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Exact Time Evolution of Genuine Multipartite Correlations for N-Qubit Systems in a Common Thermal Reservoir

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Abstract: We investigate the dynamical evolution of genuine multipartite correlations for N-qubits in a common reservoir considering a non-dissipative qubits-reservoir model. We derive an exact expression for the time-evolved density matrix by modeling the reservoir as a set of infinite harmonic oscillators with a bilinear form of interaction Hamiltonian. Interestingly, we find that the choice of two-level systems corresponding to an initially correlated multipartite state plays a significant role in potential robustness against environmental decoherence. In particular, the generalized W-class Werner state shows robustness against the decoherence for an equivalent set of qubits, whereas a certain generalized GHZ-class Werner state shows robustness for inequivalent sets of qubits. It is shown that the genuine multipartite concurrence (GMC), a measure of multipartite entanglement of an initially correlated multipartite state, experiences an irreversible decay of correlations in the presence of a thermal reservoir. For the GHZ-class Werner state, the region of mixing parameters for which there exists GMC, shrinks with time and with increase in the temperature of the thermal reservoir. Furthermore, we study the dynamical evolution of the relative entropy of coherence and von-Neumann entropy for the W-class Werner state.

Keywords: multipartite correlations; thermal reservoirs; Werner type states

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1. Introduction

Entanglement arising from the superposition principle and tensorial structure of Hilbert spaces is a striking feature of multiparty quantum systems [1]. In addition to playing a significant role in the foundation of quantum mechanics [2–4], entanglement has proved to be a crucial resource in quantum information tasks such as quantum teleportation [5], secret sharing [6], and superdense coding [7], to mention a few. Entanglement in a multipartite (more than two parties) system has also proved to be a significant resource. It plays an essential role in quantum metrology, and in some cases, protocols involving multipartite correlations are more robust than protocols involving two-party correlations [8,9]. For instance, a three-qubit maximally entangled GHZ state exhibits non-locality more strongly than two-qubit Bell states [10]. Moreover, three-party teleportation protocols have been shown to be less vulnerable to cheating as compared to two-party protocols [6,11]. Furthermore, protocols involving parties at several different spatially separated locations necessarily require multipartite correlation. Therefore, the characterization and quantification of genuine multipartite correlation has received significant interest [12–17]. Entanglement in bipartite two-qubit systems is well understood through several measures such as concurrence [18], entanglement of formation [19], negativity [20], etc. However, even for the simplest case of a three-qubit system, quantifying the underlying correlations is non-trivial [12]. For the faithful quantification of three-party correlations, which are used as a resource in teleportation and other quantum tasks, one requires a genuine correlation quantifier, which is zero for all biseparable and product states and non-zero for all non-biseparable states [14]. There exists several measures of

genuine entanglement for pure tripartite systems [17,21,22]; however, quantifying genuine entanglement for a general three-qubit system still remains a challenging task. The author of [15] has defined a measure "genuine multipartite concurrence", a quantifier of genuine multipartite entanglement, which is reduced to the Wooter's bipartite concurrence for two qubits. However, a closed expression exists only for X-type states, wherein only the diagonal and anti-diagonal elements are non-zero. Another measure widely studied for mixed three-qubit states is the tripartite negativity, which is defined through the geometric mean of negativity corresponding to the three bipartitions [23].

A realistic quantum system can never be considered isolated, as the system will inevitably interact with the environment. The interaction between the system and the environment causes a brisk decline of quantum correlations in the system. Therefore, for all practical applications, it is of significant importance to carry out a careful study of the characterization and quantification of quantum correlations under environmental decoherence. The dynamics of quantum correlations of single-qubit, bipartite, and tripartite systems in an open system have been studied extensively [24–35]. Viola et al. [36,37] worked out an exact model for a two-state quantum system coupled to a reservoir of harmonic oscillators and also showed that decoherence can dynamically be suppressed through repeated effective time-reversal operations on the combined system and bath. Correlations of bipartite systems have been extensively studied for zero temperature reservoirs [28] and finite temperature reservoirs [38]. Zeng-Zhao Li et al. [33] studied the entanglement dynamics of two coupled qubits in different environments. Their investigation showed that at zero temperature, the entanglement of two qubits saturates to a non-zero value when qubits are placed in a common environment. Recently, dynamics of a more general bipartite correlation, namely quantum discord for two initially correlated qubits in two different reservoirs, was studied for ohmic reservoirs [38], and it was shown that the preservation duration gets longer, the lower the temperature of the environment, the weaker the environmental coupling, and the larger the temperature difference between the reservoirs. The dynamics of decoherence and quantum correlation for two Bose–Einstein condensates has also been studied, where the decoherence can be controlled by manipulating the interaction between the system and the environment [39]. Jun-Hong et al. [31] explored the dynamics of a quantum register for two and three-qubit systems with dipole–dipole interactions, interacting with a common environment. Their study indicated that the environment could incoherently induce entanglement among qubits in the decoherence free space. Decoherence in quantum correlation for a three-qubit system has been studied for the case of classical environmental noise, where the quantum correlations display different decaying behaviors, depending on the system–environment interactions and different types of noise considered [40]. We note that in addition to the harmonic oscillator model, a well-known alternative to model the environment is through spin chains [41], and the resulting decoherence on the correlations and consequently on information theoretic protocols has been extensively studied [42].

In this work, we study the dynamics of genuine multipartite concurrence of various multipartite correlated states in the presence of a common thermal reservoir. The reservoir is modelled with infinite quantum harmonic oscillators, and a non-trivial bilinear form of interaction Hamiltonian is considered such that the system is non-dissipative (i.e., interaction Hamiltonian commutes with the system Hamiltonian). We show that the symmetric GHZ class of Werner state experiences irreversible loss in the genuine correlation, with the decay rate depending on the spectral density under consideration. We find the preservation time of the genuine correlation for various spectral densities and different temperature, and show that the multipartite system experiences entanglement sudden death for most of the initially mixed states. For an asymmetric form of the GHZ-class Werner state and inequivalent sets of qubits obeying a sum of frequency rule, multipartite correlation persists for a long time, and the system shows robustness against environmental decoherence, whereas the same is observed to be the case for the W-class Werner state for a set of equivalent qubits. Therefore, the choices of qubits and the initial correlated

multipartite state are of significant importance in the potential robustness of the existing multipartite correlations against environmental decoherence.

The paper is organized as follows. In Section 2, we present the N-qubit reservoir model and exactly solve for the time-evolved multipartite density matrix. In Section 3, we study the dynamics of genuine multipartite concurrence for a symmetric and asymmetric GHZ class of Werner state for various spectral densities. Furthermore, we investigate the decoherence in the W-class Werner state. Finally, in Section 4, we conclude with the summary of results and discussions.

2. Model: N-Qubits in a Common Environment

Consider a physical system of N non-interacting qubits with the energy levels separated by $2\Omega_1, 2\Omega_2, 2\Omega_3, ... \, 2\Omega_N$ immersed in a common thermal reservoir. The total Hamiltonian including N-qubits, reservoir, and the interaction term is given by,

$$\mathcal{H}_T = \mathcal{H}_S + \mathcal{H}_R + \mathcal{H}_I + \mathcal{H}_{ren}, \tag{1}$$

where $\mathcal{H}_S$ is the Hamiltonian of the N-qubits system, $\mathcal{H}_R$ is the Hamiltonian of the reservoir, and $\mathcal{H}_I$ is the Hamiltonian representing the interaction between the qubits and the reservoir. The Hamiltonian of the N-qubits system is given by,

$$\mathcal{H}_S = \sum_{i=1}^{N} \Omega_i \sigma_z^i \tag{2}$$

where $\sigma_z^i = |0\rangle_i\langle 0| + |1\rangle_i\langle 1|$ with $i \in (1, 2, ..., N)$. States $|0\rangle_i$ and $|1\rangle_i$ are the excited and ground state respectively, of the ith qubit. The thermal reservoir, which is an environment to the N-qubits, is modeled by a heat bath (at a temperature T) composed of an infinite set of harmonic oscillators with the Hamiltonian as following,

$$\mathcal{H}_R = \sum_j \omega_j a_j^\dagger a_j, \tag{3}$$

where ω_j is the frequency of jth harmonic oscillator. $a_j^\dagger$ and a_j are the bosonic creation and annihilation operators satisfying the commutation relations $[a_k, a_j^\dagger] = \delta_{kj}$ and $[a_k, a_j] = 0 = [a_j^\dagger, a_k^\dagger]$. The interaction Hamiltonian is assumed to be of the following form,

$$\mathcal{H}_I = \mathcal{H}_S \sum_j c_j (a_j^\dagger + a_j), \tag{4}$$

where c_j is the coupling constant between the system of qubits and the jth harmonic oscillator of the thermal reservoir. One observes that $[\mathcal{H}_I, \mathcal{H}_S] = 0$; therefore, the system is non-dissipative in nature. We note that this form of coupling has been previously studied in the context of quantum decoherence in trapped ions and Bose–Einstein condensates by Kuang et al. [39], for single-qubit systems by Viola and Lloyd [36], and for two-qubit systems by Yuan et al. [28].

Finally, the term $\mathcal{H}_{ren}$ is a renormalization term first considered in [43], which is given by

$$\mathcal{H}_{ren} = \mathcal{H}_S^2 \sum_j \frac{c_j^2}{\omega_j}, \tag{5}$$

and this term is crucial for the total Hamiltonian to be diagonalizable.

The Hamiltonian (1) can be exactly solved by employing the following unitary transformation,

$$U = \exp\left[\mathcal{H}_S \sum_j c_j (a_j^\dagger - a_j)\right]. \tag{6}$$

Applying U on the total Hamiltonian (1), one obtains a decoupled Hamiltonian of the form $\mathcal{H}'_T = \mathcal{H}_S + \sum_j \omega_j a_j^\dagger a_j$, whereas the initial total density operator $\rho_T(0)$ will become $\rho'_T(0) = U\rho_T(0)U^\dagger$, which evolves in time as $\rho'_T(t) = \exp(-i\mathcal{H}'_T t)\rho'_T(0)\exp(+i\mathcal{H}'_T t)$. Through the converse transformation of (6), one obtains the total density operator associated with the total Hamiltonian (1) as,

$$\rho_T(t) = \mathrm{e}^{-i\mathcal{H}_s t}\, U^{-1}\, \mathrm{e}^{-i\mathcal{H}_R t}\, U\rho_T(0)U^{-1}\, \mathrm{e}^{i\mathcal{H}_R t}\, U\, \mathrm{e}^{i\mathcal{H}_s t}\,. \tag{7}$$

Assuming that the system and reservoir are initially uncorrelated, the total initial density operator $\rho_T(0)$ can be written as a product state of the system and reservoir density operator, i.e., $\rho_T(0) = \rho_S(0) \otimes \rho_R(0)$, where $\rho_S(0)$ and $\rho_R(0)$ are the initial density operators for the system and reservoir, respectively. The reservoir density operator $\rho_R(0)$ can be expressed as $\rho_R(0) = \Pi_j \rho_j(0)$, where $\rho_j(0) = (1 - e^{-\beta\omega_j})e^{-\beta\omega_j a_j^\dagger a_j}$ corresponds to the initial density operator of the jth oscillator in thermal equilibrium at an inverse temperature β. At any time t, the reduced density matrix for the system can be obtained by tracing over the reservoir $\rho(t) = \mathrm{Tr}_R(\rho_T(t))$. It should be noted that the set $\{|a_1 a_2 a_3 \ldots a_N\rangle\}$ where $a_1, a_2, a_3 \ldots a_N \in \{0, 1\}$ forms a basis of the N-qubit system, and the state $|a_1 a_2 a_3 \ldots a_N\rangle$ is an energy eigenstate of $\mathcal{H}_S$ with eigenvalue $E_{a_1 \ldots a_N} = \sum_{i=1}^{N} (-1)^{a_i}\Omega_i$. In the computational basis, one obtains

$$\begin{aligned}\rho_{(a_1 \ldots a_N)(b_1 \ldots b_N)}(t) = \rho_{(a_1 \ldots a_N)(b_1 \ldots b_N)}(0) F_{(a_1 \ldots a_N)(b_1 \ldots b_N)}(t) \times \\ \exp\big(-i(E_{a_1 \ldots a_N} - E_{b_1 \ldots b_N})t\big),\end{aligned} \tag{8}$$

where $\rho_{(a_1 \ldots a_N)(b_1 \ldots b_N)} = \langle a_1 \ldots a_N|\rho|b_1 \ldots b_N\rangle$, and the factor $F_{(a_1 \ldots a_N)(b_1 \ldots b_N)}(t)$ depends on the reservoir part and is obtained as,

$$\begin{aligned}F_{(a_1 \ldots a_N)(b_1 \ldots b_N)}(t) = \Pi_j \,\mathrm{Tr}_R \Big[D(-\alpha_{b_1 \ldots b_N j})\, \mathrm{e}^{it\omega_j N_j}\, D(\alpha_{b_1 \ldots b_N j}) \\ \times D(-\alpha_{a_1 \ldots a_N j})\, \mathrm{e}^{-it\omega_j N_j}\, D(\alpha_{a_1 \ldots a_N j})\rho_j(0)\Big],\end{aligned} \tag{9}$$

where $N_j = a_j^\dagger a_j$, $\alpha_{a_1 \ldots a_N j} = E_{a_1 \ldots a_N} c_j/\omega_j$, and $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is a displacement operator. Using the properties of a displacement operator, namely,

$$\begin{aligned}D(\alpha_1)D(\alpha_2) &= D(\alpha_1 + \alpha_2)\exp(i\ \mathrm{Im}(\alpha_1 \alpha_2^*)) \\ \exp(\beta N_j)D(\alpha)\exp(-\beta N_j) &= \exp(\alpha\, \mathrm{e}^{\beta}\, a_j^\dagger - \alpha^*\, \mathrm{e}^{-\beta}\, a_j),\end{aligned} \tag{10}$$

the above Equation (9) is reduced to

$$\begin{aligned}F_{(a_1 \ldots a_N)(b_1 \ldots b_N)}(t) =& \Pi_j \exp\left(-i\frac{(E^2_{a_1 \ldots a_N} - E^2_{b_1 \ldots b_N})c_j^2 \sin\omega_j t}{\omega_j^2}\right) \times \\ & \mathrm{Tr}_R\left[D(\xi_{a_1 \ldots a_N b_1 \ldots b_N j})\rho_j(0)\right],\end{aligned} \tag{11}$$

where $\xi_{a_1 \ldots a_N b_1 \ldots b_N j} = (\alpha_{b_1 \ldots b_N j} - \alpha_{a_1 \ldots a_N j})(\mathrm{e}^{i\omega_j t} - 1)$. Using the following result [44],

$$\mathrm{Tr}_R\, D(\alpha)\rho_j(0) = \exp\Big(\frac{-1}{2}|\alpha|^2 \coth\frac{\beta\omega_j}{2}\Big), \tag{12}$$

the above expression (11) is simplified to

$$\begin{aligned}F_{(a_1 \ldots a_N)(b_1 \ldots b_N)}(t) =& \exp\left(-i\phi_{(a_1 \ldots a_N)(b_1 \ldots b_N)} \sum_j \frac{c_j^2 \sin(\omega_j t)}{\omega_j^2}\right) \times \\ & \exp\left(-2\tau_{(a_1 \ldots a_N)(b_1 \ldots b_N)} \sum_j \frac{c_j^2}{\omega_j^2} \sin^2\Big(\frac{\omega_j t}{2}\Big) \coth\Big(\frac{\beta\omega_j}{2}\Big)\right).\end{aligned} \tag{13}$$

where $\phi_{(a_1\ldots a_N)(b_1\ldots b_N)} = E^2_{a_1\ldots a_N} - E^2_{b_1\ldots b_N}$ and $\tau_{(a_1\ldots a_N)(b_1\ldots b_N)} = (E_{a_1\ldots a_N} - E_{b_1\ldots b_N})^2$. In the continuum limit, we take $\sum_j \to \int d\omega J(\omega)$, where $J(\omega)$ is the spectral density of the thermal reservoir, and $c_j \to c(\omega)$, and the above equation becomes,

$$F_{(a_1\ldots a_N)(b_1\ldots b_N)}(t) = \mathrm{e}^{-i\phi_{(a_1\ldots a_N)(b_1\ldots b_N)}Q_1(t)} \ \ \mathrm{e}^{-2\tau_{(a_1\ldots a_N)(b_1\ldots b_N)}Q_2(t)}, \tag{14}$$

with,

$$\begin{aligned} Q_1(t) &= \int \frac{c^2(\omega)}{\omega^2}\sin(\omega t)J(\omega)d\omega \\ Q_2(t) &= \int \frac{c^2(\omega)}{\omega^2}\sin^2(\frac{\omega t}{2})\coth(\frac{\beta\omega}{2})J(\omega)d\omega. \end{aligned} \tag{15}$$

Therefore, we have the exact evolution for the N-qubits density operator, and hence, we can study the dynamics of various correlations. We note in passing that the interaction Hamiltonian considered in the above analysis is the simplest non-trivial bilinear operator on the system and reservoir state. In contrast, one can do a similar analysis with some arbitrary function of the system Hamiltonian $f(\mathcal{H}_S)$ in the interaction term. It is important to note that in all these scenarios, the system Hamiltonian commutes with the total Hamiltonian, i.e., $[\mathcal{H}_S, \mathcal{H}_T] = 0$. Therefore, the interaction is non-dissipative, which is also evident from the above analysis, where diagonal terms are shown to be unaffected.

3. Dynamics of Genuine Multipartite Correlations for N-Qubit States

3.1. GHZ-Class Werner States

We begin with a brief review of a genuine multipartite correlation and measure genuine multipartite concurrence (GMC). It is an entanglement measure that captures the multipartite entanglement and is zero if the system is separable across any bipartition i.e., product and biseparable states. This measure works for mixed-density matrices as well, and it reduces to the Wootter's concurrence for two qubits [19]. Rafsanjani et al. provided an exact expression of GMC for an N-qubit X-state, where the density matrix contains only diagonal and anti-diagonal elements [15]. For a N-qubit system with a density matrix in the computational basis, the only non-zero elements are ρ_{ii} and $\rho_{i(n+1-i)}$, where $i = 1, ..., n$ with $n = 2^N$. Genuine multipartite concurrence for such states are given by

$$C_{GM} = 2\max\{0, |\rho_{j(n+1-j)}| - \sum_{k\neq j}^{n/2}\sqrt{\rho_{kk}\rho_{(n+1-k)(n+1-k)}}\}, \tag{16}$$

where $j = 1, 2, \ldots n/2$. In what follows, we will study evolution of genuine multipartite concurrence for the N-qubit Werner state, which is of the form,

$$\rho^W = x|\psi\rangle\langle\psi| + (1-x)\frac{I}{n}, \tag{17}$$

where $x \in [0, 1]$ is the mixing parameter, $|\psi\rangle$ is a genuine multipartite entangled pure state, and I is the n-dimensional identity matrix. It is noted that the density matrix is pure for $x = 1$ and maximally mixed for $x = 0$. Equation (17) is in the form of an X-state when the pure state $|\psi\rangle$ is a generalized GHZ state, which is given by $|GHZ\rangle_{gen} = \frac{1}{\sqrt{2}}(|i_1 i_2 \cdots i_N\rangle \pm |j_1 j_2 \cdots j_N\rangle)$, where $i_\alpha \neq j_\alpha \in \{0, 1\}$, $\alpha = 1, 2, 3$. It is important to note that the X states density matrix is physically realizable in various scenarios [45,46]. For instance, pseudo-pure states in NMR spin systems are essentially of the same form as (17). More specifically, a highly mixed thermal state of the form can be described through a traceless deviation of maximally mixed state, i.e., $\rho_t \simeq (1 - \epsilon\sum_i \sigma^i_z)/n$, which upon an action of completely positive and trace preserving map yields a density matrix $\rho_P = T(\rho_t) = 1/n - \epsilon\alpha(\rho_{pp} - 1/n)$, with ρ_{pp} representing a pure state, ϵ representing a small deviation parameter, and α, a factor determining signal loss, is essentially of the same form as the Werner state density matrix, with the mixing parameter $x = \epsilon\alpha$ [47].

For the density matrix (17), with the $|\psi\rangle = \frac{1}{\sqrt{2}}(|i_1 i_2 \cdots i_N\rangle + |j_1 j_2 \cdots j_N\rangle)$, where $i_\alpha \neq j_\alpha \in \{0,1\}$, $\alpha = 1,2,3...N$, the genuine multipartite concurrence is obtained as

$$C_{GM}(\rho^W) = \max\{0, \frac{2(n-1)x - (n-2)}{n}\}, \tag{18}$$

which for a tripartite GHZ state i.e., $n = 8$ yields,

$$C_{GM}(\rho^W) = \max\{0, \frac{7x-3}{4}\}. \tag{19}$$

Naturally, it takes the maximum value one for $x = 1$, i.e., maximally entangled GHZ states and non-zero for $x > (n-2)/2(n-1)$. The GMC is zero for $x \leq (n-2)/2(n-1)$. Therefore, it is separable across at least one bi-partition; i.e., the density matrix can be written as a convex sum of bi-separable states.

Consider an N-qubit system in a common heat bath with the initial state given by $\rho(0) = \rho^W$. As evident from the explicit form of (8), the diagonal elements will remain unchanged. For $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\cdots0\rangle + |11\cdots1\rangle)$, there are only two non-zero off-diagonal elements of the density matrix $\rho_{(00\cdots0)(11\cdots1)} = \rho^*_{(11\cdots1)(00\cdots0)}$, and their time evolution is given by

$$\rho_{(00\cdots0)(11\cdots1)}(t) = \frac{x}{2}\, e^{-i\Delta E t}\ \ e^{-i(E^2_{00\cdots0} - E^2_{11\cdots1})Q_1(t)}\ \ e^{-2(\Delta E)^2 Q_2(t)}, \tag{20}$$

where $\Delta E = E_{00\cdots0} - E_{11\cdots1}$. Using $E_{00\cdots0} = \Omega_1 + \Omega_2 + ... + \Omega_N$ and $E_{11\cdots1} = -E_{00\cdots0}$, one obtains

$$\rho_{(00...0)(11...1)}(t) = \frac{x}{2}\, e^{-i2(\sum_{i=1}^N \Omega_i)t}\, e^{-8(\sum_{i=1}^N \Omega_i)^2 Q_2(t)}. \tag{21}$$

Therefore, the genuine multipartite concurrence for $\rho(t)$ is obtained as

$$C_{GM}(\rho(t)) = \max\{0, x\, e^{-8(\sum_{i=1}^N \Omega_i)^2 Q_2(t)} - \frac{n-2}{n}(1-x)\}. \tag{22}$$

Since $Q_2(t)$ is a positive and increasing function of both time and temperature, therefore, the region for which the state has genuine multipartite entanglement shrinks with the increase in time and temperature. More precisely, at $Q_2(t)$, the state has genuine multipartite entanglement for the region $x \in \left(\frac{n-2}{n-2+ne^{-8\omega_T^2 Q_2(t)}}, 1\right]$, where $\omega_T = (\sum_{i=1}^N \Omega_i)$. A few qualitative conclusions can be made without specifying the specific form of spectral density of the reservoir and relying on the fact that $Q_2(t)$ is a positive and increasing function of time. It is immediately obvious that the region of mixing parameter x for which the state is genuine multipartite entangled decreases with the increase in time, the rate of which depends on the specific reservoir under consideration. Furthermore, for a given mixing parameter x, the correlation will decay with time, and the state encounters the sudden death of multipartite entanglement after a finite preservation time, as shown below for various spectral densities.

A simple choice of spectral density is $J(\omega) = \eta/[2\pi c^2(\omega)]$, which at zero temperature yields a linear damping factor as [39],

$$Q_2(t) = \frac{\eta t}{4}, \tag{23}$$

where η is a characteristic parameters associated with the reservoir. The genuine multipartite concurrence becomes,

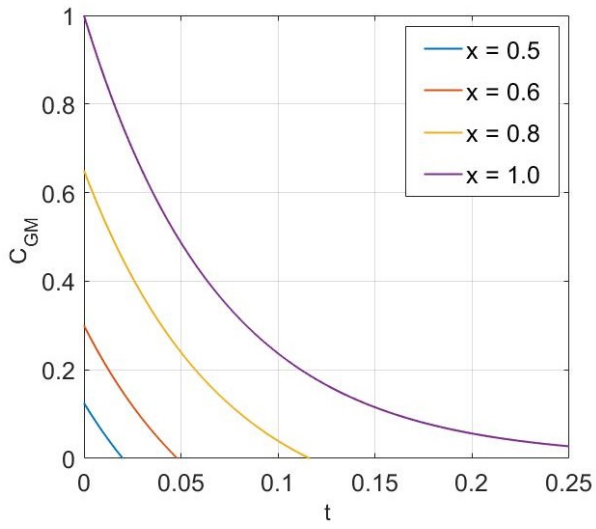
$$C_{GM}(\rho(t)) = \max\{0, x\, e^{-2(\sum_{i=1}^N \Omega_i)^2 \eta t} - \frac{n-2}{n}(1-x)\}, \tag{24}$$

which reflects the exponential degradation of multiparty correlation and has been plotted for several mixing parameter and also with the increasing parameter η in Figure 1 for the

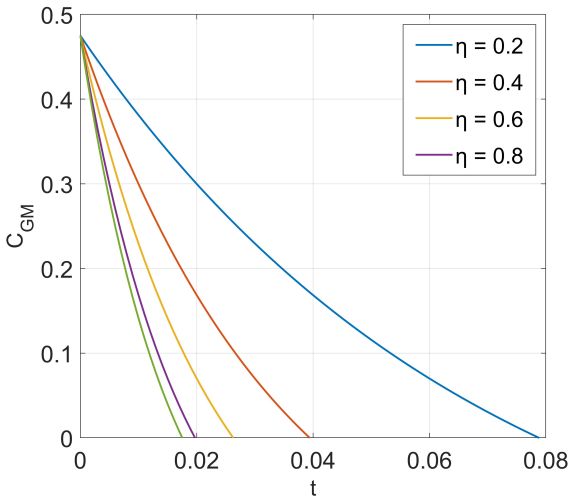
tripartite case (n = 8). As evident from the plot, the state encounters the sudden death of genuine multipartite entanglement for all the states with $x < 1$. Further, the sudden death is more rapid with the increase in the characteristic parameter η, which represents the interaction strength. The preservation time, i.e., time interval up to which correlation persists is obtained to be

$$\Delta t_P = \frac{1}{2(\sum_{i=1}^{N} \Omega_i)^2 \eta} \ln\left(\frac{nx}{(n-2)(1-x)}\right), \tag{25}$$

where the mixing parameter $x > (n-2)/2(n-1)$. As evident from the above expression, except for $x = 1$, the preservation time is finite, and therefore, the initial states in the region $(n-2)/2(n-1) < x < 1$ will encounter the sudden death of genuine multipartite entanglement.



(**a**)



(**b**)

Figure 1. Genuine multipartite concurrence plotted against time (in seconds) for the spectral density $J(\omega) = \eta/[2\pi c^2(\omega)]$ at zero temperature (for n = 8). (**a**) For various mixing parameters and a fixed interaction coefficient $\eta = 0.2$, (**b**) for various interaction coefficients η and a fixed mixing parameter $x = 0.7$. Entanglement sudden death is observed for states with $x < 1$, and it occurs more rapidly with increasing η.

Ohmic reservoirs are modeled with the following spectral density [48],

$$J(\omega) = \frac{\eta\omega}{c^2(\omega)}\, e^{-\frac{\omega}{\omega_0}}, \tag{26}$$

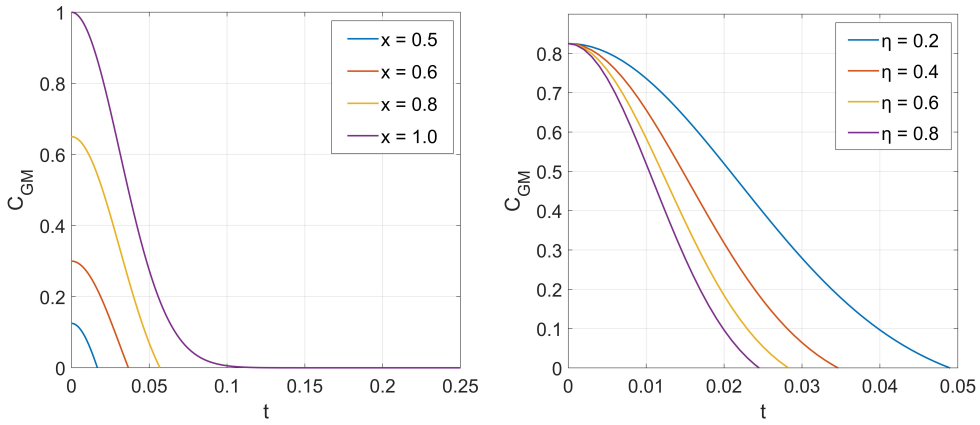
where η is a characteristic parameter of the reservoir and ω_0 is a cut-off frequency specific to the reservoir. The damping factor $Q_2(t)$ becomes

$$Q_2(t) = 2\eta \int_0^\infty \frac{1}{\omega}\, e^{-\frac{\omega}{\omega_0}} \sin^2\left(\frac{\omega t}{2}\right) \coth\left(\frac{\beta\omega}{2}\right) d\omega, \tag{27}$$

which in general is difficult to evaluate. We will evaluate the damping factor for the case of zero, low, and high-temperature regions and study the correlation in these limits. Firstly, at the zero temperature, the damping factor is obtained to be $Q_2(t) = [\eta \ln(1 + (\omega_0 t)^2)]/2$. For this case, genuine multipartite concurrence for tripartite systems (n = 8) is plotted for various mixing parameters at fixed η and various interaction parameters η at a fixed mixing parameter, as shown in Figure 2. It is observed in this case that the decay in correlation is not exponential, which is a consequence of the logarithmic factor of time in the damping factor. We observe the sudden death of genuine tripartite entanglement for states with mixing parameter $x < 1$; however, compared to the linear damping factor case, the sudden death is achieved much more slowly. The preservation time is evaluated to be

$$\Delta t_P = \frac{1}{\omega_0}\left(\exp\left(\frac{1}{4\eta\omega_T^2} \ln\left(\frac{4x}{3(1-x)}\right)\right) - 1\right)^{\frac{1}{2}}, \tag{28}$$

which is much larger than the case when the damping factor is linear in time.



(a) **(b)**

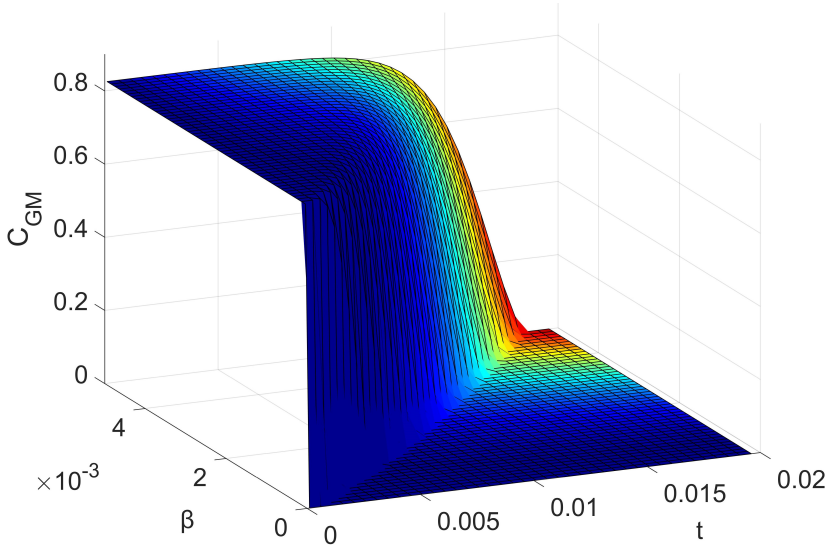
Figure 2. Genuine multipartite concurrence plotted against time (in units of $(\omega_0)^{-1}$) for ohmic spectral density at zero temperature (for n = 8). (**a**) For various mixing parameters at fixed η (**b**) for various interaction parameters η at a fixed mixing parameter $x = 0.9$. Entanglement sudden death is observed for $x < 1$; however, the decay rate is slower in this case than the exponential decay for the linear damping parameter case.

For a finite but small temperature, the damping factor for the Ohmic case becomes

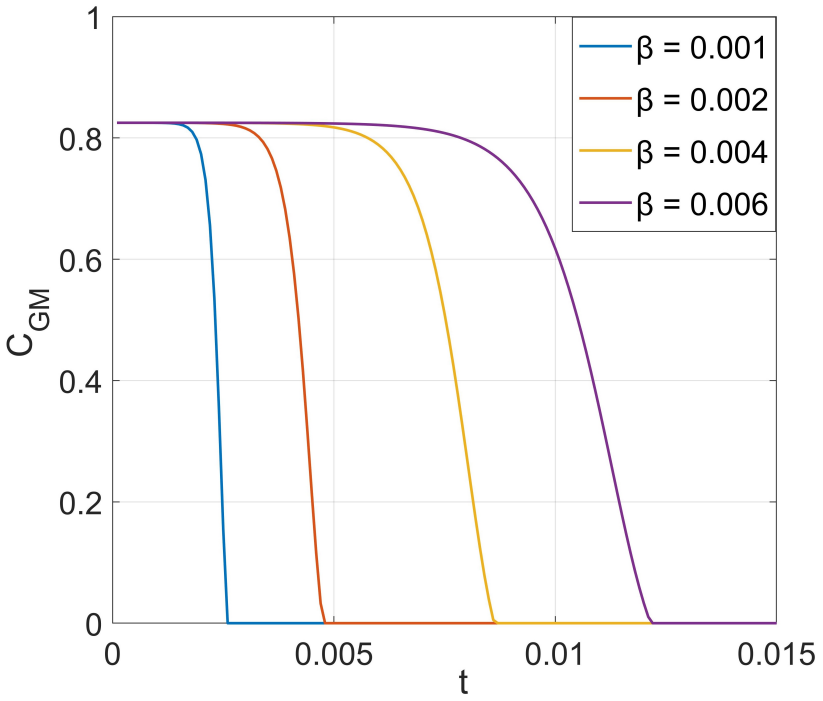
$$Q_2(t) = \eta\left[\ln\left(\frac{\beta}{\pi t}\sqrt{1 + (\omega_0 t)^2}\sinh(\frac{\pi t}{\beta})\right)\right]. \tag{29}$$

The genuine multipartite concurrence is plotted with respect to time and inverse temperature at a fixed mixing parameter $x = 0.9$, as shown in Figure 3. It is observed that at all temperatures, the correlation remains almost constant for the time interval $0 < t < T_c$, where T_c is defined as the characteristic time and then for time $t > T_c$, it decays rapidly to

zero as the system experiences the sudden death of multipartite correlation. Furthermore, as the temperature increases, the characteristic time becomes smaller, and the decay rate of tripartite concurrence gets faster; therefore, correlation sudden death occurs more quickly.



(**a**)



(**b**)

Figure 3. (**a**,**b**) Genuine multipartite concurrence with inverse temperature, and time (in seconds) for ohmic reservoir at a fixed mixing parameter $x = 0.9$ and $\omega_0 = 1$. As the temperature increases, the state experiences the sudden death of correlation sooner.

A more interesting case occurs when the generalized GHZ state under consideration for the initial density matrix is $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\cdots01\rangle + |11\cdots10\rangle)$. In this case, the non-zero diagonal elements in the density matrix are $\rho_{(00\cdots01)(11\cdots10)}$ and $\rho_{(11\cdots10)(00\cdots01)}$, and

using $E_{00\cdots01} = \sum_{i=1}^{N-1} \Omega_i - \Omega_N = -E_{11\cdots0}$, one obtains the evolution of the off-diagonal elements as,

$$\rho_{(00\cdots01)(11\cdots10)}(t) = \frac{x}{2} \exp\left(-i2(\sum_{i=1}^{N-1} \Omega_i - \Omega_N)t - 8(\sum_{i=1}^{N-1} \Omega_i - \Omega_N)^2 Q_2(t) \right). \tag{30}$$

If all the qubits are equivalent i.e., $\Omega_1 = \Omega_2 = \cdots \Omega_N$, the state suffers a similar kind of decoherence as seen earlier, i.e., the region of mixing parameters for which the state remains tripartite entangled reduces with the increase in time and temperature. Furthermore, for a given mixing parameter, the amount of correlation decays with the rate depending on the spectral density under consideration. Interestingly, for the damping factor $Q_2(t)$ given by (23), the preservation time is obtained as,

$$\Delta t_P = \frac{1}{2(\sum_{i=1}^{N-1} \Omega_i - \Omega_N)^2 \eta} \ln\left(\frac{nx}{(n-2)(1-x)} \right). \tag{31}$$

Considering the two-level systems such that $\sum_{i=1}^{N-1} \Omega_i \sim \Omega_N$, the preservation time $\Delta t_P >> 1$. Hence, one can prolong the preservation time and therefore retain the correlation for a significant time with a certain choice of qubit and initially correlated GHZ-class Werner state. In the limiting case, there exists an interesting class of inequivalent qubits such that the sum of energy gap of qubits 1 to $N-1$ is the same as that of qubit N i.e., $\sum_{i=1}^{N-1} \Omega_i = \Omega_N$, then

$$\rho_{(00\cdots01)(11\cdots10)}(t) = \frac{x}{2} = constant \quad \forall \ \beta. \tag{32}$$

Hence, in such a class of qubits, the state suffers no decoherence in the correlations. Genuine multipartite entanglement remains invariant in time for such inequivalent sets of three qubits. It is important to note that the persistence of correlation for the above mentioned constraints on qubit splitting is because the initial state $\rho(0)$ is close to an equilibrium state, and if the constraint is exactly satisfied, the initial state itself is an equilibrium state, where the equilibrium state is defined by the fact that it commutes with the total Hamiltonian, i.e., $[\rho_{eq}, \mathcal{H}_T] = 0$. In the other cases, when this constraint is not satisfied, the initial state is not an equilibrium state, and due to dephasing, it results in the loss of the off-diagonal terms, hence becoming a diagonal state in the energy basis in the large time limit. Therefore, it is at the cost of loss in correlation that the state achieves equilibration when the constraint on qubit splitting is not satisfied. For instance, the energy gap of a two-level system can be tuned in NMR by modulating the applied external magnetic field, which by coupling through the magnetic moment results in an energy gap of order μB. Therefore, it is quite evident that one can physically realize a set of three qubits such that the initially correlated state is robust in loss of correlations against environmental decoherence. Such a scenario will be beneficial for quantum tasks requiring genuine multipartite entanglement.

3.2. W-Class Werner States

In this section, we study the dynamical evolution for the W-class of Werner states. For explicitness, we limit our discussion to the tripartite case (n = 8); however, the general conclusions will hold for arbitrary N-qubit systems. Consider the initial density operator to be of the following form,

$$\rho(0) = x|W\rangle\langle W| + (1-x)\frac{I}{8}, \tag{33}$$

where the state $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ is the symmetric W state, $x \in [0,1]$ is the mixing parameter, and I is the identity matrix. Using Equation (8), one obtains the elements of density matrix at later times. The diagonal elements remains constant,

$$\begin{aligned}\rho_{(000)(000)}(t) = \rho_{(011)(011)}(t) = \rho_{(101)(101)}(t) = \rho_{(110)(110)}(t) = \rho_{(111)(111)}(t) = \frac{1-x}{8},\\ \rho_{(001)(001)}(t) = \rho_{(010)(010)}(t) = \rho_{(100)(100)}(t) = \frac{5x+3}{24},\end{aligned} \tag{34}$$

whereas the off-diagonal elements evolve as follows,

$$\begin{aligned}\rho_{(001)(010)}(t) &= \frac{x}{3}\,\mathrm{e}^{-2i(\omega_B-\omega_C)t}\;\;\mathrm{e}^{-4i\omega_A(\omega_B-\omega_C)Q_1(t)}\;\;\mathrm{e}^{-8(\omega_B-\omega_C)^2Q_2(t)},\\ \rho_{(001)(100)}(t) &= \frac{x}{3}\,\mathrm{e}^{-2i(\omega_A-\omega_C)t}\;\;\mathrm{e}^{-4i\omega_B(\omega_A-\omega_C)Q_1(t)}\;\;\mathrm{e}^{-8(\omega_A-\omega_C)^2Q_2(t)},\\ \rho_{(010)(100)}(t) &= \frac{x}{3}\,\mathrm{e}^{-2i(\omega_A-\omega_B)t}\;\;\mathrm{e}^{-4i\omega_C(\omega_A-\omega_B)Q_1(t)}\;\;\mathrm{e}^{-8(\omega_A-\omega_B)^2Q_2(t)}.\end{aligned} \tag{35}$$

As observed from the above expressions, the magnitude of each off-diagonal element has a decaying exponential factor $Q_2(t)$, the explicit form of which depends on the choice of spectral density, which is accompanied by the energy difference between the qubits. Interestingly, when all the three qubits are equivalent, i.e., $\omega_A = \omega_B = \omega_C$, there is no damping in the magnitude of the off-diagonal terms. Therefore, the state remains invariant in time, and the correlations are robust against environmental decoherence. This is in striking contrast to the case when the initial state is the GHZ-class Werner state, where to prolong the decay in correlation, one requires three inequivalent qubits.

The decoherence of a state as measured through von-Neumann entropy is defined in terms of time evolved density matrix as

$$\mathcal{D}(t) = -\operatorname{Tr}(\rho(t)\log\rho(t)). \tag{36}$$

For a specific case when the qubit A and B are identical, i.e, $\omega_A = \omega_B$ and $\omega_A - \omega_C = \Delta$, one obtains the decoherence as

$$D(t) = -\frac{3(1-x)}{4}\log\left(\frac{1-x}{8}\right) - \lambda_+\log(\lambda_+) - \lambda_-\log(\lambda_-), \tag{37}$$

where

$$\lambda_\pm = \frac{1+3x}{8} \pm \frac{x}{6}\sqrt{1+2\exp(-16\Delta^2 Q_2(t))}. \tag{38}$$

The relative entropy of coherence defined through the metric relative entropy on the state of space has been shown to be a bonafide measure of coherence [49],

$$C^{Rel}(\rho) = S(\rho^d) - S(\rho), \tag{39}$$

where the density matrix ρ^d is obtained by removing all the off-diagonal elements from ρ, and $S(\rho)$ is the entropy corresponding to the state ρ and given by,

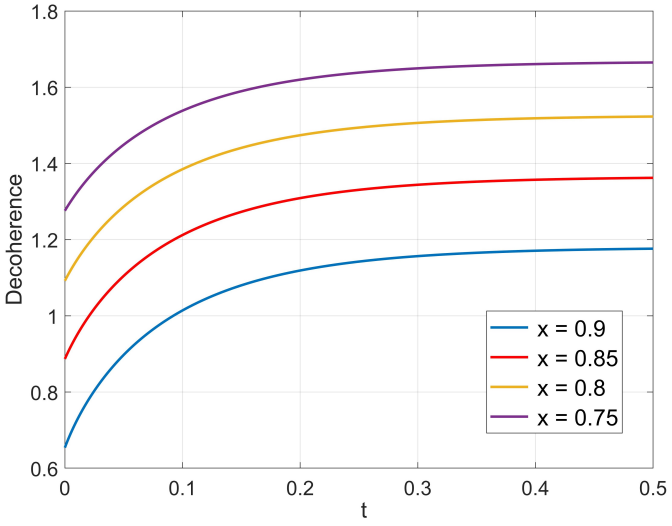
$$S(\rho) = -\sum_i \lambda_i \log(\lambda_i), \tag{40}$$

with λ_i being the eigenvalues of ρ. Evolution of the relative entropy of coherence for the W-class Werner state is obtained through the following expression,

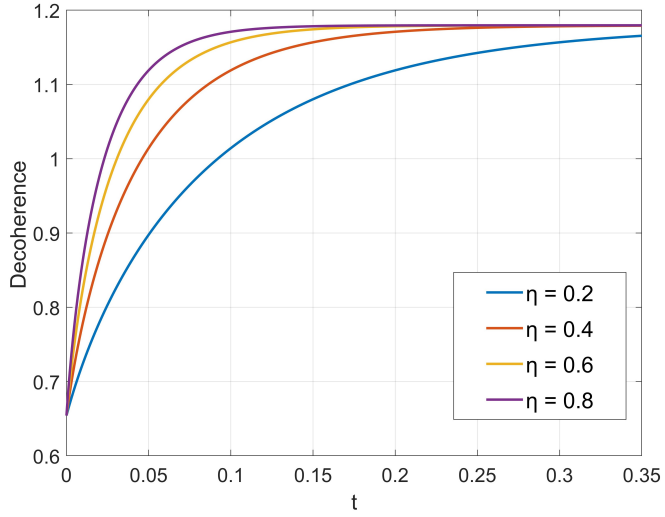
$$C^{Rel}(\rho) = \frac{1-x}{8}\log\left(\frac{1-x}{8}\right) - \frac{5x+3}{8}\log\left(\frac{5x+3}{24}\right) - \lambda_+\log(\lambda_+) - \lambda_-\log(\lambda_-) \tag{41}$$

where $\lambda_\pm$ is obtained as above for a specific choice of qubits.

In Figure 4, we plot the decoherence for the damping factor linear in time, given by (23) for various mixing parameters at a fixed interaction coefficient. It is observed that the von-Neumann entropy saturates after some time at a value depending on the mixing parameter. Furthermore, as seen from the plot for decoherence with changing interaction parameter at a fixed mixing parameter, the entropy saturates to the maximum value rapidly with the increase in η, implying a speedup in decoherence. It is important to note that similar behavior will be observed for different choices of spectral density, since that damping factor is a positive and increasing function, and the only thing differing will be the rate of decoherence. Further, as explained earlier, the choice of qubits plays a significant role in the robustness against decoherence; the more the frequency of the qubits is similar to each other, the more the system is robust against environmental decoherence.



(**a**)



(**b**)

Figure 4. Decoherence as measured by von-Neumann entropy for damping factor linear in time (in seconds) (**a**) for different mixing parameters at a fixed interaction strength $\eta = 0.2$, (**b**) for different interaction strengths at a fixed mixing parameter $x = 0.9$. It is observed that for different mixing parameters, the entropy saturates after some time, and the rate at which it is achieved increases with the increase in η.

4. Conclusions

In conclusion, we found an exact expression for a time-evolved density matrix for an N-qubit system in a common heat bath through a thermal reservoir modeled as an infinite quantum harmonic oscillator and a bilinear non-dissipative interaction Hamiltonian. Using this, we studied the dynamics of the underlying multipartite correlation for various N-qubit systems. In particular, we studied the evolution of genuine multipartite concurrence for various GHZ-class Werner states, and we showed that for an asymmetric GHZ-class Werner state, an inequivalent choice of qubits obeying a sum of frequency rule results in the robustness of the correlation against environmental decoherence. We also found the preservation time of genuine multipartite concurrence for several spectral densities and showed that the sudden death of multipartite correlation occurs for states with mixedness parameter $x < 1$, wherein the rate of decay depends on the specific reservoir under consideration. Furthermore, we studied the decoherence in the W-class Werner state through von-Neumann entropy and showed that the correlations show robustness against environmental decoherence for N equivalent qubits. The studied behavior of genuine quantum correlations in a dephasing environment will be useful in implementing quantum tasks in such scenarios. Furthermore, with a straightforward extension to qubits placed in separate environments, one can study various thermodynamic aspects, which are affected by the behavior of correlations in the presence of a dephasing environment. In particular, it would be interesting to see the transient behavior of thermodynamics quantities and their variation for different values of N for an N-qubit system, and especially for the choice of the GHZ-class Werner state, which resists dephasing.

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