_k\Big),\\ &\widetilde{z}_{j,k} = M_k x_{j,k} - \sum_{i=1}^{n} \boldsymbol{a}_{ji} M_k \widetilde{x}_{i,k} + N_k w_k, \end{aligned} \tag{17}$$

where $\mathcal{A}_{ji,k} = A_k\delta_{ji} - \boldsymbol{a}_{ji}\Big(A_k + H_{ji,k}\Big(\lambda_{i,k}C_{i,k} - p_i C^p_{i,k}\Big)\Big)$, $\widetilde{\mathcal{A}}_{ji,k} = \boldsymbol{a}_{ji}\Big(A_k - p_i H_{ji,k} C_{i,k}\Big)$, $\mathcal{B}_{ji,k} = B_k\delta_{ji} - \boldsymbol{a}_{ji} H_{ji,k} D_{i,k}$. Hereafter, the Kronecker symbol δ_{ji} is used. Let us assemble the total vectors, $x_{j,k}$, $j = \overline{1,n}$, to the extended state vector $\overline{x}_k^{\mathrm{T}} = \Big(x_{1,k}^{\mathrm{T}}, x_{2,k}^{\mathrm{T}}, \ldots, x_{n,k}^{\mathrm{T}}\Big)^{\mathrm{T}}$, and similarly the extended vectors of state error and output error are denoted as follows: $\widetilde{x}_k^{\mathrm{T}} = \Big(\widetilde{x}_{1,k}^{\mathrm{T}}, \widetilde{x}_{2,k}^{\mathrm{T}}, \ldots, \widetilde{x}_{n,k}^{\mathrm{T}}\Big)^{\mathrm{T}}$, $\widetilde{z}_k^{\mathrm{T}} = \Big(\widetilde{z}_{1,k}^{\mathrm{T}}, \widetilde{z}_{2,k}^{\mathrm{T}}, \ldots, \widetilde{z}_{n,k}^{\mathrm{T}}\Big)^{\mathrm{T}}$. Hence, the dynamics of the extended state error and the output error are described by the following system:

$$\begin{aligned} &\widetilde{x}_{k+1} = \Big(\overline{A}_k - \overline{W}_k - \overline{H}_k\Big(\overline{C}_k^{\lambda} - \overline{C}_k^{p}\Big)\Big)\overline{x}_k + \Big(\overline{W}_k - \overline{H}_k\overline{C}_k^{p}\Big)\widetilde{x}_k + (\overline{B}_k - \overline{H}_k\overline{D}_k)w_k,\\ &\widetilde{z}_k = (\overline{M}_k - \overline{V}_k)\overline{x}_k + \overline{V}_k\widetilde{x}_k + \overline{N}_k w_k, \end{aligned}$$

where matrices are as follows:

$$\overline{A}_k = I_n \otimes A_k,\ \overline{B}_k = [1,\dots,1]^{\mathrm{T}} \otimes B_k,\ \overline{M}_k = I_n \otimes M_k,\ \overline{N}_k = [1,\dots,1]^{\mathrm{T}} \otimes N_k$$
$$\overline{C}_k^{\lambda} = \operatorname*{diag}_{j=\overline{1,n}}\Big(\lambda_{j,k}C_{j,k}\Big),\ \overline{C}_k^{p} = \operatorname*{diag}_{j=\overline{1,n}}\Big(p_j C_{j,k}^{p}\Big),\ \overline{W}_k = \boldsymbol{a} \otimes A_k,$$
$$\overline{H}_k = \operatorname*{block}_{j,i=\overline{1,n}}\Big(\boldsymbol{a}_{ji}H_{ji,k}\Big),\ \overline{V}_k = \boldsymbol{a} \otimes M_k,\ \overline{D}_k = \Big[D_{1,k}^{\mathrm{T}},\dots,D_{n,k}^{\mathrm{T}}\Big]^{\mathrm{T}}.$$

Notation $\otimes$ is assigned to the Kronecker product, and the block diagonal and block matrix are denoted by $\operatorname{diag}(\cdot)$ and $\operatorname{block}(\cdot)$, respectively.

To derive the dynamics of the input-to-estimation error system, let us define the extended state vector as $\zeta_k = [\overline{x}_k^{\mathrm{T}}, \tilde{x}_k^{\mathrm{T}}]^{\mathrm{T}}$. Then, the dynamics of the input-to-estimation error system can be described by the following expression:

$$\begin{aligned} \zeta_{k+1} &= (\mathcal{A}_{0,k} + \sum_{i=1}^{n} \xi_{j,k}\mathcal{A}_{j,k})\zeta_k + \mathcal{B}_k w_k, \\ \tilde{z}_k &= \mathcal{M}_k\zeta_k + \mathcal{N}_k w_k, \end{aligned} \tag{18}$$

where $\xi_{j,k} = \lambda_{j,k} - p_j$ for $j = \overline{1,n}$, $\mathcal{M}_k = [\overline{M}_k - \overline{V}_k \quad \overline{V}_k]$, $\mathcal{N}_k = \overline{N}_k$, $G_{j,k} = \operatorname*{diag}_{i=\overline{1,n}}\delta_{ji}C_{j,k}$, and

$$\mathcal{A}_{0,k} = \begin{bmatrix} \overline{A}_k & 0 \\ \overline{A}_k - \overline{W}_k & \overline{W}_k - \overline{H}_k\overline{C}_k^{p} \end{bmatrix},\ \mathcal{A}_{j,k} = \begin{bmatrix} 0 & 0 \\ -\overline{H}_k G_{j,k} & 0 \end{bmatrix},\ \mathcal{B}_k = \begin{bmatrix} \overline{B}_k \\ \overline{B}_k - \overline{H}_k\overline{D}_k \end{bmatrix}.$$

As one can see, (18) presents the multiplicative noise system dynamics, so the statement of Theorem 2 can be applied to this object. However, inequalities (9) and (10) contain non-linear terms, therefore corresponding variables change should be used to provide an effective computational algorithm. Let us define the following set of matrices: $U_k = \begin{bmatrix} Y_k & 0 \\ 0 & \overline{H}_k \end{bmatrix}$, $X_k = R_{k+1}U_k$, $\widetilde{X}_k = \widetilde{R}_{k+1}U_k$, where $k = 0,\dots,N_h$ and Y_k are considered to be real-valued matrices of corresponding dimensions. So, after the variables change, inequalities (9) and (10) become of the form

$$\begin{gathered} \begin{bmatrix} R_k - \mathcal{M}_k^{\mathrm{T}}\mathcal{M}_k & * & * & \dots & * \\ \mathcal{N}_k^{\mathrm{T}}\mathcal{M}_k & \eta^2 I_{m_w} - \mathcal{N}_k^{\mathrm{T}}\mathcal{N}_k & * & \dots & * \\ R_{k+1}\mathcal{A}_{00,k} + X_k\mathcal{A}_{01,k} & -R_{k+1}\mathcal{B}_{00,k} - X_k\mathcal{B}_{01,k} & R_{k+1} & \dots & * \\ \sigma_1 X_k\mathcal{A}_{11,k} & 0 & 0 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_n X_k\mathcal{A}_{n1,k} & 0 & 0 & \dots & R_{k+1} \end{bmatrix} \succ 0, \\ \begin{bmatrix} \widetilde{R}_k - \mathcal{M}_k^{\mathrm{T}}\mathcal{M}_k & * & * & \dots & * \\ \mathcal{N}_k^{\mathrm{T}}\mathcal{M}_k & \gamma_2^2 I_{m_w} - \mathcal{N}_k^{\mathrm{T}}\mathcal{N}_k & * & \dots & * \\ \widetilde{R}_{k+1}\mathcal{A}_{00,k} + \widetilde{X}_k\mathcal{A}_{01,k} & -\widetilde{R}_{k+1}\mathcal{B}_{00,k} - \widetilde{X}_k\mathcal{B}_{01,k} & \widetilde{R}_{k+1} & \dots & * \\ \sigma_1 \widetilde{X}_k\mathcal{A}_{11,k} & 0 & 0 & \dots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_n \widetilde{X}_k\mathcal{A}_{n1,k} & 0 & 0 & \dots & \widetilde{R}_{k+1} \end{bmatrix} \succ 0, \end{gathered} \tag{19}$$

$$\begin{bmatrix} \eta^2 I_{m_w} - \Psi_k - \mathcal{N}_k^{\mathrm{T}}\mathcal{N}_k & * \\ R_{k+1}\mathcal{B}_{00,k} + X_k\mathcal{B}_{01,k} & R_{k+1} \end{bmatrix} \succ 0. \tag{20}$$

Boundary conditions (11) and (12) do not change.

If the obtained system of inequalities (11)–(14), (19), and (20) has a solution, then the estimator (16) designed for system (4) satisfies the condition of suboptimality in the sense of the anisotropic norm. The optimization problem can be stated as follows: $\gamma^2 \longrightarrow \min$ over (11)–(14), (19) and (20) w.r.t. $R_k, \widetilde{R}_k, X_k, \widetilde{X}_k, \eta^2, \Psi_k, \gamma_1^2, \gamma_2^2, \tau$.

5. Conclusions

In this paper, an estimation problem of a sensor network system is solved using an anisotropy-based approach. External disturbances belong to sequences of random vectors with bounded anisotropy levels of the extended input vector. By using virtual objects for every sensor, the augmented system is introduced and the input-to-error system is derived. The system is described by multiplicative noise state space description, since anisotropy-based analysis can be applied. An anisotropic norm boundedness criterion for the input-to-error system is implemented, and the non-linear system to be optimized is obtained. Appropriate variables change is used, and the optimization problem becomes linear. The parameters of the time-varying anisotropy-based estimator are calculated after proper inverse change when a suboptimal problem is numerically solved. The non-zero mean of disturbance determines the dimension rising of the convex optimization problem, since the modification of a previously developed anisotropy-based algorithm was achieved.

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