physics

Opinion

Classical Limits of Light Quanta

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Abstract: It is argued that from a formal point of view, the classical limit of light quanta or photons is not that of a point-like particle but that of a geometric ray. According to this view, standard particle-wave dualism, which is often used in schools to describe the quantum behavior of massive objects, could be replaced by a ray-wave dualism (or even a particle-ray-wave trialism), which seems to be more appropriate for massless quantum objects such as photons. We compare the limits leading from quantum electrodynamics to a classical (Hamiltonian) theory of particles for electrons with those leading from photons via Maxwell's equations to geometric ray optics. We also discuss the question to which extent Maxwell's theory for electromagnetic waves should be considered as being on the same formal level as Schrödinger's or Dirac's theory.

Keywords: quantum theory; classical limits; photon; didactics of physics

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1. Introduction

There exist many different didactical approaches to introducing quantum theory in school. A common one is emphasizing the wave-like nature of electrons, e.g., in double slit experiments. Another one is pointing out the particle-like nature of light, e.g., when light of very low intensity is detected on scintillator screens or with high-resolution charged-couple device (CCD) cameras, or in the photoelectric effect or in Compton scattering. The wave-particle duality of any form of matter is introduced as one of the characteristic features of quantum theory.

However, in their educational curriculum, pupils are commonly introduced to ray optics first, before they are confronted with wave optics. Rays describe light in a form that appeals to our daily experience. Therefore, one of the naturally arising questions is, whether or not rays are a more appropriate model for "classical photons" instead of emphasizing their particle-like nature. It has been noted that the notion of photons as particles can lead to gross misconceptions; see [1–5].

This raises another important issue: What is the classical limit of photons—optical rays or Maxwell's theory of electromagnetism, i.e., essentially wave optics? While Maxwell's theory is considered "classical physics" for many reasons (some of them are summarized in Section 5), ray optics is a particular limit of this classical theory. However, on a formal mathematical level, Maxwell's theory should be compared to Schrödinger's or Dirac's theory of electrons, which are unequivocally quantum theories of electrons; see Figure 1.

Figure 1 essentially represents the content of this paper in a schematic form. On the highest level, we have quantum electrodynamics (QED), a quantum field theory. This is the level of second quantization, in which the Hilbert space is often represented as a Fock space (i.e., a space where the base states have definite particle numbers), and it is on this level that one can speak of electrons and photons as quantum objects. The dynamical variables are field operators (to be precise, these operators have to be smeared out with test functions, i.e., they are operator-valued distributions).

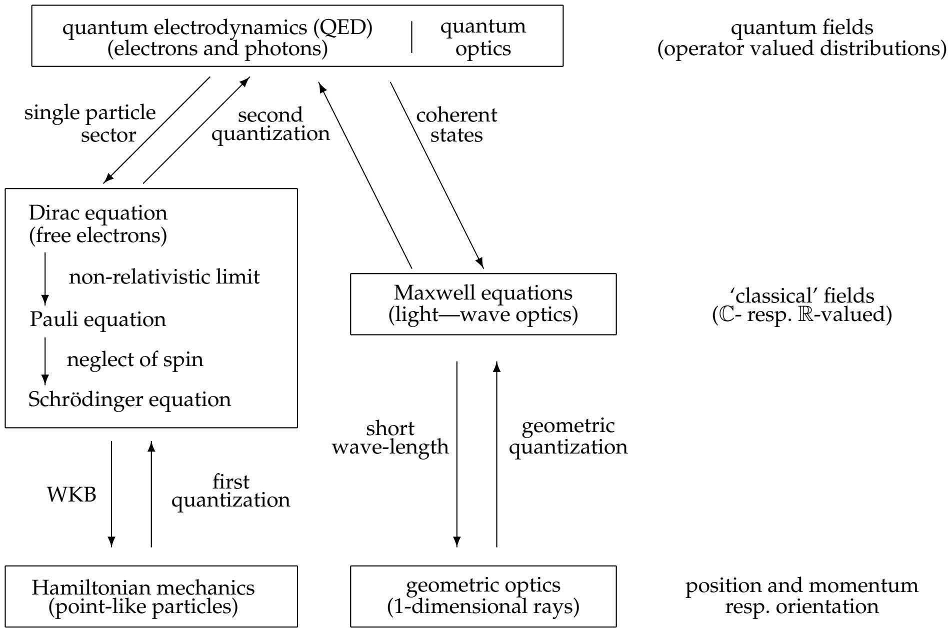

Figure 1. Comparison of the various levels of models for electrons and photons and the approximations from higher to lower levels, as well as the "quantization" procedures from lower to higher levels. For details, see text.

We place quantum optics on the same level, even though in standard applications QED and quantum optics look very different: While QED is more concerned with the scattering amplitudes of electrons, uses Dirac's interaction picture and perturbation theory as its main analytical tools, and treats photons mostly as virtual exchange particles, quantum optics prefers the Fock space representation, a non-Lorentz invariant choice of gauge, and deals mainly with non-classical properties of photon states, replacing electrons often by effective two-state systems which can absorb or emit photons.

On the middle level we have field theories. The dynamical degrees of freedom are real or complex-valued fields: the gauge field of electrodynamics and the spinor fields of Dirac's theory of electrons. While Dirac's theory for the electron (and its anti-particle, the positron) is unequivocally considered a quantum theory, Maxwell's theory for the electromagnetic field is generally considered a classical (field) theory. From a mathematical viewpoint, both theories are on the same level. However, from the viewpoint of physics and the physical interpretation of the fields, there are many differences, which are considered in Section 5. On the same formal level as Dirac's theory for the electron is Schrödinger's equation, which can be derived from Dirac's equation by taking a non-relativistic limit and neglecting the spin degree of freedom (if spin is included, one arrives at the Pauli equation).

The route from the middle level (field theory) to the highest level (quantum field theory) is second quantization: fields are replaced by operators satisfying canonical commutation relations. The way back from quantum field theory to field theory (i.e., from field operators to real or complex-valued fields), involves certain approximations: for electrons the restriction to the one-particle sector of the Fock space and for photons the restriction to so-called coherent states, i.e., eigenstates of the (annihilation part of) field operators. These relationships are discussed in more detail in the next section (Section 2).

On the lowest level, we find for the electrons a Newtonian (or rather Hamiltonian) theory of point-like particles. The classical degrees of freedom are position, $\boldsymbol{x}(t)$, and momentum, $\boldsymbol{p}(t)$, variables, which make up the classical phase space of particles. The step from here to Schrödinger's equation is (first) quantization: replacing the position and momentum variables by operators $\hat{Q}$ and $\hat{P}$ satisfying canonical commutation relations, replacing the classical Hamilton function $H(x,p)$ by a Hamilton operator $\hat{H}(\hat{Q},\hat{P})$ and postulating Schrödinger's equation for wave functions $\psi(x)$, which represent quantum states (throughout this paper operators are marked with a hat $\hat{}$). The other direction—the route from Schrödinger's equation to Hamiltonian mechanics, involves a certain approximation (the Wentzel–Kramers–Brillouin (WKB) approximation), which is essentially a short-wave approximation for the Schrödinger fields.

In a similar way, one can perform a short-wave approximation of the wave equation (or Maxwell's theory) for electromagnetic fields and arrive at geometrical optics or ray optics. This is the main reason why we propose to consider rays as the classical limit of light quanta and not point-like particles. There is no natural limit from Maxwell's theory to point-like particles. The transitions between the middle-level and lowest level, i.e., from Schrödinger to Newton on the one hand and from wave optics to ray optics, on the other hand, are considered in Section 3. What is less known is that geometric optics has a symplectic structure, similar to the phase space of Hamiltonian mechanics, and can be quantized. The quantization of ray optics leads to wave optics, i.e., back to the middle level. This is sketched in Section 4.2.

Section 5 deals with the question of whether Maxwell's theory is a quantum theory or a classical theory. Of course, eventually, this is a question of definitions and there are many convincing physical arguments to label Maxwell's theory a classical theory, however, there are also justified formal arguments to consider it as the quantized theory of geometric optics.

Finally, the conclusions (Section 6) summarize the arguments why the classical limit of photons may not only be described by a "particle-wave" duality, i.e., photons as particles (this model is often used to explain the photoelectric effect or Compton scattering) or as electromagnetic waves (explaining, e.g., interference and diffraction), but that there is a third option: classical photons as geometric rays, for example, when one considers light rays in optical systems such as lenses or mirrors.

Sections 2–4.2 are "technical sketches": they indicate the technical steps and relations between the various levels and theories. For more detailed information, the reader is referred to the literature. On the other hand, readers who are happy with the introductory remarks indicating the various relationships may also skip these sections and directly proceed to the last two sections.

2. Electrons, Photons, and the Electromagnetic Field

The starting point of our considerations is the Lagrange density of electrodynamics:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(x)\left[\gamma^{\mu}(\mathrm{i}\partial_{\mu} + eA_{\mu}(x)) - m\right]\psi(x)\,. \tag{1}$$

Here, $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the field tensor with the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ as components. The indices μ, $\nu = 0, 1, 2$, and 3, denote the time (0) and space coordinates. A_{μ} is the gauge field, $\psi(x)$ is a 4-component spinor field and $\bar{\psi}(x) = \psi^{\dagger}\gamma^{0}$ the adjoint spinor (with $\psi^{\dagger}$ the complex conjugated row vector corresponding to ψ), γ^{μ} are the γ-matrices satisfying anti-commutation relations $\{\gamma^{\mu}, \gamma^{\nu}\} = \eta^{\mu\nu}$ where $\eta^{\mu\nu}$ is the Minkowski bilinear form. e is the electric charge, m is the particle mass, and $\partial_{\mu} = \partial/\partial x_{\mu}$. In this Section only, the units are such that the Planck's reduced constant $\hbar = 1$ and the speed of light $c = 1$. The corresponding equations of motion (using Lorentz gauge $\partial_{\mu}A^{\mu} = 0$) are:

$$\Box A_{\mu}(x) = e\bar{\psi}(x)\gamma_{\mu}\psi(x) \qquad \text{and} \qquad \gamma^{\mu}(\mathrm{i}\partial_{\mu} - eA_{\mu})\psi(x) - m\psi(x) = 0\,. \tag{2}$$

The first equation is essentially Maxwell's equation expressed in terms of the gauge field and the electric current $j_{\mu} = e\bar{\psi}(x)\gamma_{\mu}\psi(x)$, the second equation is Dirac's equation.

2.1. Quantum Electrodynamics

The quantization of this field theory leads to QED: The fields are replaced by operators satisfying the above equations of motion and certain commutation relations. The free fields, in terms of which the theory is formulated perturbatively, can be expressed in terms of creation and annihilation operators in the following form:

$$\begin{aligned}
\hat{A}_\mu(x) &= \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\omega_k}} \sum_{\alpha=1}^{2} \epsilon_{\alpha\mu} \left(\hat{a}_\alpha(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} + \hat{a}^\dagger_\alpha(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx} \right) \\
\hat{\Psi}(x) &= \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\omega_k}} \sum_{s=\pm} \left(\hat{b}_s(\boldsymbol{k}) u(\boldsymbol{k},s) \mathrm{e}^{\mathrm{i}kx} + \hat{d}^\dagger_s(\boldsymbol{k},s) v(\boldsymbol{k},s) \mathrm{e}^{-\mathrm{i}kx} \right) \\
\hat{\Psi}^\dagger(x) &= \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\omega_k}} \sum_{s=\pm} \left(\hat{b}^\dagger_s(\boldsymbol{k}) \bar{u}(\boldsymbol{k},s) \mathrm{e}^{-\mathrm{i}kx} + \hat{d}_s(\boldsymbol{k},s) \bar{v}(\boldsymbol{k},s) \mathrm{e}^{\mathrm{i}kx} \right) \\
& k = (\omega_k, \boldsymbol{k}) \,, \quad \omega_k = \sqrt{\boldsymbol{k}^2 + m^2} \,, \quad x = (t, \boldsymbol{x}) \\
& kx = \boldsymbol{k} \cdot \boldsymbol{x} - \omega_k t \,,
\end{aligned}$$

where $u(\boldsymbol{k},s), v(\boldsymbol{k},s)$ are solutions of Dirac's equation and $\epsilon_{\alpha\mu}$ are two polarization vectors (the technical details are slightly more complex: in order to keep Lorentz invariance, one has to choose four polarization vectors (i.e., $\alpha = 0, \ldots, 3$) and, correspondingly, four annihilation and creation operators. In the end, one has to show that two of these—corresponding to a longitudinal photon and a pure time-like photon—do not contribute. However, such technical details shall not concern us here and the interested reader is referred to the standard textbooks on quantum field theory) of the gauge fields (orthonormal to each other and orthogonal to the momentum $\boldsymbol{k}$). $\hat{a}_\alpha(\boldsymbol{k})$ and $\hat{a}^\dagger_\alpha(\boldsymbol{k})$ are annihilation and creation operators for field modes with momentum $\boldsymbol{k}$ and polarisation ϵ_α, respectively. These field modes created by $\hat{a}_\alpha(\boldsymbol{k})^\dagger$ are called photons. In a similar way, $\hat{d}^\dagger_s(\boldsymbol{k})$ and $\hat{b}^\dagger_s(\boldsymbol{k})$ are creation operators for positrons and electrons, respectively, $\hat{d}_s(\boldsymbol{k})$ and $\hat{b}_s(\boldsymbol{k})$ are the corresponding annihilation operators. The index s accounts for the spin degree of freedom. m is zero for the gauge field and equals the electron mass for the spinor fields.

If we denote by $|\Omega\rangle$ the vacuum state of the theory, i.e., the state of lowest energy, then

$$|\alpha^\mu\rangle = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \alpha^\mu(x) \hat{A}^+_\mu(x) |\Omega\rangle \tag{3}$$

$$\text{with} \quad \hat{A}^+_\mu(x) = \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\omega_k}} \sum_{\alpha=1}^{2} \epsilon_{\mu\alpha} \left(\hat{a}^\dagger_\alpha(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx} \right) \tag{4}$$

describes a one-photon state with wave function $\alpha^\mu(x)$. In a similar way

$$|\psi\rangle = \int_{\mathbb{R}^4} \mathrm{d}^4x \, \psi(x) \hat{\Psi}^+(x) |\Omega\rangle \tag{5}$$

$$\text{with} \quad \hat{\Psi}^+(x) = \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\omega_k}} \sum_{s=\pm} \left(\hat{b}^\dagger_s(\boldsymbol{k},s) u(\boldsymbol{k},s) \mathrm{e}^{-\mathrm{i}kx} \right) \tag{6}$$

describes a single electron state with wave function $\psi(x)$.

All these expressions are quite formal and many technical details are left out. The interested reader is referred to the standard literature on quantum field theory and second quantization (e.g., [6–10]).

2.2. Quantum Optics

We consider quantum optics as a special branch of QED. In particular, equations such as Equations (3) and (4) remain true. In contrast to QED, which in its standard applications concentrates on n-point functions and scattering amplitudes for electrons and positrons, quantum optics is mainly concerned with the non-classical behavior of photonic states, i.e., with deviations of photonic states from the predictions of Maxwell's theory. This non-classical behavior is mostly due to the bosonic statistics of photons—a paradigm being the Hong-Ou-Mandel effect [11]—or due to the discrete nature of photons (see, e.g., [12]). A related application consists of the non-classical properties of intensity correlation functions (see, e.g., [13]).

Closest to the classical electromagnetic field are so-called "coherent" states, which are described by a Poisson-distribution in photon number. For simplicity, we describe these states for a given mode and polarization only, i.e., for a single harmonic oscillator.

Let us define the displacement operator

$$\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}\right) . \tag{7}$$

Acting on the vacuum state yields

$$|\alpha\rangle = \hat{D}(\alpha)|\Omega\rangle , \tag{8}$$

which is an eigenstate of the annihilation operator $\hat{a}$:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle , \tag{9}$$

and for the probability w_n of finding n photons in such a state, one obtains a Poisson distribution:

$$w_n = \exp\left(-|\alpha|^2\right)\frac{|\alpha|^{2n}}{n!} . \tag{10}$$

$|\alpha\rangle$ represents a state, for which the amplitude is given by a Gaussian distribution around the value α. It is like the ground state of a harmonic oscillator (which is a Gaussian distribution around 0) shifted to the value α. In a formal classical limit these states become sharply distributed around their classical values.

Coherent states often have similar properties as single photon states. In particular, the superposition principle is the same: The superposition of two single-photon states with, say, different polarizations or originating from different slits in a double slit experiment yields a single-photon state with the superimposed polarization or the superimposed wave-function. The same relations hold for coherent states. Only when the discreteness of photon states becomes relevant is the difference obvious (see, e.g., [12]).

Again, for details, the interested reader is referred to the literature (e.g., [14,15]).

3. The Classical Limits of Field Theory

The route from field theory to the classical requires several steps. The Dirac equation is a relativistic equation for electrons and their antiparticles, the positrons, while the Schrödinger equation is a non-relativistic equation for (spinless) electrons. The intermediate steps are (i) a non-relativistic approximation for the solutions of Dirac's equation, which leads to the Pauli equation, and (ii) the neglect of the spin degree of freedom leading to Schrödinger's equation. From Schrödinger's equation there are several (mostly equivalent) approximations, the most known being the WKB approximation, leading to Newtonian mechanics and the classical picture of an electron as a point particle.

The photon part of field theory is essentially Maxwell's theory. Neglecting the "spin" (i.e., the polarization) leads to the wave equation for the gauge field and, essentially, wave optics. A non-relativistic limit is not possible here, mainly because the photon is a massless particle which propagates with the velocity of light. Quite often the opposite limit, $c \to \infty$ is taken and any temporal dependence eliminated. However, the "classical" limit here corresponds to a short-wavelength approximation, leading to geometrical optics.

These limits are briefly described in the following sections. For technical details, the reader is again referred to the literature: for the steps from Dirac to Schrödinger, e.g., [6], for the WKB approximation, e.g., [16–18], and for the short-wave approximation, e.g., [15].

3.1. From Dirac to Schrödinger

If we multiply the Dirac equation with γ_0 it assumes formally the form of a Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi(x) = \hat{H}\psi(x) \qquad \text{with} \qquad \hat{H} = \boldsymbol{\gamma}\cdot(c\,\mathrm{i}\hbar\boldsymbol{\nabla} + e\boldsymbol{A}) + \gamma_0 mc^2 + e\phi\,. \tag{11}$$

Here, $A_\mu \simeq (\phi, \boldsymbol{A})$ and $\gamma_\mu \simeq (\gamma_0, \boldsymbol{\gamma})$. In a base where γ_0 is diagonal, the solutions for small momenta of the four components of the Dirac spinor $\psi(x)$ can be associated with the electron (two components taking care of the spin and the spatial degrees of freedom) and the positron, respectively. By making the ansatz $\psi(x) \to \tilde{\psi}(x) = \mathrm{e}^{-\mathrm{i}mc^2 t/\hbar}\psi(x)$, which essentially shifts the energy scale for the new spinor $\tilde{\psi}(x)$ to the rest energy $E_\mathrm{e} = m_\mathrm{e}c^2$ of the electron, and neglecting contributions to the energy, which are of the order $2m_\mathrm{e}c^2$, one obtains the Pauli equation for two of the components:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\tilde{\psi}(x) = \left(\frac{1}{2m}\left(\mathrm{i}\hbar\boldsymbol{\nabla} - \frac{e}{c}\boldsymbol{A}(x)\right)^2 + e\phi(x) + \mu_\mathrm{B}\boldsymbol{\sigma}\cdot\boldsymbol{B}(x)\right)\tilde{\psi}(x)\,. \tag{12}$$

σ_i are the Pauli matrices. The other two components of $\tilde{\psi}$ effectively vanish (in the sense of distributions, i.e., they oscillate so rapidly due to their high energies that they do not give contributions when smeared with 'low energy' test functions). If the external magnetic field vanishes, we obtain Schrödinger's equation:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\tilde{\psi}(x) = \left(-\frac{\hbar^2}{2m}\Delta + e\phi(x)\right)\tilde{\psi}(x)\,. \tag{13}$$

Now $\tilde{\psi}$ can be considered a one-component complex field.

3.2. From Schrödinger to Newton

There are several, largely equivalent routes from Schrödinger's equations to Newton's equations or rather to Hamiltonian mechanics. One method is the WKB approximation. One way to think of WKB is that the wavelength λ associated to a given momentum $\boldsymbol{p}$ by deBroglie's relation $|\boldsymbol{p}| = h/\lambda$ is small compared to distances, over which the potential $V(x)$ varies, so that locally one has a box-potential. So, effectively, this is a small wave-length approximation. Making the following ansatz for the wave function,

$$\psi(x) = \exp\left(\frac{\mathrm{i}}{\hbar}S(\boldsymbol{x}, t)\right), \tag{14}$$

and inserting this ansatz into Schrödinger's equation, one obtains the following differential equation for $S(\boldsymbol{x}, t)$:

$$-\frac{\partial S}{\partial t} = \frac{1}{2m}\left((\boldsymbol{\nabla}S)^2 - \mathrm{i}\hbar\Delta S(\boldsymbol{x}, t)\right) + V(\boldsymbol{x})\,. \tag{15}$$

If S is a slowly varying function (or, less formally, if we neglect terms proportional to $\hbar$), this becomes the classical Hamilton–Jacobi equation:

$$\frac{\partial S_0}{\partial t} + \frac{1}{2m}(\boldsymbol{\nabla}S_0)^2 + V(\boldsymbol{x}) = 0\,. \tag{16}$$

The classical trajectories are orthogonal to the planes of constant phase and are determined by

$$\boldsymbol{p}(t) = \boldsymbol{\nabla}S(\boldsymbol{x}(t), t) \qquad \text{or} \qquad \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \frac{1}{m}\boldsymbol{\nabla}S(\boldsymbol{x}(t), t)\,. \tag{17}$$

For the position $\boldsymbol{x}(t)$, one obtains (see, e.g., [19]):

$$m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2} \;=\; -\boldsymbol{\nabla} V(\boldsymbol{x}(t))\,. \tag{18}$$

3.3. From Maxwell to Ray Optics

In complete analogy to the WKB approximation for Schrödinger's equation, one can make the ansatz,

$$\boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{e}(\boldsymbol{x})\mathrm{e}^{\mathrm{i}k(S(\boldsymbol{x})-ct)}\,, \qquad k = \frac{2\pi}{\lambda}\,, \tag{19}$$

for the electric field (and a similar ansatz with the same phase for the magnetic field). For large values of k (or small wave-lengths λ), one can make a formal expansion of the phase $S(\boldsymbol{x})$ and the amplitude $\boldsymbol{e}(\boldsymbol{x})$ in powers of $1/k$. For the leading term S_0 in this expansion, Maxwell's equations lead to the equation

$$\boldsymbol{\nabla} S_0(\boldsymbol{x}))^2 = n^2(k,\boldsymbol{x})\,, \tag{20}$$

where $n(k,\boldsymbol{x})$ is the wavelength-dependent optical density of the medium. This is the famous eikonal equation leading to geometrical optics and Fermat's principle. One obtains trajectories of rays as the solutions of

$$\frac{\mathrm{d}\boldsymbol{x}(s)}{\mathrm{d}s} = \frac{1}{n(k,\boldsymbol{x}(s))}\boldsymbol{\nabla} S_0(\boldsymbol{x}(s))\,, \tag{21}$$

where the arc-length parametrization is used. This equation should be compared with Equation (17). For more details see, e.g., [15].

4. The Quantization of Hamiltonian Mechanics and Ray Optics

"Quantization" of Newtonian mechanics leads to quantum mechanics and Schrödinger's equation for the quantum states of, e.g., electrons. This is the subject of any course on quantum mechanics. However, ray optics can be quantized along similar lines leading to wave optics. This is less known and we briefly sketch the idea, referring the interested reader to the literature (e.g., [20]).

4.1. A Different View Onto Quantization of Hamiltonian Mechanics

The state space of Hamiltonian classical mechanics, i.e., the phase space, has a natural symplectic structure. This structure reveals itself, e.g., in the Poisson brackets, with the fundamental relation $\{q,p\} = 1$. One way to formulate the quantization procedure is to replace q and p (generalized position and momentum) by operators $\hat{Q}$ and $\hat{P}$ and to require these operators to satisfy canonical commutation relations $[\hat{Q},\hat{P}] = \mathrm{i}\hbar$.

There is, however, a different perspective onto the same procedure. The Hamiltonian equations of motion define symplectic diffeomorphisms ϕ_t, i.e., area preserving differentiable mappings of phase space onto itself: to each point (q,p) of phase space is associated the point $\phi_t(q,p) = (q(t),p(t))$, i.e., the point in phase space, where an object, which initially was at (q,p), will be after a time t. Liouville's theorem tells us that volumes in phase space will be preserved. These symplectic diffeomorphisms have a monoid structure with respect to t in the sense that $\phi_{t_2}\circ\phi_{t_1} = \phi_{t_2+t_1}$ and $\phi_0 = \mathbf{1}$. These diffeomorphisms are generated by Hamilton's equations of motion: $(\dot{q},\dot{p}) = (\partial_p H, -\partial_q H)$.

Quantization can now be formalized by looking for unitary representations of this symplectic diffeomorphisms on a Hilbert space. In quantum mechanics these are the unitary time evolution operators $\hat{U}(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)$. They satisfy Schrödinger's equation

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}t}\hat{U}(t) = \hat{H}\hat{U}(t) \qquad \hat{U}(0) = \mathbf{1}\,. \tag{22}$$

4.2. Wave Optics as the Quantization of Ray Optics

Let us first show that geometric optics can also be given a symplectic structure. Then, we indicate how this symplectic structure can be quantized.

Let us denote by $x(z)$ the point, where a geometric ray punctures a plane orthogonal to the z-direction, which is taken to be the propagation direction of the ray. Furthermore, we denote by $p(z)$ the tangent of the angle (between the z- and the x-direction) of the ray (see Figure 2). We can now represent the configuration of a ray by a two-dimensional vector $\begin{pmatrix} x(z) \\ p(z) \end{pmatrix}$. For simplicity, we consider this as a one-dimensional problem. In general, $x(z)$ and $p(z)$ will each be two-dimensional vectors (the location in the plane at point z in the propagation direction of the ray and the two tangents specifying the orientation of the ray at this plane) leading to a four-dimensional formalism.

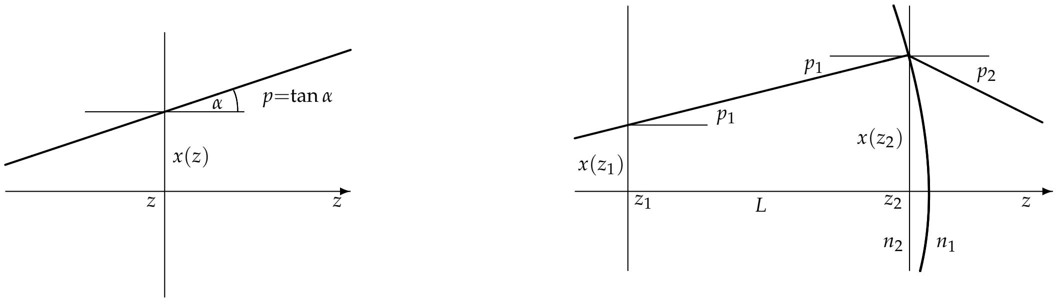

Figure 2. (**Left**): Characterization of a light ray by the parameters x and p. This refers to one dimension. In a plane orthogonal to the z-direction one has two parameters for the location and two parameters for the orientation of the ray at a position z. (**Right**): The transformations "translation" (T) by a length L and "defraction" (D) at an interface between two different optical densities n_1 and n_2. For details see text.

Essentially, there are two transformations which a ray can be subject to: (T) a "translation" describes the change in parameters when the ray simply propagates for a certain distance L without being deflected, and (D) a "defraction" of the ray, e.g., at an interface between two different optical densities n_1 and n_2. Both transformations can be described by a matrix:

$$T = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \tag{23}$$

where L is the distance by which the ray propagates in a medium of constant optical density, and $B = (n_2 - n_1)/R$ is the "refractivity" of the transition from an optical medium with density n_1 to an optical medium with optical density n_2, while R is the radius of the curvature of the interface between these two media.

Both matrices, T and D, are symplectic: they have determinant 1 and a similarity transformation with these matrices leaves the antisymmetric matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ invariant. Therefore, one finds a symplectic structure in geometric optics.

There is a general procedure to quantize spaces with a symplectic structure. Again, one looks for infinite dimensional unitary representations of the symplectic propagation matrices in a Hilbert space of square integrable functions. The time parameter t is now replaced by the propagation axis z (formally this corresponds to a limit $c \to \infty$ such that a time parameter is not involved; the transition to a relativistic formulation with finite c is an additional structure—in parallel to the transition from the non-relativistic Schrödinger equation to a relativistic Dirac formalism). The two matrices T and D are represented by unitary operators $\hat{\rho}(T)$ and $\hat{\rho}(D)$ for which the action on a square integrable function

$f(x;z)$ (square integrable in the argument x; z is an external parameter like t in quantum mechanics) is defined by

$$(\hat{\rho}(T)f)(x;z+L) = \exp\left(-\frac{\mathrm{i}\pi n}{4}\right)\sqrt{\frac{k}{2\pi L}}\int \exp\left(\mathrm{i}k\frac{(x-y)^2}{2L}\right)f(y;z)\,\mathrm{d}y \quad (24)$$

$$(\hat{\rho}(D)f)(x;z) = \exp\left(-\mathrm{i}k\frac{xBx}{2}\right)f(x;z)\,, \quad (25)$$

k being an open parameter. Note that $\hat{\rho}(T)$ essentially represents the unitary time evolution operator for a free particle in quantum mechanics with t replaced by z (and k by $\hbar$). In quantum mechanics, $\hat{\rho}(D)$ would correspond to a collision where the momentum is changed abruptly. One can easily show that these two unitary operators satisfy the same relations as the matrices T and D and, therefore, are a representation of these symplectic transformations. The applications of these formulae to various apertures leads to the integrals of Fresnels's wave optics (for details, see, e.g., [20]).

Note that the wavelength appears as a free parameter in this quantization procedure, similar to Planck's constant in quantum mechanics. Geometric ray optics does not contain the wavelength of light. Vice versa, the wavelength disappears in the leading terms of a short-wavelength approximation, which essentially is an expansion in terms of the wavelength, in a similar way as $\hbar$ disappears in a WKB approximation, which can be formulated as an expansion in terms of $\hbar$.

5. Maxwell's Theory—A Classical Theory of Electromagnetic Fields or a Quantum Theory of Ray Optics?

There are many arguments in favor of Maxwell's theory being a classical theory. Of course, as long as "classical theory" is not defined, the header question remains meaningless. Furthermore, arguably an "and" would be more appropriate in the header of this section than an "or". We list a few arguments pro and contra the assignment of "classical" or "quantum" to Maxwell's theory.

5.1. Maxwell's Equations Do Not Contain Planck's Constant

This is presumably the argument most often used by protagonists of Maxwell's theory being a classical theory. In a quantum theory $\hbar$ should appear.

The main reason that $\hbar$ does not appear in Maxwell's theory is that we are dealing with a relativistic theory of massless objects. Energy E and momentum $\boldsymbol{p}$ appear with the same power in the relativistic relation,

$$E^2 = m^2c^4 + \boldsymbol{p}^2c^2\,. \quad (26)$$

Replacing E and $\boldsymbol{p}$ by ω and $\boldsymbol{k}$ using deBroglie's relations yields:

$$\hbar^2\omega^2 = m^2c^4 + \hbar^2\boldsymbol{k}^2c^2\,, \quad (27)$$

and dividing this equation by $\hbar^2$ leaves Planck's constant only in the mass term (which essentially becomes Compton's wavelength for an object of mass m). For $m = 0$, Planck's constant vanishes from the equations. In other words, the (massless) wave equation,

$$\frac{1}{c^2}\frac{\partial^2 A_\mu(x)}{\partial t^2} - \Delta A_\mu(x) = 0\,, \quad (28)$$

contains no $\hbar$, because the $\hbar$-terms drop out of the dispersion relation

$$\frac{1}{c^2}\hbar^2\omega^2 = \hbar^2k^2\,. \quad (29)$$

For the same reason, Dirac's equation for massless spin-$\frac{1}{2}$ fermions contains no $\hbar$. Does that make massless neutrinos (as in the standard model) classical objects?

In a non-relativistic theory such as Schrödinger's theory, the energy-momentum dependence is not homogeneous: it is the classical relation

$$E = \frac{1}{2m}\boldsymbol{p}^2 + V(\boldsymbol{x})\,, \tag{30}$$

and using deBroglie's relations necessarily leaves us with $\hbar$s:

$$\hbar\omega = \frac{\hbar^2}{2m}\boldsymbol{k}^2 + V(\boldsymbol{x})\,, \tag{31}$$

which becomes Schrödinger's equation when we require this relation to hold for a wave $\psi(\boldsymbol{x})$ of frequency ω and wave vector $\boldsymbol{k}$.

5.2. The Fields in Maxwell's Theory Have no Interpretation in Terms of a Born Rule

The interpretation of fields (neither the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ nor the gauge field A_μ) is not that of "a probability amplitude of finding a photon". In other words, the absolute squares of these fields do not define a probability density for events in analogy to the Born rule in quantum mechanics. In fact, it can be shown (see, e.g., [21]) that no such relativistically invariant probability density can be constructed from these fields. There are several differences between the electronic part and the photonic part of the field theory which are responsible for this.

One of the reasons is that the relation between energy E and "number of photons" n depends on the frequency: $E = n\hbar\omega$. Furthermore, states which have a well-defined electric or magnetic field do not have a well-defined photon number. Indeed, the coherent states in quantum optics have a Poisson distribution of photon number. Expressed differently, the field operators and the number operator for photons do not commute.

On the other hand, there are relationships between the gauge field $\boldsymbol{A}$ or the electric field $\boldsymbol{E}$ and the number of photons (see, e.g., [16]). When we are dealing with fixed frequencies and consider only relative quantities (i.e., not absolute normalizations), we can interpret the absolute square of the gauge potential or the electric field as "relative frequency" and, eventually, as a probability. This holds in particular for polarization experiments, where the vector character of the electric field or the vector potential can be used as a two-dimensional probability amplitude and the absolute square (properly normalized) as a probability for "detecting an event".

5.3. Many Systems of Single Objects vs. One System of Many Objects

Quantum mechanics makes predictions about probabilities, at least in the Copenhagen interpretation. One can test such predictions by measuring relative frequencies. Ensemble interpretations of quantum mechanics claim that quantum states refer to ensembles—equally prepared systems—only [22]. The interference fringes of electrons in a double slit experiment [23] are obtained by letting many single electrons prepared with the same momentum pass through a double slit, preferably in such a way that they cannot influence each other, i.e., one at a time.

One might argue that for photons it should be possible that a single system with many photons prepared with the same quantum numbers might also allow to test probability predictions as relative frequencies. Photons are bosons and many photons can be in the same quantum state, which is not possible for electrons because of the Pauli principle. Furthermore, in contrast to electrons, photons do not interact (quantum corrections can effectively lead to photon-photon interactions via virtual electron-positron loops, but these effects are negligible and do not spoil the argument). Therefore, a single multi-particle state of photons with identical quantum numbers allows for the measurement of relative frequencies of photons in form of relative intensities. This might replace the preparation of ensembles of many single-particle states of photons. Even though correct in principle, the question remains to which extent states of classical fields in Maxwell's theory really represent such states of identical photons.

As mentioned above, this is not the case. States with more or less sharply defined field strengths correspond to coherent states of many photons. These states are superpositions of photon states with an arbitrary number of photons and do not describe a many-photon state with a definite photon number—they are not eigenstates of the photon number operator. Repeated measurements of relative frequencies in such systems do not yield the same numbers. This refers to the coherent states of laser light. Thermal light is even more diverse. Up to now, there are no good methods to prepare many-photon states with macroscopic numbers of photons in eigenstates of the number operator. Such states would exhibit an extreme quantum nature and they would differ from the states corresponding to well-defined field strengths in Maxwell's theory.

Essentially, this leads to the same conclusion as Section 5.2: the electromagnetic fields or the vector potentials do not represent wave functions for identically prepared photons, for which the integral over the absolute square has the interpretation of relative frequencies of finding definite numbers of photons in a well-defined volume. However, in the limit of large photon number N, where fluctuations can be neglected and for fields in modes with well-defined frequencies, we can obtain relative photon numbers from relative intensities of these fields. A known example being the interference fringes of monochromatic light behind a double-slit which can be calculated from the intensities of electric field strengths and which are proportional to the relative abundancies of photons.

However, the limit from macroscopic numbers of photons (of the order of 10^{18}–10^{20} photons per second in a laser pointer) to single photon sources is far from trivial. The bosonic nature of photons leads to bunching effects which make it difficult to prepare states with definite photon numbers—even photon number $n = 1$ (see, e.g., [24]).

5.4. Wave Optics is Obtained from Geometrical Optics by Geometric Quantization

As has been discussed above (Section 4.2), a formal quantization of the symplectic structure of geometrical optics leads to wave optics. In this sense it is fair to say that wave optics is the quantized theory of geometrical optics, but does "quantized theory" imply that it is a "quantum theory"?

Not necessarily: The fact that the formal mathematical relationship between geometrical optics and wave optics is the same as that between Hamiltonian mechanics and quantum mechanics does not make wave optics a quantum theory. However, wave optics has many of the usual essences or characteristic traits (the German expression "Wesenzug" has been used by [25]) of a quantum theory, such as the superposition principle, interference effects, and the quantization of modes for finite-size systems.

6. Conclusions

The mathematical analogies, discussed throughout the paper, indicate an alternative way to introduce the quantum mechanics of photons using classical models. Whether this method is adequate for schools or only for a university curriculum may be a matter of debate. Hitherto, the phenomenon of detecting single photons as described in the introduction is interpreted in schools as proof of the particle nature of light. However, these phenomena could also be explained by an indivisible ray. The photoelectric effect does not require photons to be point-like: it requires that a quantized amount of energy is transmitted locally to a single atom [2,4], which can also be explained by a ray. The Compton effect might be closer to the scattering of particles, but it can also be explained using rays to which we can assign a wave-length dependent momentum and energy. Moreover, when we refer to the point-like detection of photons on a photographic plate, this is also well explained by a ray which hits this plane.

When talking about "elementary" rays, corresponding to single light-quanta, it might be useful to introduce the notion of rays of finite length. A typical length could be in the range of meters based on the coherence length for monochromatic light. This would account for all events which are observed at "definite" times. It should be noted, however, that the notion of a coherence length is beyond the concepts of classical ray optics.

This alternative approach has some advantages:

1. The comparison of the various limits shown in Figure 1 results in a mathematically consistent and symmetric picture. From a mathematical point of view, one could argue that it makes sense to speak of particles in the case of electrons and of rays in the case of light.
2. In contrast to point-like particles, the concept of a one-dimensional ray circumvents the problem of constructing a three-dimensional probability density from the electromagnetic field (or the potential). It has been argued (see, e.g., [21]) that this is not possible and indeed, as we have indicated above, the relationships between the absolute square of these fields and the (average) number of photons depends on the frequency of the fields.
3. Massive point-like particles have a rest frame and the remaining invariance is the rotation group SO(3). This leads to the usual classification of particles according to their spin. In particular, spin-1 particles have three-dimensional representations for this degree of freedom. Photons however, being massless, have no rest frame and their invariance corresponds to the group SO(2) (for which spin-1 representations are two-dimensional). Rays also have this invariance group of rotations around the ray axis.
4. In particular, in the class room, the analogies between electrons and photons are often emphasized to such an extent that for students they appear to be similar objects [26] and the differences are not sufficiently considered. Different visualizations may prevent such identifications and help to illuminate these differences.

Of course, photons are objects of quantum electrodynamics (QED) and one may argue that classical models are inappropriate to describe quantum mechanical systems anyway. However, there are good reasons not to teach QED in school. Like any model, the notion of a photon as a wave, a particle, or an elementary ray has limitations. However, the notion of a ray allows us to evade the dichotomy of a particle-wave dualism, which in our opinion may be more appropriate for electrons instead for photons. Whether a ray-wave duality or a particle-ray-wave triality is more appropriate depends on the situations for which these models are used.

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