physics

MDPI

Article

On Majorization Uncertainty Relations in the Presence of a Minimal Length

Alexey E. Rastegin

Department of Theoretical Physics, Irkutsk State University, K. Marx St. 1, Irkutsk 664003, Russia; alexrastegin@mail.ru

Abstract: The emergence of a minimal length at the Planck scale is consistent with modern developments in quantum gravity. This is taken into account by transforming the Heisenberg uncertainty principle into the generalized uncertainty principle. Here, the position-momentum commutator is modified accordingly. In this paper, majorization uncertainty relations within the generalized uncertainty principle are considered. Dealing with observables with continuous spectra, each of the axes of interest is divided into a set of non-intersecting bins. Such formulation is consistent with real experiments with a necessarily limited precision. On the other hand, the majorization approach is mainly indicative for high-resolution measurements with sufficiently small bins. Indeed, the effects of the uncertainty principle are brightly manifested just in this case. The current study aims to reveal how the generalized uncertainty principle affects the leading terms of the majorization bound for position and momentum measurements. Interrelations with entropic formulations of this principle are briefly discussed.

Keywords: generalized uncertainty principle; minimal observable length; majorization uncertainty relations

check for updates

Citation: Rastegin, A.E. On Majorization Uncertainty Relations in the Presence of a Minimal Length. *Physics* **2022**, *4*, 1413–1425. https://doi.org/10.3390/physics4040091

Received: 22 September 2022
Accepted: 1 December 2022
Published: 14 December 2022

1. Introduction

One of the key problems of modern physics is to build a quantum theory of gravitation [1]. The existence of a minimal observable length has long been suggested due to these efforts. Let us refrain from listing them and refer to the papers [2,3]. There are proposals to investigate the testable effects of the minimal length, including astronomical observations [4,5] and experimental schemes seemingly feasible within current technology [6–8]. Papers [9–12] discussed measurements in which one may be able to probe the effects of quantum gravity. The implications of the deformed forms of the commutation relation have attracted large attention [13–18]. In particular, researchers analyzed the consequences for the harmonic oscillator [17,18], the free particle, and potentials with infinitely sharp boundaries [14]. Going beyond the linear regime in graphene, in Ref. [19], a generalized uncertainty framework compatible with quantum gravity scenarios with a minimal length was obtained.

The Heisenberg uncertainty principle [20] emphasizes fundamental limitations on the simultaneous knowledge of observables in the quantum world. Uncertainty relations in terms of the product of standard deviations were formally derived by Kennard [21] for position and momentum and later by Robertson [22] for any pair of observables. An alternative to this traditional approach is provided by entropic characterization. For the position-momentum pair, an entropic formulation was initiated by Hirschman [23] and later developed in Refs. [24,25]. With a primary focus on observables with discrete spectra, the use of entropies to characterize quantum uncertainties was explored in Refs. [26,27]. Being the subject of current research, entropic uncertainty relations are reviewed in Refs. [28–32]. The majorization approach provides another flexible way to pose uncertainty relations [33–37] with a natural transition to entropic characterization when required.

Heisenberg's uncertainty principle *per se* does not impose a restriction separately on the spreads of position or momentum. Below the scale linked to the Planck length, $\ell_{\mathrm{Pl}} = \sqrt{G\hbar/c^3} \approx 1.616 \times 10^{-35}$ m, the very structure of space-time is an open question [38]. Here, G is the Newtonian constant of gravitation, $\hbar$ is the reduced Planck's constant, and c denotes the speed of light. The Heisenberg principle is replaced here with the generalized uncertainty principle, which declares a non-zero lower bound on the position spread [39–41]. The generalized uncertainty principle can be reinterpreted as an effective variation of the Planck constant [42], with a link to Dirac's large numbers hypothesis [43]. Using the preparation scenario, entropic uncertainty relations in the presence of a minimal length were examined in Refs. [44–46]. At each stage of the scenario of successive measurements, an actual pre-measurement state depends on the results of previous measurements [47,48]. This viewpoint is closer to Heisenberg's thought experiment with microscope [49]. The generalized uncertainty principle with successive measurements of position and momentum was analyzed in Ref. [50].

This paper is devoted to majorization uncertainty relations in the presence of a minimal length. To focus on changing the majorization bound for position and momentum measurements, a consideration is restricted here to the preparation scenario. In addition, the case of high-resolution measurements with small bins is most interesting from the physical viewpoints. Hence, one naturally obtains a small dimensionless parameter, with respect to which the quantities of interest can be expanded. For practical purposes, several leading terms in expansion of the majorization bound should be taken into account. It turns out that an effect of the generalized uncertainty relation is actually revealed in this way. The paper is organized as follows. Section 2 reviews the preliminary findings and fix the notation. The derivation of basic terms of the majorization bound is presented in Section 3. Section 4 concludes the paper with a summary of the results. In Appendix A, a perturbation theory is developed to solve an auxiliary eigenvalue problem.

2. Preliminaries

In this Section, the generalized uncertainty principle and related findings are recalled. Further, basic points of the majorization approach to quantum uncertainties are discussed.

2.1. The Generalized Uncertainty Principle

The generalized uncertainty principle declares the deformed commutation relation for the position and momentum operators [13]. Some different representations of the same algebra exist. However, the physical content is determined by the physical observables. Namely, these operators provide access to the explicit information on the position and momentum measurements [51]. For convenience, the wavenumber operator, $\hat{\kappa}$, is used instead of the momentum operator, $\hbar\hat{\kappa}$. Let us consider the commutation relation,

$$\left[\hat{x},\hat{\kappa}\right] = i\left(\mathbf{1}+\beta\hat{\kappa}^2\right). \tag{1}$$

Here, the positive parameter β is rescaled by factor $\hbar^2$ from its known sense, and $\mathbf{1}$ is the identity operator. In the limit $\beta \to 0$, Equation (1) gives the known commutation relation of ordinary quantum mechanics. This is a most straightforward modification leading to the presence of a minimal length. Instead of Equation (1), more general forms of the additional term can be placed in the right-hand side [17]. Due to the results of [14], the used formulation allows us to study questions of interest with a more apparent analogy with the ordinary case. It is suitable at the first step in probing potential effects of the generalized uncertainty principle. In addition, the formulation (1) is asymmetric with respect to the role of position and momentum. It seems to be natural in topics concerning just the existence of a minimal length.

A quantum state is represented by a positive self-adjoint operator $\hat{\rho}$ with $\mathrm{Tr}(\hat{\rho}) = 1$ called the density matrix. Combining Equation (1) with the known Robertson formulation [22] leads to the inequality,

$$(\Delta\hat{x})_{\hat{\rho}}\,(\Delta\hat{\kappa})_{\hat{\rho}} \geq \frac{1}{2}\left(1+\beta\langle\hat{\kappa}^2\rangle_{\hat{\rho}}\right) \geq \frac{1}{2}\left(1+\beta(\Delta\hat{\kappa})^2_{\hat{\rho}}\right). \tag{2}$$

As in general, for any operator $\hat{Q}$ one has:

$$\langle\hat{Q}\rangle_{\hat{\rho}} = \mathrm{Tr}(\hat{Q}\hat{\rho})\,, \qquad (\Delta\hat{Q})^2_{\hat{\rho}} = \langle\hat{Q}^2\rangle_{\hat{\rho}} - \langle\hat{Q}\rangle^2_{\hat{\rho}}\,.$$

It further follows from Equation (2) that $(\Delta\hat{x})_{\hat{\rho}} \geq \sqrt{\beta}$ for every state $\hat{\rho}$. Thus, it is impossible to localize a particle below the scale corresponding to the square root of β.

It is helpful to introduce the auxiliary wavenumber operator $\hat{q}$ [14]. Let $\hat{x}$ and $\hat{q}$ be self-adjoint operators that obey $[\hat{x},\hat{q}] = i\,\mathbf{1}$. In the q-space, the action of $\hat{q}$ results in multiplying a wave function $\varphi(q)$ by q, whereas $\hat{x}\,\varphi(q) = i\,\mathrm{d}\varphi/\mathrm{d}q$. Following [14], let us define

$$\hat{\kappa} = \frac{1}{\sqrt{\beta}}\,\tan\!\left(\sqrt{\beta}\,\hat{q}\right). \tag{3}$$

So, the auxiliary wavenumber obeys the ordinary commutation relation but ranges between $\pm\,q_{\max}(\beta) = \pm\,\pi/(2\sqrt{\beta})$. The function $q \mapsto \kappa = \tan(\sqrt{\beta}q)/\sqrt{\beta}$ provides a one-to-one correspondence between $q \in (-\,q_{\max}, +\,q_{\max})$ and $\kappa \in (-\,\infty, +\,\infty)$. So, the eigenvalues of $\hat{\kappa}$ fully cover the real axis. For a pure state, one actually has the three wave functions $\phi(\kappa)$, $\varphi(q)$, and $\psi(x)$. The auxiliary wave function $\varphi(q)$ is a convenient mathematical tool as connected with $\psi(x)$ via the Fourier transform. Let the eigenkets $|q\rangle$ of $\hat{q}$ be normalized through Dirac's delta function and satisfy the completeness relation,

$$\int_{-q_{\max}}^{+q_{\max}} \mathrm{d}q\,|q\rangle\langle q| = \mathbf{1}\,. \tag{4}$$

In the q-space, the eigenfunctions of $\hat{x}$ are expressed as $\langle q|x\rangle = \exp(-iqx)/\sqrt{2\pi}$. Combining this with Equation (4), any wave function in the coordinate space reads:

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-q_{\max}}^{+q_{\max}} \exp(+iqx)\,\varphi(q)\,\mathrm{d}q\,. \tag{5}$$

Wave functions in the q- and x-spaces are connected by the Fourier transform [14], namely,

$$\varphi(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-iqx)\,\psi(x)\,\mathrm{d}x\,. \tag{6}$$

The only distinction from ordinary quantum mechanics is that each wave function $\varphi(q)$ in the q-space should be treated as 0 for all $|q| > q_{\max}(\beta)$. However, a distribution of physical wavenumber values is determined by $\phi(\kappa)$. Let us consider the probability to find the momentum between two prescribed values. Due to the one-to-one correspondence between κ and q, there is a bijection between the intervals (κ_1,κ_2), and (q_1,q_2). Thus, the probability of interest is expressed as

$$\int_{\kappa_1}^{\kappa_2} |\phi(\kappa)|^2\,\mathrm{d}\kappa = \int_{q_1}^{q_2} |\varphi(q)|^2\,\mathrm{d}q\,, \tag{7}$$

whence the probability density functions are related via $|\phi(\kappa)|^2\,\mathrm{d}\kappa = |\varphi(q)|^2\,\mathrm{d}q$.

2.2. On Majorization Uncertainty Relations

Let us proceed to a general formulation of majorization uncertainty relations. Let $y = (y_1,\cdots,y_n)$ and $z = (z_1,\cdots,z_n)$ be two n-dimensional vectors with real components.

By adding zero components, one can always reach that the two vectors have the same number of elements. One says that y is majorized by z, in symbols $y \prec z$, if [52]

$$\sum\nolimits_{j=1}^{m} y_j^{\downarrow} \leq \sum\nolimits_{j=1}^{m} z_j^{\downarrow} \tag{8}$$

for all $m = 1, \dots, n$ and

$$\sum\nolimits_{j=1}^{n} y_j^{\downarrow} = \sum\nolimits_{j=1}^{n} z_j^{\downarrow} . \tag{9}$$

The arrows down mark that the components should be taken in non-increasing order.

To pose majorization uncertainty relations, the following notions will be used [33]. The infimum of a set of vectors is defined as the vector that is majorized by every element of the set and, in turn, majorizes any vector with that property [53]. The supremum is similarly defined as the vector that majorizes every element of the set and is, in turn, majorized by any vector with that property. The procedure to calculate the desired vectors are also discussed in Refs. [33,53] with a reference to the MATHEMATICA codes prepared for these purposes. Let us refrain from discussing some subtle points related to such calculations. Even if continuous observables are dealt with, one can nevertheless restrict a consideration to a finite set of large number of bins. This holds not only due to non-zero sizes of bins but also in view of boundness of values available to be measured in practice.

To formulate majorization uncertainty relations, one should fix operators that describe each measurement of interest. By $x_{\alpha 1} < x_{\alpha 2}$, one denotes the least points of α-th bin in the position measurement. The corresponding projection operator reads:

$$\hat{\Pi}_{\alpha} = \int_{x_{\alpha 1}}^{x_{\alpha 2}} \mathrm{d}x \, |x\rangle\langle x| \, . \tag{10}$$

To the momentum measurement, let us assign a set of projection operators of the form,

$$\hat{\Lambda}_{\gamma} = \int_{\kappa_{\gamma 1}}^{\kappa_{\gamma 2}} \mathrm{d}\kappa \, |\kappa\rangle\langle \kappa| \, , \tag{11}$$

where $\kappa_{\gamma 1} < \kappa_{\gamma 2}$ are the least points of γ-th bin. The above form of the operators corresponds to an orthogonal resolution of the identity in each case. Strictly speaking, the finiteness of the detector resolution is typically addressed in terms of acceptance functions [48]. Certainly, the projection operators are obtained with an acceptance function in the form of boxcar one. A consideration is restricted to boxcar acceptance functions since the aim here is to focus on the corollaries of the generalized uncertainty principle. At the same time, the use of Gaussian acceptance functions is apparently closer to practice [48]. On the other hand, the case of high-resolution measurements with small bins is of primary interest. Moreover, little changes of the form of acceptance functions have no actual bearing on the principal possibility to observe the effects of a minimal length.

One of the advantages of the majorization approach is that uncertainty relations are expressed directly in terms of probabilities. For the prepared pre-measurement state $\hat{\rho}$, one obtains the probabilities,

$$\mathrm{Tr}(\hat{\Pi}_{\alpha}\hat{\rho}) = \int_{x_{\alpha 1}}^{x_{\alpha 2}} \langle x|\hat{\rho}|x\rangle \, \mathrm{d}x \, , \qquad \mathrm{Tr}(\hat{\Lambda}_{\gamma}\hat{\rho}) = \int_{\kappa_{\gamma 1}}^{\kappa_{\gamma 2}} \langle \kappa|\hat{\rho}|\kappa\rangle \, \mathrm{d}\kappa \, ,$$

which respectively constitute the vectors $\mathrm{p}^{\hat{x}}(\hat{\rho})$ and $\mathrm{p}^{\hat{\kappa}}(\hat{\rho})$. The majorization uncertainty relation of the paper [33] is posed as follows. It was shown that

$$\mathrm{p}^{\hat{x}}(\hat{\rho}) \otimes \mathrm{p}^{\hat{\kappa}}(\hat{\rho}) \prec \sup\Big\{ \mathrm{p}^{\hat{x} \oplus \hat{\kappa}}(\hat{\varrho}) : \, \hat{\varrho} \geq \mathbf{0}, \, \hat{\varrho}^{\dagger} = \hat{\varrho}, \, \mathrm{Tr}(\hat{\varrho}) = 1 \Big\}. \tag{12}$$

In the case of discrete observables, one *a priori* has a unitary matrix connecting two orthonormal bases. Inspecting the norms of the submatices of this unitary matrix, the majorization uncertainty relations follow straight away [34,36]. Moreover, such relations are straightforwardly converted into inequalities for Rényi and Tsallis entropies. An application of these results to neutrino flavor and mass was studied in Ref. [54] since the Pontecorvo–Maki–Nakagawa–Sakata matrix is dealt with here. In a general case, however, finding the right-hand side of Equation (12) or some of its components can be more tractable than building a suitable unitary matrix. In addition, one will not necessarily be dealing with projective measurements.

In paper [33] it is described how to calculate the first term in the right-hand side of Equation (12). Overall, one seeks for the maximum value of $\mathrm{Tr}(\hat{\Pi}_\alpha\hat{\rho})\,\mathrm{Tr}(\hat{\Lambda}_\gamma\hat{\rho})$, where the projectors are fixed and $\hat{\rho}$ is varied. The desired extremal value is realized with a pure state, for example, $|\psi_*\rangle$. As was shown in Ref. [33], the task is reduced to the eigenvalue problem,

$$\hat{\Pi}_\alpha\,\hat{\Lambda}_\gamma\,\hat{\Pi}_\alpha|\eta_*\rangle=\mu^2\,|\eta_*\rangle \tag{13}$$

with $|\eta_*\rangle=\hat{\Pi}_\alpha|\psi_*\rangle$. One should find the maximal eigenvalue of the problem (13). The first component in the right-hand side of Equation (12) is then equal to $(1+\mu_{\max})^2/4$. Further terms are more difficult to calculate. For high-resolution measurements with small bins, however, many components differ little from the first, except for the tails of the distributions and the intermediate zones. In effect, the bins should be such that many of them are lying around distribution peaks. One can leave this assumption by replacing Equation (12) with the uncertainty relation in terms of the min-entropies. For the given probability distribution $\mathrm{p}=\{p_j\}$, its min-entropy is defined as

$$H_\infty(\mathrm{p})=-\ln\bigl(\max\,p_j\bigr)\,. \tag{14}$$

The latter is obtained when the order of Rényi's entropy [55] tends to infinity. It follows from Equations (12) and (14) that

$$H_\infty\bigl(\mathrm{p}^{\hat{x}}(\hat{\rho})\bigr)+H_\infty\bigl(\mathrm{p}^{\hat{\kappa}}(\hat{\rho})\bigr)\geq 2\ln 2-2\ln(1+\mu_{\max})\,. \tag{15}$$

The next Section examines how the above relations are affected by the generalized uncertainty principle.

3. Main Results

The previous Section provides a ground to study the question how the generalized uncertainty principle affects the majorization bound for position and momentum measurements. It is natural that the analysis here begins with the case $\beta=0$.

3.1. The Case of Ordinary Commutation Relation

In Ref. [33], the problem (13) was reformulated as

$$\frac{1}{\pi}\int\limits_{-\Delta x/2}^{+\Delta x/2}\frac{\sin\bigl(\Delta\kappa(x-x')/2\bigr)}{x-x'}\,\eta_*(x')\,\mathrm{d}x'=\mu^2\,\eta_*(x)\,, \tag{16}$$

where the variables x and x' are both restricted to the range $\bigl[-\Delta x/2,+\Delta x/2\bigr]$ and $\eta_*(x)=\langle x|\eta_*\rangle$. One uses Equation (16) under the assumption that the origins of both the x and κ axes are placed into centers of the two bins for which the optimality is reached. Surely, this holds for the ordinary commutation relation. To analyze consequences of the generalized uncertainty principle, one needs also to examine Equation (16) in more detail than it was made in the paper [33].

Substituting $x=\xi\Delta x$, $s=\Delta x\Delta\kappa/(2\pi)$ and $\mu^2=s\lambda$, one rewrites Equation (16) as

$$\int_{-1/2}^{+1/2}\frac{\sin[s\pi(\xi-\xi')]}{s\pi(\xi-\xi')}\,u(\xi')\,\mathrm{d}\xi'=\lambda\,u(\xi)\,, \tag{17}$$

where $u(\xi) = \eta_*(\xi\Delta x)$. Here, the kernel is expanded as

$$\frac{\sin[s\pi(\xi - \xi')]}{s\pi(\xi - \xi')} = 1 - \frac{s^2\pi^2}{3!}\,(\xi - \xi')^2 + \frac{s^4\pi^4}{5!}\,(\xi - \xi')^4 + \cdots\,. \tag{18}$$

It then follows from Equations (A16) and (A19) that

$$\mu^2_{\max}\Big|_{\beta=0} = s\left(1 - \frac{s^2\pi^2}{36} + O(s^4)\right), \tag{19}$$

$$\eta_*(x)\big|_{\beta=0} = 1 - \frac{s^2\pi^2}{6}\left(\frac{x^2}{\Delta x^2} - \frac{1}{12}\right) + O(s^4)\,. \tag{20}$$

In the limit $s \to 0$, the term (19) tends to s as mentioned in Ref. [33]. The result (19) is useful also in the sense of characterizing a level of smallness for s. For example, for $s < 1/2$, one has:

$$\frac{s^2\pi^2}{36} < 0.069\,, \tag{21}$$

i.e., the eigenvalue correction turns out to be of several percents. Thus, the validity of perturbation expansions up to the first order does not require exceptionally high resolution. Within the given scheme, one can further derive second and higher-order perturbations, though complexity of expressions grows quickly.

3.2. The Case of Modified Commutation Relation

For $\beta \neq 0$, one keeps kernels of the Fourier transform by means of the auxiliary wavenumber. The eigenvalue problem (13) straightforwardly leads to

$$\begin{aligned}\mu^2\langle x|\eta_*\rangle &= \int_{\kappa_{\gamma 1}}^{\kappa_{\gamma 2}} \mathrm{d}\kappa \int_{x_{\alpha 1}}^{x_{\alpha 2}} \mathrm{d}x'\,\langle x|\kappa\rangle\langle\kappa|x'\rangle\,\eta_*(x') \\ &= \frac{1}{2\pi}\int_{x_{\alpha 1}}^{x_{\alpha 2}} \mathrm{d}x' \int_{q_{\gamma 1}}^{q_{\gamma 2}} \frac{2\mathrm{d}q}{1+\cos(2\sqrt{\beta}q)}\,\exp[i(x-x')q]\,\eta_*(x')\,,\end{aligned} \tag{22}$$

where one used

$$\frac{\mathrm{d}\kappa}{\mathrm{d}q} = \frac{2}{1+\cos(2\sqrt{\beta}q)}\,. \tag{23}$$

Let us now seek a possibility to translate the wavenumber axis as it was made to obtain Equation (16). Strictly speaking, this step can be used with the generalized uncertainty principle only approximately. Indeed, the standard kernels of the Fourier transform stand in Equations (5) and (6) due to the auxiliary wavenumber that ranges between $\pm\,q_{\max}(\beta)$. The latter indeed prevents translational invariance with respect to the q axis. On the other hand, the value $q_{\max}(\beta)$ corresponds to extremely high energies that are completely beyond the capabilities of modern experiments. Actually, reaching such high energies is inevitably coupled with approaching the Planck scale *per se*. Therefore, one deals with wavepackets supported in the momentum space far away from values of the mentioned order. In addition, such bins are advisably inclined to use that are small in comparison with characteristic spreading of typical wavepackets. Under these circumstances one is able to use shifts along the q and κ axes.

The eigenvalue problem (16) is then replaced with

$$\frac{1}{2\pi}\int\limits_{-\Delta x/2}^{+\Delta x/2} \eta_*(x')\,\mathrm{d}x' \int\limits_{-\Delta q/2}^{+\Delta q/2} \frac{2\exp[i(x-x')q]\,\mathrm{d}q}{1+\cos(2\sqrt{\beta}q)} = \mu^2\,\eta_*(x)\,. \tag{24}$$

Separating explicitly the term assigned to $\beta = 0$, one obtains:

$$\frac{1}{\pi}\int\limits_{-\Delta x/2}^{+\Delta x/2} \eta_*(x')\,\mathrm{d}x' \left\{ \frac{\sin\big(\Delta q(x-x')/2\big)}{x-x'} \right.$$
$$\left. + \frac{1}{2}\int\limits_{-\Delta q/2}^{+\Delta q/2} \left(\frac{2}{1+\cos(2\sqrt{\beta}q)} - 1 \right) \exp[i(x-x')q]\,\mathrm{d}q \right\} = \mu^2\,\eta_*(x)\,. \tag{25}$$

Let us take $q = y\Delta q/2$, $s = \Delta x \Delta q/(2\pi)$ and

$$\varkappa = (x-x')\,\frac{\Delta q}{2} = s\pi(\xi - \xi')\,, \tag{26}$$

then

$$\frac{1}{2}\int\limits_{-\Delta q/2}^{+\Delta q/2} \left(\frac{2}{1+\cos(2\sqrt{\beta}q)} - 1 \right) \exp[i(x-x')q]\,\mathrm{d}q = \frac{\beta\Delta q^3}{8}\, I_2(\varkappa) + O(\beta^2)\,, \tag{27}$$

with $I_n(\varkappa) = (1/2)\int_{-1}^{+1} y^n \exp(i\varkappa y)\,\mathrm{d}y$ for $n = 1,2,\ldots$. One can see from Equations (25) and (27) that

$$\int\limits_{-\Delta x/2}^{+\Delta x/2} \left(\frac{\sin\varkappa}{\pi(x-x')} + \frac{\beta\Delta q^3}{8\pi}\, I_2(\varkappa) + O(\beta^2) \right) \eta_*(x')\,\mathrm{d}x' = \mu^2\,\eta_*(x)\,.$$

By changing the variable to $\xi' = x'/\Delta x$, one finally obtains:

$$\int_{-1/2}^{+1/2} \left(\frac{s\sin\varkappa}{\varkappa} + \frac{s\beta\Delta q^2}{4}\, I_2(\varkappa) + O(\beta^2) \right) u(\xi')\,\mathrm{d}\xi' = \mu^2\, u(\xi)\,. \tag{28}$$

Substituting $\mu^2 = s\lambda$ again, Equation (28) is reduced to the form,

$$\big(\mathsf{K}^{(0)} + \varepsilon\,\mathsf{K}^{(1)} + \cdots\big) u(\xi) = \lambda\, u(\xi)\,, \tag{29}$$

with intent to use the formulas of Appendix A for $\varepsilon = \beta\Delta q^2/4$,

$$\mathsf{K}^{(0)} f(\xi) = \int_{-1/2}^{+1/2} \frac{\sin\varkappa}{\varkappa}\, f(\xi')\,\mathrm{d}\xi'\,,$$
$$\mathsf{K}^{(1)} f(\xi) = \int_{-1/2}^{+1/2} \frac{\varkappa^2\sin\varkappa + 2\varkappa\cos\varkappa - 2\sin\varkappa}{\varkappa^3}\, f(\xi')\,\mathrm{d}\xi'\,.$$

It follows from Equation (A15) that the factor of $\beta\Delta q^2/4$ in the first-order correction reads as

$$\langle \widetilde{w}_0, \mathsf{K}^{(1)}\widetilde{w}_0 \rangle = \int_{-1/2}^{+1/2} \mathrm{d}\xi \int_{-1/2}^{+1/2} \mathrm{d}\xi'\, \frac{\varkappa^2\sin\varkappa + 2\varkappa\cos\varkappa - 2\sin\varkappa}{\varkappa^3}\, \widetilde{w}_0(\xi)\,\widetilde{w}_0(\xi')\,, \tag{30}$$

where

$$\widetilde{w}_0(\xi) = 1 - \frac{s^2\pi^2}{6}\left(\xi^2 - \frac{1}{12}\right) + O(s^4)$$

due to Equation (A19). Again, a consideration is aimed to be restricted to the case of sufficiently high resolution. The calculations show that

$$\frac{\varkappa^2\sin\varkappa + 2\varkappa\cos\varkappa - 2\sin\varkappa}{\varkappa^3} = \frac{1}{3} - \frac{s^2\pi^2}{10}\,(\xi - \xi')^2 + \cdots\,, \tag{31}$$

and further,

$$\langle\widetilde{w}_0,\mathsf{K}^{(1)}\widetilde{w}_0\rangle=\int_{-1/2}^{+1/2}\mathrm{d}\xi\int_{-1/2}^{+1/2}\mathrm{d}\xi'\left\{\frac{1}{3}-\frac{s^2\pi^2}{10}\,(\xi-\xi')^2-\frac{s^2\pi^2}{18}\left(\xi^2-\frac{1}{12}\right)\right.$$
$$\left.-\frac{s^2\pi^2}{18}\left(\xi'^2-\frac{1}{12}\right)+\cdots\right\}=\frac{1}{3}-\frac{s^2\pi^2}{60}+O(s^4)\,. \tag{32}$$

Summing up, the result is obtained in the form,

$$\frac{\mu_{\max}^2}{s}=\lambda=1+\frac{\beta\Delta q^2}{12}-\frac{s^2\pi^2}{12}\left(\frac{1}{3}+\frac{\beta\Delta q^2}{20}\right)+\cdots\,. \tag{33}$$

The latter allows us to probe how the generalized uncertainty principle affects majorization uncertainty relations for position and momentum.

3.3. Discussion

Thus, expressions have been obtained for the leading term in majorization uncertainty relations in the presence of a minimal length. As is seen from the right-hand side of Equation (33), for measurements with sufficiently high resolution the maximal eigenvalue grows with an increase in β. Hence, the lower bound of the uncertainty relation (15) will decrease. This tendency is interesting in comparison with other uncertainty relations. It also follows from Equation (33) that the effects of changing Δx and Δq on the actual level of uncertainty differ. The position bin governs $\mu_{\max}$ only via s, whereas the auxiliary-wavenumber bin Δq does also through the terms involving β. At the fixed s, changes of a typical size of momentum bins are more influential on the amount of uncertainties. Certainly, these findings are stipulated by the initial choice of the deformed commutation relation.

Substituting Equation (33) into Equation (15) leads straight to

$$H_\infty\big(\mathrm{p}^{\hat{x}}(\hat{\rho})\big)+H_\infty\big(\mathrm{p}^{\hat{\kappa}}(\hat{\rho})\big)$$
$$\geq\,2\ln 2-2\ln\left\{1+\sqrt{s}\left(1+\frac{\beta\Delta q^2}{24}-\frac{s^2\pi^2}{24}\left(\frac{1}{3}+\frac{\beta\Delta q^2}{20}\right)+\cdots\right)\right\}. \tag{34}$$

It is instructive to compare Equation (34) with the uncertainty relation,

$$H_1\big(\mathrm{p}^{\hat{x}}(\hat{\rho})\big)+H_1\big(\mathrm{p}^{\hat{\kappa}}(\hat{\rho})\big)\geq\ln\left(\frac{e\pi}{\Delta x\Delta\kappa}\right)+\big\langle\ln(\mathbf{1}+\beta\hat{\kappa}^2)\big\rangle_{\hat{\rho}}\,, \tag{35}$$

proved in Ref. [45]. By $H_1(\mathrm{p})$, the Shannon entropy of the corresponding probability distribution is meant. Since the left-hand side of Equation (34) includes the two min-entropies, it differs from entropic uncertainty relations of Refs. [45,50]. Therein, the entropic parameters are connected due to the use of inequalities between the corresponding norms of a function and its Fourier transform. Hence, the obtained inequalities cannot involve min-entropies for both the observables. In this regard, the discussion here completed the consideration of the paper [45]. Another difference is that the right-hand side of Equation (34) decreases with β, at least for high-resolution measurements. In contrast, the correction term in the right-hand side of Equation (35) increases. This distinction reflects that the min-entropies depend only on the maximal probabilities. Naturally, the scope of Equation (34) is restricted to measurements with sufficiently high resolution. In the meantime, only such measurements are recognized as capable to verify uncertainty relations of various forms.

A natural question arises about the right-hand side of Equation (33). What are, in order of magnitude, the terms depending on β? To answer the question, let us make some plausible assumptions about the typical values of Δx and Δq. Surely, they are mainly determined by the capabilities of modern experimental techniques. Any detailed discussion is beyond the scope of this paper. Instead, one can refer to the concrete experimental results of the verification of the Heisenberg uncertainty principle [56]. On average, typical bins can be estimated as $\Delta x \sim 100$ nm and $\Delta\kappa \sim 10^7$ 1/m. The latter also holds for Δq in view of $\kappa = \tan(\sqrt{\beta}q)/\sqrt{\beta}$ and given that

$$\sqrt{\beta} \sim \ell_{\mathrm{Pl}} = 1.616 \times 10^{-35}\ \mathrm{m}\,.$$

The calculations then give $s \sim 1/(2\pi) \approx 0.159$ and

$$\frac{\beta\Delta q^2}{s^2\pi^2} \sim 10^{-55}\,. \tag{36}$$

This value characterizes a ratio of the second term to the third one in the right-hand side of Equation (33). It is not surprising that the effects of the generalized uncertainty principle are estimated as extremely small.

Direct observational evidence for a foamed structure of space-time at the Planck scale seem to be currently unfeasible with an elementary particle as probe. In this way, the theoretical results of the form of Equation (34) are also unable to assist the presence of a minimal length in testing. Instead, paper [7] considered the use of a macroscopic probe for exploring space-time "roughness" at the relevant scale. It was found that, within the given level of ultrahigh vacuum and cryogenic technology, the proposed tabletop experiment could already be sensitive sufficiently. A witness for space-time "roughness" is provided the frequency of a certain event with a single photon turns out to be significantly less than the expected level. Applications of uncertainty relations to experiments of such a kind deserve to be studied in a separate investigation.

4. Conclusions

We have considered majorization uncertainty relations for position and momentum measurements in the presence of a minimal length. In particular, the uncertainty relation in terms of min-entropies was also formulated. In general, the proposed approach develops the treatment of Ref. [33] in combination with the generalized uncertainty principle. It was advisable from the physical viewpoint to focus on position and momentum measurements with sufficiently high resolution. In this way, one has derived corrections to leading terms of the majorization bound for the corresponding observables discretized into bins. Naturally, the changes of interest are determined by the parameter β that controls the modified commutation relation (1). The presented results allow us to reveal typical behavior and to estimate an order of corrections induced by the presence of a minimal length.

The obtained expressions with β are closely related to the structure of the modified commutation relation. These terms reveal some features that one could expect from the physical viewpoint. It is natural enough that lower bounds on uncertainty quantifiers will rather increase with the growth of β. The commutation relation (1) is indeed asymmetric in handling with position and momentum. The correction terms reflect this property, and the shortening of the momentum bins has a greater effect than the shortening of the position ones. At the same time, all of the mentioned changes lie in a so narrow range that they can hardly be probed within the capabilities of the modern experiment. A relative weight of the correction terms was estimated at a level undetectable in practice. It is apparent that a real successful experiment to show the presence a minimal length would be an exceptional advance.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Solution of the Eigenvalue Problem

It is useful to consider the eigenvalue problem in a form posed as

$$\int_{-1/2}^{+1/2} k(\xi,\xi')\, u(\xi')\, \mathrm{d}\xi' = \lambda\, u(\xi)\,, \tag{A1}$$

where the kernel is expanded as

$$k(\xi,\xi') = k^{(0)}(\xi,\xi') + \varepsilon\, k^{(1)}(\xi,\xi') + \varepsilon^2 k^{(2)}(\xi,\xi') + \cdots\,. \tag{A2}$$

The case (18) is gained for $\varepsilon = s^2$, $k^{(0)}(\xi,\xi') = 1$, $k^{(1)}(\xi,\xi') = A(\xi-\xi')^2$ with $A = -\pi^2/6$, and so on. The operator in the left-hand side of Equation (A1) is a Hilbert–Schmidt one, whenever

$$\int_{-1/2}^{+1/2} \mathrm{d}\xi \int_{-1/2}^{+1/2} \mathrm{d}\xi'\, \left|k(\xi,\xi')\right|^2 < \infty\,. \tag{A3}$$

It is known that Hilbert–Schmidt integral operators are both continuous and compact. The current study deals with symmetric real kernels, i.e., $k(\xi,\xi') = k(\xi',\xi)$ and $k(\xi,\xi') = k(\xi,\xi')^*$. Then, the integral operators of interest are self-adjoint.

For $\varepsilon = 0$, one obtains from Equation (17) the eigenvalue problem,

$$\int_{-1/2}^{+1/2} w(\xi')\, \mathrm{d}\xi' = \nu\, w(\xi)\,, \tag{A4}$$

with the eigenvalues $\nu_0 = 1$ and $\nu_1 = 0$. The function $w_0(\xi) = 1$ corresponding to $\nu_0 = 1$ is normalized as

$$\int_{-1/2}^{+1/2} w_0(\xi)^2 \mathrm{d}\xi = 1\,. \tag{A5}$$

The eigenvalue $\nu_1 = 0$ is degenerate, and there is a countably infinite set $\{w_n(\xi)\}_{n=1}^{\infty}$ of eigenfunctions such that

$$\int_{-1/2}^{+1/2} w_n(\xi)\, \mathrm{d}\xi = 0\,, \qquad \int_{-1/2}^{+1/2} w_m(\xi)\, w_n(\xi)\, \mathrm{d}\xi = \delta_{mn}\,, \tag{A6}$$

where δ_{mn} is the Kronecker delta. The first of these formulas implies $\langle w_0, w_n\rangle = 0$ for $n = 1,2,\ldots$ as well. One can recall here the Legendre polynomials $P_n(y)$ with the generating function,

$$\frac{1}{\sqrt{1-2yt+t^2}} = \sum_{n=0}^{\infty} P_n(y)\, t^n\,. \tag{A7}$$

Taking $y = 2\xi$, one can herewith write

$$w_n(\xi) = \sqrt{2n+1}\, P_n(2\xi) \qquad (n = 0,1,2,\ldots)\,. \tag{A8}$$

Apparently, the choice (A8) is not unique, but it is sufficient for the purposes here.

In general, the quantities of interest are represented by expansions,

$$\lambda_n = \lambda_n^{(0)} + \varepsilon\, \lambda_n^{(1)} + \varepsilon^2 \lambda_n^{(2)} + \cdots\,, \tag{A9}$$

$$u_n(\xi) = u_n^{(0)}(\xi) + \varepsilon\, u_n^{(1)}(\xi) + \varepsilon^2 u_n^{(2)}(\xi) + \cdots\,. \tag{A10}$$

For the problem (17), one has $\lambda_n^{(0)} = \delta_{n0}$ and $u_n^{(0)}(\xi) = w_n(\xi)$. One also deals with the Hilbert–Schmidt operators of the form,

$$\mathsf{K}^{(n)} f(\xi) = \int_{-1/2}^{+1/2} k^{(n)}(\xi, \xi')\, f(\xi')\, \mathrm{d}\xi' \,, \tag{A11}$$

$$\mathsf{K} = \mathsf{K}^{(0)} + \varepsilon\, \mathsf{K}^{(1)} + \varepsilon^2 \mathsf{K}^{(2)} + \cdots . \tag{A12}$$

Let us write the total equation as

$$\begin{aligned} &\big(\mathsf{K}^{(0)} + \varepsilon\, \mathsf{K}^{(1)} + \cdots\big)\big(u_n^{(0)}(\xi) + \varepsilon\, u_n^{(1)}(\xi) + \cdots\big) \\ &= \big(\lambda_n^{(0)} + \varepsilon\, \lambda_n^{(1)} + \cdots\big)\big(u_n^{(0)}(\xi) + \varepsilon\, u_n^{(1)}(\xi) + \cdots\big)\,, \end{aligned} \tag{A13}$$

whence

$$\mathsf{K}^{(1)} u_n^{(0)}(\xi) + \mathsf{K}^{(0)} u_n^{(1)}(\xi) = \lambda_n^{(1)} u_n^{(0)}(\xi) + \lambda_n^{(0)}\, u_n^{(1)}(\xi) \tag{A14}$$

in the first order. Due to self-adjointness, one finally obtains:

$$\lambda_n^{(1)} = \big\langle u_n^{(0)}, \mathsf{K}^{(1)} u_n^{(0)} \big\rangle\,. \tag{A15}$$

For the problem (17), the first-order correction to the eigenvalue reads as

$$\lambda_n^{(1)} = (2n+1) \int_{-1/2}^{+1/2} \mathrm{d}\xi \int_{-1/2}^{+1/2} \mathrm{d}\xi'\, k^{(1)}(\xi, \xi')\, P_n(2\xi)\, P_n(2\xi')\,.$$

For $n = 0$, this formula gives

$$\lambda_0^{(1)} = \int_{-1/2}^{+1/2} \mathrm{d}\xi \int_{-1/2}^{+1/2} \mathrm{d}\xi'\, k^{(1)}(\xi, \xi') = \frac{A}{6}\,, \tag{A16}$$

provided that $k^{(1)}(\xi, \xi') = A(\xi - \xi')^2$.

In order to examine Equation (29), one needs to know the corresponding eigenfunction of the problem (17). For $n = 0$, the first-order correction to the eigenfunction is obtained from $\langle w_0, u_0^{(1)} \rangle = 0$. The latter can be rewritten as

$$\mathsf{K}^{(0)} u_0^{(1)}(\xi) = \int_{-1/2}^{+1/2} u_0^{(1)}(\xi')\, \mathrm{d}\xi' = 0\,, \tag{A17}$$

since $w_0(\xi) = 1$ and $k^{(0)}(\xi, \xi') = 1$. Combining Equation (A14) with Equation (A17) finally gives

$$u_0^{(1)}(\xi) = \mathsf{K}^{(1)} w_0(\xi) - \lambda_0^{(1)} w_0(\xi)\,. \tag{A18}$$

By $k^{(1)}(\xi, \xi') = A(\xi - \xi')^2$, one has:

$$u_0^{(1)}(\xi) = A\left(\xi^2 + \frac{1}{12}\right) - \frac{A}{6} = A\left(\xi^2 - \frac{1}{12}\right). \tag{A19}$$

References

1. Rovelli, C. *Quantum Gravity*; Cambridge University Press: Cambridge, UK, 2004. [CrossRef]
2. Garay, L.J. Quantum gravity and minimum length. *Int. J. Mod. Phys. A* **1995**, *10*, 145–165. [CrossRef]
3. Hossenfelder, S. Minimal length scale scenarios for quantum gravity. *Living Rev. Relativ.* **2013**, *16*, 2. [CrossRef] [PubMed]
4. Amelino-Camelia, G.; Ellis, J.; Mavromatos, N.E.; Nanopoulos, D.V.; Sarkar, S. Tests of quantum gravity from observations of γ-ray bursts. *Nature* **1998**, *393*, 763–765. [CrossRef]
5. Jacob, U.; Piran, T. Neutrinos from gamma-ray bursts as a tool to explore quantum-gravity-induced Lorentz violation. *Nature Phys.* **2007**, *3*, 87–90. [CrossRef]
6. Pikovski, I.; Vanner, M.R.; Aspelmeyer, M.; Kim, M.; Brukner, Č. Probing Planck-scale physics with quantum optics. *Nature Phys.* **2012**, *8*, 393–397. [CrossRef]

7. Bekenstein, J.D. Is a tabletop search for Planck scale signals feasible? *Phys. Rev. D* **2012**, *86*, 124040. [CrossRef]
8. Marin, F.; Marino, F.; Bonaldi, M.; Cerdonio, M.; Conti, L.; Falferi, P.; Mezzena, R.; Ortolan, A.; Prodi, G.A.; Taffarello, L.; et al. Gravitational bar detectors set limits to Planck-scale physics on macroscopic variables. *Nature Phys.* **2013**, *9*, 71–73. [CrossRef]
9. Das, S.; Vagenas, E. Universality of quantum gravity corrections. *Phys. Rev. Lett.* **2008**, *101*, 221301. [CrossRef]
10. Ali, A.F.; Khalil, M.M.; Vagenas, E.C. Minimal length in quantum gravity and gravitational measurements. *EPL (Europhys. Lett.)* **2015**, *112*, 20005. [CrossRef]
11. Bousso, K.; Halpern, I.; Koeller, J. Mass bounds for compact spherically symmetric objects in generalized gravity theories. *Phys. Rev. D* **2016**, *94*, 064047. [CrossRef]
12. Howl, R.; Hackermüller, L.; Bruschi, D.E.; Fuentes, I. Gravity in the quantum lab. *Adv. Phys. X* **2018**, *3*, 1383184. [CrossRef]
13. Kempf, A.; Mangano, G.; Mann, R.B. Hilbert space representation of the minimal length uncertainty relation. *Phys. Rev. D* **1995**, *52*, 1108–1118. [CrossRef] [PubMed]
14. Pedram, P. New approach to nonperturbative quantum mechanics with minimal length uncertainty. *Phys. Rev. D* **2012**, *85*, 024016. [CrossRef]
15. Bosso, P. Rigorous Hamiltonian and Lagrangian analysis of classical and quantum theories with minimal length. *Phys. Rev. D* **2018**, *97*, 126010. [CrossRef]
16. Chung, W.S.; Hassanabadi, H. A new higher order GUP: One dimensional quantum system. *Eur. Phys. J. C* **2019**, *79*, 213. [CrossRef]
17. Bosso, P.; Luciano, G.G. Generalized uncertainty principle: From the harmonic oscillator to a QFT toy model. *Eur. Phys. J. C* **2021**, *81*, 982. [CrossRef]
18. Kim, M.S.; Hwang, M.-R.; Jung, E.; Park, D.K. Rényi and von Neumann entropies of thermal state in generalized uncertainty principle-corrected harmonic oscillator. *Mod. Phys. Lett. A* **2021**, *36*, 2150250. [CrossRef]
19. Iorio, A.; Pais, P.; Elmashad, I.A.; Ali, A.F.; Faizal, M.; Abou-Salem, L.I. Generalized Dirac structure beyond the linear regime in graphene. *Int. J. Mod. Phys. D* **2018**, *27*, 1850080. [CrossRef]
20. Heisenberg, W. Über den anschaulichen inhalt der quanten theoretischen kinematik und mechanik. *Z. Phys.* **1927**, *43*, 172–198. [CrossRef]
21. Kennard, E.H. Zur quantenmechanik einfacher bewegungstypen. *Z. Phys.* **1927**, *44*, 326–352. [CrossRef]
22. Robertson, H.P. The uncertainty principle. *Phys. Rev.* **1929**, *34*, 163–164. [CrossRef]
23. Hirschman, I.I. A note on entropy. *Amer. J. Math.* **1957**, *79*, 152–156. [CrossRef]
24. Beckner, W. Inequalities in Fourier analysis. *Ann. Math.* **1975**, *102*, 159–182. [CrossRef]
25. Białynicki-Birula, I.; Mycielski, J. Uncertainty relations for information entropy in wave mechanics. *Commun. Math. Phys.* **1975**, *44*, 129–132. [CrossRef]
26. Deutsch, D. Uncertainty in quantum measurements. *Phys. Rev. Lett.* **1983**, *50*, 631–633. [CrossRef]
27. Maassen, H.; Uffink, J.B.M. Generalized entropic uncertainty relations. *Phys. Rev. Lett.* **1988**, *60*, 1103–1106. [CrossRef] [PubMed]
28. Wehner, S.; Winter, A. Entropic uncertainty relations—A survey. *New J. Phys.* **2010**, *12*, 025009. [CrossRef]
29. Białynicki-Birula, I.; Rudnicki, Ł. Entropic uncertainty relations in quantum physics. In *Statistical Complexity*; Sen, K.D., Ed.; Springer: Dordrecht, The Netherlands, 2011; pp. 1–34. [CrossRef]
30. Coles, P.J.; Berta, M.; Tomamichel, M.; Wehner, S. Entropic uncertainty relations and their applications. *Rev. Mod. Phys.* **2017**, *89*, 015002. [CrossRef]
31. Toscano, F.; Tasca, D.S.; Rudnicki, Ł.; Walborn, S.P. Uncertainty relations for coarse-grained measurements: An overview. *Entropy* **2018**, *20*, 454. [CrossRef]
32. Hertz, A.; Cerf, N.J. Continuous-variable entropic uncertainty relations. *J. Phys. A Math. Theor.* **2019**, *52*, 173001. [CrossRef]
33. Partovi, M.H. Majorization formulation of uncertainty in quantum mechanics. *Phys. Rev. A* **2011**, *84*, 052117. [CrossRef]
34. Puchała, Z.; Rudnicki, Ł.; Życzkowski, K. Majorization entropic uncertainty relations. *J. Phys. A Math. Theor.* **2013**, *46*, 272002. [CrossRef]
35. Friedland, S.; Gheorghiu, V.; Gour, G. Universal uncertainty relations. *Phys. Rev. Lett.* **2013**, *111*, 230401. [CrossRef]
36. Rudnicki, Ł.; Puchała, Z.; Życzkowski, K. Strong majorization entropic uncertainty relations. *Phys. Rev. A* **2014**, *89*, 052115. [CrossRef]
37. Rastegin, A.E.; Życzkowski, K. Majorization entropic uncertainty relations for quantum operations. *J. Phys. A Math. Theor.* **2016**, *49*, 355301. [CrossRef]
38. Amati, D.; Ciafaloni, M.; Veneziano, G. Can spacetime be probed below the string size? *Phys. Lett. B* **1989**, *216*, 41–47. [CrossRef]
39. Scardigli, F. Generalized uncertainty principle in quantum gravity from micro-black hole gedanken experiment. *Phys. Lett. B* **1999**, *452*, 39–44. [CrossRef]
40. Bambi, C. A revision of the generalized uncertainty principle. *Class. Quantum Grav.* **2008**, *25*, 105003. [CrossRef]
41. Tawfik, A.N.; Diab, A.M. A review of the generalized uncertainty principle. *Rep. Prog. Phys.* **2015**, *78*, 126001. [CrossRef]
42. Ali, A.F.; Elmashad, I.; Mureika, J. Universality of minimal length. *Phys. Lett. B* **2022**, *831*, 137182. [CrossRef]
43. Dirac, P.A.M. Cosmological models and the large numbers hypothesis. *Proc. R. Soc. Lond. A* **1974**, *338*, 439–446.
44. Abdelkhalek, K.; Chemissany, W.; Fiedler, L.; Mangano, G.; Schwonnek, R. Optimal uncertainty relations in a modified Heisenberg algebra. *Phys. Rev. D* **2016**, *94*, 123505. [CrossRef]
45. Rastegin, A.E. On entropic uncertainty relations in the presence of a minimal length. *Ann. Phys.* **2017**, *382*, 170–180. [CrossRef]

46. Hsu, L.-Y.; Kawamoto, S.; Wen, W.-Y. Entropic uncertainty relation based on generalized uncertainty principle. *Mod. Phys. Lett. A* **2017**, *32*, 1750145. [CrossRef]
47. Srinivas, M.D. Optimal entropic uncertainty relation for successive measurements in quantum information theory. *Pramana J. Phys.* **2003**, *60*, 1137–1152. [CrossRef]
48. Distler, J.; Paban, S. Uncertainties in successive measurements. *Phys. Rev. A* **2013**, *87*, 062112. [CrossRef]
49. Rastegin, A.E. Entropic uncertainty relations for successive measurements of canonically conjugate observables. *Ann. Phys.* **2016**, *528*, 835–844. [CrossRef]
50. Rastegin, A.E. Entropic uncertainty relations for successive measurements in the presence of a minimal length. *Entropy* **2018**, *20*, 354. [CrossRef]
51. Bosso, P.; Petruzziello, L.; Wagner, F. The minimal length is physical. *Phys. Lett. B* **2022**, *834*, 137415. [CrossRef]
52. Bhatia, R. *Matrix Analysis*; Springer: New York, NY, USA, 1997. [CrossRef]
53. Partovi, M.H. Correlative capacity of composite quantum states. *Phys. Rev. Lett.* **2009**, *103*, 230502. [CrossRef]
54. Rastegin, A.E.; Shemet, A.M. Flavor-mass majorization uncertainty relations and their links to the mixing matrix. *Mod. Phys. Lett. A* **2021**, *36*, 2150211. [CrossRef]
55. Rényi, A. On Measures of Entropy and Information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (Berkeley/Los Angeles, CA, USA, June 20 – July 30, 1960). Volume 1: Contributions to the Theory of Statistics*; Neyman, J., Ed.; University of California Press: Berkeley/Los Angeles, CA, USA, 1961; pp. 547–561. Available online: https://projecteuclid.org/proceedings/berkeley-symposium-on-mathematical-statistics-and-probability/proceedings-of-the-fourth-berkeley-symposium-on-mathematical-statistics-and-probability-volume-1-contributions-to-the-theory-of-statistics/toc/bsmsp/1200512153 (accessed on 25 November 2022).
56. Nairz, O.; Arndt, M.; Zeilinger, A. Experimental verification of the Heisenberg uncertainty principle for fullerene molecules. *Phys. Rev. A* **2002**, *65*, 032109. [CrossRef]