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Spin Current in *BF* Theory

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Abstract: In this paper, a current that is called spin current and corresponds to the variation of the matter action in *BF* theory with respect to the spin connection A which takes values in Lie algebra $\mathfrak{so}(3,\mathbb{C})$, in self-dual formalism is introduced. For keeping the 2-form B^i constraint (covariant derivation) $DB^i = 0$ satisfied, it is suggested adding a new term to the *BF* Lagrangian using a new field ψ^i, which can be used for calculating the spin current. The equations of motion are derived and the solutions are dicussed. It is shown that the solutions of the equations do not require a specific metric on the 4-manifold M, and one just needs to know the symmetry of the system and the information about the spin current. Finally, the solutions for spherically and cylindrically symmetric systems are found.

Keywords: *BF* theory; local Lorentz symmetry; local Lorentz currents

1. Introduction

The *BF* theory on 4-manifold M is a topological theory, which includes constraints when terms turn to gravity theory [1]. The fundamental variables are 2-form $B \in \Omega^2(M;\mathfrak{so}(3,1))$ and spin connection ω, which takes values in Lie algebra $\mathfrak{so}(3,1)$, and all derivatives are linear and applied only on ω, which makes it easy for canonical formalism, finding the phase space, Hamiltonian equations, quantization, etc. [1]. This theory does not require a metric to be formulated, as the metric is a derived quantity from the solutions of B. That gives motivation to formulate Einstein's gravity as a theory of 2-forms rather than the metric tensors, and so no pre-existing geometrical structure is needed to obtain the gravity. Let $F(\omega) \in \Omega^2(M;\mathfrak{so}(3,1))$ be the curvature of ω. The pure *BF* theory action is $\int_M \mathrm{Tr}(B \wedge F(\omega))$, which is invariant (symmetric) under local Lorentz transformation (regarded as gauge group) and under arbitrary diffeomorphisms of M, and does not need using a metric. The equations of motion are $F(\omega) = 0$ and $d_\omega B = 0$, where d_ω stands for covariant derivative with respect to the connection ω, thus, B defines a twisted de Rham cohomology class $[B] \in H^2_{DR}(M,\mathfrak{so}(3,1))$, and the solution of $F(\omega) = 0$ is unique up to gauge and diffeomorphism transformations. There are no local degrees of freedom because the system has so much symmetry that all solutions are locally equivalent under gauge transformation of the group $SO(3,1)$ and under diffeomorphisms of M. Hence, the pure *BF* theory is a topological theory [2,3].

In constrained *BF* theory, the Lagrangian includes the constraint term $\varphi_{IJKL}B^{IJ} \wedge B^{KL}$. The traceless matrix φ plays the role of a Lagrangian multiplier that imposes the constraint on the 2-form B^{IJ}, so that its solutions are given in terms of 1-forms $e^I = e^I_\mu dx^\mu$, that is $B^{IJ} = e^I \wedge e^J$, where the capital letters $I, J, \ldots = 0,1,2,3$ are the Lorentz indices and Greek letters $\mu, \nu, \ldots = 0,1,2,3$ are the space-time tangent indices. The frame fields $e^I_\mu dx^\mu$ are regarded as gravitational fields, therefore, the constrained *BF* theory turns to general relativity theory; the reason is that when φ_{IJKL} is not constant (like cosmological constant), the term $\varphi_{IJKL}B^{IJ} \wedge B^{KL}$ breaks the diffeomorphisms invariance of *BF* action, thus, there are non-equivalent local solutions and so local degrees of freedom exist as known in general relativity in the vacuum. Since the field φ_{IJKL} is not a physical variable, the equations of motion of general relativity do not to include it (see Appendix A). The problem with

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constrained *BF* theory is that the equation of motion, $\delta S/\delta B = 0$, of the action variation with respect to B contains the non-physical variable φ_{IJKL}, but one can remove it by taking the trace of the equations, but there is also a problem with the trace operation, as it reduces the equations to one equation, which is not enough for getting a solution. For that reason, the solutions of *BF* theory using the equation $\delta S/\delta\omega = 0$ are searched for. In general, the equations of motion of constrained *BF* theory including matter give a relation between the curvature $F^{IJ}(\omega)$ and the frame fields $\Sigma^{IJ} = e^I \wedge e^J$ (the Plebanski 2-form), in matrix notation, that is $F = \chi\Sigma + \xi\bar{\Sigma}$, where the bar indicates anti-frame field, and χ, ξ are symmetric matrices of scalar fields [4]. Therefore, the problem turns to finding χ and ξ.

Let us start with the definition of the spin current J and discuss its conservation in *BF* theory including matter (in general, a matter Lagrangian is not specified). The spin current J appears in the equations of motion as a source for $d_\omega B$ by the equation $*d_\omega B + J = 0$ ('$*$' is Hodge star operator), and, in order to get $d_\omega B = 0$ in this study, a new term is added to *BF* Lagrangian, like $\mathrm{Tr}(\psi B \wedge F(\omega))$, using a new field ψ, which is seen as a redefinition $B \to B + \psi B$. One finds that the equation of motion of ψ is the same conservation equation $D_\mu J^\mu = 0$ of the spin current vector field J, where D_μ is the covariant derivative. Furthermore, by choosing $d_\omega B = 0$ in the equations of motion, the spin current becomes a source for the field ψ instead of B and one gets a new formula (definition) for the spin current using ψ, and since the spin current regards symmetry of the system, the field ψ also regards that symmetry. One can see that the equations of *BF* theory can be solved only by solving the spin current equation, $\delta S/\delta\omega = 0$, $J \neq 0$, with $d_\omega B = 0$ and without solving the equation $\delta S/\delta B = 0$, which includes the Lagrangian multiplier ϕ_{IJKL} (a non-physical variable), and without using a gravitational metric on M, so that one just needs to use the spin current and know the symmetry of the system. That means that the *BF* equations can be solved only by using the coupling term $\int_M \omega_\mu^{IJ} J_{IJ}^\mu$, which makes them easy to solve, and makes the theory similar to the gauge theory. Furthermore, since ω^{IJ} is 1-form and J_{IJ} is a vector field, the term $\omega_\mu^{IJ} J_{IJ}^\mu$ is naturally defined on M without needing to use additional structures (like a metric), thus, solving the system equation using only that coupling term gives a topological theory, i.e, the theory turns to finding 1-forms and vector fields, similarly to Chern–Simons theory, which includes the Wilson loops as a source for the gauge field. That makes it easy to solve the equations in different cases of the spin current, e.g, point charge, straight line current, circular current, etc. The lines of the spin current can be described using any coordinates system, e.g, Euclidean coordinates, etc., so the *BF* theory can be studied in any coordinates system, but in order to avoid an effect of the coordinates on the lines of spin currents, the coordinates are left to be flat (not curved). Furthermore, since the spin current is the source for the field ψ, this field has singularities on the lines of that spin current. One can see that the solution of B^{IJ} can always be written as $e^I \wedge e^J$, so one gets the gravity theory. Finally, an example of explicit solution of the equations in the case of a spherical and cylindrical symmetric systems in static case just by finding the field ψ are given; actually, the field ψ is used for obtaining the spin current.

2. Spin Current in *BF* Theory

Let M be a connected oriented smooth 4-manifold and $P \to M$ be an $SO(3,1)$-principal bundle with a spin connection ω, which is locally a 1-form with values in $\mathfrak{so}(3,1)$ and $F \in \Omega^2(M;\mathfrak{so}(3,1)_P)$ is its curvature. The *BF* theory action is invariant under global and local Lorentz transformation, which gives a conserved current, and it is called here a spin current. Before discussing the conservation of the spin current, let us introduce the self-dual formalism.

Definition 1. *The self-dual projection is a homomorphism*

$$\mathfrak{so}(3,1)_P = P \times_{SO(3,1)} \mathfrak{so}(3,1) \to \mathfrak{so}(3,\mathbb{C})_P = P \times_{SO(3,1)} \mathfrak{so}(3,\mathbb{C})$$

defined by

$$\left(P^i_{IJ}\right) : \mathfrak{so}(3,1) \hookrightarrow \mathfrak{so}(3,1)_{\mathbb{C}} = \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C}) \to \mathfrak{so}(3,\mathbb{C}),$$

for $P \times_{SO(3,1)} \mathfrak{so}(3,1) = (P \times \mathfrak{so}(3,1))/SO(3,1)$, *where Latin letters* $i, j, \ldots = 1,2,3$, *with using the matrices [5]*

$$P^i_{IJ} = \frac{1}{2}\varepsilon^i{}_{jk}, \text{ for } I = i, J = j, \text{ and } P^i_{0j} = -P^i_{j0} = -\frac{i}{2}\delta^i_j, \text{ for } I = 0, J = j \neq 0, \tag{1}$$

where $\varepsilon^i{}_{jk}$ *is the totally anti-symmetric Levi-Civita tensor, and* δ^i_j *is the Kronecker delta.*

This relates to the fact that the complexified Lie algebra of $SO(3,1)$ *has the decomposition* $\mathfrak{so}(3,1)_{\mathbb{C}} = \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(3,\mathbb{C})$ *[6]. The new connection is locally an* $\mathfrak{so}(3,\mathbb{C})$*-valued 1-form* A *on* M *whose components are*

$$A^i_\mu = P^i_{IJ}\omega^{IJ}_\mu = \frac{1}{2}\varepsilon^i{}_{jk}\omega^{jk}_\mu - i\omega^{0i}_\mu, \tag{2}$$

and its curvature is

$$F^i(A) = P^i_{IJ}F^{IJ}(\omega) = dA^i + \varepsilon^i{}_{jk}A^j \wedge A^k, \tag{3}$$

where d *is the exterior derivative. The two form* B^{IJ} *is mapped to* $B^i = P^i_{IJ}B^{IJ}$. *The covariant derivative* $D_\mu = \nabla_\mu + \omega^{IJ}_\mu$ *acting on sections of* $TM \otimes \mathfrak{so}(3,1,\mathbb{C})_P$ *becomes* $D_\mu = \nabla_\mu + A_\mu$, *with* $A^{ij}_\mu = \varepsilon^{ij}{}_k A^k_\mu$, *where* ∇_μ *is the affine connection on the tangent (T) space* TM.

Using the new variables one can write the Lagrangian of matter (without specifying matter fields), $L_{\text{matter}}(e^I, \omega^{IJ})$, *as* $L_{\text{matter}}(B^i, A^i, \bar{B}^i, \bar{A}^i)$, *where* $\bar{B}^i$ *and* $\bar{A}^i$ *(anti-self-dual representation) are the complex conjugation of* B^i *and* A^i. *The Urbantke formula (Equation (A4), Appendix B) writes the metric* $g_{\mu\nu}$ *using only the constrained* B^i *without using the constrained* $\bar{B}^i$. *Furthermore, the self-dual connection* A^i *is compatible with* B^i *via* $d_A B^i = 0$, *while the anti-self-dual connection* $\bar{A}^i$ *is compatible with* $\bar{B}^i$ *via* $d_{\bar{A}}\bar{B}^i = 0$. *Here* d_A *is the exterior covariant derivative with respect to the connection* A. *By that one may suppose*

$$\frac{\delta}{\delta \bar{B}^i} L_{\text{matter}} = 0, \quad \frac{\delta}{\delta \bar{A}^i} L_{\text{matter}} = 0, \tag{4}$$

or just writing $L_{\text{matter}}(B^i, A^i)$.

Definition 2. *Let* A *be the self-dual connection on the* $\mathfrak{so}(3,\mathbb{C})$*-bundle* $\mathfrak{so}(3,\mathbb{C})_P \to M$. *Let* L_{matter} *be the Lagrangian of matter fields on* M. *Then the spin current* J^μ_i *is defined as*

$$J^\mu_i = \frac{\delta}{\delta A^i_\mu} L_{\text{matter}}.$$

The matter action, S_{matter}, is required to be invariant under any infinitesimal local Lorentz transformation $\omega^{IJ}_\mu \mapsto \omega^{IJ}_\mu + D_\mu \Lambda^{IJ}$ for infinitesimal transformation parameter $\Lambda^{IJ} \in \Omega^0(M; \mathfrak{so}(3,1)_P)$. Now, let us assume S_{matter} has this property. Then one gets the following.

Lemma 1. *The spin current* J^μ_i *given by* $J^\mu_i = \frac{\delta}{\delta A^i_\mu} L_{\text{matter}}$ *in gravity theory is conserved [7].*

Proof. Since S_{matter} is invariant under infinitesimal gauge transformation Λ^{IJ}, it is invariant under (one may suggest the condition (4))

$$A^i_\mu \mapsto A^i_\mu + D_\mu \Lambda^i \quad \text{for} \quad \Lambda^i = P^i_{IJ}\Lambda^{IJ} \in \Omega^0(M; \mathfrak{so}(3,\mathbb{C})_P).$$

The variation

$$S_{\text{matter}}\left(A^i + D\Lambda^i\right) - S_{\text{matter}}\left(A^i\right) = \int\limits_M d^4x\left(D_\mu\Lambda^i\right)\frac{\delta}{\delta A^i_\mu}L_{\text{matter}}$$
$$= -\int\limits_M d^4x\Lambda^i D_\mu\left(\frac{\delta}{\delta A^i_\mu}L_{\text{matter}}\right) + \int\limits_{\partial M} d^3x^\mu\Lambda^i\frac{\delta}{\delta A^i_\mu}L_{\text{matter}}$$

vanishes for arbitrary Λ^i only when $D_\mu\left(\frac{\delta}{\delta A^i_\mu}L_{\text{matter}}\right) = 0$, where Λ^i is considered to vanish on the boundary ∂M. Thus, the current $J^\mu_i = \frac{\delta}{\delta A^i_\mu}L_{\text{matter}}$ is conserved. Actually, the previous calculation based on the idea that A^i and $\bar{A}^i$ transform independently under infinitesimal local Lorentz transformation $\omega^{IJ}_\mu \mapsto \omega^{IJ}_\mu + D_\mu\Lambda^{IJ}$, therefore, there is another current that associates with the connection $\bar{A}^i$ when the matter Lagrangian depends also on $\bar{A}^i$. □

One finds the same for the general relativity (GR) action; by using the variables (Σ^i, A^i), one obtains the equation

$$\int\limits_M d^4x\left(D_\mu\Lambda^i\right)\frac{\delta}{\delta A^i_\mu}S_{\text{GR}} = 0 \Rightarrow -\int\limits_M d^4x\Lambda^i D_\mu\frac{\delta}{\delta A^i_\mu}S_{\text{GR}} = 0.$$

In $3+1$ decomposition of the space-time manifold $M = \Sigma\times\mathbb{R}$, let Σ_t be space-like slice of constant time t, with the coordinates x^a, $a = 1,2,3$, let 0 be the time index. In the Hamilton–Jacobi system, by using the variables (E^a_i, A^i_a, A^i_0) on the slice of constant time Σ_t, the equation becomes

$$\begin{aligned}&-\int\limits_{\Sigma\times\mathbb{R}} d^4x\Lambda^i D_a\left(\frac{\delta}{\delta A^i_a}S_{\text{GR}}\right) - \int\limits_{\Sigma\times\mathbb{R}} d^4x\Lambda^i D_0\left(\frac{\delta}{\delta A^i_0}S_{\text{GR}}\right)\\ &= -\text{const.}\times\int\limits_{\Sigma\times\mathbb{R}} d^4x\Lambda^i(D_aE^a_i + D_0(D_aE^a_i)) = 0,\end{aligned} \tag{5}$$

which is satisfied when $D_aE^a_i = 0$, where E^a_i is conjugate momentum to A^i_a, and the relations $E^a_i = \text{const.}\times\frac{\delta}{\delta A^i_a}S_{\text{GR}}$ and $D_aE^a_i = \text{const.}\times\frac{\delta}{\delta A^i_0}S_{\text{GR}}$ [5,8,9] are used. In this paper, one fixes $D_aE^a_i = 0$.

Remark 1. *To note is that the current J^μ_i is similar to the currents in Yang–Mills theory of the gauge fields, and one can see this clearly when regards the connection A^i_μ as a gauge field, by that the current J^μ_i relates to the local Lorentz invariance (local symmetry). The metric $g_{\mu\nu} = e^I_\mu e^J_\nu\eta_{IJ}$ (η_{IJ} is Lorentz metric) is invariant under arbitrary local Lorentz transformations, like $e^I_\mu(x) \mapsto U^I{}_J(x)e^J_\mu(x)$, for $U(x)\in SO(3,1)$, therefore, the local Lorentz symmetry is an internal degree of freedom.*

Definition 3. *The action of BF theory including matter (without cosmological constant) on $SO(3,1)$-principal bundle $P\to M$ is defined to be [10]*

$$S = S_{\text{topological}} + S_{\text{constraints}} + S_{\text{matter}},$$

with

$$S_{\text{topological}} = \int\limits_M B_i\wedge F^i(A), \quad \textit{and} \quad S_{\text{constraints}} = \frac{1}{2}\int\limits_M \varphi_{ij}B^i\wedge B^j,$$

where $\varphi \in \Gamma(M; \mathrm{End}(\mathfrak{so}(3,\mathbb{C})_P))$ is a traceless matrix of scalar fields φ_{ij}. Actually, it is not required to be symmetric since a new term to be added to BF Lagrangian (see the discussion below Equation (23)). The connection A on the Lie algebra bundle $\mathfrak{so}(3,\mathbb{C})_P$, which is locally a 1-form with values in $\mathfrak{so}(3,\mathbb{C})$ and its curvature $F(A) \in \Omega^2(M; \mathfrak{so}(3,\mathbb{C})_P)$ are defined in the Equations (2) and (3). The index contraction is done by using δ_{ij}, the Killing form on $\mathfrak{so}(3,\mathbb{C})$.

Hence,

$$S = \int_M \left(B_i \wedge F^i(A) + \frac{1}{2}\varphi_{ij} B^i \wedge B^j \right) + S_{\text{matter}}. \tag{6}$$

Since the matrix φ is traceless, one can write $\varphi_{ij} = m_{ij} - (m_{11} + m_{22} + m_{33})\delta_{ij}/3$, for some not traceless matrix (m_{ij}). The variation of the action with respect to m_{ij} produces a quadratic equation in B^i whose solution turns the theory into general relativity. These are

$$B^i \wedge B^j = \frac{1}{3}\delta^{ij} B_i \wedge B^j.$$

The solutions to this are all of the following form $B^i = P^i_{IJ} e^I \wedge e^J$, in which the gravitational fields e^I_μ are considered as frame fields [11]. Using the self-dual formula (Equation (1)), the constrained 2-form B^i is written as

$$B^i = \frac{1}{2}\varepsilon^i{}_{jk} e^j \wedge e^j - i e^0 \wedge e^i = \Sigma^i,$$

this is $B^i\big|_{\text{constrained}} = \Sigma^i$, using the notation $\Sigma^i = P^i_{IJ} e^I \wedge e^J$.

The equation of motion with respect to B^i is

$$F^i(A) + \varphi^i{}_j B^j + \frac{\delta S_{\text{matter}}}{\delta B_i} = 0,$$

or

$$F^i(A) = -\varphi^i{}_j B^j - \frac{\delta S_{\text{matter}}}{\delta B_i}. \tag{7}$$

Since $F^i(A) \in \Omega^2(M; \mathfrak{so}(3,\mathbb{C})_P)$ is 2-form with values in $\mathfrak{so}(3,\mathbb{C})$, the $\frac{\delta}{\delta B_i} S_{\text{matter}}$ is also 2-form with values in $\mathfrak{so}(3,\mathbb{C})$.

Lemma 2. *In constrained B^i, the variation $\frac{\delta}{\delta B_i} S_{\text{matter}} \in \Omega^2(M; \mathfrak{so}(3,\mathbb{C})_P)$ has the form*

$$\frac{\delta S_{\text{matter}}}{\delta B_i}\bigg|_{\text{constraint}} = T^i{}_j \Sigma^j + \xi^i{}_j \bar{\Sigma}^j,$$

for some matrices $T^i{}_j, \xi^i{}_j \in \Gamma(M; \mathrm{End}(\mathfrak{so}(3,\mathbb{C})_P))$, with $T^i{}_j = T_j{}^i$ and $\xi^i{}_j = \bar{\xi}_j{}^i$ (see Appendix B, for more details).

Therefore, in the vacuum, $T^i{}_j = 0$ is set. Using this formula in Equation (7) implies

$$F^i(A) = -\varphi^i{}_j \Sigma^j - T^i{}_j \Sigma^j - \xi^i{}_j \bar{\Sigma}^j,$$

or

$$F^i(A) = \psi^i{}_j \Sigma^j - \xi^i{}_j \bar{\Sigma}^j, \tag{8}$$

for some matrix $\psi = -\varphi - T \in \Gamma(M; \mathrm{End}(\mathfrak{so}(3,\mathbb{C})_P))$ [12].

Since $\mathrm{Tr}(\varphi) = 0$, so $\mathrm{Tr}(\psi) = -\mathrm{Tr}(T)$, Equation (8) yields

$$\Sigma^{\mu\nu}_i F^i_{\mu\nu} = -\mathrm{Tr}(T) \quad \text{for} \quad \Sigma^i_{\mu\nu}\Sigma^{\mu\nu}_j = \delta^i_j, \text{ and } \quad \Sigma^i_{\mu\nu}\bar{\Sigma}^{\mu\nu}_j = 0. \tag{9}$$

Thus, in the vacuum, $T^i{}_j = 0$, one has: $\Sigma_i^{\mu\nu} F^i_{\mu\nu} = 0$. $\Sigma_i^{\mu\nu} F^i_{\mu\nu}$is called the Tr$F$ here. Equation (9) does not contain the non-physical variable φ, but the problem with it is that the trace process decreasing the number of equations. Therefore, $\Sigma_i^{\mu\nu} F^i_{\mu\nu} = -\mathrm{Tr}(T)$ is a condition on the solutions. The (0,2) tensor $\Sigma_j^{\mu\nu}$ is inverse of the 2-form $\Sigma^j_{\mu\nu}$ (Appendix C).

The equation of motion with respect to the connection A^i is

$$DB^i + \frac{\delta}{\delta A^i} S_{\text{matter}} = 0,$$

or

$$\varepsilon^{\mu\nu\rho\sigma} D_\nu B^i_{\rho\sigma} + J^{\mu i} = 0. \tag{10}$$

where $DB^i = dB^i + \varepsilon^i{}_{jk} A^j B^k$.

One can see that $\varepsilon^{\mu\nu\rho\sigma} D_\nu B^i_{\rho\sigma} = 0$ cannot be chosen when $J^{\mu i} \neq 0$, but the condition $\varepsilon^{\mu\nu\rho\sigma} D_\nu B^i_{\rho\sigma} = 0$ leads to the constraint $D_a E^{ai} = 0$ which is satisfied in the Hamilton–Jacobi system, Equation (5). One gets $D_a E^{ai} = 0$ from $\varepsilon^{\mu\nu\rho\sigma} D_\nu B^i_{\rho\sigma} = 0$ by setting $\mu = 0$, so $\varepsilon^{0abc} D_a B^i_{bc} = 0$, then $\varepsilon^{0abc} \equiv \varepsilon^{abc}$ is used to get $\varepsilon^{abc} D_a B^i_{bc} = 2 D_a E^{ai} = 0$, where $E^{ai} = \varepsilon^{abc} B^i_{bc}/2$ is conjugated to the connection A^i_a on space-like slice of constant time on which the coordinates x^a are used. Furthermore, the condition $DB^i = 0$ is necessary when the connection A^i is flat, by that the 2-form B^i belongs to the twisted de Rham cohomology classes $H^2_{DR}(M, \mathfrak{so}(3,\mathbb{C})_P)$, and this is necessary for getting a topological theory. One can solve that problem by adding new terms to the BF action (6) with which there are many possibilities for controlling Equation (10) for $J^{\mu i} \neq 0$ with choosing $DB^i = 0$. Only some simple possibilities are chosen below in order to get simple results.

By acting by D_μ on Equation (10), one gets:

$$\varepsilon^{\mu\nu\rho\sigma} D_\mu D_\nu B^i_{\rho\sigma} + D_\mu J^{\mu i} = 0,$$

and using $D_\mu J^{\mu i} = 0$, one obtains

$$\varepsilon^{\mu\nu\rho\sigma} D_\mu D_\nu B^i_{\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} [D_\mu, D_\nu] B^i_{\rho\sigma}/2 = 0,$$

but $[D_\mu, D_\nu] = F_{\mu\nu}(A)$, therefore,

$$\varepsilon^{\mu\nu\rho\sigma} (F_{\mu\nu}(A))^i{}_j B^j_{\rho\sigma} = 0.$$

Then using $(F_{\mu\nu}(A))_{ij} = \varepsilon_{ijk} F^k_{\mu\nu}(A)$, implies

$$\varepsilon^{\mu\nu\rho\sigma} \varepsilon_{ijk} F^j_{\mu\nu}(A) B^k_{\rho\sigma} = 0. \tag{11}$$

As it is shown below, one can regard Equation (11) as an equation of motion with respect to a new field ψ^i, with the possibility of choosing $DB^i = 0$ with $J^{\mu i} \neq 0$.

In order to include the constraints $D_\mu J^{\mu i} = 0$ and $DB^i = 0$ in BF theory, the following action is suggested.

Definition 4. *A new term is added to the BF action (6) to get*

$$S = \int_M \left(B_i \wedge F^i(A) + \frac{1}{2} \varphi_{ij} B^i \wedge B^j \right) + \int_M \varepsilon_{ijk} \psi^i B^j \wedge F^k(A) + S_{\text{matter}}, \tag{12}$$

in which $\int_M \varepsilon_{ijk} \psi^i B^j \wedge F^k(A)$ *is added, for some vector field* $\psi^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$. *One can relate the new term to a redefinition like* $B^i \to B^i + \varepsilon^i{}_{jk} \psi^j B^k$ *in pure BF Lagrangian* $B_i \wedge F^i(A)$.

The equation of motion of this action with respect to the field ψ^i is

$$\varepsilon_{ijk}\varepsilon^{\mu\nu\rho\sigma}B^j_{\mu\nu}F^k_{\rho\sigma}(A)=0, \tag{13}$$

which is the same as Equation (11). By using $(F_{\mu\nu}(A))_{ij}=\varepsilon_{ijk}F^k_{\mu\nu}(A)$, one gets

$$\varepsilon^{\mu\nu\rho\sigma}B^j_{\mu\nu}(F_{\rho\sigma}(A))^i{}_j=0,$$

but $[D_\mu,D_\nu]=F_{\mu\nu}(A)$. Therefore,

$$\varepsilon^{\mu\nu\rho\sigma}[D_\rho,D_\sigma]B^i_{\mu\nu}=0,$$

but $\varepsilon^{\mu\nu\rho\sigma}D_\rho B^i_{\mu\nu}=0$ $(DB^i=0)$ is chosen as suggested before, thus,

$$\frac{\delta}{\delta\psi^i}S_{\text{matter}}=0$$

is satisfied.

The equation of motion of this action with respect to the connection A^k is

$$DB^k+D\left(\varepsilon^k{}_{ij}\psi^iB^j\right)+\frac{\delta}{\delta A^k}S_{\text{matter}}=0,$$

or

$$\varepsilon^{\mu\nu\rho\sigma}D_\nu B^k_{\rho\sigma}+\varepsilon^{\mu\nu\rho\sigma}D_\nu\left(\varepsilon^k{}_{ij}\psi^iB^j_{\rho\sigma}\right)+\frac{\delta}{\delta A^k_\mu}S_{\text{matter}}=0. \tag{14}$$

In this equation, one can choose the condition $\varepsilon^{\mu\nu\rho\sigma}D_\nu B^k_{\rho\sigma}=0$, which is equivalent to $D_\mu\Sigma^{i\mu\nu}=0$ in constrained B^i, since $B^i\big|_{\text{constraint}}=\Sigma^i=P^i_{IJ}e^I\wedge e^J$ and [13,14]

$$\frac{1}{2!}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\Sigma^i_{\rho\sigma}=\left(*\Sigma^i\right)^{\mu\nu}=\left(-i\Sigma^i\right)^{\mu\nu}=-i\Sigma^{\mu\nu i},\quad e=\det(e^I_\mu), \tag{15}$$

where the Hodge duality theory between the forms and the tensor fields is used; here, $\Sigma^i_{\mu\nu}$ is 2-form and $\Sigma^{i\mu\nu}$ is (0,2)-tensor field. One can see that $\Sigma^{\mu\nu}_i$ is inverse of $\Sigma^i_{\mu\nu}$, so that $\Sigma^{\mu\nu}_j\Sigma^i_{\mu\nu}=\delta^i_j$ (see Appendix C for more details).

With that, the term $\varepsilon^{\mu\nu\rho\sigma}D_\nu B^k_{\rho\sigma}$ in constrained B^i becomes $-iD_\nu\Sigma^{\mu\nu i}$, and so one can choose the condition $D_\nu\Sigma^{\mu\nu i}=0$, which is locally equivalent to $DB^i=0$ in constrained B^i.

The remaining equation of (14) in constrained B^i is

$$D\left(\varepsilon^k{}_{ij}\psi^i\Sigma^j\right)+\frac{\delta}{\delta A^k}S_{\text{matter}}=0,$$

or

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon^k{}_{ij}\left(D_\nu\psi^i\right)\Sigma^j_{\rho\sigma}+\frac{\delta}{\delta A^k_\mu}S_{\text{matter}}=0,$$

hence,

$$-2i\left(\varepsilon^k{}_{ij}D_\nu\psi^i\right)\Sigma^{\mu\nu j}+J^{\mu k}=0, \tag{16}$$

in which the spin current $J^\mu_i=\delta S_{\text{matter}}/\delta A^i_\mu$ is used and the condition $D_\nu\Sigma^{\mu\nu j}=0$ is imposed. Below, the condition $D_\mu J^{\mu k}=0$ is discussed. Here, both $\Sigma^{\mu\nu j}$ and $J^{\mu k}$ are tensor fields.

Remark 2. *To note is that Equation (16) is similar to the current* $J^{\mu n}=\left(\partial^\mu\varphi^i\right)T^n_{ij}\varphi^j$ *in scalar field theory with symmetry and generators* T^n_{ij}*, so one has* $J^{0n}=\left(\partial^0\varphi^i\right)T^n_{ij}\varphi^j=\pi^iT^n_{ij}\varphi^j$*, where*

π^i is conjugate momentum to φ_i. Similarly, Equation (16) gives $J^{0i} = 2i(D_\mu\psi_k)\varepsilon^{ki}{}_j\Sigma^{j\mu 0}$(for $\mu = a = 1,2,3$), so here $D_a\psi_i$ is conjugate momentum to Σ^{ia0}, noting that the indices raising in $\Sigma^{j\mu\nu}$ is done by using a metric $g_{\mu\nu}$.

The equation of motion of the action (12) with respect to B^i in constrained *BF* (like deriving Equation (8)) is

$$F^i(A) + \varphi^i{}_j\Sigma^j + \varepsilon^i{}_{jk}\psi^j F^k(A) + T^i{}_j\Sigma^j + \xi^i{}_j\bar{\Sigma}^j = 0. \tag{17}$$

Multiplying by $\Sigma_i^{\mu\nu}$, summing over the indices and using $\Sigma_i^{\mu\nu}\bar{\Sigma}^j_{\mu\nu} = 0$, one obtains:

$$\Sigma_i^{\mu\nu}F^i_{\mu\nu} + \varphi^i{}_j\Sigma_i^{\mu\nu}\Sigma^j_{\mu\nu} + \varepsilon^i{}_{jk}\psi^j\Sigma_i^{\mu\nu}F^k_{\mu\nu} + T^i{}_j\Sigma_i^{\mu\nu}\Sigma^j_{\mu\nu} = 0. \tag{18}$$

Then, using $\Sigma_i^{\mu\nu}\Sigma^j_{\mu\nu} = \delta^j_i$ to obtain

$$\Sigma_i^{\mu\nu}F^i_{\mu\nu} + \mathrm{Tr}(\varphi) + \varepsilon^i{}_{jk}\psi^j\Sigma_i^{\mu\nu}F^k_{\mu\nu} + \mathrm{Tr}(T) = 0.$$

Since $\mathrm{Tr}(\varphi) = 0$ and $\varepsilon^i{}_{jk}\Sigma_i^{\mu\nu}F^k_{\mu\nu} = 0$ (see Equations (13) and (15)), one finds:

$$\Sigma_i^{\mu\nu}F^i_{\mu\nu} + \mathrm{Tr}(T) = \mathrm{Tr}(F) + \mathrm{Tr}(T) = 0. \tag{19}$$

Equation (17) allows us to write $F^i(A)$ in terms of Σ^i and $\bar{\Sigma}^i$, and since $\varepsilon^i{}_{jk}\Sigma_i^{\mu\nu}F^k_{\mu\nu} = 0$, one can write

$$F^i(A) = \chi^i{}_j\Sigma^j + \chi'^i{}_j\bar{\Sigma}^j, \tag{20}$$

for some symmetric matrix (χ^{ij}) and skew-hermitian matrix (χ'^{ij}). Using this equation in Equation (19), one obtains:

$$\mathrm{Tr}(\chi^i{}_j) + \mathrm{Tr}(T^i{}_j) = 0. \tag{21}$$

In addition to this relation, there is another relation between the vector field ψ^i and the symmetric matrix $\chi^i{}_j$ when $T^i{}_j \neq 0$ and $J^{\mu i} \neq 0$, from the conservation of the current (16), $D_\nu J^{\nu i} = 0$, one has (for $D_\mu\Sigma^{\mu\nu i} = 0$):

$$\begin{aligned}
\frac{i}{2}D_\nu J^{\nu i} &= \left(D_\nu D_\mu\psi_k\right)\varepsilon^{ki}{}_j\Sigma^{\mu\nu j} = \frac{1}{2}\left(\left[D_\nu, D_\mu\right]\psi_k\right)\varepsilon^{ki}{}_j\Sigma^{\mu\nu j} \\
&= -\frac{1}{2}F^\ell_{\mu\nu}(A)\varepsilon_{\ell k}{}^m\psi_m\varepsilon^{ki}{}_j\Sigma^{\mu\nu j} = -\frac{1}{2}F^\ell_{\mu\nu}(A)\varepsilon_{\ell km}\psi^m\varepsilon^{kij}\Sigma^{\mu\nu}_j \\
&= \frac{1}{2}F^\ell_{\mu\nu}(A)\varepsilon_{k\ell m}\psi^m\varepsilon^{kij}\Sigma^{\mu\nu}_j = \frac{1}{2}F^\ell_{\mu\nu}(A)\psi^m\left(\delta^i_\ell\delta^j_m - \delta^i_m\delta^j_\ell\right)\Sigma^{\mu\nu}_j \\
&= \frac{1}{2}F^i_{\mu\nu}(A)\psi^j\Sigma^{\mu\nu}_j - \frac{1}{2}F^j_{\mu\nu}(A)\psi^i\Sigma^{\mu\nu}_j,
\end{aligned}$$

and using Equation (20), one gets:

$$\begin{aligned}
\frac{i}{2}D_\nu J^{\nu i} &= \frac{1}{2}\chi^i{}_m\Sigma^m_{\mu\nu}\psi^j\Sigma^{\mu\nu}_j - \frac{1}{2}\chi^j{}_m\Sigma^m_{\mu\nu}\psi^i\Sigma^{\mu\nu}_j = \frac{1}{2}\chi^i{}_m\psi^j\delta^m_j - \frac{1}{2}\chi^j{}_m\psi^i\delta^m_j \\
&= \frac{1}{2}\chi^i{}_j\psi^j - \frac{1}{2}\psi^i tr\left(\chi^i{}_j\right) = 0, \quad \text{for } J^{\mu i} \neq 0.
\end{aligned} \tag{22}$$

This is another relation between the vector field ψ^i and the symmetric matrix $\chi^i{}_j$ in existence of matter $T^i{}_j \neq 0$ with $J^{\mu i} \neq 0$. In this case, the matrix $\chi^i{}_j$ has to satisfy $\det\left(\chi^i{}_j - \mathrm{Tr}(\chi^i{}_j)\right) = 0$ in order to get $\psi^i \neq 0$; of course, this condition is not needed in the vacuum $T^i{}_j = 0$, $J^{\mu i} = 0$.

Using Equation (20) in (17), one obtains:

$$\chi^i{}_j\Sigma^j + \varphi^i{}_j\Sigma^j + \varepsilon^i{}_{jk}\psi^j\chi^k{}_\ell\Sigma^\ell + T^i{}_j\Sigma^j + (\dots)^i{}_j\bar{\Sigma}^j = 0.$$

That yields

$$\chi^{i\ell} + \varphi^{i\ell} + \varepsilon^{i}{}_{jk}\psi^{j}\chi^{k\ell} + T^{i\ell} = 0. \tag{23}$$

This equation relates to the equation of motion $\delta S/\delta B = 0$, and it includes the Lagrangian multiplier φ^{ij}, which is a non-physical variable that makes (23) difficult to solve. Therefore, one needs to find χ and ψ using the other equations of motion obtained above. One can see that φ is not required to be symmetric matrix since the third term in (23) is not symmetric in general. The symmetric matrix T^{ij} is assumed to be given using the matter Lagrangian (Appendix B), thus, the total unknown variables are $3+5+8=16$ of the vector ψ, the symmetric matrix χ (with (21)) and the traceless matrix φ. Equation (23) gives 9 equations, therefore, there are $16-9=7$ unknown variables, but when $J^{\mu i} \neq 0$, they reduce to 6 unknown variables (regarding Equation (22)). However, if one chooses a solution for which the symmetric matrix χ^{ij} becomes diagonal, like

$$\chi = \left(K^{i}\delta^{i}_{j}\right) = \mathrm{diag}(K^{1}, K^{2}, K^{3}), \tag{24}$$

for some scalar functions K^1, K^2 and K^3 on M. Thus, the unknown variables reduce to 4 variables and to 3 variables when $J^{\mu i} \neq 0$.

Remark 3. *The field ψ^i is a solution of $D_\mu D^\mu \psi^i = 0$ (see Appendix C), so if $D_\mu v^i = 0$, then $\psi^i + v^i$ is another solution, and that makes the components ψ^1, ψ^2 and ψ^3 of the vector field ψ^i independent variables, therefore, one can regard them as the degrees of freedom of the system and solve the equations of motions in terms of them. Note that $\psi^i \mapsto \psi^i + v^i$ ($Dv^i = 0$) does not change the current $J^{vi} = 2i\big(D_\mu \psi_k\big)\varepsilon^{ki}{}_{j}\Sigma^{\mu\nu j}$.*

The Bianchi identity $DF^i = 0$ implies $\left(D\chi^{i}{}_{j}\right) \wedge \Sigma^j = 0$ (for $D\Sigma^i = 0$), hence $(dK^i)\delta^i{}_j \wedge \Sigma^j = 0$, where $D\delta^{ij} = 0$ is used along with the covariant derivative $Dv^i = dv^i + \varepsilon^{i}{}_{jk}A^j v^k$. Therefore, one obtains:

$$\varepsilon^{\mu\nu\rho\sigma}(\partial_\nu K^i)\Sigma^{i}_{\rho\sigma} = 0, \tag{25}$$

where $\partial_\nu \equiv d/dx_\nu$. In $3+1$ decomposition of the space-time manifold $M = \Sigma \times \mathbb{R}$, let Σ_t be the space-like slice of constant time t with the coordinates x^a with $a = 1, 2, 3$, and 0 is the time index. The equation $\epsilon^{\mu\nu\rho\sigma}D_\mu \Sigma^{i}_{\nu\rho} = 0$ ($DB^i = 0$) decomposes into two equations,

$$D_a E^{ai} = 0 \quad \text{and} \quad \epsilon^{abc} D_b B^{i}_{c} = 0, \tag{26}$$

in which the vector field E^i and the 1-form B^i:

$$E^{ai} = \epsilon^{0abc}\Sigma^{i}_{bc}/2 = \epsilon^{abc}\Sigma^{i}_{bc}/2, \quad B^{i}_{c} = \Sigma^{i}_{0c}, \tag{27}$$

are introduced on the space-like slice Σ_t (the field E^{ai} is conjugate to the connection A^{i}_{a}).

Equation (25) decomposes into (for $\partial_0 K^i = 0$)

$$\begin{aligned} &\varepsilon^{0abc}(\partial_a K^i)\Sigma^{i}_{bc} = (\partial_a K^i)\varepsilon^{abc}\Sigma^{i}_{bc} = 2(\partial_a K^i)E^{ai} = 0, \\ &\varepsilon^{abc0}(\partial_b K^i)\Sigma^{i}_{c0} = -\varepsilon^{abc}(\partial_b K^i)B^{i}_{c} = 0. \end{aligned} \tag{28}$$

One can solve them by writing (for non-zero curvature $F^i(A)$)

$$E^{ai} = \frac{1}{2}\varepsilon^{abc}(\partial_b K^i)r^{i}_{c}, \qquad B^{i}_{c} = (\partial_c K^i)u^i,$$

for some $r^i \in \Omega^1(M; \mathfrak{so}(3,\mathbb{C})_P)$ and $u^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$. The functions K^i are scalars, the indices are just for distinguishing each from the others. Thus, one gets the solutions

$$\Sigma^{i}_{ab} = (\partial_{[a}K^i)r^{i}_{b]}, \quad \text{and} \quad \Sigma^{i}_{0a} = (\partial_a K^i)u^i. \tag{29}$$

Equation (26) implies $Dr^i = 0$ and $Du^i = 0$.

In the static case $J^a_i = 0$ (zero current) with $J^0_i \neq 0$(non-zero charge), the spin current formula $J^\nu_k = 2i\varepsilon_{kij}\left(D_\mu \psi^i\right)\Sigma^{\mu\nu j}$ decomposes into two equations:

$$J^b_k = 2i\varepsilon_{kij}\Big(D_a\psi^i\Big)\Sigma^{abj} = 0, \quad \text{and} \quad J^0_k = 2i\varepsilon_{kij}\Big(D_a\psi^i\Big)\Sigma^{a0j} \neq 0. \tag{30}$$

One can solve the first equation in terms of ψ^i by writing $\Sigma^{abi} = fv^{[a}D^{b]}\psi^i$, for some vector $v \in \Gamma(M; T\Sigma)$ that satisfies $v^a D_a\psi^i = 0$, and f is scalar function on M. Including f in v, one can just write $\Sigma^{abi} = v^{[a}D^{b]}\psi^i$. Let us note that $D^a\psi^i = g^{ab}D_b\psi^i$ without a need the used metric g_{ab} to be specified. Regarding the second equation of (30), when $J^{0i} = 0$, one gets the solution, $\Sigma^{0ai} = -\Sigma^{a0i} = fD^a\psi^i$. Furthermore, when $J^{0i} \neq 0$, we let $\Sigma^{0ai} = fD^a\xi^i$ for a vector field $\xi^i \neq \psi^i$. To note is that no specific metric g_{ab} is required for raising and lowering the indices $a, b, \ldots$ on Σ_t, so let it be the metric coming from pulling back of the Lorentz metric, where Σ_t is kept to be immersed in $\mathbb{R}^4$.

Lemma 3. *By comparing the solutions $\Sigma^{abi} = v^{[a}D^{b]}\psi^i$ and $\Sigma^{0ai} = fD^a\xi^i$ of Equations (30) with the solutions (29), and in order to get a correspondence between that solutions, one finds that*

$$\begin{aligned} &\psi^i = K^i b^i, \quad r^i_c = v_c b^i, \quad \xi^i = K^i u^i, \\ &Db^i = 0, \quad Du^i = 0, \quad dv = 0, \quad f = 1, \end{aligned} \tag{31}$$

for some vector fields, $b^i, u^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$.

By that, one obtains the solutions,

$$\begin{aligned} E^i &= \frac{1}{2}\epsilon^{abc}\Sigma^i_{ab}\partial_c = \frac{1}{2}\epsilon^{abc}v_a\Big(\partial_b K^i\Big)b^i\partial_c \in \Gamma(M; T\Sigma \otimes \mathfrak{so}(3,\mathbb{C})_P), \\ B^i &= \Sigma^i_{0a}dx^a = \Big(dK^i\Big)u^i \in \Gamma(M; T^*\Sigma \otimes \mathfrak{so}(3,\mathbb{C})_P), \end{aligned} \tag{32}$$

without needing to use a specific metric.

Remark 4. *Regarding the solutions of Equations (29) and (30), let us note that for every two solutions of Σ^i_{ab} and Σ^{abi}, the metric g_{ab} satisfies $\Sigma^i_{ab} = g_{aa'}g_{bb'}\Sigma^{a'b'i}$. Furthermore, the metric used in $\Sigma^i_{ab} = g_{aa'}g_{bb'}\Sigma^{a'b'i}$ is not necessarily the same metric used in $D^a\psi^i = g^{ab}D_b\psi^i$ for getting the solutions of (30). It is convenient to start from a solution of Σ^i_{ab}, and by using a metric g_{ab}, to obtain the corresponding solution of Σ^{abi}.*

Remark 5. *In solution (32), one can see that Σ^{IJ} can be written as $e^I \wedge e^J$, as required in constraint BF theory to get gravity theory, that is, according to self-dual projection, there are vector fields b^I and K^I satisfying $K^i b^i = P^i_{IJ}b^I(K^J b^J)$, therefore, $\Sigma^{IJ}_{ab} = \Big(b^I v_{[a}\Big)D_{b]}\big(K^J b^J\big)$, then one can write $e^I_a = v_a b^I$ and $e^J_b = \big(\partial_b K^J\big)b^J$. Furthermore, from $\Sigma^{IJ}_{0a} = u^I\big(\partial_a K^J\big)u^J$, one gets: $e^J_a = \big(\partial_a K^J\big)u^J$ and $e^I_0 = v_0 u^I$, for $v_0 = 1$. A more general case is to find three vector fields b^I_1, b^I_2 and K^I satisfying $K^i b^i = P^i_{IJ}b^I_1(K^J b^J_2)$, therefore, $\Sigma^{IJ}_{ab} = \Big(b^I_1 v_{[a}\Big)D_{b]}\Big(K^J b^J_2\Big)$, then one can write $e^I_a = v_a b^I_1$ and $e^J_b = \big(\partial_b K^J\big)b^J_2$. Furthermore, from $\Sigma^{IJ}_{0a} = u^I_1\big(\partial_a K^J\big)u^J_2$, one gets $e^J_a = \big(\partial_a K^J\big)u^J_2$ and $e^I_0 = v_0 u^I_1$, for $v_0 = 1$. By that, (32) can be written as $\Sigma^i = P^i_{IJ}\Sigma^{IJ}$ for $\Sigma^{IJ} = e^I \wedge e^J$. However, to note is that solution (32) is a general solution and one has to find a special solution, like to let b^i and u^i be constant fields.*

Using the solution of Σ^{0ai} in Equation (30), one obtains:

$$\begin{aligned} J^0_k = Q_k &= -2i\epsilon_{kij}\left(b^i\partial_a K^i\right)\left(u^j\partial^a K^j\right) \\ &= -i\epsilon_{kij}\left(b^i\partial_a K^i u^j\partial^a K^j - b^j\partial_a K^j u^i\partial^a K^i\right) \\ &= -i\epsilon_{kij}\left(\partial_a K^i\right)\left(\partial^a K^j\right)(b^i u^j - b^j u^i). \end{aligned} \tag{33}$$

One can see that $J^0_k \neq 0$ takes place only when $b \neq u$. Therefore, in the vacuum it must be $b = u$. If $J^a_k \neq 0$, it must be $\psi^i = g^i b^i$ with $g^i \neq K^i$.

If the charges $J^{0i} \neq 0$ are given as functions on M, then letting $b^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$ be constant field on M, one can determine the scalar functions K^i using Equation (33), and obtaining the vector $v \in \Gamma(M; T\Sigma)$ using $v^a\partial_a K^i = 0$. However, to satisfy $Db^i = db^i + \varepsilon^i_{jk}A^j b^k = 0$ for a constant vector field b^i, the connection A^i_μ must be written as $A^i_\mu = A_\mu b^i$. Furthermore, $u^i = b^i + f(x)b^i + a^i$ is chosen, for a constant $a^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$ satisfying $a_i b^i = 0$, the function f is needed for satisfying $Du^i = 0$. Examples of determining b^i and u^i in spherical and cylindrical symmetries are given below. Then, one obtains B^{ai} and E^{ai} using Equation (32), and obtain the matrix χ using Equation (24), thus, obtaining the curvature $F = \chi\Sigma$. Note that $v^a E^i_a = 0$, $v_a B^{ai} = 0$ and $v^a\partial_a K^i = 0$ depend on the symmetry of the system, for example, spherical symmetry, cylindrical symmetry, and so on. Thus, one sees that the equations of motion of *BF* theory can be solved without needing to use a gravitational metric on the manifold M.

3. Solutions for Spherically Symmetric System

It was shown above that one can solve the equations of motion in *BF* theory by using a complex vector field $\psi^i = K^i b^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$, which allows us to obtain v, E^i, B^i and J^{0i}, according to Equations (31)–(33). Here, the solutions to be found for spherically symmetric system in the vacuum ($u = b$) and then apply it for matter located at a point. As it was seen above, the solution of the system regards the symmetry of that system, since one searches for a vector $v \in \Gamma(M; T\Sigma)$ that satisfies $v^a D_a\psi^i = 0$, $v^a E^i_a = 0$ and $v^a B^i_a = 0$. For example, in spherical symmetry, the spherical coordinates (r, θ, φ) to be used on the space-like slice $\Sigma_t = \Sigma = \mathbb{R}^3$. Letting the vector field ψ^i to depend only on the radius r, one gets (for $Db^i = 0$):

$$D^2\psi^i = D^2(K^i b^i) = b^i\nabla^2 K^i = b^i\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}K^i\right) = 0 \Rightarrow K^i = \frac{c^i}{r}, \tag{34}$$

so $\psi^i = c^i b^i/r$, for some constants $c^i \in \mathbb{R}$. Actually, one can include c^i in b^i and just write $\psi^i = b^i/r$. Therefore,

$$D\psi^i = b^i(dK^i) = b^i\left(dr\partial_r + d\theta\partial_\theta + d\varphi\partial_\varphi\right)\frac{1}{r} = -\frac{b^i}{r^2}dr,$$

thus, the 1-form v ($dv = 0$, $g^{ab}v_a D_b\psi^i = 0$) is

$$v = a_1 d\theta + a_2 d\varphi,\ a_1, a_2 \in \mathbb{R},$$

where, in the spherical symmetry, a_1 and a_2 kept to not depend on the coordinates θ and φ. The values of the constants a_1 and a_2 are not significant since $a^i = g^{ij}a_j$ is the Killing vector, thus, set $a_1 = a_2 = 1$. The used metric g_{ab} here is the standard metric in the spherical coordinates, because no any other metric is defined. In what follows in this Section, the indices r, θ and φ denote the spherical components.

Using Equation (32), one gets the solutions of the 1-form B^i and the vector field E^i,

$$B^i = (dK^i)u^i = -\frac{b^i}{r^2}dr, \quad for \quad u = b,$$

$$E^i = \frac{1}{2}\varepsilon^{abc}\Sigma^i_{bc}\partial_a = \frac{1}{2}\varepsilon^{abc}v_b\left(\partial_c K^i\right)b^i\partial_a = -\varepsilon^{\varphi\theta r}\frac{b^i}{2r^2}\partial_\varphi - \varepsilon^{\theta\varphi r}\frac{b^i}{2r^2}\partial_\theta$$
$$= \varepsilon^{\varphi r\theta}\frac{b^i}{2r^2}\partial_\varphi + \varepsilon^{\theta r\varphi}\frac{b^i}{2r^2}\partial_\theta.$$

By that, one gets:

$$\Sigma^i_{0r} = -\frac{b^i}{r^2}, \quad \Sigma^i_{r\theta} = \frac{b^i}{2r^2}, \quad \Sigma^i_{r\varphi} = \frac{b^i}{2r^2}, \tag{35}$$

while the other components like $\Sigma^i_{0\theta}, \Sigma^i_{0\varphi}, \ldots$ are zeros.

One obtains the matrix χ using Equation (24) with the solution (34),

$$\chi = \left(K_j\delta^i_j\right) = \frac{1}{r}\mathrm{diag}(c^1, c^2, c^3),$$

where the constants c^i have to be determined in order to satisfy the condition $\mathrm{Tr}\chi = 0$ (in the vacuum), so $\sum\limits_{i=1}^{3} c_i = 0$. Thus, one gets the curvature $F = \chi\Sigma + \chi'\bar{\Sigma}$ (with setting $\chi' = 0$ in the vacuum [15]),

$$F^i_{0r} = \chi^i{}_j\Sigma^j_{0r} = -\frac{c^i b^i}{r^3}, \qquad F^i_{r\theta} = \chi^i{}_j\Sigma^j_{r\theta} = \frac{c^i b^i}{2r^3}, \qquad F^i_{r\varphi} = \chi^i{}_j\Sigma^j_{r\varphi} = \frac{c^i b^i}{2r^3}. \tag{36}$$

Now, let us calculate the connection A^i and the field b^i, which satisfies $Db^i = 0$. Using $F^i = dA^i + \varepsilon^i{}_{jk}A^j \wedge A^k$, one obtains:

$$F^i_{0r} = \frac{1}{2}\left(\partial_0 A^i_r - \partial_r A^i_0\right) + \varepsilon^i{}_{jk}A^j_0 A^k_r,$$
$$F^i_{\theta r} = \frac{1}{2}\left(\partial_\theta A^i_r - \partial_r A^i_\theta\right) + \varepsilon^i{}_{jk}A^j_\theta A^k_r,$$
$$F^i_{\varphi r} = \frac{1}{2}\left(\partial_\varphi A^i_r - \partial_r A^i_\varphi\right) + \varepsilon^i{}_{jk}A^j_\varphi A^k_r.$$

Since a spherically symmetric system is under consideration, the connection A^i are considered depending on r only. If the gauge $A^i_r = 0$ is chosen, then:

$$F^i_{0r} = -\frac{1}{2}\partial_r A^i_0, \qquad F^i_{\theta r} = -\frac{1}{2}\partial_r A^i_\theta, \qquad F^i_{\varphi r} = -\frac{1}{2}\partial_r A^i_\varphi,$$

and, therefore, using the solution (36), one obtains:

$$-\frac{1}{2}\partial_r A^i_0 = -\frac{c^i b^i}{r^3}, \qquad \frac{1}{2}\partial_r A^i_\theta = \frac{c^i b^i}{2r^3}, \qquad \frac{1}{2}\partial_r A^i_\varphi = \frac{c^i b^i}{2r^3}.$$

However, $Db^i = db^i + \varepsilon^i_{jk}A^j b^k = 0$ and $\partial_\mu b^i = 0$ for $\mu \neq r$, therefore,

$$\partial_r b^i + \varepsilon^i{}_{jk}A^j_r b^k = \partial_r b^i = 0, \qquad \partial_0 b^i + \varepsilon^i{}_{jk}A^j_0 b^k = \varepsilon^i{}_{jk}A^j_0 b^k = 0,$$
$$\partial_\theta b^i + \varepsilon^i{}_{jk}A^j_\theta b^k = \varepsilon^i{}_{jk}A^j_\theta b^k = 0, \qquad \partial_\varphi b^i + \varepsilon^i{}_{jk}A^j_\varphi b^k = \varepsilon^i{}_{jk}A^j_\varphi b^k = 0.$$

Therefore, the field $b^i \in \Gamma(M; \mathfrak{so}(3, \mathbb{C})_P)$ is constant, and one gets the solution,

$$A^i_0 = -\frac{c^i b^i}{r^2}, \qquad A^i_\theta = -\frac{c^i b^i}{2r^2}, \qquad A^i_\varphi = -\frac{c^i b^i}{2r^2},$$

where $\varepsilon^i{}_{jk}b^j b^k = 0$ is sued. Thus, in this solution the field $b^i \in \Gamma(M; \mathfrak{so}(3, \mathbb{C})_P)$ is constant on $M = \Sigma \times \mathbb{R}$. Next is to find b^i in the case of matter located at a point.

Solutions for Matter Located at a Point

If there is matter located at a point in $\Sigma_t = \Sigma = \mathbb{R}^3$, one, thus, has a spherically symmetric system in a static case $J^0_i \neq 0$, $J^a_i = 0$. Let the origin $(0,0,0) \in \mathbb{R}^3$ to be that point, therefore, the charge (33) is given by $Q^i(x) = Q^i_0 \delta^3(x)$, so $\int\limits_{\mathbb{R}^3} Q^i_0 \delta^3(x) = Q^i_0 = \text{const.}$ (conservation of the charges). In order to get the same solution as in Equations (35) and (36), the field b^i is kept to be a constant, and in Equation (33),

$$J^0_k(x) = Q_k(x) = -i\epsilon_{kij}\left(\partial_a K^i\right)\left(\partial^a K^j\right)(b^i u^j - b^j u^i), \tag{37}$$

$Q^i(x) = Q^i_0 \delta^3(x)$ is used.

For spherical symmetry, the functions K^i are given by $K^i = c^i/r$ (Equation (34)), therefore,

$$Q_k(x) = -2ig^{rr}\left(\partial_r K^i \partial_r K^j\right)\left(\epsilon_{kij} b^i u^j\right) = -i2\frac{c^i c^j}{r^4}\left(\epsilon_{kij} b^i u^j\right).$$

Therefore, in order to get $Q^i = Q^i_0 \delta^3(x)$, one replaces $1/r^4$ with $1/(r^4 + \epsilon^4)$, for some infinitesimal parameter $\epsilon \to 0^+$, and looks for a solution for the field u^j like

$$u^j = b^j + \epsilon f b^j + \epsilon a^j/(-2i\pi^2\sqrt{2}),$$

for some function f on M that is needed for satisfying $Du^i = 0$ and a constant vector field $a^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$. With that, one obtains (for $i, j \neq k$):

$$Q_k(r) = -2i\frac{c^i c^j}{r^4 + \epsilon^4}\epsilon_{kij} b^i \left(b^j + \epsilon f b^j + \frac{\epsilon}{-2i\pi^2\sqrt{2}} a^j\right) = \frac{1}{\pi^2\sqrt{2}}\frac{\epsilon}{r^4 + \epsilon^4}\epsilon_{kij}(c^i b^i)(c^j a^j).$$

Comparing with $Q^i(x) = Q^i_0 \delta^3(x)$, one finds $\epsilon^k{}_{ij}(c^i b^i)(c^j a^j) = Q^k_0 = \text{const.}$, and, by imposing $(c_i a_i)(c^i a^i) = 1$ with $(c_i b_i)(c^i a^i) = 0$:

$$(cb)^2 = (c_i b_i)(c^i b^i) = Q_{0k} Q^k_0,$$

thus, $c^i b^i = e^i \sqrt{Q_{0k} Q^k_0}$ are chosen for $\|e^i\| = 1$, so that the constant field b^i is determined by $\sqrt{Q_{0k} Q^k_0}$ with free $SO(3,\mathbb{C})$ rotation.

By that (for $r > 0$, $\epsilon \to 0^+$), one obtains the same solutions as in Equations (35) and (36), but with $c^i b^i = e^i \sqrt{Q_{0k} Q^k_0}$ for $\|e^i\| = 1$ and $\sum\limits_{i=1}^{3} c_i = 0$. Since $c^i b^i$ is a finite value, it is not sufficient to let the constants c_i take arbitrary values, so chosen them to be $(c_i) = (1, 1, -2)$.

By that, an example for the possibility of solving the equations of motion in *BF* theory without needing to use a gravitational metric on M is given, so that one needs just to use a vector field $\psi^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$, which is defined in the spin current $J^\mu_k = 2i\varepsilon_{kij}\left(D_\nu \psi^i\right)\Sigma^{\mu\nu j}$ of matter, Equation (16). Furthermore, it is shown that the solutions depend on the symmetry of the system, since one needs some vector v that satisfies $v^a \partial_a K^i = 0$, $v^a E^i_a = 0$, $v_a B^{ai} = 0$ for obtaining the solutions.

4. Solutions for Cylindrically Symmetric System

In a cylindrically symmetric system, the matter is considered to be homogeneously located along the Z-axis. Similar to the above-considered spherical symmetry, one searches for the field $\psi^i = K^i b^i \in \Gamma(M; \mathfrak{so}(3,\mathbb{C})_P)$, and then calculates v, E^i, B^i and J^{0i}, according to Equations (31)–(33). The vector $v \in \Gamma(M; T\Sigma)$ satisfies $v^a D_a \psi^i = 0$, $v^a E^i_a = 0$ and $v^a B^i_a = 0$, thus, it is the Killing vector. Here, the solution in the vacuum ($u = b$) to be found and, then, to be to be applied to a matter located homogeneously along the Z-axis. The needed information for solving the equations of motion is only the spin charge $Q^i(x)$, Equation (33). As it was mentioned before, there is no need to use a gravitational metric, a standard metric to be used instead. In cylindrical symmetry, the cylindrical coordinates

(ρ, φ, z) on the space-like slice $\Sigma_t = \Sigma = \mathbb{R}^3$ of constant time t to be used. Letting the vector field ψ^i to depend only on the radius ρ, one gets (for $D^2\psi^i = 0$ and $Db^i = 0$):

$$D^2\psi^i = D^2(K^i b^i) = b^i \nabla^2 K^i = b^i \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}K^i\right) = 0 \Rightarrow K^i = c^i \log(\rho), \tag{38}$$

so $\psi^i = c^i b^i \log(\rho)$, for some constants $c^i \in \mathbb{R}$. Therefore,

$$D\psi^i = c^i b^i (dK^i) = c^i b^i \left(d\rho\partial_\rho + d\varphi\partial_\varphi + dz\partial_z\right)\log(\rho) = \frac{c^i b^i}{\rho} d\rho,$$

thus, the 1-form v ($dv = 0$, $g^{ab} v_a D_b \psi^i = 0$) is

$$v = a_1 d\varphi + a_2 dz,\ a_1, a_2 \in \mathbb{R},$$

where, in the cylindrical symmetry, a_1 and a_2 are kept not depending on the coordinates z and φ. The values of the constants a_1 and a_2 are not significant since $a^i = g^{ij} a_j$ is Killing vector, thus, are set as $a_1 = a_2 = 1$. The used metric g_{ab} here is the standard metric in the cylindrical coordinates because no any other metric is defined. In what follows in this Section, the indices ρ, θ and φ denote the cylindrical components.

Using Equation (32), one gets the solutions of the 1-form B^i and the vector field E^i, namely,

$$B^i = \frac{c^i u^i}{\rho} d\rho = \frac{c^i b^i}{\rho} d\rho, \quad \text{for} \quad u = b,$$

$$E^i = \frac{1}{2}\varepsilon^{abc}\Sigma^i_{bc}\partial_a = \frac{1}{2}\varepsilon^{abc} v_b \left(\partial_c K^i\right) b^i \partial_a = \varepsilon^{\varphi z\rho}\frac{c^i b^i}{2\rho}\partial_\varphi + \varepsilon^{z\varphi\rho}\frac{c^i b^i}{2\rho}\partial_z.$$

By that, one gets:

$$\Sigma^i_{0\rho} = -\Sigma^i_{\rho 0} = \frac{c^i b^i}{\rho}, \quad \Sigma^i_{\rho z} = -\Sigma^i_{z\rho} = -\frac{c^i b^i}{2\rho}, \quad \Sigma^i_{\rho\varphi} = -\Sigma^i_{\varphi\rho} = -\frac{c^i b^i}{2\rho}, \tag{39}$$

while the other components like $\Sigma^i_{0z}, \Sigma^i_{0\varphi}, \ldots$ are zeros.

The matrix χ is obtained using Equation (24) with the solution (38),

$$\chi = \left(K_j \delta^i_j\right) = \log(\rho)\mathrm{diag}(c^1, c^2, c^3),$$

where the constants c^i have to be determined in order to satisfy the condition $\mathrm{Tr}\chi = 0$ (in the vacuum), so $\sum_{i=1}^{3} c_i = 0$. Thus, one gets the curvature $F = \chi\Sigma + \chi'\bar{\Sigma}$ (with setting $\chi' = 0$ in the vacuum [15]),

$$F^i_{0\rho} = \chi^i{}_j \Sigma^j_{0\rho} = \frac{\log(\rho)}{\rho} c^i c^i b^i, \qquad F^i_{\rho z} = \chi^i{}_j \Sigma^j_{\rho z} = -\frac{\log(\rho)}{2\rho} c^i c^i b^i, \qquad F^i_{\rho\varphi} = \chi^i{}_j \Sigma^j_{\rho\varphi} = -\frac{\log(\rho)}{2\rho} c^i c^i b^i. \tag{40}$$

Using the gauge $A^i_\rho = 0$, with letting A^i depend only on ρ, one obtains

$$A^i_0 = -(\log(\rho))^2 (c^i)^2 b^i, \qquad A^i_z = -\frac{1}{2}(\log(\rho))^2 (c^i)^2 b^i, \qquad A^i_\varphi = -\frac{1}{2}(\log(\rho))^2 (c^i)^2 b^i,$$

where the field $b^i \in \Gamma(M; \mathfrak{so}(3, \mathbb{C})_P)$ is constant on $M = \Sigma \times \mathbb{R}$.

As in the spherical symmetry case, one finds b^i by using the spin charge $Q^i(x)$, which is given by Equation (33). Since the system is static and the matter homogeneously located along the Z-axis, the spin charge $Q^i(\rho, \varphi, z)$ is given by $Q^i(\rho, \varphi, z) = Q^i_0 \delta(\rho)/2\pi\rho$, which yields $\int_0^\infty \int_0^{2\pi} \rho d\varphi d\rho Q^i_0 \delta(\rho)/2\pi\rho = Q^i_0$ for each point of Z. Here, Q^i_0 is the point charge located at each point of the Z-axis.

In order to get the same solution, as in Equations (39) and (40), the field b^i is kept to be constant, and in Equation (33):

$$J^0_k(x) = Q_k(x) = -i\epsilon_{kij}\left(\partial_a K^i\right)\left(\partial^a K^j\right)(b^i u^j - b^j u^i), \tag{41}$$

$Q^i(x) = Q^i_0\delta(\rho)/2\pi\rho$ is used.

In cylindrical symmetry, the functions K^i are given by $K^i = c^i\log(\rho)$, Equation (38), therefore,

$$Q_k(x) = -2ig^{\rho\rho}\left(\partial_\rho K^i\partial_\rho K^j\right)\left(\epsilon_{kij}b^i u^j\right) = -2i\frac{c^i c^j}{\rho^2}\left(\epsilon_{kij}b^i u^j\right).$$

Therefore, in order to get $Q^i = Q^i_0\delta(\rho)/2\pi\rho$, one replaces $1/\rho^2$ with $1/\rho\rho^{1-\epsilon}$, for some infinitesimal parameter $\epsilon \to 0^+$, and chooses a solution for the field u^j like

$$u^j = b^j + \epsilon f b^j + \epsilon a^j/(-4i\pi),$$

for some scalar function f on M that is needed for satisfying $Du^i = 0$, with a constant vector field $a^i \in \Gamma(M;\mathfrak{so}(3,\mathbb{C})_P)$. By that, one obtains (for $i,j \neq k$)

$$Q_k(r) = -2i\frac{c^i c^j}{\rho\rho^{1-\epsilon}}\epsilon_{kij}b^i\left(b^j + \epsilon f b^j + \frac{\epsilon}{-4i\pi}a^j\right) = \frac{1}{2\pi\rho}\frac{\epsilon}{\rho^{1-\epsilon}}\epsilon_{kij}(c^i b^i)(c^j a^j).$$

Comparing with $Q^i = Q^i_0\delta(\rho)/2\pi\rho$, one finds: $\epsilon^k{}_{ij}(c^i b^i)(c^j a^j) = Q^k_0 = \text{const.}$, and, by imposing $(c_i a_i)(c^i a^i) = 1$ with $(c_i b_i)(c^i a^i) = 0$:

$$(cb)^2 = (c_i b_i)(c^i b^i) = Q_{0k}Q^k_0,$$

thus, choosing $c^i b^i = e^i\sqrt{Q_{0k}Q^k_0}$ for $\|e^i\| = 1$, so the constant field $c^i b^i$ is determined by $\sqrt{Q_{0k}Q^k_0}$ with free $SO(3,\mathbb{C})$ rotation.

By that (for $\rho > 0, \epsilon \to 0^+$), one obtains the same solutions as in Equations (39) and (40), but with $c^i b^i = e^i\sqrt{Q_{0k}Q^k_0}$ for $\|e^i\| = 1$ and $\sum_{i=1}^{3} c_i = 0$. Since $c^i b^i$ is a finite value, it is not sufficient to let the constants c_i take arbitrary values, so chosen to be $(c_i) = (1,1,-2)$.

5. Conclusions

In this paper, the *BF* theory has been studied including matter by redefinition of the 2-form B^i as $B^i + \varepsilon^i{}_{jk}\psi^j B^k$, so that one can get $DB^i = 0$, with D being a covariant derivative and Latin letters $i,j,\ldots = 1,2,3$, in the case of non-zero spin current of matter fields. The new field ψ^i is defined using the spin current vector, $J^{\nu i} = (D_\mu\psi_k)\varepsilon^{ki}{}_j\Sigma^{\mu\nu j}$, $\Sigma^{\mu\nu j}$ being the frame field and Greek letters $\mu,\nu,\ldots = 0,1,2,3$, are the space-time tangent indices. It is shown that one can solve the *BF* equations by using only the spin current of matter, that is, it is enough to solve the equations $\delta S/\delta A^i = 0$ (for the action variation with the spin connection A^i), $DB^i = 0$ and $J^{\nu i} = (D_\mu\psi_k)\varepsilon^{ki}{}_j\Sigma^{\mu\nu j}$ without using a gravitational metric on the M 4-manifold and without a need to solve the equation $\delta S/\delta B^i = 0$, which includes the Lagrangian multiplier ϕ_{ij} (a non-physical variable), so that one gets ϕ_{ij} by using the solutions in $\delta S/\delta B^i = 0$. It is found that to obtain the solutions of *BF* theory, it is enough to use (find) the field ψ^i and the Killing vector v (satisfies $v^a D_a\psi^i = 0$) in Euclidean coordinates, where it is convenient to describe the spin currents and their lines in Euclidean coordinates with no need to describe them in curved coordinates. Furthermore, it is possible to obtain the solutions of *BF* theory using only the charges $J^{0i} \neq 0$ when they are given as functions on M in the static case (discussion below Equation (33)). It is shown that the singularities appear in solution of ψ^i, that is related to the idea that the spin current $J^{\mu i}$ is the source for ψ^i, therefore, ψ^i has singularities on the line of that spin current, and the singularities appear by that and not by using a gravitational metric. It is

found that the solutions of *BF* theory equations depend on the symmetry of the system and every two solutions of $(\Sigma^i_{ab}, \Sigma^i_{0a})$ and $(\Sigma^{abi}, \Sigma^{0ai})$ determine a metric (Remark 4), and those solutions are able to be written as $e^I \wedge e^J$ using 1-forms e^I, where $I, J, \ldots = 0, 1, 2, 3$ are the Lorentz indices. Finally, the solutions of *BF* theory are applied to spherically and cylindrically symmetric systems in static case of matter.

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Appendix A. Verifying the General Relativity Equations in the Vacuum Using *BF* Equations

Satisfying of the equation of motion of classical general relativity (GR) in the vacuum is tested using the solutions in *BF* theory; the equations do not include the non-physical variable ϕ. The GR equations in the vacuum are [5]:

$$D_a E^a_i = 0, \quad E^a_i F^i_{ab} = 0, \quad C = \varepsilon_{ijk} E^{ai} E^{bj} F^k_{ab} = 0. \tag{A1}$$

Here, the gauge $e^0_a = 0$ is used with the metric $g_{ab} = \delta_{ij} e^i_a e^j_b$. Note that $D_a E^a_i = 0$ is satisfied when $DB^i = 0$ is satisfied (see the discussion below Equation (10)).

Using Equation (20), in the vacuum $\chi^i{}_j = 0$, it reduces to $F^i(A) = \chi^i{}_j \Sigma^j$, on the spacelike surface of constant time Σ_t, it becomes $F^i(A)_{ab} = \chi^i{}_j \Sigma^j_{ab}$. Using $E^{ai} = \varepsilon^{abc} \Sigma^i_{bc}$, one obtains:

$$F^i(A)_{ab} = \chi^i{}_j \Sigma^j_{ab} = \chi^i{}_j \varepsilon_{abc} E^{jc}.$$

Multiplying it by E^a_i and summing over the contracted indices, one gets:

$$E^a_i F^i(A)_{ab} = E^a_i \chi^i{}_j \varepsilon_{abc} E^{jc} = \chi_{ij} \varepsilon_{abc} E^{ai} E^{jc} = 0,$$

where the fact that the matrix χ_{ij} is symmetric is used. Therefore, the second constraint (A1) is satisfied. Using $F^i(A)_{ab} = \chi^i{}_j \varepsilon_{abc} E^{jc}$ in C, yields

$$C = \varepsilon_{ijk} E^{ai} E^{bj} \chi^k{}_{i'} \varepsilon_{abc} E^{i'c} = \chi_k{}^{i'} \varepsilon^{ijk} \varepsilon_{abc} E^a_i E^b_j E^c_{i'}.$$

Then using $E^a_i = e e^a_i$, where e^a_i is the inverse of the gravitational field e^i_a and $e = \det(e^i_a)$, one obtains:

$$C = e^3 \chi_k{}^{i'} \varepsilon^{ijk} \varepsilon_{abc} e^a_i e^b_j e^c_{i'} = e^3 \chi_k{}^{i'} \varepsilon^{ijk} e^{-1} \varepsilon_{iji'} = 2e^2 \mathrm{Tr}\chi.$$

However, in the vacuum $\mathrm{Tr}\chi = 0$ (Equation (21)), thus, $C = \varepsilon_{ijk} E^{ai} E^{bj} F^k_{ab} = 0$ is satisfied. By that one finds that the general relativity constraints are satisfied in the vacuum using the equations of motion of *BF* theory. In existence of matter, the first two equations are still satisfied, the third equation becomes $C = \varepsilon_{ijk} E^{ai} E^{bj} F^k_{ab} = 2e^2 \mathrm{Tr}\chi = -2e^2 \mathrm{Tr}(T^i{}_j)$.

Appendix B. Calculating $\frac{\delta S_{\text{matter}}}{\delta \Sigma^i}$

Starting from

$$\left.\frac{\delta S_{\text{matter}}}{\delta B^\ell_{\sigma_1\rho_1}}\right|_{\text{constraint}} = \frac{\delta S_{\text{matter}}}{\delta \Sigma^\ell_{\sigma_1\rho_1}} = \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g_{\mu_1\nu_1}} \frac{\sqrt{-g}}{-2} \frac{\delta g_{\mu_1\nu_1}}{\delta \Sigma^\ell_{\sigma_1\rho_1}} = T^{\mu_1\nu_1} \frac{\sqrt{-g}}{-2} \frac{\delta g_{\mu_1\nu_1}}{\delta \Sigma^\ell_{\sigma_1\rho_1}}, \tag{A2}$$

where $T^{\mu\nu}$ is energy-momentum tensor. Using

$$\delta g_{\mu_1\nu_1} = \frac{1}{\sqrt{-g}} \delta\left(\sqrt{-g} g_{\mu_1\nu_1}\right) - \frac{1}{\sqrt{-g}} g_{\mu_1\nu_1} \delta\left(\sqrt{-g}\right),$$

one obtains:

$$\sqrt{-g}\frac{\delta g_{\mu_1\nu_1}}{\delta\Sigma^{\ell}_{\sigma_1\rho_1}} = \frac{\delta\left(\sqrt{-g}g_{\mu_1\nu_1}\right)}{\delta\Sigma^{\ell}_{\sigma_1\rho_1}} - g_{\mu_1\nu_1}\frac{\delta(\sqrt{-g})}{\delta\Sigma^{\ell}_{\sigma_1\rho_1}} = I_1 - I_2. \tag{A3}$$

Using the Urbantke formula [12],

$$\sqrt{-g}g_{\mu_1\nu_1} = \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{ijk}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\Sigma^k_{\sigma\nu_1}, \tag{A4}$$

one gets:

$$\begin{aligned}\delta\left(\sqrt{-g}g_{\mu_1\nu_1}\right) &= \delta\left(\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{ijk}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\Sigma^k_{\sigma\nu_1}\right)\\ &= \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{ijk}\left(\delta\Sigma^i_{\mu_1\mu}\right)\Sigma^j_{\nu\rho}\Sigma^k_{\sigma\nu_1} + \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{ijk}\Sigma^i_{\mu_1\mu}\left(\delta\Sigma^j_{\nu\rho}\right)\Sigma^k_{\sigma\nu_1} + \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{ijk}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\left(\delta\Sigma^k_{\sigma\nu_1}\right).\end{aligned}$$

Calculating I_1 in (A3):

$$\begin{aligned}I_1 &= \frac{\delta}{\delta\Sigma^{\ell}_{\sigma_1\rho_1}}\left(\sqrt{-g}g_{\mu_1\nu_1}\right)\\ &= \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{ijk}\Sigma^j_{\nu\rho}\Sigma^k_{\sigma\nu_1}\left(\delta^i_\ell\delta^{\sigma_1}_{\mu_1}\delta^{\rho_1}_{\mu}\right) + \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{ijk}\Sigma^i_{\mu_1\mu}\Sigma^k_{\sigma\nu_1}\left(\delta^j_\ell\delta^{\sigma_1}_{\nu}\delta^{\rho_1}_{\rho}\right)\\ &\quad + \varepsilon^{\mu\nu\rho\sigma}\varepsilon_{ijk}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\left(\delta^k_\ell\delta^{\sigma_1}_{\sigma}\delta^{\rho_1}_{\nu_1}\right)\\ &= \varepsilon^{\rho_1\nu\rho\sigma}\varepsilon_{\ell jk}\Sigma^j_{\nu\rho}\Sigma^k_{\sigma\nu_1}\left(\delta^{\sigma_1}_{\mu_1}\right) + \varepsilon^{\mu\sigma_1\rho_1\sigma}\varepsilon_{i\ell k}\Sigma^i_{\mu_1\mu}\Sigma^k_{\sigma\nu_1} + \varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{ij\ell}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\left(\delta^{\rho_1}_{\nu_1}\right)\\ &= \varepsilon^{\rho_1\nu\rho\sigma}\varepsilon_{\ell ij}\Sigma^i_{\nu\rho}\Sigma^j_{\sigma\nu_1}\left(\delta^{\sigma_1}_{\mu_1}\right) + \varepsilon^{\mu\sigma_1\rho_1\sigma}\varepsilon_{i\ell j}\Sigma^i_{\mu_1\mu}\Sigma^j_{\sigma\nu_1} + \varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{ij\ell}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\left(\delta^{\rho_1}_{\nu_1}\right)\\ &= \varepsilon^{\rho_1\nu\rho\sigma}\varepsilon_{\ell ij}\Sigma^i_{\nu\rho}\Sigma^j_{\sigma\nu_1}\left(\delta^{\sigma_1}_{\mu_1}\right) - \varepsilon^{\mu\sigma_1\rho_1\sigma}\varepsilon_{\ell ij}\Sigma^i_{\mu_1\mu}\Sigma^j_{\sigma\nu_1} + \varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{\ell ij}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\left(\delta^{\rho_1}_{\nu_1}\right).\end{aligned}$$

To calculate I_2, one uses

$$\sqrt{-g} = \frac{i}{6}\varepsilon^{\mu\nu\rho\sigma}\delta_{ij}\Sigma^i_{\mu\nu}\Sigma^j_{\rho\sigma},$$

hence,

$$\begin{aligned}I_2 = g_{\mu_1\nu_1}\frac{\delta}{\delta\Sigma^{\ell}_{\sigma_1\rho_1}}\left(\sqrt{-g}\right) &= \frac{i}{6}g_{\mu_1\nu_1}\varepsilon^{\mu\nu\rho\sigma}\delta_{ij}\left(\delta^i_\ell\delta^{\sigma_1}_{\mu}\delta^{\rho_1}_{\nu}\right)\Sigma^j_{\rho\sigma} + \frac{i}{6}g_{\mu_1\nu_1}\varepsilon^{\mu\nu\rho\sigma}\delta_{ij}\Sigma^i_{\mu\nu}\left(\delta^j_\ell\delta^{\sigma_1}_{\rho}\delta^{\rho_1}_{\sigma}\right)\\ &= \frac{i}{6}g_{\mu_1\nu_1}\varepsilon^{\sigma_1\rho_1\rho\sigma}\delta_{i\ell}\Sigma^i_{\rho\sigma} + \frac{i}{6}g_{\mu_1\nu_1}\varepsilon^{\mu\nu\sigma_1\rho_1}\delta_{i\ell}\Sigma^i_{\mu\nu} = \frac{i}{3}g_{\mu_1\nu_1}\varepsilon^{\mu\nu\sigma_1\rho_1}\delta_{i\ell}\Sigma^i_{\mu\nu}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sqrt{-g}\frac{\delta g_{\mu_1\nu_1}}{\delta\Sigma^{\ell}_{\sigma_1\rho_1}} &= I_1 - I_2\\ &= \varepsilon^{\rho_1\nu\rho\sigma}\varepsilon_{\ell ij}\Sigma^i_{\nu\rho}\Sigma^j_{\sigma\nu_1}\left(\delta^{\sigma_1}_{\mu_1}\right) - \varepsilon^{\mu\sigma_1\rho_1\sigma}\varepsilon_{\ell ij}\Sigma^i_{\mu_1\mu}\Sigma^j_{\sigma\nu_1} + \varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{\ell ij}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\left(\delta^{\rho_1}_{\nu_1}\right)\\ &\quad - \frac{i}{3}g_{\mu_1\nu_1}\left(\varepsilon^{\mu\nu\sigma_1\rho_1}\delta_{i\ell}\Sigma^i_{\mu\nu}\right).\end{aligned} \tag{A5}$$

By this, Equation (A2) reads:

$$\begin{aligned}\left.\frac{\delta S_{\text{matter}}}{\delta B^{\ell}_{\sigma_1\rho_1}}\right|_{\text{constraint}} &= \frac{-1}{2}T^{\mu_1\nu_1}(I_1 - I_2)\\ &= \frac{1}{2}T^{\mu_1\nu_1}\varepsilon^{\mu\sigma_1\rho_1\sigma}\varepsilon_{\ell ij}\Sigma^i_{\mu_1\mu}\Sigma^j_{\sigma\nu_1} - \frac{1}{2}T^{\mu_1\nu_1}\varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{\ell ij}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\left(\delta^{\rho_1}_{\nu_1}\right)\\ &- \frac{1}{2}T^{\mu_1\nu_1}\varepsilon^{\rho_1\nu\rho\sigma}\varepsilon_{\ell ij}\Sigma^i_{\nu\rho}\Sigma^j_{\sigma\nu_1}\left(\delta^{\sigma_1}_{\mu_1}\right) + \frac{i}{6}T^{\mu_1\nu_1}g_{\mu_1\nu_1}\varepsilon^{\mu\nu\sigma_1\rho_1}\delta_{i\ell}\Sigma^i_{\mu\nu}\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{A6}$$

Using $T^{\mu_1\nu_1} = T^{IJ}e^{\mu_1}_I e^{\nu_1}_J$, the first term of (A6) reads:

$$\begin{aligned}2I_1 &= T^{\mu_1\nu_1}\varepsilon^{\mu\nu\sigma_1\rho_1}\varepsilon_{ij\ell}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\nu_1} = T^{IJ}e^{\mu_1}_I e^{\nu_1}_J\varepsilon^{\mu\nu\sigma_1\rho_1}\varepsilon_{ij\ell}P^i_{KL}P^j_{K_1L_1}e^K_{\mu_1}e^L_\mu e^{K_1}_\nu e^{L_1}_{\nu_1}\\ &= T^{IJ}\delta^K_I\delta^{L_1}_J\varepsilon^{\mu\nu\sigma_1\rho_1}\varepsilon_{ij\ell}P^i_{KL}P^j_{K_1L_1}e^L_\mu e^{K_1}_\nu = T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}\varepsilon_{ij\ell}P^i_{IL}P^j_{K_1J}e^L_\mu e^{K_1}_\nu\\ &= T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}\varepsilon_{ij\ell}P^i_{IL}P^j_{K_1J}\left(P^{LK_1}_n\Sigma^n_{\mu\nu} + \bar{P}^{LK_1}_n\bar{\Sigma}^n_{\mu\nu}\right),\end{aligned}$$

where

$$e^L_{[\mu}e^{K_1}_{\nu]} = P^{LK_1}_n\Sigma^n_{\mu\nu} + \bar{P}^{LK_1}_n\bar{\Sigma}^n_{\mu\nu}$$

is used.

One uses the following property of the self-dual projection:

$$-\frac{1}{2}\varepsilon_{ij\ell}P^i_{IJ}P^j_{KL} = \frac{1}{4}(\eta_{JK}P_{\ell IL} - \eta_{JL}P_{\ell IK}) - \frac{1}{4}(I \leftrightarrow J), \tag{A7}$$

which can be easily checked when $I = 0$ and J, K, L are spatial indices, and when $I = K = 0$ and J, L are spatial indices, so the Lorentz invariance asserts that this property is also satisfied when I, J, K, L are all spatial indices. By using this property, one obtains:

$$\begin{aligned}2I_1 &= T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}\frac{-1}{2}(\eta_{LK_1}P_{\ell IJ} - \eta_{LJ}P_{\ell IK_1} - \eta_{IK_1}P_{\ell LJ} + \eta_{IJ}P_{\ell LK_1})e^L_\mu e^{K_1}_\nu\\ &= T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}\frac{-1}{2}\left(P_{\ell IJ}g_{\mu\nu} - \eta_{LJ}P_{\ell IK_1}e^L_\mu e^{K_1}_\nu - \eta_{IK_1}P_{\ell LJ}e^L_\mu e^{K_1}_\nu + \eta_{IJ}\Sigma_{\ell\mu\nu}\right)\\ &= T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}\frac{-1}{2}\left(-\eta_{LJ}P_{\ell IK_1}e^L_\mu e^{K_1}_\nu - \eta_{IK_1}P_{\ell LJ}e^L_\mu e^{K_1}_\nu + \eta_{IJ}\Sigma_{\ell\mu\nu}\right)\\ &= \frac{1}{2}T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}(\eta_{LJ}P_{\ell IK_1} + \eta_{IK_1}P_{\ell LJ})e^L_{[\mu}e^{K_1}_{\nu]} - \frac{1}{2}\eta_{IJ}T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}\Sigma_{\ell\mu\nu},\end{aligned}$$

hence,

$$\begin{aligned}2I_1 &= \frac{1}{2}T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}(\eta_{LJ}P_{\ell IK_1} + \eta_{IK_1}P_{\ell LJ})\left(P^{LK_1}_i\Sigma^i_{\mu\nu} + \bar{P}^{LK_1}_i\bar{\Sigma}^i_{\mu\nu}\right)\\ &- \frac{1}{2}\eta_{IJ}T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}\Sigma_{\ell\mu\nu}.\end{aligned}$$

Finally,

$$\begin{aligned}I_1 &= \frac{1}{4}T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}(\eta_{LJ}P_{\ell IK_1} + \eta_{IK_1}P_{\ell LJ})\left(P^{LK_1}_i\Sigma^i_{\mu\nu} + \bar{P}^{LK_1}_i\bar{\Sigma}^i_{\mu\nu}\right)\\ &- \frac{1}{4}\eta_{IJ}T^{IJ}\varepsilon^{\mu\nu\sigma_1\rho_1}\Sigma_{\ell\mu\nu}.\end{aligned}$$

Furthermore, the second term of (A6) becomes

$$\begin{aligned}
2I_2 &= -T^{\mu_1\nu_1}\varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{ij\ell}\Sigma^i_{\mu_1\mu}\Sigma^j_{\nu\rho}\delta^{\rho_1}_{\nu_1} = -T^{IJ}e^{\mu_1}_I e^{\rho_1}_J \varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{ij\ell}P^i_{KL}P^j_{K_1L_1}e^K_{\mu_1}e^L_\mu e^{K_1}_\nu e^{L_1}_\rho \\
&= -T^{IJ}\delta^K_I e^{\rho_1}_J \varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{ij\ell}P^i_{KL}P^j_{K_1L_1}e^L_\mu e^{K_1}_\nu e^{L_1}_\rho = -T^{IJ}e^{\rho_1}_J \varepsilon^{\mu\nu\rho\sigma_1}\varepsilon_{ij\ell}P^i_{IL}P^j_{K_1L_1}e^L_\mu e^{K_1}_\nu e^{L_1}_\rho \\
&= -T^{IJ}\varepsilon^{\mu\nu\rho\sigma_1}\frac{-1}{2}\left(\eta_{LK_1}P_{\ell IL_1} - \eta_{LL_1}P_{\ell IK_1} - \eta_{IK_1}P_{\ell LL_1} + \eta_{IL_1}P_{\ell LK_1}\right)e^{\rho_1}_J e^L_\mu e^{K_1}_\nu e^{L_1}_\rho = \\
&\frac{1}{2}T^{IJ}\varepsilon^{\mu\nu\rho\sigma_1}\left(P_{\ell IL_1}g_{\mu\nu}e^{L_1}_\rho - P_{\ell IK_1}g_{\mu\rho}e^{K_1}_\nu - \eta_{IK_1}P_{\ell LL_1}e^L_\mu e^{K_1}_\nu e^{L_1}_\rho + \eta_{IL_1}P_{\ell LK_1}e^L_\mu e^{K_1}_\nu e^{L_1}_\rho\right)e^{\rho_1}_J \\
&= \frac{1}{2}T^{IJ}\varepsilon^{\mu\nu\rho\sigma_1}\left(-\eta_{IK_1}P_{\ell LL_1} + \eta_{IL_1}P_{\ell LK_1}\right)e^{\rho_1}_J e^L_\mu e^{K_1}_\nu e^{L_1}_\rho.
\end{aligned}$$

Using $\varepsilon^{\mu\nu\rho\sigma_1}e^L_\mu e^{K_1}_\nu e^{L_1}_\rho = e\varepsilon^{LK_1L_1M}e^{\sigma_1}_M$, where e is the determinant of (e^I_μ), one obtains:

$$\begin{aligned}
2I_2 &= \frac{1}{2}eT^{IJ}\left(-\eta_{IK_1}P_{\ell LL_1} + \eta_{IL_1}P_{\ell LK_1}\right)e^{\rho_1}_J\varepsilon^{LK_1L_1M}e^{\sigma_1}_M \\
&= \frac{1}{2}eT^{IJ}\left(-\eta_{IK_1}P_{\ell LL_1} + \eta_{IL_1}P_{\ell LK_1}\right)\varepsilon^{LK_1L_1M}e^{[\rho_1}_J e^{\sigma_1]}_M \\
&= \frac{1}{2}eT^{IJ}\left(-\eta_{IK_1}P_{\ell LL_1} + \eta_{IL_1}P_{\ell LK_1}\right)\varepsilon^{LK_1L_1M}e^{\rho_1}_{[J}e^{\sigma_1}_{M]} \\
&= \frac{1}{2}eT^{IJ}\left(-\eta_{IK_1}P_{\ell LL_1} + \eta_{IL_1}P_{\ell LK_1}\right)\varepsilon^{LK_1L_1M}\left(P_{iJM}\Sigma^{i\rho_1\sigma_1} + \bar{P}_{iJM}\bar{\Sigma}^{i\rho_1\sigma_1}\right).
\end{aligned}$$

One uses the self-dual property,

$$P^i_{IJ}\varepsilon^{IJKL} = -2iP^{iKL},$$

to get

$$2I_2 = -ieT^{IJ}\left(\eta_{IK_1}P^{K_1M}_\ell + \eta_{IL_1}P^{L_1M}_\ell\right)\left(P_{iJM}\Sigma^{i\rho_1\sigma_1} + \bar{P}_{iJM}\bar{\Sigma}^{i\rho_1\sigma_1}\right).$$

Furthermore, using

$$\frac{1}{2!}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\Sigma^i_{\rho\sigma} = \left(*\Sigma^i\right)^{\mu\nu} = \left(-i\Sigma^i\right)^{\mu\nu} = -i\Sigma^{\mu\nu i}, \quad e = \sqrt{-g},$$

one obtains:

$$I_2 = -\frac{1}{4}T^{IJ}\left(\eta_{IK_1}P^{K_1M}_\ell + \eta_{IL_1}P^{L_1M}_\ell\right)\left(-P_{iJM}\varepsilon^{\rho_1\sigma_1\rho\sigma}\Sigma^i_{\rho\sigma} + \bar{P}_{iJM}\varepsilon^{\rho_1\sigma_1\rho\sigma}\bar{\Sigma}^i_{\rho\sigma}\right).$$

One gets the third term of (A6) by the replacing $\sigma_1 \leftrightarrow \rho_1$ in I_2, and, with reversing its sign:

$$I_3 = +\frac{1}{4}T^{IJ}\left(\eta_{IK_1}P^{K_1M}_\ell + \eta_{IL_1}P^{L_1M}_\ell\right)\left(-P_{iJM}\varepsilon^{\sigma_1\rho_1\rho\sigma}\Sigma^i_{\rho\sigma} + \bar{P}_{iJM}\varepsilon^{\sigma_1\rho_1\rho\sigma}\bar{\Sigma}^i_{\rho\sigma}\right) = I_2,$$

therefore,

$$I_2 + I_3 = +\frac{1}{2}T^{IJ}\left(\eta_{IK_1}P^{K_1M}_\ell + \eta_{IL_1}P^{L_1M}_\ell\right)\left(-P_{iJM}\varepsilon^{\sigma_1\rho_1\rho\sigma}\Sigma^i_{\rho\sigma} + \bar{P}_{iJM}\varepsilon^{\sigma_1\rho_1\rho\sigma}\bar{\Sigma}^i_{\rho\sigma}\right).$$

The fourth term of (A6) is

$$I_4 = \frac{i}{6}T^{\mu_1\nu_1}g_{\mu_1\nu_1}\varepsilon^{\mu\nu\sigma_1\rho_1}\Sigma_{\mu\nu\ell}.$$

Using $T^{\mu\nu}g_{\mu\nu} = T^{IJ}\eta_{IJ} = T$, Equation (A6) reads:

$$\begin{aligned}
&\left.\frac{\delta S_{\text{matter}}}{\delta B^{\ell}_{\sigma_1\rho_1}}\right|_{\text{constraint}} = I_1 + I_2 + I_3 + I_4 \\
&= \frac{1}{4}T^{IJ}\left(\eta_{LJ}P_{\ell IK} + \eta_{IK}P_{\ell LJ}\right)\left(P_i^{LK}\varepsilon^{\mu\nu\sigma_1\rho_1}\Sigma^i_{\mu\nu} + \bar{P}_i^{LK}\varepsilon^{\mu\nu\sigma_1\rho_1}\bar{\Sigma}^i_{\mu\nu},\right) - \frac{1}{4}T\varepsilon^{\mu\nu\sigma_1\rho_1}\Sigma_{\ell\mu\nu} \\
&+ \frac{1}{2}T^{IJ}\left(\eta_{IK}P^{KM}_{\ell} + \eta_{IL}P^{LM}_{\ell}\right)\left(-P_{iJM}\varepsilon^{\sigma_1\rho_1\rho\sigma}\Sigma^i_{\rho\sigma} + \bar{P}_{iJM}\varepsilon^{\sigma_1\rho_1\rho\sigma}\bar{\Sigma}^i_{\rho\sigma}\right) \\
&+ \frac{i}{6}T\varepsilon^{\mu\nu\sigma_1\rho_1}\Sigma_{\mu\nu\ell}.
\end{aligned}$$

For brevity, this form is written as

$$\left.\frac{\delta S_{\text{matter}}}{\delta B^{\ell}_{\sigma_1\rho_1}}\right|_{\text{constraint}} = \varepsilon^{\mu\nu\sigma_1\rho_1}T_{\ell i}\Sigma^i_{\mu\nu} + \varepsilon^{\mu\nu\sigma_1\rho_1}\xi_{\ell i}\bar{\Sigma}^i_{\mu\nu},$$

or

$$\left.\frac{\delta S_{\text{matter}}}{\delta B^{i}}\right|_{\text{constraint}} = \frac{\delta S_m}{\delta \Sigma^i} = T_{ij}\Sigma^j + \xi_{ij}\bar{\Sigma}^j, \tag{A8}$$

with the complex matrices T and ξ given by

$$\begin{aligned}
T_{ij} &= \frac{1}{4}T^{IJ}\left(\eta_{LJ}P_{iIK} + \eta_{IK}P_{iLJ}\right)P_j^{LK} - \frac{1}{2}T^{IJ}\left(\eta_{IK}P_i^{KM} + \eta_{IL}P_i^{LM}\right)P_{jJM} \\
&\quad + T\left(\frac{i}{6} - \frac{1}{4}\right)\delta_{ij} \\
&= -\frac{1}{2}\left(P_{iIK}P_{jJ}{}^{K}\right)T^{IJ} + \left(\frac{i}{6} - \frac{1}{4}\right)T\delta_{ij},
\end{aligned} \tag{A9}$$

and

$$\begin{aligned}
\xi_{ij} &= \frac{1}{4}T^{IJ}\left(\eta_{LJ}P_{iIK} + \eta_{IK}P_{iLJ}\right)\bar{P}_j^{LK} + \frac{1}{2}T^{IJ}\left(\eta_{IK}P_i^{KM} + \eta_{IL}P_i^{LM}\right)\bar{P}_{jJM} \\
&= \frac{3}{2}\left(P_{iIK}\bar{P}_{jJ}{}^{K}\right)T^{IJ}.
\end{aligned}$$

The self-dual projection matrices P^i_{IJ} are given in Equation (1),

$$P^i_{IJ} = \frac{1}{2}\varepsilon^i{}_{jk}, \text{ for } I = i, J = j, \text{ and } P^i_{0j} = -P^i_{j0} = -\frac{i}{2}\delta^i_j, \text{ for } I = 0, J = j \neq 0. \tag{A10}$$

and $\bar{P}^i_{IJ}$ are their complex conjugate. One finds:

$$\begin{aligned}
&\left(P_{iIK}P^K_{jJ}\right)T^{IJ} = \left(P_{iI0}P^0_{jJ} + P_{iI\ell}P^{\ell}_{jJ}\right)T^{IJ} \\
&= P_{i\ell 0}P^0_{jm}T^{\ell m} + P_{i0\ell}P^{\ell}_{j0}T^{00} + P_{im\ell}P^{\ell}_{j0}T^{m0} + P_{i0\ell}P^{\ell}_{jm}T^{0m} + P_{in\ell}P^{\ell}_{jm}T^{nm} \\
&= -P_{i\ell 0}P_{jm0}T^{\ell m} + P_{i0\ell}P^{\ell}_{j0}T^{00} + P_{im\ell}P^{\ell}_{j0}T^{m0} + P_{i0\ell}P^{\ell}_{jm}T^{0m} + P_{in\ell}P^{\ell}_{jm}T^{nm} \\
&= -\frac{i}{2}\delta_{i\ell}\frac{i}{2}\delta_{jm}T^{\ell m} + \frac{-i}{2}\delta_{i\ell}\frac{-i}{2}\delta^{\ell}_j T^{00} + \frac{1}{2}\varepsilon_{im\ell}\frac{-i}{2}\delta^{\ell}_j T^{m0} + \frac{-i}{2}\delta_{i\ell}\frac{1}{2}\varepsilon^{\ell}_{jm}T^{0m} + \frac{1}{2}\varepsilon_{in\ell}\frac{1}{2}\varepsilon^{\ell}_{jm}T^{nm} \\
&= \frac{1}{4}T_{ij} - \frac{1}{4}\delta_{ij}T^{00} + \frac{-i}{4}\varepsilon_{imj}T^{m0} + \frac{-i}{4}\varepsilon_{jmi}T^{0m} + \frac{1}{4}\left(\delta_{ij}\delta_{nm} - \delta_{im}\delta_{nj}\right)T^{nm} \\
&= \frac{1}{4}T_{ij} - \frac{1}{4}\delta_{ij}T^{00} + \frac{-i}{4}\varepsilon_{imj}T^{0m} + \frac{-i}{4}\varepsilon_{jmi}T^{0m} + \frac{1}{4}\delta_{ij}\delta_{nm}T^{nm} - \frac{1}{4}T_{ji} \\
&= -\frac{1}{4}\delta_{ij}T^{00} + \frac{1}{4}\delta_{ij}\delta_{nm}T^{nm} = \frac{1}{4}\delta_{ij}\eta_{00}T^{00} + \frac{1}{4}\delta_{ij}\eta_{nm}T^{nm} = \frac{1}{4}\delta_{ij}\eta_{IJ}T^{IJ} = \frac{1}{4}\delta_{ij}T.
\end{aligned}$$

Using this in Equation (A9), one gets:

$$T_{ij} = -\frac{1}{8}\delta_{ij}T + \left(\frac{i}{6} - \frac{1}{4}\right)T\delta_{ij} = \left(\frac{i}{6} - \frac{3}{8}\right)T\delta_{ij}. \tag{A11}$$

Appendix C. Calculating ψ^i

Here, it is verifired that the (0,2)-tensor field $\Sigma^{\mu\nu i}$ defined in

$$-i\Sigma^{\mu\nu i} = \frac{1}{2!}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\Sigma^i_{\rho\sigma}, \quad e = \det(e^I_\mu),$$

is inversion of the 2-form $\Sigma^i_{\mu\nu}$, that is $\Sigma^i_{\mu\nu}\Sigma^{\mu\nu}_j = \delta^i_j$. Multiplying by $\Sigma_{\mu\nu j}$ and summing up over contracted indices, one gets:

$$-i\Sigma_{\mu\nu j}\Sigma^{\mu\nu i} = \frac{1}{2}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\Sigma_{\mu\nu j}\Sigma^i_{\rho\sigma}.$$

Then, using $\Sigma^i_{\mu\nu} = P^i_{IJ}e^I_\mu e^J_\nu$, one finds:

$$-i\Sigma_{\mu\nu j}\Sigma^{\mu\nu i} = \frac{1}{2}e^{-1}\varepsilon^{\mu\nu\rho\sigma}P_{jIJ}P^i_{KL}e^I_\mu e^J_\nu e^K_\rho e^L_\sigma.$$

Using $\varepsilon^{\mu\nu\rho\sigma}e^I_\mu e^J_\nu e^K_\rho e^L_\sigma = e\varepsilon^{IJKL}$, one gets:

$$-i\Sigma_{\mu\nu j}\Sigma^{\mu\nu i} = \frac{1}{2}e^{-1}P_{jIJ}P^i_{KL}e\varepsilon^{IJKL} = \frac{1}{2}P_{jIJ}P^i_{KL}\varepsilon^{IJKL}.$$

Then, one uses the self-dual projection property,

$$P^i_{IJ}\varepsilon^{IJKL} = -2iP^{iKL},$$

to obtain

$$-i\Sigma_{\mu\nu j}\Sigma^{\mu\nu i} = \frac{1}{2}P_{jIJ}(-2iP^{iIJ}) = P_{jIJ}(-iP^{iIJ}).$$

Therefore,

$$\Sigma_{\mu\nu j}\Sigma^{\mu\nu i} = P_{jIJ}P^{iIJ} = \delta^i_j,$$

where the self-dual projection property, $P_{jIJ}P^{iIJ} = \delta^i_j$ is used. The sum is over the contracted indices.

Now, let us calculate the vector field ψ^i, which is given in $J^\mu_k = 2i\varepsilon_{kij}\left(D_\nu\psi^i\right)\Sigma^{\mu\nu j}$, therefore, it is related to the current $J^{\mu i}$, or, in other words, the current is source for ψ^i. If $J^\mu_k = 2i\varepsilon_{kij}\left(D_\nu\psi^i\right)\Sigma^{\mu\nu j}$ is multiplied by $\Sigma^k_{\mu\rho}$ and summed up over contracted indices, one obtains:

$$\begin{aligned}\Sigma^k_{\mu\rho}J^\mu_k &= 2i\varepsilon_{kij}\left(D_\nu\psi^i\right)\Sigma^k_{\mu\rho}\Sigma^{\mu\nu j} = 2i\varepsilon_{kij}\left(D_\nu\psi^i\right)\left(P^k_{IJ}e^I_\mu e^J_\rho\right)\left(P^j_{KL}e^{\mu K}e^{\nu L}\right)\\ &= 2i\varepsilon_{kij}P^k_{IJ}P^j_{KL}\left(D_\nu\psi^i\right)\eta^{IK}e^J_\rho e^{\nu L} = -2i\eta^{IK}\varepsilon_{ikj}P^k_{IJ}P^j_{KL}\left(D_\nu\psi^i\right)e^J_\rho e^{\nu L}.\end{aligned} \tag{A12}$$

Using Equation (A7):

$$\varepsilon_{ikj}P^k_{IJ}P^j_{KL} = -\frac{1}{2}(\eta_{JK}P_{iIL} - \eta_{JL}P_{iIK}) + \frac{1}{2}(\eta_{IK}P_{iJL} - \eta_{IL}P_{iJK}).$$

Multiplying by η^{IK} and summing over I, K, one gets:

$$\eta^{IK}\varepsilon_{ikj}P^k_{IJ}P^j_{KL} = -\frac{1}{2}(P_{iJL} - 0) + \frac{1}{2}(4P_{iJL} - P_{iJL}) = P_{iJL}.$$

Using this formula in (A12), one finds:

$$\Sigma^k_{\mu\rho} J^\mu_k = P^k_{IJ} e^I_\mu e^J_\rho J^\mu_k = -2iP_{iJL}\left(D_\nu \psi^i\right) e^J_\rho e^{\nu L}.$$

Multiplying by e^ρ_K(inverse of e^I_ρ), one gets $P^k_{IK} e^I_\mu J^\mu_k = -2iP_{iKL}\left(D_\nu \psi^i\right) e^{\nu L}$. From this equation, one can see that $J^{\mu i} = 2ig^{\mu\nu} D_\nu \psi^i + v^{\mu i}$, for some vector $v^{\mu i}$ satisfying $v^{\mu i} e^I_\mu = 0$. However, $D_\mu \Sigma^{\mu\nu i} = 0$, so $D_\nu e^I_\mu = 0$, therefore, $e^I_\mu D_\nu v^{\mu i} = 0$, and thus, one can choose $D_\nu v^{\mu i} = 0$ which allows us to calculate $v^{\mu i}$ and then, to calculate ψ^i in terms of $J^{\mu i}$ and A^i_μ. The equation $D_\mu J^{\mu i} = 0$ implies $D_\mu D^\mu \psi^i = D^2 \psi^i = 0$. Let us note that a specific metric $g_{\mu\nu}$ defined by $e^I_\mu = g_{\mu\nu} e^{\nu I}$ is used, but, according to the Remark 4, there is a metric for every two solutions of $(\Sigma^i_{ab}, \Sigma^i_{0a})$ and $(\Sigma^{abi}, \Sigma^{0ai})$, and also the equation $D^2 \psi^i = 0$ is invariant under any coordinates transformation $x \to x'$, $\psi'^i(x') = \psi^i(x)$, so that the same equation $D^2 \psi^i = 0$ is valid for other metric $g'_{\mu\nu}$.

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