

Article

Simultaneity and Time Reversal in Quantum Mechanics in Relation to Proper Time

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Abstract: In Newtonian physics, the equation of motion is invariant when the direction of time ($t \rightarrow -t$) is flipped. However, in quantum physics, flipping the direction of time changes the sign of the Schrödinger equation. An anti-unitary operator is needed to restore time reversal in quantum physics, but this is at the cost of not having a consistent definition of time reversal applicable to all fundamental theories. On the other hand, a quantum system composed of a pair of entangled particles behaves in such a manner that when the state of one particle is measured, the second particle ‘simultaneously’ acquires a determinate state. A notion of absolute simultaneity seems to be inferred by quantum mechanics, even though it is forbidden by the postulates of relativity. We aim to point out that the above two problems can be overcome if the wavefunction is defined with respect to proper time, which in fact is the real physical time instead of ordinary time.

Keywords: proper time; time reversal; simultaneity; wavefunction



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1. Introduction

Time reversal and simultaneity are time-dependent concepts. In Newtonian physics, the equation of motion is second order in time; thus, if $x(t)$ is a solution, then so is $x(-t)$. The equation of motion is said to be time-reversal invariant.

In quantum physics, Schrödinger’s equation is first order in time, and thus flipping the direction of time ($t \rightarrow -t$) changes the sign of the equation. However, if the wavefunction $\psi(x, t)$ satisfies Schrödinger’s equation, then $\psi^*(x, -t)$ is a solution of the complex conjugate of Schrödinger’s equation. The complex conjugation compensates for the change in the sign of time ($t \rightarrow -t$). Specifically, time reversal in quantum physics is defined by an anti-unitary operator [1] that maps a wavefunction $\psi(x, t)$ into its complex conjugate $\psi^*(x, -t)$ with an opposite sign of time. The anti-unitary operator restores the invariance of time reversibility but at the cost of not having a general definition of time reversal applicable to all fundamental theories [2].

Some authors [2–4] have questioned the standard definition of time reversal in quantum mechanics. For these authors, time reversal should normally be represented by simply flipping the sign of time ($t \rightarrow -t$). In particular, according to Albert [3], the fact that Schrödinger’s equation is first order in time entails that the evolution of quantum states cannot possibly be invariant under time reversal; otherwise, it would be a theory where nothing ever happens.

On the other hand, the laws of physics are invariant under Lorentz transformation. In particular, simultaneity has no meaning in special relativity, independent of any frame of reference, and there should be no preferred frame of reference [5,6]; thus, absolute simultaneity has no sense.

However, a quantum system composed of a pair of entangled particles behaves in such a manner that the quantum state of one particle cannot be described independently of that of the other [7]. Standard quantum mechanics postulates that neither of the particles has a determinate state until it is measured. Because both particles are correlated, it is necessary

that when the state of one particle is measured, the second particle should ‘simultaneously’ acquire a determinate state.

Time reversal and simultaneity have been widely debated in the different interpretations and candidate theories of quantum mechanics, including the Copenhagen interpretation [8], Everett many-world theory [9], de Broglie–Bohm pilot-wave theory [10,11], and GRW spontaneous collapse theory [12].

The Copenhagen interpretation considers standard quantum mechanics as only an instrument that allows us to determine the effects of microscopic objects belonging to an unknowable quantum realm on macroscopic instruments.

Everett’s many-world theory claims [13] that when a measurement is conducted on a particle in a superposition state, deterministic branching takes place where, on one branch, a first detector detects the particle while a second detector does not, and at the ‘same instant’, but on the other branch (i.e., another world), the first detector does not detect the particle while the second detector detects it. Unfortunately, there seems to be no clear meaning of a ‘same instant’ for a multitude of disconnected worlds.

The de Broglie–Bohm theory considers that a corpuscle, such as an electron, always has a well-determined position on a definite trajectory through physical space. However, its movement is influenced by an associated wave function, giving rise to wave-like properties. For a multi-particle system, the theory explicitly formulates the non-local dependence of a particle’s evolution at a given instant on the positions of all other particles at the same instant, implying absolute simultaneity. This would be acceptable if the de Broglie–Bohm theory were Lorentz invariant, but unfortunately, it is not [14].

The GRW spontaneous collapse theory [12] modifies Schrodinger’s equation with stochastic terms that have the effect of making a wavefunction obey Schrodinger’s equation most of the time, except for exceedingly rare and random instants when it undergoes a spontaneous collapse. The collapse modifies instantaneously and simultaneously all the spatial arguments of the wavefunction. However, an instantaneous collapse in one Lorentz frame may not be instantaneous in another.

The basic problem that remains is the inconsistency [15] of almost all of the above models with relativity, and in particular, nonlocality [16] and simultaneity. The notion of simultaneity is related to nonlocality, a notion that has been addressed in [17], and that we keep for future research. In this study, we concentrate on simultaneity and time reversibility.

This paper aims to point out that the above two problems seem to emanate from not choosing the appropriate notion of time in quantum mechanics. In Section 2, we review the formalism of Minkowski spacetime and, in particular, the geometrical representation of proper time. In Section 3, we define the wavefunction with respect to proper time, leading to a notion of ‘proper-time-simultaneity’. We also propose deriving the corresponding equations of motion with respect to proper time and discuss the proper-time-reversal invariance with respect to these equations. In Section 4, we consider the non-relativistic limit and aim to restore a general definition of time reversal that simply consists of flipping the sign of time. In Section 5, we propose the derivation of the continuity equation with respect to proper time. Section 6 presents a simple example illustrating the evolution of a particle in a box according to the equations of motion with respect to proper time.

2. Invariant Spacetime Structure

We propose considering the evolution of a quantum system (a particle or a collection of particles) from the perspective of proper time τ , by using the hyperbolic spacetime structure inside a light cone associated with the particle. Specifically, we use the formalism of Minkowski spacetime [18] as defined in a geometrical manner byourgoulhon [19].

The Minkowski spacetime $\mathcal{M}$ is an affine space of four dimensions on $\mathbb{R}$ endowed with a bilinear metric tensor g defined in an underlying vector space E of signature $(+, -, -, -)$. In vector space E , a set C composed of all null vectors forms a light cone C composed of two sheets, C^+ and C^- , defining the future and past light cones, respectively.

Given the above-defined spacetime $\mathcal{M}$ and an arbitrary origin $O \in \mathcal{M}$, a family of affine subspaces $(S_\tau)_{\tau \in R}$ is defined such that each subspace S_τ corresponds to the set of points of ε that can be connected to the origin O by a time-like vector $\overrightarrow{ON}$ of modulus τ , where $\tau \in R$:

$$S_\tau = \{N \in \mathcal{M}, \overrightarrow{ON} \cdot \overrightarrow{ON} = -\tau^2 < 0\} \quad (1)$$

Henceforth, we are interested in physical systems that follow time-like or null world-lines and do not consider the set of space-like vectors. In spacetime (ε, g) , a point $N \in \varepsilon$ is said to belong to the subspace S_τ iff $\overrightarrow{ON} \cdot \overrightarrow{ON} = -\tau^2$. Each set of points S_τ consists of two subsets or two sheets, S_τ^+ and S_τ^- belonging to the interiors of the future C^+ and past C^- light cones, respectively.

$$S_\tau^+ = \{N \in S_\tau, \tau \geq 0\} \quad (2)$$

$$S_\tau^- = \{N \in S_\tau, \tau < 0\} \quad (3)$$

Let (x^0, x^1, x^2, x^3) be the coordinates of $N \in S_\tau$ in the affine frame defined by origin O and an appropriate basis. Then, $\overrightarrow{ON} \cdot \overrightarrow{ON} = -\tau^2$ can be expressed as follows:

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -\tau^2 \quad (4)$$

where $x^0 = t$, $x^1 = x/c$, $x^2 = y/c$, $x^3 = z/c$.

Equation (4) is a three-dimensional hyperboloid of the two sheets S_τ^+ and S_τ^- spanned by the free extremities of the time-like vectors $\overrightarrow{ON}$.

The algebraic value τ of the time-like vector $\overrightarrow{ON}$ is the proper time for the physical system. It generates a family of affine subspaces, $(S_\tau)_{\tau \in R}$, defined by Equation (1). This family consists of three-dimensional hyperboloids associated, on one hand, with future-directed proper times $\tau \geq 0$ and, on the other hand, with past-directed proper times $\tau < 0$.

The sheet S_τ of each hyperboloid forms a ‘spatial-hypersurface’ that we shall simply call a ‘slice’ associated with a corresponding proper time $\tau \in R$. All points on any given slice S_τ are associated with the same proper time τ , which is indeed invariant to all observers from the perspective of any corresponding inertial frame of reference.

Using Equation (4), the proper time τ can be expressed as:

$$\tau = \pm \|\overrightarrow{ON}\|/c = \pm \sqrt{t^2 - (x^2 + y^2 + z^2)/c^2} = \pm \sqrt{t^2 - (\vec{x})^2/c^2} \quad (5)$$

where the ‘+’ sign corresponds to a future-directed proper time and the ‘−’ sign corresponds to a past-directed proper time with respect to the origin (i.e., event) O , $\|\overrightarrow{ON}\|$ is the modulus of the vector $\overrightarrow{ON}$, and $\vec{x}$ is the three-dimensional space coordinate.

For each proper time τ , the free extremity N of the vector $\overrightarrow{ON}$ spans the hyperbolic slice S_τ . The hyperbolic slice S_τ is a piecewise twice continuously differentiable curve of Minkowski spacetime (ε_I, g) composed of a set of hyperbolic points (u, τ) . The hyperbolic coordinate u represents the orientation of a ray R_u (i.e., a straight line) passing through the point $O \in \mathcal{M}$. All points (u, τ) on the same ray R_u share the same hyperbolic coordinate u . On the other hand, all points (u, τ) on the same slice S_τ share the same invariant proper time coordinate τ .

Thus, the rays $(R_u)_u$ and hyperbolic slices $(S_\tau)_\tau$ define a hyperbolic frame of reference $(O; R_u, S_\tau)$ where a given hyperbolic point (u, τ) is the intersection between the corresponding ray R_u and the slice S_τ . Therefore, each slice S_τ can be defined by the following set of points:

$$S_\tau = \{u_\tau = (u, \tau)\} \quad (6)$$

As each slice S_τ is associated with a corresponding proper time τ , all points u belonging to that slice S_τ may be considered to be ‘simultaneous’ in the sense of proper time. In other words, each slice S_τ is a class of ‘proper-time-simultaneity’ made up of a set of points that are associated with the same proper time instant τ .

The passage from one slice S_{τ_1} into a subsequent slice S_{τ_2} represents the ‘transition’ from a first proper time τ_1 to a consequent proper time τ_2 . Thus, proper time provides an invariant time ordering of the set of slices with respect to the event O .

We note that the hyperbolic slice S_τ can be parameterised by a bijective function φ from parameter λ into the points on that slice S_τ such that any point u on slice S_τ is given by $u = \varphi(\lambda) \equiv u(\lambda)$. Parameter λ can be chosen as the standard time coordinate t or standard space position $\vec{x} = (x, y, z)$. Therefore, for simplicity, each slice S_τ can be expressed as follows:

$$S_\tau = \{\vec{x}_\tau = (\vec{x}, \tau)\} \quad (7)$$

3. Wavefunction and Its Evolution through Proper Time

A quantum system composed of a pair of entangled particles is represented by a wavefunction that simultaneously defines both particles such that when the state of one particle is measured, the second particle should acquire a determinate state at the same time. However, in special relativity, there is no meaning of a ‘same time instant’ for spatially separated positions.

This inconsistency between quantum mechanics and special relativity may be solved by postulating “the existence of an underlying temporally invariant wavefunction such that the standard wavefunction is but an approximation of the former”. According to this hypothesis, the standard wavefunction can be deduced from the hypothetical temporally invariant wavefunction and not vice versa. In other words, it is necessary to first define a temporally invariant wavefunction and then deduce a standard wavefunction.

In Section 2, it is shown that all the points on a given hyperboloid slice S_τ are equidistant from the origin O of a Lorentz coordinate system and are thus invariant with respect to proper time for all inertial observers. Proper time is generally defined with respect to the world line, followed by a particle. However, here, we take advantage of the geometrical representation of proper time, as outlined in the preceding section, to provide invariant time ordering of the set of hyperboloid slices for particles moving freely from a central origin O . That is, the worldlines of free-moving particles connecting the origin O to different points on a given hyperboloid slice S_τ have the same value of proper time. In this case, an invariant ‘flow of physical time’ can be represented by a continuous and subsequent ordering of different hyperboloid slices with respect to a central event O . Furthermore, each hyperboloid slice S_τ can be used to represent a class of ‘proper-time-simultaneity’. It should be clear that this invariant flow of proper time is restricted to free-moving systems within a light cone defined with respect to an origin O and should not be confused with the concept of absolute time, which is banished by relativity.

In view of the above, it seems straightforward to suppose that the temporally invariant wavefunction (or, for short, ‘invariant wavefunction’) for a given quantum system composed of free-moving particles should be associated to a hyperboloid slice S_τ referenced with respect to proper time. An invariant wavefunction defined with respect to proper time τ would then be relativistically invariant for all corresponding inertial observers. Hereafter, the expression ‘quantum system’ refers to a physical system composed of free-moving particle(s).

First, we define an invariant unit state vector $|\varphi(\tau)\rangle$ as a function of proper time $\tau \in \mathbb{R}$ related to the corresponding slice S_τ . Each slice S_τ can be defined using Equation (7) by the set of points or events $\{\vec{x}_\tau = (\vec{x}, \tau)\}$ that lie at the same proper time value τ from some central event O (origin of a Lorentz coordinate system). Then, slice S_τ is considered to represent a position basis $\{|\vec{x}_\tau, \rangle\}$, which can be associated with a corresponding Hilbert

space $\mathcal{H}$ with elements $|\vec{x}_\tau\rangle$ labelled by a continuous variable $\vec{x}_\tau$ normalized using the Dirac δ -function:

$$\langle \vec{x}'_\tau | \vec{x}_\tau \rangle = \delta(\vec{x}'_\tau - \vec{x}_\tau) \quad (8)$$

The invariant unit state vector $|\varphi(\tau)\rangle$ in the Hilbert space $\mathcal{H}$ associated with slice S_τ can then be expanded as an integral function of the base elements $|\vec{x}_\tau\rangle$ as follows:

$$|\varphi(\tau)\rangle = \int d\vec{x}_\tau \varphi(\vec{x}_\tau) |\vec{x}_\tau\rangle \equiv \int d\vec{x} \varphi(\vec{x}, \tau) |\vec{x}, \tau\rangle \quad (9)$$

In Equation (9), the invariant state vector $|\varphi(\tau)\rangle$ of a physical system is described as a superposition of position basis elements $|\vec{x}_\tau\rangle$, each of which corresponds to a definite point $(\vec{x}, \tau)$ on slice S_τ , where τ is a constant. The expanding coefficients or ‘weights’ $\varphi(\vec{x}_\tau) \equiv \varphi(\vec{x}, \tau)$ represent a complex-valued invariant wavefunction where all the arguments are defined at the same proper time value τ with respect to a central event O.

The state vector $|\varphi(\tau)\rangle$ belongs to the Hilbert space $\mathcal{H}$ and represents the vector sum or resultant of the decomposed position states. As all the superposed arguments are defined at the same invariant proper time instant τ , there is a sense of calculating their resultant.

The arguments $\vec{x}_\tau$ of the invariant wavefunction $\varphi(\vec{x}_\tau)$ are associated with points $(\vec{x}, \tau)$ of the corresponding slice S_τ . Thus, the position state of a physical system at any given proper time instant τ is represented by an invariant wavefunction $\varphi(\vec{x}, \tau)$ that has a corresponding relativistic energy $E(\vec{x}, \tau)$ at that specific proper time τ .

As indicated above, the standard wavefunction is considered an approximation of the invariant wavefunction, and thus the latter cannot be derived from the former; that is, applying a relativistic transformation to the arguments of the standard wavefunction does not lead to an invariant wavefunction. In fact, had we started from a standard wavefunction $\psi(\vec{x}, t)$ and transformed the standard time into proper time according to the relativistic expression of Equation (5), we would have simply obtained a wavefunction equivalent to the standard wavefunction, but dependent on different values of proper time, as follows:

$$\psi(\vec{x}, t) = \psi(\vec{x}, \pm \sqrt{\tau^2 + (\vec{x})^2/c^2}) = \psi(\vec{x}, \tau(\vec{x})) \quad (10)$$

Proper time in Equation (10) depends on $\vec{x}$, and therefore there is no unique proper time value for all arguments of the wavefunction. This clearly shows that a temporally invariant wavefunction cannot be deduced from a standard wavefunction. Thus, as indicated above, it must first be defined.

In fact, what is proposed in this paper is not a relativistic transformation of a wavefunction from standard time into proper time, but a change of perspective in which the arguments of the invariant wavefunction $\varphi(\vec{x}_\tau) \equiv \varphi(\vec{x}, \tau)$ of Equation (9) are defined immediately from the start as arguments associated to elementary events that lie at the same proper time value τ from a central event O. On the other hand, as the invariant wavefunction $\varphi(\vec{x}, \tau)$ depends on proper time τ , its evolution should also be defined with respect to proper time τ instead of ordinary time t . In other words, the evolution of the invariant wavefunction $\varphi(\vec{x}, \tau)$ should be defined with respect to subsequent hyperboloid slices. This may be achieved by associating relativistic energy $E(\vec{x}, \tau)$ to the system represented by the invariant wavefunction $\varphi(\vec{x}, \tau)$.

In particular, the relativistic energy $E(\vec{x}, \tau)$ of a system with a definite momentum P (or velocity v) with respect to an inertial frame of reference (Lorentz coordinate system) is defined as follows:

$$E = \sqrt{m^2c^4 + P^2c^2} = mc^2 / \sqrt{1 - v^2/c^2} \quad (11)$$

Equation (11) can also be expressed as:

$$\sqrt{1 - v^2/c^2} = mc^2/E \quad (12)$$

Momentum and energy operators are generators of translations in space $\vec{x}$ and time t , respectively, and they operate on the wavefunction to quantify the rate of change of their states. Thus, even though the momentum and energy operators are defined as functions of differentials in space $\vec{x}$ and time t , respectively, the corresponding momentum and energy observables do not necessarily depend explicitly on the space and/or time variables. For example, momentum and energy observables are stationary for free particles.

On the other hand, the differential quantum operator associated with energy E is given by:

$$i\hbar\partial/\partial t \equiv E \quad (13)$$

To define the energy operator for a system with respect to proper time, we use Equation (5) to express the differential of proper time $\delta\tau$ as a function of the differentials of ordinary time δt and space $\delta\vec{x}$ as follows:

$$\delta\tau = \pm\sqrt{\delta t^2 - \delta\vec{x}^2/c^2} \quad (14)$$

The '+/−' signs designate vectors inside the upper/lower light cones with respect to a central event O. In the upper light cone, $\delta\tau \geq 0$, whereas in the lower light cone, $\delta\tau < 0$.

Using relation (14), the differential $\delta\tau$ can be expressed as follows:

$$\delta\tau = \pm\delta t\sqrt{1 - (\delta\vec{x}/\delta t)^2/c^2} = \pm\delta t\sqrt{1 - v^2/c^2} \quad (15)$$

Injecting Equation (12) into Equation (15), we obtain

$$\delta\tau = \pm\delta t(mc^2/E) \quad (16)$$

By substituting Equation (16) into Equation (13), we obtain the following energy operator with respect to proper time:

$$i\hbar\partial/\partial\tau \equiv \pm E^2/mc^2 \quad (17)$$

The term E^2/mc^2 represents a 'characteristic proper energy' of the system associated with the evolution of the invariant wavefunction through proper time. In the absence of potential energy, the characteristic proper energy is equal to $mc^2 + P^2/m$. The operator of this characteristic proper energy may be called the 'proper Hamiltonian' $\hat{H}$ defined as follows:

$$\hat{H} = \hat{E}^2/mc^2 \quad (18)$$

To describe the evolution of the invariant wavefunction $\varphi(\vec{x}, \tau)$ with respect to proper time, we apply Equations (17) and (18) to the invariant wavefunctions $\varphi(\vec{x}, \tau)$, as follows:

$$i\hbar\frac{\partial\varphi(\vec{x}, \tau)}{\partial\tau} = \pm\hat{H}\varphi(\vec{x}, \tau) = \pm(\hat{E}^2/mc^2)\varphi(\vec{x}, \tau) \quad (19)$$

The solutions of the above system of equations are:

$$\varphi(\tau, x) = \varphi_0 e^{\pm i(E^2/\hbar mc^2)\tau} \quad (20)$$

where φ_0 is an initial distribution and where the '+' sign (respectively, '−' sign) designates a future-directed (respectively, past-directed) proper time evolution of the invariant wavefunction $\varphi(\vec{x}, \tau)$ with respect to a central event O.

For simplicity, we exclude the writing of a hat on top of the operators. By expanding the expression in Equation (19) into a system of two equations while using the relativistic energy E of Equation (11), we obtain:

$$i\hbar \frac{\partial \varphi(\vec{x}, \tau)}{\partial \tau} = +(mc^2 + P^2/m) \varphi(\vec{x}, \tau) \quad \tau \geq 0 \quad \text{future-directed} \quad (21)$$

$$i\hbar \frac{\partial \varphi(\vec{x}, \tau)}{\partial \tau} = -(mc^2 + P^2/m) \varphi(\vec{x}, \tau) \quad \tau < 0 \quad \text{past-directed} \quad (22)$$

Equations (21) and (22) describe the evolution of the invariant wavefunction $\varphi(\vec{x}, \tau)$ according to two different dynamics: Equation (21) corresponds to future-directed dynamics taking place within a future-light cone, while Equation (22) corresponds to past-directed dynamics taking place within a past-light cone. It is important to note that the above system is composed of two separate equations applicable in opposite directions of proper time. Strictly speaking, the predictive equation is not identical to the retrodictive equation, implying inherent dissymmetry between the two directions of time. However, the system of equations as a whole may be considered to be time-reversal invariant in the sense that if we reverse the direction of proper time in any one of the two equations, we directly obtain the other equation.

In fact, the above system of equations can be expressed as follows:

$$i\hbar \frac{\partial \varphi(\vec{x}, |\tau|)}{\partial |\tau|} = +(mc^2 + P^2/m) \varphi(\vec{x}, |\tau|) \quad \text{future-directed} \quad (23)$$

$$-i\hbar \frac{\partial \varphi(\vec{x}, -|\tau|)}{\partial |\tau|} = -(mc^2 + P^2/m) \varphi(\vec{x}, -|\tau|) \quad \text{past-directed} \quad (24)$$

Where $|\tau|$ denotes the absolute value of τ . Because $\tau \geq 0$ in the first equation and $\tau < 0$ in the second equation, they have been replaced by $+\tau$ and $-\tau$, respectively.

Wavefunction $\varphi(\vec{x}, |\tau|)$ is the solution of Equation (23). If we reverse the time direction of proper time ($|\tau| \rightarrow -|\tau|$) in the first equation, we consistently obtain Equation (24), with the wavefunction $\varphi(\vec{x}, -|\tau|)$ as a solution. Each equation is, in the conventional sense, the time reversal of the other, and thus the system of equations as a whole may reasonably be considered time-reversal invariant. Normally, each equation by itself cannot be time-reversal invariant because it is only applicable in a unique direction of time.

In fact, the system of Equations (23) and (24) can be expressed as a single equation. This can be achieved by first expressing Equation (17) as a product of the two terms before applying it to the invariant wavefunction $\varphi(\vec{x}, \tau)$, as follows:

$$\left(i\hbar \frac{\partial}{\partial \tau} + E^2/mc^2\right) \left(i\hbar \frac{\partial}{\partial \tau} - E^2/mc^2\right) \varphi(\vec{x}, \tau) = 0 \quad (25)$$

Developing Equation (17) we obtain the following equation of motion:

$$\left(-\hbar^2 \frac{\partial^2}{\partial \tau^2} - (E^2/mc^2)^2\right) \varphi(\vec{x}, \tau) = 0 \quad (26)$$

The above Equation (26) is second order in time and is thus time-reversal invariant. However, we lose the refinement of the different dynamics defined in the system of Equations (23) and (24) with respect to the opposite directions of proper time.

4. Time Reversal in the Non-Relativistic Limit

Equation (5) can be expressed as follows:

$$\tau = \sqrt{t^2 - x^2/c^2} = t\sqrt{1 - v^2/c^2} \approx t(1 - v^2/2c^2) \quad (27)$$

For a free particle in the non-relativistic limit, proper time can roughly be approximated by ordinary time $\tau \approx t$. Thus, the invariant wavefunction $\varphi(\vec{x}, \tau)$ can roughly be approximated by the standard wavefunction $\varphi(\vec{x}, \tau) \approx \varphi(\vec{x}, t) \equiv \psi(\vec{x}, t)$, at least within a small spatial extension. By introducing these approximations into Equation (19) and the fact that $\delta\tau = \pm\delta t(mc^2/E)$, we obtain:

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = +E\psi(\vec{x}, t) \quad \text{futur - directed} \quad (28)$$

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = -E\psi(\vec{x}, t) \quad \text{past - directed} \quad (29)$$

The energy E for a free particle in the nonrelativistic limit can be expressed as follows:

$$E = \sqrt{m^2c^4 + P^2c^2} = mc^2\sqrt{1 + v^2/c^2} \approx mc^2 + mv^2/2 \quad (30)$$

By introducing the approximation of Equation (30) into the above system of Equations (28) and (29), we obtain

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = (mc^2 + mv^2/2)\psi(\vec{x}, t) \quad \text{for } t \geq 0 \quad (31)$$

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = -(mc^2 + mv^2/2)\psi(\vec{x}, t) \quad \text{for } t \leq 0 \quad (32)$$

The term mc^2 is a constant that corresponds to the rest energy. This has no consequences for the evolution of the physical system and can be omitted. Moreover, the nonrelativistic kinetic energy $mv^2/2$ can be expressed as $p^2/2m$, leading to the following system of equations:

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = (p^2/2m)\psi(\vec{x}, t) \quad \text{for } t \geq 0 \quad (33)$$

$$i\hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} = (p^2/2m)\psi(\vec{x}, t) \quad \text{for } t \leq 0 \quad (34)$$

The system of Equations (33) and (34) concerns two separate Schrödinger equations describing the evolution of the wavefunction $\psi(\vec{x}, t)$ according to the future and past directed dynamics, respectively. This system may be considered as time-reversal invariant in the conventional sense, insofar as the system in its globality is concerned. Reversing the direction of time in any one of the two equations directly leads to the other equation.

This system of Equations (33) and (34) seems to restore a general definition of time reversal in terms of a simple unitary operator consisting of flipping the sign of time ($t \rightarrow -t$), which is applicable to classical as well as quantum dynamics.

5. Invariant Continuity Equation

Injecting the momentum operator $p = -i\hbar\vec{\nabla}$ into Equations (23) and (24), we obtain

$$i\hbar \frac{\partial \varphi(\vec{x}, \tau)}{\partial \tau} = +\left(mc^2 - \frac{\hbar^2}{m}\nabla^2\right)\varphi(\vec{x}, \tau) \quad \tau \geq 0 \quad \text{future - directed} \quad (35)$$

$$i\hbar \frac{\partial \varphi(\vec{x}, \tau)}{\partial \tau} = -\left(mc^2 - \frac{\hbar^2}{m}\nabla^2\right)\varphi(\vec{x}, \tau) \quad \tau < 0 \quad \text{past - directed} \quad (36)$$

To derive the continuity equation, we consider the future-directed evolution according to Equation (35) Multiplying Equation (35) by the conjugate invariant wavefunction

$\varphi^*(\vec{x}, \tau) \equiv \varphi^*$ and multiplying the complex conjugate of Equation (35) by the invariant wavefunction $\varphi(\vec{x}, \tau) \equiv \varphi$ gives:

$$i\hbar\varphi^*\frac{\partial\varphi}{\partial\tau} = +\varphi^*\left(mc^2 - \frac{\hbar^2}{m}\nabla^2\right)\varphi \quad (37)$$

$$-i\hbar\varphi\frac{\partial\varphi^*}{\partial\tau} = +\varphi\left(mc^2 - \frac{\hbar^2}{m}\nabla^2\right)\varphi^* \quad (38)$$

Subtracting the second Equation (38) from the first Equation (37) yields the following result:

$$i\hbar\frac{\partial\varphi^*\varphi}{\partial\tau} = \frac{\hbar^2}{m}\left(\varphi\nabla^2\varphi^* - \varphi^*\nabla^2\varphi\right) \quad (39)$$

The above Equation (39) can be simplified, as follows:

$$\frac{\partial|\varphi|^2}{\partial\tau} = -\frac{i\hbar}{m}\vec{\nabla} \cdot \left(\varphi\vec{\nabla}\varphi^* - \varphi^*\vec{\nabla}\varphi\right) \quad (40)$$

Equation (40) can be written as a continuity equation:

$$\frac{\partial\rho}{\partial\tau} + \vec{\nabla} \cdot \vec{j} = 0 \quad (41)$$

where

$$\vec{j} = \frac{i\hbar}{m}\left(\varphi\vec{\nabla}\varphi^* - \varphi^*\vec{\nabla}\varphi\right) \quad (42)$$

Note that the above expression for current $\vec{j}$ is twice the value of the standard current. The continuity Equation (41) is proper-time-reversal invariant. When proper time is reversed, the velocity, and thus the current $\vec{j}$, is reversed. Thus, the continuity equation is invariant under time reversal because both sides of the equation change their signs.

Integrating the continuity equation (41) over the volume of the entire space, we obtain the following:

$$\frac{d}{d\tau} \int \rho dx dy dz = \int \frac{\partial\rho}{\partial\tau} dx dy dz = - \int \vec{\nabla} \cdot \vec{j} dx dy dz = \int \vec{j} d^2s \quad (43)$$

In the last equality, Gauss's theorem is used to transform the volume integral into a surface integral over s . The last integral is equal to zero as the current $\vec{j}$ vanishes at the boundary of surface s at infinity. Thus, Equation (43) becomes:

$$\frac{d}{d\tau} \int \rho dx dy dz = 0 \quad (44)$$

Therefore, the integral of $\rho = \varphi\varphi^*$ over the entire space is conserved at each proper time instant τ .

The continuity Equation (41), as well as the equations of motion Equations (21) and (22) can be applied to all existing quantum theories, such as many-world theories, de Broglie-Bohm theories, and collapse theories.

For example, in the case of the de Broglie-Bohm theory, let $X(\tau)$ be the actual position of the particle. The invariant wavefunction $\varphi(\vec{x}, \tau)$ can be expressed as a function of its amplitude $R(\vec{x}, \tau)$ and phase $S(\vec{x}, \tau)$ as follows:

$$\varphi(\vec{x}, \tau) = R(\vec{x}, \tau)e^{iS(\vec{x}, \tau)/\hbar} \quad (45)$$

We introduce the above formulation into the current $\vec{j}$ of Equation (42), and we obtain:

$$\vec{j} = \frac{2\hbar}{m} R^2 \vec{\nabla} S = \frac{2\hbar}{m} \rho \vec{\nabla} S \quad (46)$$

The current $\vec{j}$ can be expressed as the density multiplied by the velocity $\vec{U}$ of the particle, which is related to the phase $S(\tau, x, y, z)$ of the wavefunction according to the following equation:

$$\vec{U} = \frac{d\vec{x}}{d\tau} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau} \right) = \frac{2\hbar}{m} \vec{\nabla} S \Big|_{X(\tau)} \quad (47)$$

The gradient of the phase $\vec{\nabla} S$ is evaluated at the actual location of the particle $X(\tau)$. Here, $\vec{U}$ is an invariant velocity corresponding to the spatial components of the 4-relativistic velocity.

It is to be noted that the above formulation is not restricted to a single particle. The origin of a Lorentz coordinate system used to construct the slices may be a central event O corresponding to the emission of a plurality of entangled particles sent off in different directions and at different speeds. In this case, the quantum system is composed of n entangled particles defined by an n -particle wavefunction $\varphi(\vec{x}_1, \dots, \vec{x}_i, \dots, \vec{x}_n, \tau)$ with respect to the slices constructed out of the original central event common to all these particles.

Thus, for an n -particle wave function $\varphi(\vec{x}_1, \dots, \vec{x}_i, \dots, \vec{x}_n, \tau)$ where $\vec{x}_i$ represents the position of the i th particle, Equation (47) can be generalised as follows:

$$\vec{U}_i = \frac{d\vec{x}_i}{d\tau} = \frac{2\hbar}{m} \vec{\nabla}_i S(\vec{x}_1, \dots, \vec{x}_i, \dots, \vec{x}_n, \tau) \Big|_{X_i(\tau)} \quad (48)$$

Equation (48) is Lorentz invariant, and it defines the velocity $\vec{U}_i$ of an i^{th} particle at a given proper time, τ , with respect to the positions of all other particles at the same proper time τ . Thus, it makes sense that the motion of the i^{th} particle at a given proper time τ depends on the positions of all other particles at the same proper time τ . In fact, at any given proper time τ , the positions of all the particles are associated to the same slice no matter what the distance is between these particles.

6. Simple Application

We shall consider here a simple example with respect to the future-directed evolution of the invariant wavefunction $\varphi(x, \tau)$ in two dimensions $(x, \tau) \in R^2$ according to the following equation:

$$i\hbar \frac{\partial \varphi(x, \tau)}{\partial \tau} = H\varphi(x, \tau) = \left(E^2/mc^2 \right) \varphi(x, \tau) \quad (49)$$

For an isolated system, we suppose that the energy operator $H = E^2/mc^2$ does not explicitly depend on proper time. The eigenfunctions of the energy operator may be defined independently of proper time, as follows:

$$H\varphi_n(x) \equiv \left(E^2/mc^2 \right) \varphi_n(x) = \left(E_n^2/mc^2 \right) \varphi_n(x) \quad (50)$$

The solutions of Equation (50) defines a set $\{E_n^2/mc^2, \varphi_n(x)\}$ of real eigenvalues E_n^2/mc^2 representing the energy levels of the system and eigenfunctions $\varphi_n(x)$.

On the other hand, the proper time solution of the future-directed evolution Equation (49) is:

$$\varphi(\tau, x) = \varphi_0 e^{-i(E^2/\hbar mc^2)\tau} \quad (51)$$

where φ_0 represents the initial invariant wavefunction at $\tau = 0$.

A given invariant wavefunction $\varphi(x, \tau)$ at $\tau = 0$ is defined as the superposition of stationary states:

$$\varphi_0 = \varphi(x, 0) = \sum_n a_n \varphi_n(x) \quad (52)$$

Expressions of Equations (51) and (52) show that the proper time evolution of an arbitrary invariant wavefunction $\varphi(x, \tau)$ can be expressed as

$$\varphi(x, \tau) = \sum_n a_n e^{-i(E^2/\hbar mc^2)\tau} \varphi_n(x) \quad (53)$$

Thus, the evolution with respect to proper time of an isolated system can be obtained after determining the eigenvalues and eigenfunctions of the proper energy.

As a simple example, we consider a particle that moves within a square potential well [20] with the following evenly defined potential $V(x)$:

$$V(x) = \begin{cases} 0 & \text{for } |x| < a \\ V_0 > 0 & \text{otherwise} \end{cases} \quad (54)$$

In the position representation, the governing equation is:

$$H\varphi(x) = (E^2/mc^2)\varphi(x) \quad (55)$$

where

$$H = mc^2 + P^2/m + V(x) \equiv mc^2 + (-\hbar^2/m) \frac{d^2}{dx^2} + V(x) \quad (56)$$

By introducing the expression of Equation (56) into Equation (55), we obtain

$$\frac{d^2\varphi}{dx^2} = -\left(\frac{E^2 - m^2c^4 - Vmc^2}{\hbar^2c^2}\right)\varphi \quad (57)$$

By considering the definition of $V(x)$, Equation (57) is reduced to the following pair of equations:

$$\frac{d^2\varphi}{dx^2} = -\left(\frac{E^2 - m^2c^4}{\hbar^2c^2}\right)\varphi \quad \text{for } |x| < a \quad (58)$$

$$\frac{d^2\varphi}{dx^2} = -\left(\frac{E^2 - m^2c^4 - V_0mc^2}{\hbar^2c^2}\right)\varphi \quad \text{otherwise} \quad (59)$$

The solutions of the first Equation (58) are:

$$\begin{cases} \varphi(x) = B\cos(kx) & \text{for even parity} \\ \varphi(x) = B\sin(kx) & \text{for odd parity} \end{cases} \quad (60)$$

where

$$k = \sqrt{(E^2 - m^2c^4)/\hbar^2c^2} = P/\hbar \quad (61)$$

In the nonrelativistic limit, $P \approx mv$, and thus we obtain the nonrelativistic expression of k :

$$k \approx \sqrt{m^2v^2/\hbar^2} = \sqrt{2mE/\hbar^2} \quad (62)$$

where $E = mv^2/2$.

The second Equation (59) can be written as:

$$\frac{d^2\varphi}{dx^2} = \left(\frac{V_0mc^2 + m^2c^4 - E^2}{\hbar^2c^2}\right)\varphi \quad (63)$$

The particle is bounded by the potential well, $E^2 < V_0 mc^2$, and thus:

$$\left(\frac{V_0 mc^2 + m^2 c^4 - E^2}{\hbar^2 c^2} \right) > 0 \quad (64)$$

Then, the solutions to the second Equation (59) are

$$\varphi(x) = Ae^{\pm Kx} \quad (65)$$

where

$$K = \sqrt{(V_0 mc^2 + m^2 c^4 - E^2) / \hbar^2 c^2} \quad (66)$$

In the case of even parity, by considering the continuity of $\frac{d\varphi}{dx}$ and $\varphi(x)$ at $x = a$, we obtain:

$$k \tan(ka) = K = \sqrt{(V_0 mc^2 + m^2 c^4 - E^2) / \hbar^2 c^2} = \sqrt{(V_0 m / \hbar^2) - k^2} \quad (67)$$

The expression of Equation (67) can be rewritten as:

$$k \tan(ka) = K = \sqrt{(W^2 / k^2 a^2) - 1} \quad (68)$$

where

$$W = \sqrt{V_0 m a^2 / \hbar^2} \quad (69)$$

W and k are dimensionless variables.

The square well traps the particle regardless of how small V_0 and a are. The number of solutions increases as W increases.

In the case of odd parity:

$$k \cot(ka) = -K = -\sqrt{\frac{V_0 m}{\hbar^2} - k^2} \quad (70)$$

The solutions are $W = (2r + 1)\pi/2$ for $r = 1, 2, \dots$

In the case of an infinitely narrow potential well, W tends to infinity; therefore, $\tan(ka)$ tends to zero.

For example, for odd parity states, we have:

$$\cot(ka) = \infty \quad (71)$$

Thus, the distinct solutions are:

$$k_n = n\pi/a \quad \text{where } n \in N^* \quad (72)$$

Thus, using equation (5.13), we get

$$E_n = \sqrt{\frac{\hbar^2 n^2 \pi^2 c^2}{a^2} + m^2 c^4} = mc^2 \sqrt{1 + \frac{\hbar^2 n^2 \pi^2}{a^2 m^2 c^2}} \quad (73)$$

In the non-relativistic limit

$$E_n \approx mc^2 \left(1 + \frac{\hbar^2 n^2 \pi^2}{2a^2 m^2 c^2} \right) = mc^2 + \frac{\hbar^2 n^2 \pi^2}{2ma^2} \quad (74)$$

The term mc^2 is a constant; thus, we obtain the classical expression:

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2} \quad (75)$$

Inside the well, the solutions are:

$$\varphi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \quad (76)$$

The example above describes the proper-time evolution of a particle in a box. In the non-relativistic limit, the evolution becomes identical to that of the standard formalism [20].

7. Conclusions

Proper time is considered to be the only real physical time. Each proper time instant is associated with a specific hyperboloid slice. The ‘flow’ of proper time is thus represented by subsequent hyperboloid slices with respect to a central event. The quantum state of a physical system is defined with respect to proper time, which is relativistically invariant to all corresponding inertial observers. For a quantum system composed of a pair of entangled particles, it makes sense to define the state of spatially separated particles at the same proper time instant. The evolution of the wavefunction is then described with respect to the subsequent hyperboloid slices. It should be stressed that the invariant wavefunction representing the quantum state is hypothetically constructed right from the beginning with respect to the proper time, and not as a result of a relativistic transformation of the arguments of a standard wavefunction.

In the non-relativistic limit, the dynamics of the invariant wavefunction yield two separate Schrödinger equations according to future- and past-directed dynamics, thus restoring a universal definition of time reversal in terms of a simple unitary operator consisting of flipping the sign of time. This suggests that the original hypothesis of defining the wavefunction with respect to proper time is reasonable.

In a future work, we shall consider the applicability of the above formalism defined in a configuration space to other types of quantum representations, such as the Wigner distribution, which is defined in a phase space.

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References

1. Wigner, E. *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*; Academic Press: New York, NY, USA, 1959.
2. Loprez, C. Seeking for a Fundamental Quantum Arrow of Time. *Front. Phys.* **2018**, *6*, 104. [[CrossRef](#)]
3. Albert, D.Z. *Time and Chance*; Harvard University Press: Cambridge, MA, USA, 2000.
4. Callender, C. Is Time ‘Handed’ in a Quantum World? In *Proceedings of the Aristotelian Society*; Blackwell Science Ltd.: Oxford, UK, 2000.
5. Einstein, A. Zur Elektrodynamik bewegter Körper. *Ann. Phys.* **1905**, *17*, 891. [[CrossRef](#)]
6. Maudlin, T. *Philosophy of Physics*; Space and Time; Princeton University Press: Princeton, NJ, USA, 2012.
7. Einstein, A.; Podolsky, B.; Rosen, N. Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? *Phys. Rev.* **1935**, *47*, 777–780. [[CrossRef](#)]
8. Bohr, N. The Quantum Postulate and the Recent Development of Atomic Theory. *Nature* **1928**, *121*, 580–590. [[CrossRef](#)]
9. Everett, H. *The Theory of the Universal Wave Function*; Princeton University: Princeton, NJ, USA, 1956.
10. Broglie, L.D. Remarques sur la théorie de l’onde-pilote. *Acad. Sci. Paris* **1951**, *233*, 641–644.
11. Bohm, D. A Suggested Interpretation of the Quantum Theory in Terms of ‘Hidden’ Variables. *Phys. Rev. D* **1986**, *85*, 166. [[CrossRef](#)]
12. Ghirardi, G.C.; Rimini, A.; Weber, T. United Dynamics for Microscopic and Macroscopic Systems. *Phys. Rev. D* **1986**, *34*, 470. [[CrossRef](#)] [[PubMed](#)]
13. Wallace, D. *Emergent Multiverse*; Oxford University Press: Oxford, UK, 2012.
14. Bricmont, J. *Making Sense of Quantum Mechanics*; Springer: Berlin/Heidelberg, Germany, 2016.
15. Yasmineh, S. Foundations of Quantum Mechanics. *Encyclopedia* **2022**, *2*, 1082–1090. [[CrossRef](#)]

16. Bell, J. On the Einstein Podolsky Rosen Paradox. *Physics* **1964**, *1*, 195–200. [[CrossRef](#)]
17. Yasmineh, S. Ontology of a Wavefunction from the Perspective of an Invariant Proper Time. *Front. Phys.* **2021**, *9*, 734. [[CrossRef](#)]
18. Minkowski, H. *Raum und Zeit. Jahresberichte der Deutschen Mathematiker Vereinigung*; Vieweg+ Teubner Verlag: Wiesbaden, Germany, 1909.
19. Gourgoulhon, E. *Special Relativity in General Frames*; Springer: Berlin, Germany, 2013.
20. Binney, J.; Skinner, D. *The Physics of Quantum Mechanics*; Oxford University Press: Oxford, UK, 2014.