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Review

Fractal Entropy of Nuclear Medium Probed by K^0_S Mesons Produced in AuAu Collisions at RHIC

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Abstract: In this paper, we review our findings concerning fractal entropy of microscopic configurations corresponding to the production of K^0_S mesons in AuAu collisions in the z-scaling approach. The entropy is expressed via structural and fragmentation fractal dimensions, and model parameter c_{AuAu} is interpreted as a specific heat of produced medium. These parameters are related to the respective momentum fractions of the colliding nuclei, the momentum fractions of the scattered constituents that fragment into the produced hadrons, and the multiplicity density of negative particles in the central interaction region. The dependence of the entropy on the collision energy over the range of 7.7–200 GeV for most central and most peripheral events is studied as a function of the transverse momentum of the produced K^0_S mesons. A non-trivial dependence of the entropy on the collision energy with decreasing transverse momentum is found. This reflects the irregularity of the behavior of the specific heat, c_{AuAu}, and can point to a manifestation of phase transition in nuclear matter.

Keywords: high energy heavy ion collisions; entropy in heavy ion physics; fractality in heavy ion physics; RHIC

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1. Introduction

Entropy is one of the fundamental concepts and terms used for the description of multiparticle systems. This term was used for the thermodynamic description of gas, liquid, and solid states of matter. The concept of entropy was introduced by Rudolf Clausius [1] in thermodynamics to measure the amount of energy in a system that cannot produce work. Ludwig Boltzmann proposed [2] that the nature of entropy is related to the probability of the state of atoms or molecules in a system and presented an atomic interpretation in the foundational works of statistical mechanics and gas dynamics. Later, Boltzmann [3], Josiah Willard Gibbs [4], and James Clerk Maxwell [5] gave a statistical interpretation of entropy.

Boltzmann considered entropy as a measure of the number of possible microscopic states of individual atoms or molecules of a system. He assumed that the system looks like an ensemble of ideal gas particles. The measure, S, was determined to be proportional to the logarithm of the number of micro-states W, $S \sim \ln W$. The new concept, namely the probability distribution, was introduced to describe macroscopic observables of a system in terms of the microscopic interactions, which fluctuate around an average configuration. This measure was called entropy, which is central to the second law of thermodynamics. The law states that the entropy of isolated systems left to spontaneous evolution cannot decrease with time. The entropy of the system reached in a state of thermodynamic equilibrium is highest.

The second law of thermodynamics and the principle of maximum entropy production [6,7] for isolated systems have been used to describe one-particle classical (Maxwell–Boltzmann) and quantum (Bose–Einstein and Fermi–Dirac) distributions for many-particle systems. Various thermodynamic properties, such as heat capacity, thermal expansion, compressibility, and others, are defined by physical variables (the state variables—pressure, temperature, volume, etc.) that characterize a state of thermodynamic equilibrium. The entropy is a function of state. All other functions (internal energy, enthalpy, Gibbs's potentials, etc.) of state can be expressed via the entropy. Clausius created the term entropy as an extensive thermodynamic variable. One of the main properties of the entropy in the classical thermodynamics is additivity. It means that the entropy of the sum of independent sub-systems is equal to the sum of their entropies.

A generalization of the standard Boltzmann–Gibbs entropy, S_{BG}, in statistical thermodynamics was proposed for the case of non-extensive (non-additive) systems by Constantino Tsallis [8]. Since then, the corresponding type of entropy is known as the q-entropy or Tsallis entropy, S_q. The hypothesis of non-extensivity is based on the assumption that the strong interaction in a thermodynamically anomalous system leads to new degrees of freedom and to quite a different approach in statistical physics of the non-Boltzmann type.

Another generalization of S_{BG} in the statistical mechanics based on a new entropic functional was supposed by Giorgio Kaniadakis [9,10]. The corresponding entropy is now known as the k-entropy or Kaniadakis entropy, S_k. The approach based on a new one-parameter (k) deformation of the exponential and logarithm functions. Both S_q and S_k entropies are one-parameterized functions of probability distribution. A more general form of entropy—the two-parameterized function, $S_{q,r}$—was proposed by Bhudev Sharma and Dharam Mittal [11]. The new parameters—q and r—define the generalized exponential and logarithm functions. In the hope that a thermodynamic framework might be extended to strongly interacting statistical systems, a number of generalized entropies has been proposed (see [12,13] and references therein). These entropies and the corresponding statistical distributions have been adopted successfully in the description of a variety of systems in the fields of quantum statistics, plasma physics, particle physics, astrophysics, cosmology, condensed matter, quantum physics, seismology, genomics, economy, and epidemiology, among many others (see [8,14,15] and reference therein).

The notion of entropy is intimately connected with phase transitions. Nowadays, it is generally accepted that the phase transitions are associated with transformations of thermodynamic quantities, such as entropy, energy density, and a set of response functions, namely specific heat, compressibility, and susceptibility, with changes in the collision energy, centrality, momentum, and type of detected particles. This gives the opportunity to study the properties of a produced matter in different phase states and under various phase transitions. Special interest represents a growth in fluctuations and correlations, as well as the discontinuity of the specific heat and various response functions along the transition coexistence lines and near a critical point (CP). These features are considered to reflect a change in symmetry for a system in the critical region.

The symmetry of the interaction near CP manifests itself as self-similarity and demonstrates the scaling and universality of physical observables [16,17]. General features for the description of various critical phenomena are power laws, interaction symmetries, and dimensions of space. The Beam Energy Scan program has been proposed [18,19] to study the phase diagram of nuclear matter. The search for phase transition and CP signatures is one of the priority tasks of the relativistic heavy-ion programs performed at the SPS (Super Proton Synchrotron), RHIC (Relativistic Heavy Ion Collider), and LHC (Large Hadron Collider) [20,21].

In the present paper, we use a new entropy named fractal entropy to describe the general features of K^0_S-meson production in AuAu collisions. This entropy is connected with ideas of the fractal character of colliding objects and the fractal nature of fragmentation processes in the final state. The fractal entropy depends on the corresponding fractal dimensions and a parameter c_{AuAu} interpreted as the specific heat of produced matter. To investigate the critical phenomena, we analyzed data obtained by the STAR Collaboration at RHIC over a wide range of the nucleon-nucleon center-of-mass energy, $\sqrt{s_{NN}}$ = 7.7–200 GeV, and centralities of (0 to 5)% (most central collisons) up to (60 to 80)% (most peripheral). The analysis of experimental data [22–25] is performed in the framework of z-scaling (see [26–29] and reference therein). The concept is based on the hypothesis that hadron production at high energies is governed by self-similarity. The principles of Lorentz invariance, locality, and the fractality of hadron production are exploited to give qualitative and quantitative descriptions of constituent interactions at a microscopic level. The new entropy, which takes into account the fractality of hadron production, introduced in the z-scaling approach, appears to be an important ingredient of data analysis in high-energy particle and nuclear physics.

2. Concept of z-Scaling

Here, we briefly describe the basic ideas of the z-scaling concept and give the physical meaning of the main ingredients of this approach [26–29]. We assume that at high energy the collisions of A_1 and A_2 nuclei can be described in terms of their constituents. The momentum distribution of a produced inclusive particle h in the process $A_1 + A_2 \to h + X$ is expressed via the characteristics of a binary collision of the constituents with masses $x_1 M_1$ and $x_2 M_2$. The quantities x_1 and x_2 are the momentum fractions carried by the interacting constituents of the incoming nuclei with masses M_1 and M_2 and 4-momenta P_1 and P_2, respectively. It is also assumed that the objects with masses m_a/y_a and $(x_1 M_1 + x_2 M_2 + m_b/y_b)$ are produced in the scattered and recoil directions in such a collision. The quantities y_a and y_b represent the corresponding momentum fractions carried by the inclusive particle with mass m_a and its hadron counterpart with mass m_b moving in opposite direction. The momentum conservation law in the binary collision gives the following condition:

$$(x_1 P_1 + x_2 P_2 - p/y_a)^2 = M_X^2. \tag{1}$$

The recoil mass is considered in the form $M_X = x_1 M_1 + x_2 M_2 + m_b/y_b$. The produced particle with mass m_b ensures the conservation of additive quantum numbers.

Using the scenario of binary constituent interactions, we also assume that the inclusive hadron production has the property of self-similarity. This means that the inclusive invariant cross section, $Ed^3\sigma/d^3p$, for an inclusive hadron h with the momentum p and energy E can be expressed in terms of dimensionless scale-dependent quantities—the scaling function $\psi(z)$ and a self-similarity variable z.

In the present approach, the self-similarity of hadron interactions at a constituent level is studied by the variable z defined as follows

$$z = z_0 \Omega^{-1}, \tag{2}$$

where $z_0 = \sqrt{s_\perp}/[(dN_{\mathrm{ch}}/d\eta|_{\eta\approx 0})^c\, m_N]$. The variable z is proportional to the transverse kinetic energy, $\sqrt{s_\perp}$, of a selected constituent sub-process consumed on the production of the inclusive particle with mass m_a and its counterpart with mass m_b. The quantity $dN_{\mathrm{ch}}/d\eta|_{\eta\approx 0}$ is the corresponding (pseudorapidity) multiplicity density of charged particles produced in the reaction central region at the mid-rapidity, $\eta \approx 0$. The multiplicity density is related to a state of the produced medium. The parameter c characterizes the properties of this medium. The constant m_N is fixed at the value of the nucleon mass. The symbol Ω stands for the maximum of the function,

$$\Omega = (1 - x_1)^{\delta_1}(1 - x_2)^{\delta_2}(1 - y_a)^{\epsilon_a}(1 - y_b)^{\epsilon_b}, \tag{3}$$

under the kinematic constraint (1). This quantity connects the momentum fractions characterizing the properties of hadron production at a microscopic level with the structural and fragmentation characteristics of the reaction. The structure of the colliding hadrons/nuclei and fragmentation of the objects produced in binary collisions of their constituents are described by δ_1, δ_2 and ϵ_a, ϵ_b, respectively. These parameters are interpreted as fractal dimensions. The quantity Ω is proportional to the relative number of all such configurations in the inclusive process, which contain a state defined by the fractions x_1, x_2, y_a, and y_b. For given values of δ_1, δ_2, ϵ_a, and ϵ_b, the selected sub-process is singled out of all constituent sub-processes by the values of the momentum fractions x_1, x_2, y_a, and y_b, which maximize the function $\Omega(x_1, x_2, y_a, y_b)$ under condition (1). The quantity Ω^{-1} characterizes a resolution at which the selected sub-process is singled out of the inclusive reaction. The value of z_0 is proportional to the transverse kinetic energy in the selected sub-process consumed on the production of the inclusive particle with mass m_a and its counterpart with mass m_b. According to its definition, the scaling variable z is a function of the momentum fractions x_1, x_2, y_a, y_b, and the multiplicity density and depends on the parameters $\delta_1, \delta_2, \epsilon_a, \epsilon_b$, and c.

The scaling function $\psi(z)$ is expressed in terms of the experimentally measured inclusive invariant cross section, $Ed^3\sigma/dp^3$, the (pseudorapidity) multiplicity density, $dN/d\eta$, and the total inelastic cross section, σ_{in}, as follows:

$$\psi(z) = \frac{\pi}{(dN/d\eta)\,\sigma_{\text{in}}} J^{-1} E \frac{d^3\sigma}{dp^3}, \tag{4}$$

where J is the Jacobian of the transformation from $\{p_T^2, y\}$ to $\{z, \eta\}$ pairs of variables. The multiplicity density, $dN/d\eta$, in Equation (4) concerns particular hadrons species; it depends on the center-of-mass energy, on various multiplicity selection criteria, and on the production angles, at which the inclusive spectra are measured. The integral of $\psi(z)$ over the interval $(0, \infty)$ is normalized to unity. The integral allows the interpretation of the scaling function as a probability density of hadron distribution in the inclusive process with the corresponding z-value.

3. Fractal Entropy $S_{\delta,\epsilon}$

Definition (2) corresponds to the physical meaning of the self-similarity variable z as the transverse kinetic energy, $\sqrt{s_\perp}$, per one configuration of the collision system in which the inclusive process can be realized, including constituent and fragmentation degrees of freedom. The number of such configurations is $W = W_0\,[dN_{\text{ch}}/d\eta|_{\eta\approx0}]^c\,\Omega$, where W_0 is unknown constant. We define the fractal entropy of hadron production in AA collision in the z-scaling approach [26–29] as follows:

$$S_{\delta,\epsilon} = c\,\ln\left(dN_{\text{ch}}/d\eta|_{\eta\approx0}\right) + \ln\Omega + \ln W_0. \tag{5}$$

The multiplicity density $dN_{\text{ch}}/d\eta|_{\eta\approx0}$ of charged particles produced in the central pseudorapidity range characterizes the temperature of the created matter. The parameter c is interpreted as the specific heat of the medium. The relative volume Ω in the space of the momentum fractions is given by Equation (3).

The principle of maximum entropy applied to the entropy $S_{\delta,\epsilon}$ with additional constraint is used to determine the scaling variable z. The constraint (1) follows from the momentum conservation law for the binary collisions of hadron constituents. The parameters δ_1, δ_2, ϵ_a, and ϵ_b are obtained from the scaling behavior of $\psi(z)$ as a function of the self-similarity variable z. For such established fractal dimensions, the momentum fractions x_1, x_2, y_a, and y_b are determined from the conditions for the maximum of the entropy $S_{\delta,\epsilon}$ under kinematic constraint (1).

Let us note that the principle of the maximal entropy $S_{\delta,\epsilon}$ corresponds to the minimal resolution, Ω^{-1}, needed for the determination of a selected sub-process. As seen from Equation (3), the volume Ω is a product of the complements of the momentum fractions raised to generally non-integer, fractional numbers. We consider that the fractal properties of the colliding objects, interaction of constituents, and the fragmentation process relate to fundamental symmetries of particle origin at small scales.

4. Anomalous Behavior of the Fractal Entropy

The growth of the particle multiplicity $dN_{\text{ch}}/d\eta$ in heavy-ion collisions with the energy $\sqrt{s_{NN}}$ is commonly associated with the increase in entropy. The fractal entropy (Figure 1) reveals unusual behavior, namely anomalous decreases in a certain energy range for the most central collisions. The effect is enhanced as the momentum of inclusive neutral kaons decreases.

The dependence of $S_{\delta,\epsilon}$ for K_S^0-meson production at the transverse momentum $p_T = 0.3$, 0.7, 1.0, 1.5, 2.0, and 3.0 GeV/c (where c denotes the speed of light) in most central and most peripheral AuAu collisions in the rapidity range $|y| < 0.5$ as a function of $\sqrt{s_{NN}}$ is shown in Figure 1a,b, respectively. A relative scale for presentation of the entropy $S_{\delta,\epsilon}$ is used. As one can see from Figure 1a, there is an anomalous behavior of the entropy $S_{\delta,\epsilon}$ in the soft region of p_T for 0–5% most central collisions. The fractal entropy reaches a local maximum in the energy range $\sqrt{s_{NN}}$ = 11.5–19.6 GeV at $p_T = 0.3$ GeV/c. Then, an abrupt fall in $S_{\delta,\epsilon}$ at $\sqrt{s_{NN}}$ = 27–39 GeV with a gradual increase at higher energies is observed. Similar behavior is also visible at $p_T = 0.6$ and 1.0 GeV/c. The entropy $S_{\delta,\epsilon}$ increases monotonously with the collision energy at $p_T \geq 2$ GeV/c. As one can see from Figure 1b, the anomalous behavior of the fractal entropy disappears in 60–80% peripheral collisions at all analyzed energies and transverse momenta. Let us note that estimation of errors for $S_{\delta,\epsilon}$ is at the level of 2–5% and does not destroy the non-monotonic behavior of the fractal entropy in the most central and the monotonic behavior for most peripheral collisions with decreasing p_T in the energy range $\sqrt{s_{NN}}$ = 7.7–200 GeV.

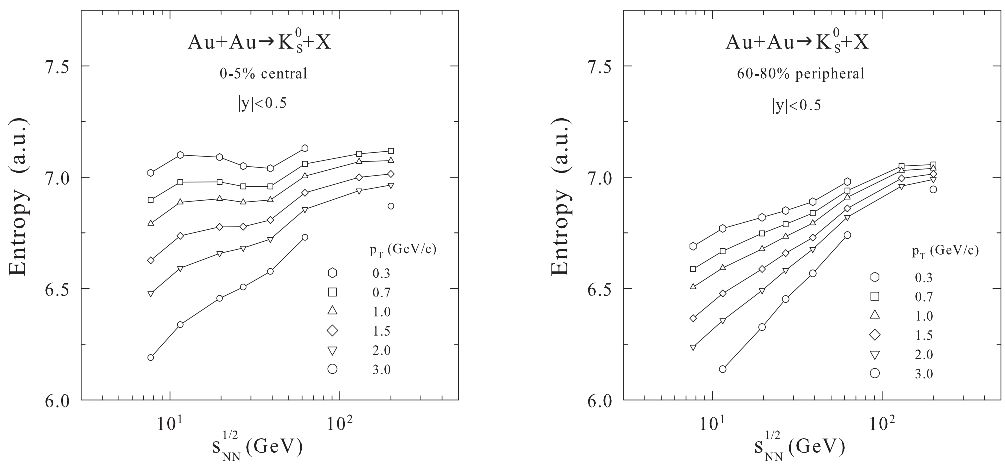

Figure 1. The dependence of the entropy (5) on the nucleon-nucleon center-of-mass, energy $\sqrt{s_{NN}}$, for K_S^0-meson transverse momentum p_T = 0.3, 0.7, 1.0, 1.5, 2.0, and 3.0 GeV/c in the 0–5% (most central) (**a**) and 60–80% (most peripheral) (**b**) AuAu collisions in the rapidity interval $|y| < 0.5$. Figures are taken from Ref. [30].

There are important observations in the soft region ($p_T < 2$ GeV/c) of particle production in heavy-ion collisions at the RHIC that are related to the properties of the matter produced [31]. The results of the experimental measurements brought arguments that well-equilibrated matter, including strangeness, is created at the RHIC. Some data analyses have indicated that a strongly coupled quark–gluon–plasma (QGP) liquid seem to be created [18–21]. The spectra for various particle types are well described using hydrodynamical models at $p_T < 2$ GeV/c. (see [21] and references therein). The measured properties of relevant quantities indicated a creation of nearly-perfect quark–gluon liquid in 200 A GeV AuAu collisions at the RHIC [32]. This accelerator provides enough energy for bulk observables to be sensitive to the properties of QGP. One of them is the small viscosity to entropy ratio, which is a typical feature of the QGP liquid [33].

We argue that the fractal entropy studied in the paper can trace the phase transition changes of the system. Let us consider the nuclear system in a gas phase at a constituent level at low $\sqrt{s_{NN}}$. In a phase transition region, the nuclear system is expected to change from the gas phase to the QGP "liquid" phase at higher energy. This is connected with a decrease in its entropy. Above the phase transition, a strongly coupled liquid QGP phase is believed to develope in the system, as it is inferred from analyses of the RHIC data [18,19]. The fractal entropy of the liquid phase is expected to increase with the collision energy above the phase transition region. Similar behavior can be seen in Figure 1a in the soft p_T

region.The relatively high values of the fractal entropy $S_{\delta,\epsilon}$ at $\sqrt{s_{NN}} = 11.5$–19.6 GeV in the low-p_T region represent a large number of microscopic configurations of the produced matter in a gas state. According to the principle of maximum entropy, these configurations contain the selected one that leads to the production of K^0_S mesons with given transverse momentum. We observe an abrupt decrease in the fractal entropy $S_{\delta,\epsilon}$ in the region $\sqrt{s_{NN}} \sim 30$–40 GeV at low p_T for 0–5% most central collisions. This anomaly is considered as an indication of a phase transition of the produced matter between the gas state at low energies and the QGP liquid phase at higher $\sqrt{s_{NN}}$. In this region, the number of the constituent configuration of the system decreases due to the phase transition from the gas phase to the liquid phase. As follows from Figure 1, such kinds of anomalous behavior diminish at higher p_T, and it is not seen at all in 60–80% peripheral collisions. The irregularity in fractal entropy in the central collisions depends on the type of the inclusive particle. The anomaly is not observed for pions and negative hadrons (see [29] and references therein). This would indicate that non-strange particles do not reveal sensitivity to phase–phase transition effects, such as abrupt changes in fractal entropy.

5. Self-Similarity of K^0_S Production

The anomaly of the entropy $S_{\delta,\epsilon}$ as a function of the collision energy at low p_T depicted in Figure 1a corresponds to the scaling behavior of K^0_S-meson production shown in Figure 2. The scaling is obtained for nearly constant values of the structural fractal dimension δ and for the abruptly falling energy dependence of the model parameter c_{AuAu}, interpreted as the specific heat of the nuclear medium produced in AuAu collisions. The symbols in Figure 2 correspond to the transverse momentum distributions measured in most central AuAu collisions [22,23,25]. The solid line describes the z-scaling curve for proton-proton (pp) interactions. One can observe a "collapse" of data points in z-presentation. This indicates the universality of the scaling function. The points are well described by the fitting curve in the range of z = 0.1–7. The general properties of such dependence, found in previous analyses for other hadrons, namely, a power law at high z and flattening at low z, are confirmed. The scaling function $\psi(z)$ demonstrates smooth behavior over a wide range of the self-similarity variable z. It corresponds to the various centralities, collision energies, and transverse momenta of the inclusive K^0_S mesons produced in AuAu collisions.

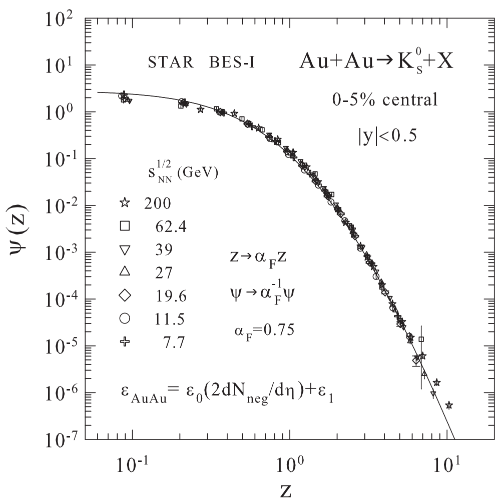

Figure 2. The scaling function $\psi(z)$ for K^0_S mesons produced in the 0–5% (most central) AuAu collisions at $\sqrt{s_{NN}} = 7.7$, 11.5, 19.6, 27, 39, 62.4, and 200 GeV in the rapidity range $|y| < 0.5$. The symbols correspond to the data [22,23,25] of the Beam Energy Scan (BES) program by the STAR Collaboration at RHIC. The solid line is a reference curve for pp collisions [27]. Figure is taken from Ref. [30].

The nucleus fractal dimension δ_A is expressed in terms of the nucleon fractal dimension and the atomic number A as $\delta_A = A\delta$. The dependence of the nucleon fractal dimension δ, used in the z-presentation of spectra of K^0_S mesons produced in AuAu collisions is shown in Figure 3a. The nucleon fractal dimension was found constant $\delta \simeq 0.5$ for $\sqrt{s_{NN}} \geq 19.6$ GeV. At lower collision energies, such as $\sqrt{s_{NN}} = 11.5$ and 7.7 GeV, the scaling hypothesis is consistent, with a slight decrease in δ. The nucleus fractal dimension characterizes the fractal structure of the colliding nuclei. The fractality, concerning also nuclei constituents, is part of a broader context of scaling and structural self-similarity. Note that for structureless objects and objects with invisible internal structure, the structural dimensions should be zero. It seems therefore natural that both dimensions δ_A and δ decrease at low $\sqrt{s_{NN}}$. The fractal fragmentation dimension (3) is found to be an increasing function of the multiplicity density for produced negative particles in the AuAu system [29]. This result is also in agreement with the self-similarity of K^0_S-meson production. We expect that such kinds of multiplicity dependence will be valid in the asymptotic region ($z > 10$) as well.

The interesting observation is that the parameter c_{AuAu} shown in Figure 3b decreases abruptly from the value of 0.16 at $\sqrt{s_{NN}} \leq 11.5$ GeV to about 0.10 at $\sqrt{s_{NN}}$ =39 GeV. This is followed by a kink and a flattening as the energy increases. The values of the specific heat parameter c_{AuAu} are consistent with the scaling behavior of $\psi(z)$ depicted in Figure 2.

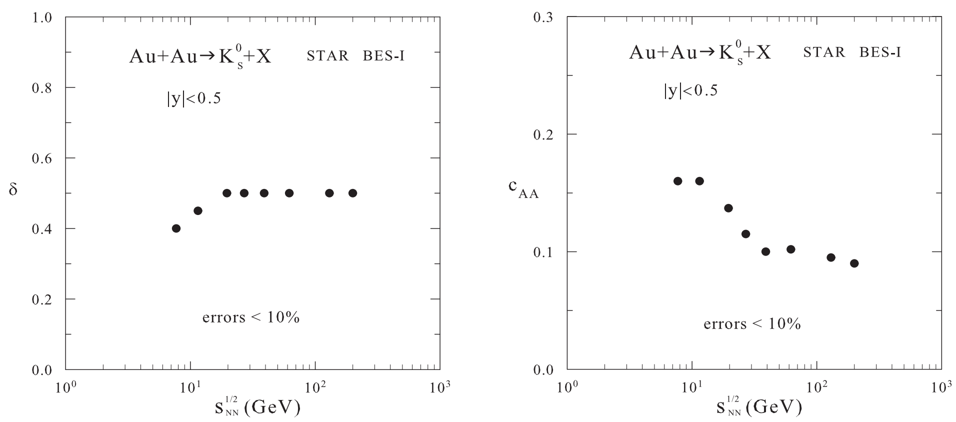

Figure 3. The nucleon structural dimension δ (**a**) and the specific heat parameter c_{AA} (**b**) for K^0_S-meson production in AuAu collisions in dependence on the collision energy at $|y| < 0.5$. Figures are taken from Ref. [30].

Specific heat is one of the thermodynamic quantities used to characterize properties of a multi-particle system. It is known [16,17] that in very different systems, specific heat as a function of temperature exhibits power behavior near phase transitions and discontinuity at CP. In classical thermodynamics, the isochoric heat capacity c_V is expressed via the entropy in the following way: $c_V = -TdS/dT|_V$ at a constant volume V with T denoting the temperature. It is considered that a specific heat should reveal growth at $T \to T_c$, and discontinuity or singular behavior at a CP itself. According to the above expression for c_V, the larger thermal fluctuations produces smaller values of specific heat.

One can see that the specific heat parameter c_{AuAu} for the AuAu system rapidly decreases in the range of $\sqrt{s_{NN}}$ = 11.5–39 GeV as the collision energy increases. This gives us the possibility to interpret such behavior of c_{AuAu} for K^0_S-meson production in AuAu collisions as the growth of thermal fluctuations at high energies, and its suppression at low energies. Similar results for the energy dependence of a specific heat are found in Ref. [34]. Nevertheless, we should note that the basic assumptions used in Ref. [34] are completely different.

As one can see, the definition of entropy is an essential ingredient in the thermodynamic analysis of datasets obtained in heavy-ion experiments. The various forms of deformed entropy (Tsallis, Kaniadakis, Sharma–Mittal, and others) proposed as a generalization of the standard Boltzmann–Gibbs entropy, S_{BG}, in statistical thermodynamics are parameter-dependent quantities. These parameters describe some properties (long-range correlations, non-Gaussian fluctuations) of non-additive systems at their thermal contact. In order to obtain the consistency of the thermodynamic descriptions of multi-particle systems and give them a physical meaning, a redefinition of the classical relations obtained within Tsallis, Kaniadakis, and Sharma–Mittal statistics are needed. In the z-scaling approach the fractal entropy takes into account both the fractal structure of colliding ions and the fractal nature of the fragmentation process. The corresponding fractal dimensions are found from the scaling behavior of $\psi(z)$ as a function of the self-similarity variable z. It should be noted that the energy dependence of the specific heat parameter c_{AuAu} for K^0_S mesons differs from the constant behavior obtained for negative hadrons [29]. This emphasizes an important role of the strange probes in studying the properties of nuclear matter. We conclude that the observed irregularities in the behavior of physical quantities, such as specific heat and entropy, could indicate the existence of phase transitions in nuclear matter.

6. Discussion

The concept of the fractal entropy proposed for the description of hadron production in heavy-ion collisions in the z-scaling approach is applied to K^0_S-meson production in AuAu collisions at RHIC energies. The non-monotonic growth of the entropy $S_{\delta,\epsilon}$ with $\sqrt{s_{NN}}$ is found for $p_T < 1$ GeV/c in the most central collisions. The observed feature is found to present along with the scaling behavior of $\psi(z)$ function and the anomalous dependence of the specific heat parameter c_{AuAu} on the collision energy in the region 11–39 GeV. We suggest that these properties are connected with a phase transition in nuclear matter. The obtained results support the hypothesis that strange particles are more sensitive probes of the nuclear matter than the non-strange ones. The determination of the phase diagram of the nuclear matter produced in heavy-ion collisions is one of the key tasks of programs performed at the SPS, RHIC, and LHC [20,21]. In the z-scaling approach, the principle of the maximum entropy formulated for the isolated system was used to determine the momentum fractions x_1, x_2, y_a, and y_b. We observed a decrease in the entropy $S_{\delta,\epsilon}$ in certain energy regions that could be interpreted as a freeze-out of some degrees of freedom in the produced nuclear matter. It is characteristic for the transition from a "gas" phase, considered at the constituent level at low $\sqrt{s_{NN}}$, to the "liquid" phase at higher energy. The subsequent increase in the fractal entropy with energy can be connected with the growth of the number of its micro-states. This would correspond to the increase in the entropy of the strongly coupled QCD "liquid" phase.

It is of interest to study of the observed anomalous behavior of $S_{\delta,\epsilon}$ at transverse momenta below 300 MeV/c, both for K^0_S mesons and other strange hadrons, such as for $K^{\pm}, K^{0*}, \phi$ mesons, for the maximum available energy and collision centrality region. These results would provide new information and clarify on the physical meaning of the asymptotic behavior of the scaling function in the low- and high-p_T regions, as well as verify the hypothesis of the scaling function shape universality.

Most secondaries in the soft region corresponding to the large multiplicity density and accompanying the production of a trigger particle are considered to reflect properties of the matter produced [31]. A small viscosity-to-entropy ratio has been identified as a special feature of the perfect quark–gluon liquid to be created in heavy-ion collisions at the highest RHIC energy [32,33]. In this sense, entropy changes are assumed to reflect the evolution details of the system. Let us stress that the type of probe (or trigger particle) plays an important role in studying both the phase transition region and properties of different phases themselves. There are different assumptions on the phases of nuclear matter produced in heavy-ion collisions [35–37]. Nevertheless, at present there is no convincing experimental evidence on the location of phase boundaries, the CP, specific properties of phases, or even the type of phase transitions.

In connection with the anomalous behavior found in the energy dependence of the fractal entropy discussed in the present paper, some interesting features reported by the STAR Collaboration to be mentioned. Among them are a non-monotonic energy dependence of the ratio of net-proton cumulants C_2 and C_4 in the range of $\sqrt{s_{NN}}$ = 7.7–62.4 GeV for the (0–5)% central AuAu collisions [38]; a non-monotonic variation with collision energy of the moments of the net-baryon number distribution and kurtosis times variance of the net-proton number [39]; a power-law behavior of scaled factorial moments for the intermittency of identified charged hadrons in AuAu collisions; and a non-monotonic energy dependence of scaling exponent ν that reaches a minimum around $\sqrt{s_{NN}} = 27$ GeV in most central AuAu collisions [40]. The nuclear compound yield ratio $N_t\, N_p / N_d^2$ (with N_t, N_p, and N_d denoting the triton, proton, and deutron yields, respectively), predicted to be sensitive to the local density fluctuation of neutrons, monotonically decreases with increasing charged-particle multiplicity $dN_{\mathrm{ch}}/d\eta$ and exhibits a scaling behavior in AuAu collisions. Relative to the coalescence baseline mechanism, enhancements in the yield ratios are observed in (0–10)% centrality collisions at 19.6 and 27 GeV with a significance of 2.3 and 3.4 standard deviations, respectively [41]. All the above-mentioned experimentally determined properties can have physical consequences for determining the structure of the phase diagram of nuclear matter.

Searching for the corresponding signatures of phase transition and CPs in heavy-ion collisions over a wide range of energy, centrality, ion type, momentum, and type of probes, as well as the development of methods adequate to the task, are of a high priority. We consider the z-scaling approach to be one of such methods. The verification of the anomalous behavior of the fractal entropy with high accuracy is of significant interest for the study of the phase diagram of nuclear matter. However, the interpretation of the non-monotonic entropy dependence requires more evidence to be understood as a manifestation of a phase transition in the nuclear medium. Further analyses of high-precision data are needed to support such a conclusion. Moreover, other possible explanations for the observed anomaly in the energy dependence of the fractal entropy must be ruled out.

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