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Article

Measuring the Density Matrix of Quantum-Modeled Cognitive States

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Abstract: Inspired by the principles of quantum mechanics, we constructed a model of students' misconceptions about heat and temperature, conceptualized as a quantum system represented by a density matrix. Within this framework, the presence or absence of misconceptions is delineated as pure states, while the probability of mixed states is also considered, providing valuable insights into students' cognition based on the mental models they employ when holding misconceptions. Using the analysis model previously employed by Lei Bao and Edward Redish, we represented these results in a density matrix. In our research, we utilized the Zeo and Zadnik Thermal Concept Evaluation among 282 students from a private university in Northeast Mexico. Our objective was to extract information from the analysis of multiple-choice questions designed to explore preconceptions, offering valuable educational insights beyond the typical Correct–Incorrect binary analysis of classical systems. Our findings reveal a probability of 0.72 for the appearance of misconceptions, 0.28 for their absence, and 0.43 for mixed states, while no significant disparities were observed based on gender or scholarship status, a notable difference was observed among programs ($p < 0.05$). These results are consistent with the previous literature, confirming a prevalence of misconceptions within the student population.

Keywords: quantum cognition; density matrix; heat and temperature; misconceptions; engineering students

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1. Introduction

When dealing with the study of non-ideal quantum systems, it is common to consider Hamiltonians that include non-Hermitian terms, representing energy loss due to the interaction with an external environment. This approach is widely used to describe quantum systems coupled to sinks or sources and can arise in various contexts, such as the study of optical waveguides, Feshbach resonances, multiphoton ionization, and open quantum systems [1–3].

However, it is also important to note that there are quantum systems with Hermitian Hamiltonians, whose solutions offer a deep and accurate understanding of a wide range of physical phenomena. A prominent example is the quantum harmonic oscillator, whose stationary states are described by Hermitian operators and whose solutions have been fundamental in the development of quantum theory.

In this article, we will consider only Hermitian quantum systems to find the density matrix of a system, highlighting its importance in the analysis of real physical systems and its applications in various fields of quantum physics.

The models used in quantum mechanical formalism have proven to be very beneficial in effectively describing psychological, economic, financial, social or political processes [4–9]. In the literature, numerous models not only clarify a particular phenomenon but also facilitate the extension of these models to analogous phenomena [10,11].

In quantum formalism, a particle is in a superposition of multiple states, so we have employed a quantum analogy to develop a new mathematical model for social challenges. "The approach does not at all imply that there are quantum mechanical processes occurring at a macroscopic level. Rather, the use of the formalism to social science problems in general, in effect, does bring in the use of a different mathematical structure" [12]. For this, we have revisited models that offer a framework for constructing cognitive models, employing the principles of quantum probability theory in their mathematical formulation.

In accordance with Pothos [13] in the realm of behavioral science, quantum probability theory operates without delving into the intricacies of physics. This methodology mirrors the broader trend in cognitive science, which often sidesteps neurophysiological intricacies to concentrate on delineating behavioral structures. Essentially, quantum theory serves as a proposition regarding the underlying principles of cognition. By exploring a quantum behavioral model, the proposition suggests that fundamental processes in quantum theory can be loosely correlated with mental processes, while this might appear ambitious, it is no more so than proposing that cognition adheres to classical probability theory principles, for instance.

Various studies have delved into cognition, with a particular focus on analyzing misconceptions, especially within the realms of science and engineering. Consequently, numerous tools have been developed to detect and rectify these misconceptions [14]. Misconceptions occur when an individual's knowledge and beliefs diverge from scientific accuracy, also referred to as preconceived notions, unscientific beliefs, or conceptual misunderstandings [15]. While there exists a variety of misconceptions within science and engineering, our study concentrates on those surrounding the concepts of heat and temperature.

To investigate these misconceptions, we will adopt Lei Bao and Edward Redish's "Model Analysis" framework, adapting it to suit our specific needs. Within our model, we establish a matrix of cognitive states where states represent the presence or absence of erroneous concepts. State 1 signifies the presence of misconceptions, while state 2 indicates their absence when answered in accordance with formal definitions. We assign probabilities ω_1 and ω_2 to these states, where ω_1 denotes the likelihood of misconceptions appearing, and ω_2 represents their absence when adhering to formal conceptual definitions.

Aim and Structure of the Study

Section 2 introduces the model proposed in quantum cognition theory. Section 3 outlines the materials and methods employed in our research. Following this, Section 4 presents the results obtained, which are subsequently discussed in Section 5. Finally, in Section 6, we will present the results obtained in our study, followed by conclusions derived from these findings. In addition, we will discuss possible directions for future work related to our research.

2. Model Used in the Study

In this study, we explore the potential of the density matrix as an essential tool in educational decision making. By providing a deeper understanding of the data, this tool opens up new possibilities for personalized instruction, assessment of student achievement, and overall improvement of educational quality. We are confident that this approach will not only contribute to the advancement and innovation in the study of human cognition, but will also provide a solid foundation for future research in the educational field.

Quantum cognition provides a framework for developing cognitive models by utilizing the mathematical principles of quantum probability theory [16,17]. Quantum probability theory is a mathematical approach that assigns probabilities to quantum events, which can provide a new perspective in understanding human cognition. In the context of quantum cognition, an event is usually linked to the outcome of a judgment process, and the cognitive state of the individual is described to determine the probability of a specific choice. This connection between quantum probability theory and human cognition opens up new avenues for exploring how the mathematical tools of quantum mechanics help us model our understanding of decision making and cognitive behavior.

Within the context of quantum cognition, probabilities between 0 and 1 are assigned to represent the uncertainty inherent in the decision-making process, analogous to quantum theory. These probabilities reflect the probabilistic interpretation of quantum mechanics, where the probability amplitude, a complex number, is used to calculate the probability that a system is in a particular state when measured. In this framework, probabilities not only represent the possibility of a specific choice, but also incorporate information about the possible dynamics, state transformations, and measurements that may take place in the system. Thus, quantum cognition provides a powerful approach to model uncertainty in decision making, enabling a more complete understanding of cognitive processes and the evaluation of optimal strategies in complex and changing environments.

This approach corresponds with the model that we want to propose in this work, establishing the states of the system as the appearance or non-appearance of misconceptions, which will be treated as a semi-classical quantum system.

The density matrix in quantum theory is a mathematical representation that describes the quantum state of a physical system. It combines possible quantum states $|\psi_\alpha\rangle$ with associated probabilities ω_α. This tool is particularly useful for systems that are not in a pure state, but rather in a superposition of multiple states. The density matrix is represented by

$$\rho = \sum_\alpha \omega_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|, \tag{1}$$

where ρ fulfills being Hermitian, normalized ($Tr(\rho) = 1$), and a positive operator.

These three properties essentially ensure that the eigenvalues of ρ are positive, real numbers which sum to 1, and thus have the interpretation of probabilities. If the states $|\psi_i\rangle$ are orthonormal then the ω_i play the role of the probabilities for the initial state to be one of the $|\psi_i\rangle$ [18]. This representation allows a clear and efficient description of the probabilities associated with the different possibilities of the quantum system. It significantly simplifies the calculations and improves the understanding of the results, as it provides an intuitive way to understand the probability distribution of the system in terms of its basic components.

When the student's state is probed by presenting a particular question or scenario, the student will often respond by activating a single mental model. We consider that the student's mental state has been momentarily collapsed by the probe into the selected model state. The process by which this selection is made can be quite complex. When it is difficult to make that choice, a student may fall into a state of explicit confusion in which several models appear to be equally plausible but generate contradictory results, and the student cannot determine which is most appropriate to use. Multiple-choice questions can extract information about such states.

In this paper, we adopt an interpretation of student mental models, discussed in the citations of [19–22], where quantum states are handled in terms of their internal coherences rather than as errors. Students may employ a variety of mental models rather than a single one, which is reflected in their ability to answer equivalent questions in a variety of ways. This situation is common in introductory physics and other educational contexts. The choice of a particular model depends on the educational history and mental state of the student, suggesting a probabilistic approach to understanding their behavior. Students may be in a pure model state if they use a single model consistently, while they are considered to be in a mixed model state if they employ a combination of models. Model selection can be complex, and in cases of confusion, students may consider multiple models simultaneously.

Both states (state 1: appearance of misconceptions; state 2: no appearance of misconceptions) coexist simultaneously, but only one is observed in the intervention, suggesting they are in a superposition. This concept, fundamental in quantum probability theory, implies that a system is in an indefinite superposition state until a measurement is made.

In contrast, in classical probability theory, it is assumed that a system is in a defined state at any given moment [23].

For this, we identify pure states as the appearance or non-appearance of misconceptions, while mixed states represent situations in which a student may know the formal definition of the concept, but due to factors such as incorrect interpretation of information or confusion of context, they are led to an answer guided by misconceptions. Therefore, it is crucial to identify and address these cases to promote a more precise and profound understanding of scientific concepts. To calculate these mixed states, which allow us to obtain this information, we utilize the model described previously by Lei Bao and Edward Redish.

Incorporating this quantum metaphor, the integration of an additional instrument is proposed to show the student's status as pure or mixed: the density matrix defined in the model analysis [22]. This innovative approach involves the construction of a matrix D for each k of the total number of students (N), which summarizes the essence of their educational situation:

$$D_k = \begin{pmatrix} n_1 & \sqrt{n_1 n_2} \\ \sqrt{n_1 n_2} & n_2 \end{pmatrix}. \tag{2}$$

In its current form, the density matrix corresponding to an individual student encompasses only the student's scores, n_1 and n_2. These values n_1 and n_2 coincide with the initial weights of the density matrix, that is, the probability that the student responded using model 1 that was represented in the variable ω_1 and the probability that the student responded using model 2 in the variable ω_2. Typically, these are simply graded based on class averages of right and wrong, respectively, which translates into whether or not erroneous preconceptions appear.

However, this scenario changes when the density matrices of all students within a class are consolidated. The construction of the density matrix for a class involves the sum of the density matrices belonging to each individual student:

$$D = \frac{1}{N} \sum_{k=1}^{N} D_k. \tag{3}$$

Conventional statistical approaches that simply sum class scores effectively only capture the diagonal of the density matrix, thus discarding valuable information about the combination of student models. The density matrix of the class model plays a vital role in capturing and preserving essential structural details about individual student models within a class. Through the analysis of this matrix, educators can glean insights into the distribution of student models and the coherence of their model utilization. The matrix's configuration mirrors various scenarios that students might demonstrate in utilizing their models, ranging from consistent usage of the same model to employing different models or experiencing mixed model states. These scenarios embody statistical characteristics of the student body, distinct from the probabilistic nature of individual student model states. The diagonal elements of the class model density matrix indicate the proportion of responses associated with specific models employed by the class, while the off-diagonal elements depict the consistency of individual students in utilizing their models. Large off-diagonal elements suggest either low consistency or significant variation in how students apply their models [19].

According to various authors [19–21], traditional methodologies, such as factor analysis, which retain individual student responses within a large matrix of questions and answers, have the objective of obtaining information from the patterns in the students' responses. However, its effectiveness depends on the assumption of consistent student patterns. For example, consider a scenario in which a test is given with N students, half of whom consistently adhere to model 1 and the other half to model 2. In such cases, factor analysis will reveal two distinct factors. However, if all students in the class use both models equally (50% of the time each), the factor analysis will not identify any strong factors. This method inherently assumes student consistency, leading to a loss of information about students' model combinations. In such cases, qualitative research can complement the approach by revealing common models, thus allowing insights into the extent to which students employ mixed models, using the student response matrix as a valuable resource.

3. Materials and Methods

3.1. Research Designs

The Thermal Concepts Evaluation questionnaire was applied, with which we can identify in each of the questions if the students respond from model 1 or model 2, that is, if misconceptions appear or if the answer coincides with the expected model of the formal analysis. The probability that the student responded using model 1 was represented in the variable ω_1 and the probability that the student responded using model 2 in the variable ω_2, which are interpreted as the initial weights of the density matrix.

3.2. Participants and Application Procedure

The questionnaire was applied to the sample in the spring of 2023, which consisted of 282 students, which are described in Table 1. Programs to which the students in the sample belong: Architecture, Art and Design (AAYD), Health Sciences (CS), Law and Social Sciences (DSYC), Education and Humanities (EYH), Engineering and Technology (IYT), and Business (N).

Table 1. Participants.

Program	***n***
Architecture, Art and Design (AAYD)	43
Health Sciences (CS)	21
Law and Social Sciences (DSYC)	28
Education and Humanities (EYH)	15
Engineering and Technology (IYT)	86
Business (N)	89
Gender	***n***
Female	132
Male	150
Scholarship	***n***
Yes	182
No	100
N	282

The students participated voluntarily, and they were well-informed about their role in this research. Throughout the study, the confidentiality of their identity and personal information was rigorously maintained.

Although the sample size is correct for the calculations performed in the study, systematic error potentially caused by the sample size was taken into account, which can be observed in Equation (4).

3.3. Questionnaires

The Thermal Concepts Evaluation (TCE) questionnaire developed by Yeo and Zadnik was applied, which consists of 26 questions that allow us to analyze the ideas that students have about heat and temperature [24].

- Students' conception of heat.
- Students' conception of temperature.
- Students' conception about heat transfer and temperature change.
- Students' conception about "thermal properties" of materials.

Table 2 describes the alternative misconceptions studied in the questionnaire.

Table 2. Students conceptions about heat and temperature [24].

Alternative Conceptions of Heat	Item Questions
Heat is a substance	10, 22
Heat is not energy	22
Heat and cold are different, rather than opposite ends of a continuum	10, 13, 18, 23, 24
Heat and temperature are the same thing	15, 18
Heat is proportional to temperature	7, 11, 15
Heat is not a measurable, quantifiable concept	7
Alternative conceptions of Temperature	**Item Questions**
Temperature is the "intensity" of heat	15
Skin or touch can determine temperature	16
Perceptions of hot and cold are unrelated to energy transfer	10, 18, 21, 22
When temperature at boiling remains constant, something is "wrong".	5
Boiling point is the maximum temperature a substance can reach	19
A cold body contains no heat	7, 10, 11, 22, 26
The temperature of an object depends on its size	1, 9, 14
There is no limit on the lowest temperature	25
Alternative conceptions of heat transfer and temperature change	**Item Questions**
Heating always results in an increase in temperature	3, 4, 5
Heat only travels upward.	20
Heat rises.	20
Heat and cold flow like liquids.	10, 13
Objects of different temperature that are in contact with each other or in contact with air at different temperature, do not necessarily move toward the same temperature. (Thermal equilibrium is not a concept).	1, 2, 3, 6
Heat flows more slowly through conductors making them feel hot	25
The kinetic theory does not really explain heat transfer. (Explanations are recited but not believed).	18, 20, 21
Alternative conceptions of "thermal properties" of materials	**Item Questions**
Temperature is a property of a particular material or object	9, 14, 16, 24
Metal has the ability to attract, hold, intensify or absorb heat and cold	9, 14, 16, 20
Objects that readily become warm do not readily become cold	25
Different materials hold the same amount of heat	11
The boiling point of water is 100 °C (only)	4, 8, 19
Ice is at 0 °C and/or cannot change temperature	1
Water cannot be at 0 °C	2, 11
Steam is more than 100 °C	6, 19
Materials like wool have the ability to warm things up.	17, 23
Some materials are difficult to heat: they are more resistant to heating	26
Bubbles mean boiling The bubbles in boiling water contain "air", "oxygen", or "nothing"	12

Correct answers align with the physical concepts taught in the classroom, reflecting their correspondence with everyday life. As a result, these answers are related to state 2, because alternative misconceptions are not detected. On the other hand, incorrect answers, due to the design of the test, are associated with alternative misconceptions, represented in state 1.

Questions were added to the questionnaire that could support us in the analysis of the research, such as career, gender, and scholarship status.

3.4. Statistical Analysis and Data Processing

We use software R version 3.6.1 to derive the statistical results presented in this work [25]. In order to evaluate the significance of each factor (gender, scholarship and program) and given that the data collected did not meet the assumptions of normality and homoscedasticity, a multiple Kruskal–Wallis analysis was performed.

4. Results

Table 3 shows the means obtained by the students in state 1 and state 2, which in our model would be related to the weight of each of the cognition states, where ω_1 represents the average number of questions answered with alternative concepts, and ω_2 represents the average number of questions answered correctly in the TCE questionnaire.

Table 3. Probabilities of the appearance or non-appearance of preconceptions by program, gender and scholarship.

Program	ω_1	ω_2
Architecture, art and design (AAYD)	0.72	0.28
Health sciences (CS)	0.68	0.32
Law and social sciences (DSYC)	0.77	0.23
Education and humanities (EYH)	0.72	0.28
Engineering and technology (IYT)	0.71	0.29
Business (N)	0.73	0.27
Gender	ω_1	ω_2
Female	0.73	0.27
Male	0.71	0.29
Scholarship	ω_1	ω_2
Yes	0.71	0.29
No	0.72	0.28
Total	0.72	0.28

In Table 3, a clear predominance of state 1 over state 2 is highlighted, suggesting a higher prevalence of misconceptions among students compared to a correct understanding of the concepts of heat and temperature. It is observed that the values of state 1 range from 0.68 to 0.77, with the lowest value for the CS program and the highest for the DSYC program. On the other hand, the values of state 2 represent the complement of these results. It is important to note that these values represent only the averages obtained from the questionnaires, excluding the values of mixed states at this point.

Performing the Kruskal–Wallis Test for each of the variables analyzed in the study (program, scholarship and gender), we can observe in Table 4 that only for the program, the p-value is less than 0.05. Therefore, we can conclude that only for that variable is significant difference among the groups.

Table 4. Kruskall–Wallis Test.

Variable	χ^2	**DF**	p
Program	11.764	5	0.03817 *
Gender	3.4603	1	0.06286
Scholarship	0.0001161	1	0.9914

* $p < 0.05$.

In the following subsections, the results obtained in each of the analyzed categories will be examined in detail: program, gender, and scholarship.

4.1. Analysis by Program

In the analysis by program, we can see in Table 5 that the students who obtained the highest score in state 2 are the CS students with an average of 0.32 ± 0.12, followed by IYT (0.29 ± 0.11), AAYD (0.28 ± 0.09), EYH (0.28 ± 0.07), and N (0.27 ± 0.08), while those who obtained the lowest weighting are the DYCS students with an average of 0.23 ± 0.09.

Table 5. Analysis by program of the non-appearance of preconceptions, mean and standard deviation.

Program	n	ω_2	**sd**
Architecture, Art and Design (AAYD)	43	0.28	0.09
Health Sciences (CS)	21	0.32	0.12
Law and Social Sciences (DSYC)	28	0.23	0.09
Education and Humanities (EYH)	15	0.28	0.07
Engineering and Technology (IYT)	86	0.29	0.11
Business (N)	89	0.27	0.08

The program analysis allows us to conclude that only DYCS students exhibit a sample different from the other programs. Therefore, if you have a population with students who have similar characteristics to this program, appropriate adjustments should be made to the values obtained in the study.

4.2. Analysis by Gender

The analysis by gender in Table 6 shows us that men are more analytical than women, since their average score in state 2 is higher (F: 0.27 ± 0.09; M: 0.29 ± 0.11); although, there is not a significant difference between both groups.

Table 6. Analysis by gender of the non-appearance of preconceptions, mean and standard deviation.

Gender	n	ω_2	**sd**
F	132	0.27	0.09
M	150	0.29	0.11

In the gender analysis, we can see that there are no significant differences within the sample composed of 47% women and 53% men. Therefore, we assume that regardless of gender, the proposed model is applicable.

4.3. Analysis by Scholarship

When scholarships are taken into account, Table 7 shows that the means are identical in both scenarios, indicating that there are no statistically significant differences between students with or without scholarships.

Table 7. Analysis by scholarship of the non-appearance of preconceptions, mean and standard deviation.

Scholarship	n	ω_2	**sd**
Yes	182	0.28	0.10
No	100	0.28	0.09

4.4. Consistency of the Results Obtained

In order to compare our results with those obtained in other studies, we decided to analyze studies that applied the TCE in similar populations, as well as studies that applied questionnaires with the same purpose, such as the Force Concept Inventory (FCI) and Cognitive Reflection Test (CRT).

The CRT is widely employed to differentiate individuals' inclination towards two modes of thinking [26]. The CRT is especially utilized for evaluating one's intuitive–analytic cognitive style [27]. For this comparison, we interpret the intuitive model as that model that uses preconceptions in state 1 and the analytical model as that corresponding to state 2. This allows us to compare the results obtained when applying the TCE test with those obtained in the CRT test, thus verifying whether the weights are specific to the test or consistent with findings in the literature.

Force Concept Inventory (FCI) measures students' understanding of the fundamental principles of Newtonian physics using simple language and questions that include distracting, common-sense answers [28]. This questionnaire serves to identify and classify misconceptions, so it was used within the articles compared to measure the use of state 1 and state 2 in these concepts.

In our comparison of articles, we sought those that featured a population similar to that employed in our study. This choice was made to ensure comparability within the same context, retaining studies that, like ours, involved university students or an equivalent group. Additionally, we considered studies that utilized different instruments from ours but shared a comparable approach concerning state 1 and state 2. This allowed us to create a comparative Table 8 encompassing works employing TCE, CRT, or FCI. For all tests, ω_2 is defined as the number of correct questions, divided by the total number of questions. On the other hand, ω_1 refers to questions that were answered based on alternative conceptions, divided by the total number of questions.

Table 8. Results of similar test appearance (ω_1) or non-appearance (ω_2) of misconception (TCE, CRT and FCI).

Author	Variable Output	n	ω_1	ω_2
Our Study	TCE	282	0.72	0.28
Luera [29]	TCE	47	0.65	0.35
Baser [30]	TCE	42	0.68	0.32
	TCE	40	0.70	0.30
Frederick * [26]	CRT	1774	0.65	0.35
Toplak [31]	CRT	346	0.77	0.23
Bialek [32]	CRT	1573	0.89	0.11
Welsh [33]	CRT	58	0.59	0.41
Bao [19]	FCI	778	0.73	0.27
Smith [20]	FCI	109	0.78	0.22

* Modified sample.

Regardless of the comparative studies, it is evident that System 1 exhibits dominance and has a higher weight, which aligns with our results. There is a clear dominance of misconceptions, as the probability ω_1 (59–89%) is generally higher than ω_2 (11–41%).

4.5. Sources of Variability

We consider the stochastic variability that arises from the use of multiple instruments and calculate the average of the errors obtained by each instrument. In this calculation, we assumed that all instruments had the same weight. We show this systematic error in Table 9 with these values.

Table 9. Systematic error.

Instrument	Mean
CRT	0.2528
FCI	0.2460
TCE	0.3134
Systematic Error	**0.0674**

In the context of TCE, we calculate the standard deviation, assuming that when using the same instrument with similar populations, these values exhibit stochastic behavior.

Considering these sources of variability, we can define our error range as the mean value, accounting for the stochastic variability resulting from the comparison of several studies using the same instrument, and an additional systematic error arising from the comparison of different instruments. This leads us to the following calculation:

$$\omega_2 \pm \mathrm{Err}_{\mathrm{stochastic}} \pm \mathrm{Err}_{\mathrm{systematic}} = 0.28 \pm 0.0351 \pm 0.0674. \quad (4)$$

In the following Figure 1, we can see the means of each of the studies. The solid lines represent the means obtained by state 2 in each questionnaire (Blue: TCE; Red: CR;, Green: FCI). The error bars shown represent twice the standard error; when it was not mentioned in the cited article, it was estimated using proportions with the sample size. The gray interval represents the mean-centered error region as Equation (4) (include systematic and stochastic error).

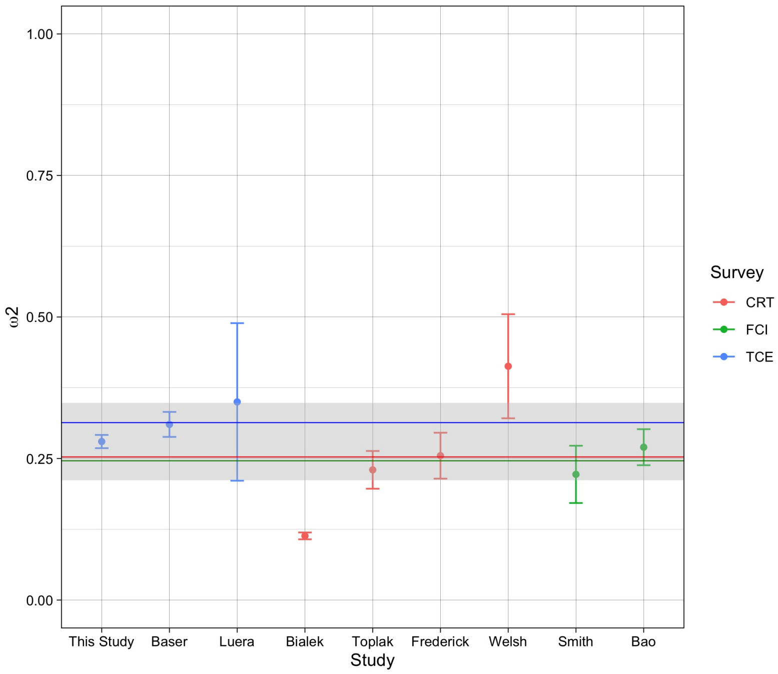

Figure 1. Results of similar tests (non-appearance of misconception in TCE, CRT and FCI) [19,20,26,29–33]. Means and confidence intervals (CIs) for the measurements outlined in Table 3 at a 95% confidence level (CL). Solid colored lines depict the average for each test (TCE, CRT, FCT). The gray shaded area represents this measurement, accounting for both stochastic and systematic errors as described in Equation (4).

When comparing the three instruments, we can observe that the CRT is hypersensitive, generating a high stochastic variability in the scores. Even with this variability, the dominance of state 1 over state 2 is still visible in all three questionnaires.

Therefore, in comparison with other studies, we can assume that the weights obtained in our study, being consistent with those compared in the literature, can be established as these necessary constant probabilities for the density matrix. The results obtained in this section demonstrate the consistency of the values, regardless of the variables that were analyzed. In the next subsection, we will describe the density matrix based on the values presented.

4.6. Density Matrix

In the density matrix, the weights or prior probabilities w_i are determined by considerations external to the studied system, unrelated to it; their origin can be either quantum or classical. It is generally assumed that these probabilities w_i are constants, so the temporal evolution of the density matrix is determined by that of the states it contains, that is, by the rules of quantum mechanics [34]. With the results obtained in this section, the two models described in Section 3 were replicated, which allows us to provide the density matrix according to the canonical model and the classical model.

Canonical Ensemble

We were able to observe that there was no significant difference related to factors, so this would lead us to define our density matrix as follows:

$$\rho = 0.72|\psi_1\rangle\langle\psi_1| + 0.28|\psi_2\rangle\langle\psi_2|, \tag{5}$$

where the states of the system we can defined by

$|\psi_1\rangle$ is the vector representing the first quantum state (system 1).

$|\psi_2\rangle$ is the vector representing the second quantum state.

And its matrix representation is given by

$$\rho = \begin{pmatrix} 0.72 & 0 \\ 0 & 0.28 \end{pmatrix}. \tag{6}$$

The density matrix shown in Equation (6) shows on its diagonal the values of the pure states of the system. As we can observe, this model only provides us with information about the probabilities of the appearance of misconceptions in students, but it does not allow us to analyze the probabilities of mixed states.

Furthermore, for the analysis of the model, as we describe in Equation (2), a density matrix is constructed per student, which becomes the density matrix of a class by adding them and dividing them by the total number of students as in Equation (3):

$$\rho = \frac{1}{282}\begin{pmatrix} 203.31 & 122.59 \\ 122.59 & 78.69 \end{pmatrix}. \tag{7}$$

Simplifying the density of states matrix, we obtain

$$\rho = \begin{pmatrix} 0.72 & 0.43 \\ 0.43 & 0.28 \end{pmatrix}. \tag{8}$$

In this case, the density matrix in Equation (8) shows the probabilities of the pure states along its main diagonal, but goes further by encompassing additional information about the system by including mixed-state probabilities.

4.7. Factor Analysis

Factor analysis is a technique commonly used in educational and psychological research to extract information from a correlation matrix constructed from students' scores on different items of a test. The factors derived from this matrix reveal how the different elements of the test are related in terms of consistency between students' responses. The goal of factor analysis is not to explain the underlying reasons for these relationships, but rather to identify patterns and associations between test items based on student responses. In educational research, researchers often ask multiple equivalent questions about the same concept but with different contexts in assessment instruments. However, the influence of context can generate variation in the way students answer these equivalent questions, resulting in low consistency in student scores within groups of questions considered equivalent by experts. Interpretation of these findings depends on the student learning models that researchers employ. When students have mixed knowledge models, low consistency between equivalent item scores is primarily due to the influence of context on their knowledge [19]. In Table 10, we can observe the following cases: if the student responds in alignment with a specific thinking model, the correlation matrix would be obtained in the first form, and in the case where the student presents one or more thinking models, the second case would occur.

Table 10. Factor analysis cases [19].

	Factor Analysis	
	Case 1	**Case 2**
Correlation matrix	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -0.33 & -0.33 & -0.33 \\ -0.33 & 1 & -0.33 & -0.33 \\ -0.33 & -0.33 & 1 & -0.33 \\ -0.33 & -0.33 & -0.33 & 1 \end{pmatrix}$
Eigenvalues	$\sigma_1^2 = 4, \sigma_{2,3,4}^2 = 0$	$\sigma_{1,2,3,4}^2 \neq 0$

Following Bao's idea [19], we proceed to select a set of questions from the Test of Misconceptions (TCE), specifically questions P9, P14, P16, and P20, which contain the same misconception among their answer choices: "Metal has the ability to attract, retain, intensify, or absorb heat and cold." If we assume that students have this misconception deeply rooted in their thinking, then upon examining the correlation matrix, we would observe a structure similar to case 1, where only one consistent pattern (one significant eigenvalue) is identified due to uniformity in responses. On the other hand, if we consider that students have a mixed-state thinking regarding this misconception, meaning they respond variably depending on the context of the question, the correlation matrix would reflect a structure similar to case 2. In this scenario in Equation (9), upon calculating the eigenvalues, we would find more than one significant value, indicating the presence of multiple consistent patterns in responses.

$$\begin{pmatrix} & P9 & P14 & P16 & P20 \\ P9 & 1.00 & 0.09 & -0.03 & -0.06 \\ P14 & 0.09 & 1.00 & -0.02 & 0.04 \\ P16 & -0.03 & -0.02 & 1.00 & -0.03 \\ P20 & -0.06 & 0.04 & -0.03 & 1.00 \end{pmatrix}. \tag{9}$$

$$Eigenvalues \quad \sigma_1^2 = 1.1025367 \quad \sigma_2^2 = 1.0496245 \quad \sigma_3^2 = 0.9806959 \quad \sigma_4^2 = 0.8671430$$

The presence of mixed mental states among students, as suggested by this analysis, underscores the importance of employing modeling and analysis techniques that account for this variability, such as mixed-state modeling, for a more accurate interpretation of TCE results and a deeper understanding of how students handle and apply misconceptions in different learning contexts.

5. Discussion

It is worth noting that in some cases, other authors, such as Wood [27], reported a higher value for state 2, but the context of their study differed from ours. Furthermore, in Frederick's article [26], the original population included individuals who were not comparable to our sample; therefore, we considered only data relevant to our comparable sample. Since they included mainly high percentile profile students, their samples are not representative of our target population. In the literature, we found other contexts in which state 2 dominated, which although they coincide with the results obtained in this study, they are not comparable due to the level of studies of the participants [21,35,36].

Pathare and Pradhan [37] have revealed some common likely causes of misconceptions:

- Students have some presumed models formed even before they encounter a scientific notion.
- A word/phrase which means one thing in daily parlance may mean something else in scientific terminology.
- Students fail to understand the limitations of the applicability of a concept or a law and hence they over-generalize.
- The explanation of a concept delivered by a teacher is accepted by the student as it is. One of the important reasons for this is the examination-driven system which depends heavily on skills of memory and recall and underemphasizes understanding.
- Teachers often fail to give students an overview of the topic necessary for understanding it. This may lead to the formation of alternative models which are different from the relevant scientifically accepted models.

Therefore, state 1 predominates over state 2 since "misconceptions are deeply ingrained intuitive ideas" [27]. The reason it is common to find that state 1 dominates over state 2 is that state 2 requires more energy than state 1 [38]. This dynamic is because state 1 operates automatically and quickly, based on intuitions and misconceptions, making it the default choice for everyday tasks. In contrast, state 2, which is used for tasks that require more cognitive effort and deeper processing, consumes more mental resources and requires conscious effort. Therefore, the predominance of system 1 in our decision making and behavior is largely due to its energy efficiency, as it allows us to make quick and automated decisions in most situations.

Although these articles allowing us to conclude the dominance of state 1 over state 2, it is essential to highlight the additional information provided by the mixed states of the model. The values that lie between one state and another help us identify the confusion experienced by students when learning a concept, leading them to respond based on one model or another depending on the context of the problem. This may be due to some erroneous preconception that remains ingrained in their consciousness. It would be difficult to argue that decision errors between models arise from a lack of engagement or attention to the corresponding questions, as is the case in similar decision-making scenarios [9]. Therefore, the classical model is limited, and the quantum model is necessary.

When students exhibit mixed model states, the low consistency between students' scores on different equivalent items is primarily due to the context dependency of their knowledge [19]. This leads them to interpret the concept differently depending on the contextualization of the item, despite having knowledge of the theoretical concept that involves the correct response.

Limitations of the Study

The values discovered for the weights of state 1 and state 2 can be applied in situations similar to those of the present study. However, in different settings, appropriate adjustments would need to be made to fit the population in question. These values provide us with an understanding of students' cognitive behavior prior to any intervention aimed at enabling them to interpret concepts analytically. As a future step, we will seek to design an intervention that allows students to interpret physical concepts from the perspective of their state 2, avoiding falling into misconceptions guided by intuition. This could result in a more significant contribution to the field of education.

Although misconceptions are often guided heuristically, we cannot determine their precise cognitive origins. Therefore, a more controlled intervention would be necessary to define the mental processes that involve the emergence of these misconceptions. Just as it is also necessary to investigate further in order to find the probabilities of mixed states.

The study was carried out with first-year university students. If general conclusions are to be obtained, a study must be carried out with more controlled variables than those carried out here, taking into account different ages and student backgrounds.

The presented model is not the final quantum version; rather, it is an initial state that serves as a crucial stepping stone. Since this study focuses on establishing the foundations for subsequent calculations of matrices involving the probability of mixed states, the accuracy and robustness of these calculations may be subject to refinements in future iterations. Therefore, it is essential to recognize that the limitations of the current model lie in its transitional nature, and further advancements and adjustments are anticipated for a more comprehensive understanding in later stages of research in this topic.

6. Conclusions

In general, misconceptions dominate clearly over correct concepts, since all our values obtained through the TCE are consistent with findings from other articles and studies, such as the FCI and CRT. The proposed models prove their utility for similar populations, especially when limited information is available about the students. Once the corresponding systematic error has been accounted for, the values obtained align with the information found in the scientific literature.

The values obtained as weights of state 1 and state 2 did not show significant differences when analyzing by gender or scholarship and significant differences were only found when analyzing the different programs. For this reason, we assume that in similar populations, similar results will be obtained. In the study conducted in this population and similar populations compared, we can observe that state 1 (70%) dominates over state 2 (30%).

The additional information that we can obtain from our model, in these mixed states, allows us to measure the degree of confusion in the student, which enables us to know at what level of transition they are, from a state 1 dominated by preconceptions to a state 2, where they show dominance of concepts.

As a future work perspective, the possibility of designing an intervention with the aim of conducting a post-test, following the approach of various articles, is considered. This would allow for an analysis of the change compared to the initial state studied, potentially providing a deeper understanding of the effects of such interventions on similar populations.

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