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Lorentz Boosts and Wigner Rotations: Self-Adjoint Complexified Quaternions

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Abstract: In this paper, Lorentz boosts and Wigner rotations are considered from a (complexified) quaternionic point of view. It is demonstrated that, for a suitably defined self-adjoint complex quaternionic 4-velocity, pure Lorentz boosts can be phrased in terms of the quaternion square root of the relative 4-velocity connecting the two inertial frames. Straightforward computations then lead to quite explicit and relatively simple algebraic formulae for the composition of 4-velocities and the Wigner angle. The Wigner rotation is subsequently related to the generic *non-associativity* of the composition of three 4-velocities, and a *necessary and sufficient* condition is developed for the associativity to hold. Finally, the authors relate the composition of 4-velocities to a specific implementation of the Baker–Campbell–Hausdorff theorem. As compared to ordinary 4×4 Lorentz transformations, the use of self-adjoint complexified quaternions leads, from a computational view, to storage savings and more rapid computations, and from a pedagogical view to to relatively simple and explicit formulae.

Keywords: special relativity; quaternions; Lorentz boosts; composition of velocities; Wigner angle

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1. Introduction

The use of Hamilton's quaternions [1–6] as applied to special relativity has a very long, complicated, and rather fraught history—largely due to a significant number of rather sub-optimal notational choices being made in the early literature [7,8], which was then compounded by the introduction of multiple mutually disjoint ways of representing the Lorentz transformations [7–10]. Early formulations were heavily influenced by Minkowski's $x^4 = ict$ notation, with its imaginary time component, essentially using anti-self-adjoint complexified quaternions, with the result that there were a lot of extra and ultimately superfluous factors of (complex) i floating around. This made dealing with Lorentz transformations and Wigner angles more tricky than necessary. Here, c denotes the speed of light and t is the time. Subsequent developments have, if anything, even further confused the situation [11–13]. For additional background, see also Refs. [14–16]. Ref. [16] adopts a particularly unusual notation where the ordinary complex i is denoted by the @ symbol.

One reason for being particularly interested in these issues is due to various attempts to simplify the discussion of the interplay between the Thomas rotation [17–25], the relativistic composition of 3-velocities [26–30], and the very closely related Wigner angle [31–34]. In an earlier article [34], the authors considered ordinary quaternions and found that it was useful to work with the relativistic half velocities w, defined by $v = 2w/(1+w^2)$ so that $w = v/(1+\sqrt{1-v^2}) = v/2 + \mathcal{O}(v^3)$, where c is set to 1. In the current paper, the authors re-phrase things in terms of self-adjoint complex quaternionic 4-velocities, arguing for a number of simple compact formulae relating Lorentz transformations and the Wigner angle.

A second reason for being interested in quaternions is purely a matter of computational efficiency. Quaternions are often used to deal with 3-dimensional spatial rotations, and while mathematically the use of quaternions is completely equivalent to working with the usual 3×3 orthogonal matrices, that is the group $SO(3)$, the use of quaternions implies significant savings in storage and significant gains in computational efficiency.

A third reason for being interested in quaternions is purely a matter of pedagogy. Quaternions give one a different viewpoint on the usual physics of special relativity, and, in particular, the Lorentz transformations. Using quaternions leads to novel simple results for boosts where they are represented by the square-root of the relative 4-velocity, and simple novel results for the Wigner angle.

At a deeper level, the authors formally connect a composition of 4-velocities to a symmetric version of the Baker–Campbell–Hausdorff (BCH) theorem [35–39]. Unfortunately, while certainly elegant, most results based on the BCH expansion seem to not always be computationally useful.

2. Quaternions

It is useful to consider three distinct classes of quaternions [1–11,34]:

- ordinary classical quaternions,
- complexified quaternions,
- self-adjoint complexified quaternions.

While the discussion in Ref. [34] focussed on the ordinary classical quaternions, and so was implicitly a space plus time formalism, in this paper, the authors focus on the self-adjoint complexified quaternions, in order to develop an integrated space-time formalism. To set the framework, let us consider the discussion just below.

2.1. Ordinary Classical Quaternions

The ordinary classical quaternions are generalizations of the complex numbers that can be written in the form [1–6,34]:

$$\mathbf{q} = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}. \tag{1}$$

Here the corefficients a, b, c, and d are real numbers $\{a, b, c, d\} \in \mathbb{R}$, whereas $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the quaternion units which satisfy Hamilton's famous relation

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2}$$

These ordinary quaternions form a four–dimensional non-commutative number system, commonly denoted $\mathbb{H}$ in honour of Hamilton, that is generally treated as an extension of the complex numbers $\mathbb{C}$. Mathematically, the ordinary quaternions are sometimes described as a *skew field*, or a *division algebra*. Technically, "skew field" is the same as "division algebra", though sometimes this usage is refined so that "skew field" is taken to be the same as "associative non-commutative division algebra". Note that $\mathbb{R}$ and $\mathbb{C}$ are commutative, so they are not skew, while the octonians $\mathbb{O}$ are non-associative. The quaternions are often cited as the premier example of a "skew field".

Now define the quaternion conjugate of an ordinary quaternion $\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ to be $\mathbf{q}^\star = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$, and define the norm of $\mathbf{q}$ to be

$$\mathbf{q}\mathbf{q}^\star = |\mathbf{q}|^2 = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}. \tag{3}$$

For now, focus on the *pure quaternions* (with no real part). That is, in terms of the usual vector dot-product, one considers quaternions of the specific form $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k} = (a, b, c) \cdot (\mathbf{i}, \mathbf{j}, \mathbf{k})$. Define, in general, $\mathbf{v} = \vec{v} \cdot (\mathbf{i}, \mathbf{j}, \mathbf{k})$. In this instance, the product of two pure quaternions $\mathbf{p}$ and $\mathbf{q}$ is given in terms of the usual vector dot-product and vector cross-product by the relation

$$\mathbf{p}\mathbf{q} = -\vec{p}\cdot\vec{q} + (\vec{p}\times\vec{q})\cdot(\mathbf{i},\mathbf{j},\mathbf{k}). \tag{4}$$

In particular, note that this yields $\mathbf{q}^2 = -\vec{q}\cdot\vec{q} = -|\mathbf{q}|^2$. Furthermore, the commutator and anti-commutator of two pure quaternions is given by:

$$[\mathbf{p},\mathbf{q}] = 2(\vec{p}\times\vec{q})\cdot(\mathbf{i},\mathbf{j},\mathbf{k}), \quad \text{and} \quad \{\mathbf{p},\mathbf{q}\} = -2\,\vec{p}\cdot\vec{q}. \tag{5}$$

2.2. Complexified Quaternions

In counterpoint, the complexified quaternions are numbers that can be written in the form

$$\mathbf{Q} = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}, \tag{6}$$

where a, b, c, and d are now *complex* numbers $\{a,b,c,d\}\in\mathbb{C}$. It is important to note at this stage that, although it is quite common to embed the complex numbers into the quaternions by identifying the complex unit i with the quaternion unit $\mathbf{i}$, it is here *essential* that one distinguish between i and $\mathbf{i}$ when dealing with the complexified quaternions in $\mathbb{C}\otimes\mathbb{H}$. (Note that Ref. [16] adopts the unusual convention that $i \to @$ and $(\mathbf{i},\mathbf{j},\mathbf{k}) \to (i,j,k)$. Additionally there one deals with anti-self-adjoint complexified quaternions.) As well as the previously defined $\star$ operation, there are now two *additional* conjugates one can perform on the complexified quaternions: In addition to the quaternion conjugate, $\mathbf{q}^\star = a - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k}$, one can define the ordinary complex conjugate, $\overline{\mathbf{q}} = \bar{a} + \bar{b}\,\mathbf{i} + \bar{c}\,\mathbf{j} + \bar{d}\,\mathbf{k}$, and a third type of (adjoint) conjugate given by $\mathbf{q}^\dagger = (\overline{\mathbf{q}})^\star$. Note that this now leads to potentially three distinct notions of "norm":

$$|\mathbf{q}|^2 = \mathbf{q}^\star\mathbf{q} = a^2+b^2+c^2+d^2 \in \mathbb{C}, \qquad \bar{\mathbf{q}}\mathbf{q} = \bar{a}a - \bar{b}b - \bar{c}c - \bar{d}d \in \mathbb{R}, \tag{7}$$

and

$$\mathbf{q}^\dagger\mathbf{q} = \bar{a}a + \bar{b}b + \bar{c}c + \bar{d}d \in \mathbb{R}^+. \tag{8}$$

These complexified quaternions are commonly called "biquaternions" in the literature [40–45]. Unfortunately, as the word "biquaternion" has at least two *other* different possible meanings, it is safer to simply call these quantities the complexified quaternions. Some authors unfortunately use the word bi-quaternion to informally refer to the split-bi-quaternions, or the dual-quaternions (which really should be called the dual-bi-quaternions), or even the octonions [40–51]. About the only thing on which there is universal agreement is that the bi-quaternions are taken to be 8-dimensional.

2.3. Self-Adjoint Complexified Quaternions

One of the fundamental issues with trying to reformulate special relativity in terms of quaternions is that, although both space–time and quaternions are intrinsically four–dimensional, the norm of an ordinary quaternion $\mathbf{q} = a_0 + a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ is given by $|\mathbf{q}|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$, whereas, in contrast, the Lorentz invariant "norm" of a spacetime 4-vector A^μ is given by the expression $||A||^2 = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$. In order to address this fundamental issue, consider self-adjoint complexified quaternions satisfying $\mathbf{q} = \mathbf{q}^\dagger$. That is, consider complexified quaternions with with the real scalar part and the imaginary vectorial part:

$$\mathbf{q} = a_0 + ia_1\,\mathbf{i} + ia_2\,\mathbf{j} + ia_3\,\mathbf{k} = a_0 + i(a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}). \tag{9}$$

Here, $i\in\mathbb{C}$ is the usual complex unit and $a_0,a_1,a_2,a_3\in\mathbb{R}$. Self-adjoint quaternions have the norm,

$$|\mathbf{q}|^2 = \mathbf{q}^\star\mathbf{q} = (a_0 - i(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}))(a_0 + i(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})) = a_0^2 - a_1^2 - a_2^2 - a_3^2. \tag{10}$$

Thence, from the quaternionic point of view, the most natural signature choice is the $(+---)$ "mostly negative" convention. This particular norm is real, but need not be positive, and is physically and mathematically appropriate for describing the Lorentz invariant norm of a spacetime 4-vector in a quaternionic framework.

Indeed, writing $\mathbf{q}_1 = a_0 + i(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})$ and $\mathbf{q}_2 = b_0 + i(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k})$, for the Lorentz invariant inner product of two 4-vectors, one can write:

$$\eta(\mathbf{q}_1, \mathbf{q}_2) = \frac{1}{2}(\mathbf{q}_1^\star \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1) = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \in \mathbb{R}. \tag{11}$$

3. Lorentz Transformations

3.1. General Form

Quaternionic Lorentz transformations are now uniquely characterized by the two key features:

- they must be linear mappings from self-adjoint 4-vectors to self-adjoint 4-vectors, and
- they must preserve the Lorentz invariant inner product.

Taking $\mathbf{L}$ to be a complexified quaternion, the first condition suggests looking at the linear mapping

$$\mathbf{q} \to \mathbf{L}\,\mathbf{q}\,\mathbf{L}^\dagger \tag{12}$$

because this transformation will preserve the self-adjointness of $\mathbf{q}$. In fact, it is the only way to build a linear mapping from self-adjoint complexified quaternions to self-adjoint complexified quaternions using only the quaternion algebra.

The second condition then requires:

$$\begin{aligned} \mathbf{q}_1^\star \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1 &= \{\mathbf{L}^{\star\dagger}\ \mathbf{q}_1^\star\ \mathbf{L}^\star\}\ \{\mathbf{L}\mathbf{q}_2\mathbf{L}^\dagger\} + \{\mathbf{L}^{\star\dagger}\ \mathbf{q}_2^\star \mathbf{L}^\star\}\ \{\mathbf{L}\ \mathbf{q}_1\ \mathbf{L}^\dagger\} \\ &= \mathbf{L}^{\star\dagger}\ (\mathbf{q}_1^\star\ \{\mathbf{L}^\star\ \mathbf{L}\}\mathbf{q}_2 + \mathbf{q}_2^\star\{\mathbf{L}^\star\ \mathbf{L}\}\ \mathbf{q}_1)\ \mathbf{L}^\dagger. \end{aligned} \tag{13}$$

Now, if $\mathbf{L}^\star\mathbf{L} = 1$, that is $\mathbf{L}^\star = \mathbf{L}^{-1}$, this simplifies to

$$\mathbf{q}_1^\star \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1 = \mathbf{L}^{\star\dagger}(\mathbf{q}_1^\star\ \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1)\ \mathbf{L}^\dagger. \tag{14}$$

However, then, noting that $(\mathbf{q}_1^\star \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1) \in \mathbb{R}$, one has

$$\mathbf{q}_1^\star \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1 = (\mathbf{L}^{\star\dagger}\mathbf{L}^\dagger)\ (\mathbf{q}_1^\star\ \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1) = (\mathbf{L}\mathbf{L}^\star)^\dagger\ (\mathbf{q}_1^\star\ \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1) = \mathbf{q}_1^\star \mathbf{q}_2 + \mathbf{q}_2^\star \mathbf{q}_1. \tag{15}$$

So, a necessary and sufficient condition for the quaternionic mapping $\mathbf{q} \to \mathbf{L}\,\mathbf{q}\,\mathbf{L}^\dagger$ to preserve the quaternionic form of the Lorentz invariant inner product is

$$\mathbf{L}^\star\mathbf{L} = 1; \qquad \text{that is,} \qquad \mathbf{L}^\star = \mathbf{L}^{-1}. \tag{16}$$

Note that this condition implies that the set of quaternionic Lorentz transformations forms a group under quaternion multiplication. This now uniquely characterizes the Lorentz group.

3.2. Rotations

The rotations form a well-known subgroup of the Lorentz group, and, in a quaternionic form, a rotation about the $\hat{\mathbf{n}}$ axis, where $\hat{\mathbf{n}} = \vec{n} \cdot (\mathbf{i}, \mathbf{j}, \mathbf{k})$, can be represented by

$$\mathbf{R} = \exp(\theta\,\hat{\mathbf{n}}) = \cos\theta + \hat{\mathbf{n}}\ \sin\theta. \tag{17}$$

This observation goes back to the days of Hamilton, and the fact that 3-dimensional rotations can be represented in this quite straightforward manner, is one of the reasons so much effort was put into development of the quaternion formalism. From the point of view of the (ordinary) quaternions (not complexified in this case), one has:

$$\mathbf{R}^{-1} = \exp(-\theta\hat{\mathbf{n}}) = \cos\theta - \hat{\mathbf{n}}\ \sin\theta = \mathbf{R}^{\dagger} = \mathbf{R}^{\star}, \tag{18}$$

since now $\mathbf{q} = \bar{\mathbf{q}}$ and $\mathbf{q}^{\dagger} = \mathbf{q}^{\star}$.

Indeed, the characterization $\mathbf{R}^{-1} = \mathbf{R}^{\dagger} = \mathbf{R}^{\star}$ is both necessary and sufficient for a quaternion to represent a rotation.

3.3. Factorization—Quaternionic Polar Decomposition

One can now see how to factorize a general Lorentz transformation into the product of a boost and a rotation. This is effectively a quaternionic form of the notion of a "polar decomposition" that one usually encounters in matrix algebra (the discussion has been made somewhat pedestrian in the interests of pedagogical clarity).

Without any loss of generality, one may always write:

$$\mathbf{L} = \sqrt{\mathbf{L}\mathbf{L}^{\dagger}}\Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big). \tag{19}$$

To do this one just needs to know that the product of two complexified quaternions is again a complexified quaternion, and that the multiplication of complexified quaternions is associative.

Now, $\mathbf{L}\mathbf{L}^{\dagger}$ is self-adjoint—so $\sqrt{\mathbf{L}\mathbf{L}^{\dagger}}$ is self-adjoint, and, in turn, $(\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}$ is self-adjoint. Consequently,

$$\Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)\Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)^{\dagger} = \Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)\Big(\mathbf{L}^{\dagger}(\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\Big) = 1. \tag{20}$$

Hence,

$$\Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)^{-1} = \Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)^{\dagger}. \tag{21}$$

Furthermore,

$$(\mathbf{L}\mathbf{L}^{\dagger})^{\star} = (\mathbf{L}^{\dagger})^{\star}\mathbf{L}^{\star} = (\mathbf{L}^{\star})^{\dagger}\mathbf{L}^{\star} = (\mathbf{L}^{-1})^{\dagger}\mathbf{L}^{-1} = (\mathbf{L}^{\dagger})^{-1}\mathbf{L}^{-1} = (\mathbf{L}\mathbf{L}^{\dagger})^{-1}. \tag{22}$$

That is, $(\mathbf{L}\mathbf{L}^{\dagger})$ is a Lorentz transformation (in fact, a self-adjoint Lorentz transformation), and, consequently, $\sqrt{\mathbf{L}\mathbf{L}^{\dagger}}$ is also a (self-adjoint) Lorentz transformation. However, then, by the group property, $\Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)$ must also be a Lorentz transformation, so

$$\Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)^{-1} = \Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)^{*}. \tag{23}$$

However, this now implies that $\mathbf{R} = \Big((\mathbf{L}\mathbf{L}^{\dagger})^{-1/2}\mathbf{L}\Big)$ must be a rotation, andthis shows that in general,

$$\mathbf{L} = \sqrt{\mathbf{L}\mathbf{L}^{\dagger}}\ \mathbf{R}. \tag{24}$$

Indeed, in the next section one can see that the self-adjoint Lorentz transformation $\mathbf{B} = \sqrt{\mathbf{L}^{\dagger}\mathbf{L}}$ is actually a boost and that, in general, one has:

$$\mathbf{L} = \mathbf{B}\ \mathbf{R}. \tag{25}$$

4. Lorentz Boosts

One now shows how quaternions can be used to obtain a pure Lorentz transformation—a boost, a Lorentz transformation that depends only on the relative velocity, without any rotation—from the square root of the relative 4-velocity connecting the two inertial frames. In order to proceed, one must first obtain an explicit expression for the square root of a four–velocity $\mathbf{V}$.

4.1. 4-Velocity, and Square Root of 4-Velocity

Represent a position 4-vector $\vec{X} = (t, x, y, z) = (t, \vec{x})$ by the self-adjoint quaternion

$$\mathbf{X} = t + i(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}). \tag{26}$$

Consider a parameterized curve $\mathbf{X}(\lambda)$. For two points on the curve one can define

$$\Delta\mathbf{X} = \Delta t + i(\Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} + \Delta z\,\mathbf{k}), \tag{27}$$

and, hence,

$$\Delta\tau^2 = (\Delta\mathbf{X})^\star\,\Delta\mathbf{X} = \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2) \tag{28}$$

is just the proper time between those two events. Taking appropriate limits, one can reparameterize the curve $\mathbf{X}(\lambda) \to \mathbf{X}(\tau)$ as a function of the proper time.

Differentiating with respect to the proper time gives a quaternionic notion of 4-velocity,

$$\mathbf{V} = \gamma(1 + iv\,\hat{\mathbf{n}}) \qquad \text{with} \qquad |\mathbf{V}|^2 = \mathbf{V}^\star\mathbf{V} = 1. \tag{29}$$

Here, as usual, $\gamma = 1/\sqrt{1-v^2}$, and v is the usual 3-speed (recall that herein $c \to 1$). By construction, $\mathbf{V}$ is a self-adjoint complexified quaternion. To explicitly find the square root, first, one presents an elementary discussion. Let us introduce the notion of rapidity in the usual manner by setting $\xi = \tanh^{-1} v$. Then, 4-velocities can be written in the form

$$\mathbf{V} = \gamma\,(1 + iv\hat{\mathbf{n}}) = \cosh\xi + i\,\hat{\mathbf{n}}\,\sinh\xi = \mathrm{e}^{i\xi\hat{\mathbf{n}}}. \tag{30}$$

The square root of the 4-velocity is then easily seen to be $\sqrt{\mathbf{V}} = \mathrm{e}^{i\xi\hat{\mathbf{n}}/2}$.

Explicitly, using hyperbolic half-angle formulae, from $\gamma = \cosh\xi$ and $v = \tanh\xi$ one gets:

$$\cosh(\xi/2) = \sqrt{\frac{\gamma+1}{2}}, \qquad \text{and} \qquad \sinh(\xi/2) = \sqrt{\frac{\gamma-1}{2}}. \tag{31}$$

Thence, explicitly,

$$\sqrt{\mathbf{V}} = \sqrt{\frac{\gamma+1}{2}} + i\hat{\mathbf{n}}\,\sqrt{\frac{\gamma-1}{2}}. \tag{32}$$

In terms of the relativistic half velocity, implicitly defined by $v = 2w/(1+w^2)$, so that one has $w = v/(1+\sqrt{1-v^2})$, it is easy to check that

$$w = \sqrt{\frac{\gamma-1}{\gamma+1}}, \qquad \text{and} \qquad \gamma_w = \sqrt{\frac{\gamma+1}{2}}. \tag{33}$$

So, one can write:

$$\sqrt{\mathbf{V}} = \gamma_w\,(1 + i\hat{\mathbf{n}}\,w). \tag{34}$$

There are many other ways of getting to the same result. The current discussion has been designed to be as straightforward and explicit as possible.

4.2. Lorentz Boosts in Terms of Relative 4-Velocity

Now, that one has an expression for the square root of a 4-velocity $\mathbf{V}$, one can show how a pure Lorentz transformation is obtained from quaternion conjugation by $\sqrt{\mathbf{V}}$.

Without any loss of generality, define $\mathbf{V} = \gamma(1 + i\mathbf{i}v)$, which is the four–velocity for an object travelling with speed v in the $\hat{x}$ direction, and represent the four–vector $\vec{X} = (t, x, y, z)$

by the self-adjoint quaternion $\mathbf{X} = (t + i\mathbf{i}x + i\mathbf{j}y + i\mathbf{k}z) = (t + i(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z))$. Now, consider the transformation of $\mathbf{X}$ given by $\mathbf{X} \mapsto \sqrt{\mathbf{V}}\,\mathbf{X}\,\sqrt{\mathbf{V}}$. That is,

$$\begin{aligned}\mathbf{X} &\mapsto \sqrt{\mathbf{V}}(t + i\mathbf{i}x + i\mathbf{j}y + i\mathbf{k}z)\sqrt{\mathbf{V}} \\ &= \sqrt{\mathbf{V}}(t + i\mathbf{i}x)\sqrt{\mathbf{V}} + \sqrt{\mathbf{V}}(i\mathbf{j}y + i\mathbf{k}z)\sqrt{\mathbf{V}} \\ &= (t + i\mathbf{i}x)\mathbf{V} + (i\mathbf{j}y + i\mathbf{k}z)\sqrt{\mathbf{V}^\star}\sqrt{\mathbf{V}},\end{aligned} \tag{35}$$

where in the last equality one can use the fact that $\sqrt{\mathbf{V}}$ commutes with $\mathbf{i}$, and the anti-commutativity of $\mathbf{i}$ with $\mathbf{j}$ and $\mathbf{k}$ to write $\sqrt{\mathbf{V}}\,\mathbf{j} = \mathbf{j}\sqrt{\mathbf{V}^\star}$, and $\sqrt{\mathbf{V}}\,\mathbf{k} = \mathbf{k}\sqrt{\mathbf{V}^\star}$. Explicit calculation of $\sqrt{\mathbf{V}^\star}\sqrt{\mathbf{V}}$ yields:

$$\sqrt{\mathbf{V}^\star}\sqrt{\mathbf{V}} = \exp(-i\xi\mathbf{i}/2)\exp(i\xi\mathbf{i}/2) = 1. \tag{36}$$

That is, $\sqrt{\mathbf{V}}$ is a unit quaternion. Thus,

$$\sqrt{\mathbf{V}}\,\mathbf{X}\,\sqrt{\mathbf{V}} = (t + i\mathbf{i}x)\mathbf{V} + (i\mathbf{j}y + i\mathbf{k}z). \tag{37}$$

Using the above expression for $\mathbf{V}$, we find $(t + i\mathbf{i}x)\mathbf{V} = \gamma\{(t + vx) + i\mathbf{i}(x + vt)\}$, giving a final result:

$$\mathbf{X} = (t + i\mathbf{i}x + i\mathbf{j}y + i\mathbf{k}z) \mapsto \gamma\{(t + vx) + i\mathbf{i}(x + vt)\} + (i\mathbf{j}y + i\mathbf{k}z), \tag{38}$$

which are the well–known inverse Lorentz transformations—boosts in the $(-\hat{x})$ direction. Although, for the purpose of simplifying the presentation, one defines the $\mathbf{i}$ axis to lie in the direction of the boost, it should be clear that this argument is in fact completely general, one merely needs to rotate the $\mathbf{i}$ axis into the direction of the general boost $\hat{\mathbf{n}}$. This is compatible with the general definition of the 4-vector $\vec{X} \leftrightarrow \mathbf{X}$.

That is, a boost—a pure Lorentz transformation that depends only on relative velocities but without any rotation—corresponds to:

$$\mathbf{X} \to \sqrt{\mathbf{V}}\,\mathbf{X}\,\sqrt{\mathbf{V}} = \mathrm{e}^{i\xi\hat{\mathbf{n}}/2}\,\mathbf{X}\,e^{i\xi\hat{\mathbf{n}}/2}. \tag{39}$$

A related formula can be found in Ref. [16], modulo the notational changes $i \to @$, and $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \to (i, j, k)$, and switching to anti-self-dual 4-vectors, which consequently require extraneous factors of the symbol @.

4.3. Combination of 4-Velocities

Starting in the rest frame of some object, where $\mathbf{V}_0 = 1$, successively apply two boosts, $\mathbf{V}_1 = \gamma_1(1 + i\hat{\mathbf{n}}_1 v_1)$ and $\mathbf{V}_2 = \gamma_2(1 + i\hat{\mathbf{n}}_2 v_2)$, in directions $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ with velocities v_1 and v_2, respectively. The result of this is to shift one's rest frame to a frame moving with 4-velocity $\mathbf{V}_{1\oplus 2}$, which is equivalent to relativistically combining the two 4-velocities $\mathbf{V}_1$ and $\mathbf{V}_2$. This method has the added benefit that it obtains an expression for the γ–factor of the frame, $\gamma_{1\oplus 2}$, and, hence, its speed without having to take the norm of $\mathbf{V}_{1\oplus 2}$, thereby avoiding lots of tedious algebra.

One begins by boosting the initial rest frame starting with with $\mathbf{V}_0 = 1$ as in Section 4.2:

$$\mathbf{V}_{1\oplus 2} = \sqrt{\mathbf{V}_2}\,\sqrt{\mathbf{V}_1}\,\mathbf{V}_0\,\sqrt{\mathbf{V}_1}\,\sqrt{\mathbf{V}_2} = \sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2}. \tag{40}$$

Similarly,

$$\mathbf{V}_{2\oplus 1} = \sqrt{\mathbf{V}_1}\,\sqrt{\mathbf{V}_2}\,\mathbf{V}_0\,\sqrt{\mathbf{V}_2}\,\sqrt{\mathbf{V}_1} = \sqrt{\mathbf{V}_1}\,\mathbf{V}_2\sqrt{\mathbf{V}_1}. \tag{41}$$

That is, the relativistic combination of 4-velocities simply amounts to

$$\mathbf{V}_{1\oplus 2} = \sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2}, \qquad \mathbf{V}_{2\oplus 1} = \sqrt{\mathbf{V}_1}\,\mathbf{V}_2\sqrt{\mathbf{V}_1}. \tag{42}$$

This makes it obvious that $\mathbf{V}_{1\oplus 2} \neq \mathbf{V}_{2\oplus 1}$ unless $[\mathbf{V}_1, \mathbf{V}_2] = 0$, which, in turn, requires the 3-velocities $\vec{v}_1$ and $\vec{v}_2$ to be parallel. In the special case, where the 4-velocities do commute, one has:

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1\mathbf{V}_2 = \mathbf{V}_2\mathbf{V}_1 = \mathbf{V}_{2\oplus 1}. \tag{43}$$

In terms of rapidities, the general case is:

$$\mathbf{V}_{1\oplus 2} = \mathrm{e}^{i\xi_2 \hat{\mathbf{n}}_2/2}\, \mathrm{e}^{i\xi_1 \hat{\mathbf{n}}_1}\, e^{i\xi_2 \hat{\mathbf{n}}_2/2}, \qquad \mathbf{V}_{2\oplus 1} = \mathrm{e}^{i\xi_1 \hat{\mathbf{n}}_1/2}\, \mathrm{e}^{i\xi_2 \hat{\mathbf{n}}_2}\, e^{i\xi_1 \hat{\mathbf{n}}_1/2}. \tag{44}$$

Viewed in this way, the relativistic combination of 4-velocities can be interpreted as an application of the symmetrized version of the BCH expansion [35–39]. Indeed, taking logarithms:

$$\xi_{1\oplus 2}\, \hat{\mathbf{n}}_{1\oplus 2} = -i \ln\Big\{\mathrm{e}^{i\xi_2 \hat{\mathbf{n}}_2/2}\, \mathrm{e}^{i\xi_1 \hat{\mathbf{n}}_1}\, \mathrm{e}^{i\xi_2 \hat{\mathbf{n}}_2/2}\Big\}, \tag{45}$$

$$\xi_{2\oplus 1}\, \hat{\mathbf{n}}_{2\oplus 1} = -i \ln\Big\{\mathrm{e}^{i\xi_1 \hat{\mathbf{n}}_1/2}\, \mathrm{e}^{i\xi_2 \hat{\mathbf{n}}_2}\, \mathrm{e}^{i\xi_1 \hat{\mathbf{n}}_1/2}\Big\}. \tag{46}$$

Unfortunately this formal result, despite being quite elegant, is not really computationally effective. One could, for instance, fully expand the expression in Equation (44) and isolate the real part to deduce

$$\xi_{1\oplus 2} = \xi_{2\oplus 1} = \cosh^{-1}(\cosh\xi_1 \cosh\xi_2 + \sinh\xi_1 \sinh\xi_2 \cos\theta), \tag{47}$$

where θ is the angle between $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$. As expected, for collinear 3-velocities this reduces to

$$\xi_{1\oplus 2} = \xi_{2\oplus 1} \to |\xi_1 \pm \xi_2|. \tag{48}$$

To check this for consistency, note that

$$\gamma_{1\oplus 2} = \eta(\mathbf{V}_0, \mathbf{V}_{1\oplus 2}) = \frac{1}{2}(\mathbf{V}_0^\star \mathbf{V}_{1\oplus 2} + \mathbf{V}_{1\oplus 2}^\star \mathbf{V}_0) = \frac{1}{2}(\mathbf{V}_{1\oplus 2} + \mathbf{V}_{1\oplus 2}^\star). \tag{49}$$

Thence,

$$\gamma_{1\oplus 2} = \frac{1}{2}\Big(\sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2} + \sqrt{\mathbf{V}_2^\star}\,\mathbf{V}_1^\star\sqrt{\mathbf{V}_2^\star}\Big). \tag{50}$$

As soon as $\gamma_{1\oplus 2}$ is real and $\big(\sqrt{\mathbf{V}_2}\big)^{-1} = \big(\sqrt{\mathbf{V}_2}\big)^\star$, then:

$$\gamma_{1\oplus 2} = \Big(\sqrt{\mathbf{V}_2}\Big)^\star \gamma_{1\oplus 2}\sqrt{\mathbf{V}_2} = \frac{1}{2}(\mathbf{V}_1\mathbf{V}_2 + \mathbf{V}_2^\star\mathbf{V}_1^\star) = \frac{1}{2}(\mathbf{V}_1\mathbf{V}_2 + (\mathbf{V}_1\,\mathbf{V}_2)^\star). \tag{51}$$

It is then easy (indeed, almost trivial) to see that

$$\gamma_{1\oplus 2} = \gamma_1\gamma_2(1 + \vec{v}_1 \cdot \vec{v}_2) = \gamma_{2\oplus 1}. \tag{52}$$

While this very easily yields the magnitude of the combined 3-velocities $|\vec{v}_{1\oplus 2}| = |\vec{v}_{2\oplus 1}|$, isolating the direction of the combined 3-velocities is much more subtle. Note that $\hat{v}_{1\oplus 2} \neq \hat{v}_{2\oplus 1}$ in general, see Equations (45) and (46).

The rapidity formalism is also particularly useful for quickly double-checking formal relationships such as

$$\sqrt{\mathbf{V}^\star} = \mathrm{e}^{-i\xi\hat{\mathbf{n}}/2} = \Big(\sqrt{\mathbf{V}}\,\Big)^\star = \sqrt{\mathbf{V}^{-1}} = \Big(\sqrt{\mathbf{V}}\,\Big)^{-1}. \tag{53}$$

One can also use this formalism to write a general Lorentz transformation in the form

$$\mathbf{L} = \mathbf{B}\,\mathbf{R} = e^{i\xi\hat{\mathbf{n}}/2}\, e^{\theta\hat{\mathbf{m}}/2}. \tag{54}$$

Here, $\hat{\mathbf{n}}$ is the direction of the boost $\mathbf{B}$ and $\hat{\mathbf{m}}$ is the axis of the rotation $\mathbf{R}$.

5. Wigner Rotation

One now derives an explicit quaternionic formula for the Wigner rotation; for relevant background, see references [31–34]. Note that for 4-velocities:

$$\mathbf{V} = \mathbf{V}^\dagger, \qquad \mathbf{V}^{-1} = \mathbf{V}^\star, \qquad \sqrt{(\mathbf{V})^{-1}} = \left(\sqrt{\mathbf{V}}\right)^{-1}, \tag{55}$$

while for rotations one has:

$$\mathbf{R}^\dagger = \mathbf{R}^\star; \qquad \mathbf{R}^{-1} = \mathbf{R}^\dagger = \mathbf{R}^\star. \tag{56}$$

Now, note

$$\mathbf{V}_{1\oplus 2} = \sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2} = \sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1}\;\sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_2} \tag{57}$$

Define a quaternion $\mathbf{R}$ by taking

$$\sqrt{\mathbf{V}_{1\oplus 2}}\;\mathbf{R} = \sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1}, \tag{58}$$

and then checking to see that this quaternion does in fact correspond to a rotation. First,

$$\mathbf{R} = \sqrt{\mathbf{V}_{1\oplus 2}^{-1}}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1}, \qquad \mathbf{R}^\dagger = \sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_{1\oplus 2}^{-1}}. \tag{59}$$

Now,

$$\begin{aligned}
\mathbf{R}\mathbf{R}^\dagger &= \sqrt{\mathbf{V}_{1\oplus 2}^{-1}}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_{1\oplus 2}^{-1}} \\
&= \sqrt{\mathbf{V}_{1\oplus 2}^{-1}}\sqrt{\mathbf{V}_2}\mathbf{V}_1\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_{1\oplus 2}^{-1}} \\
&= \sqrt{\mathbf{V}_{1\oplus 2}^{-1}}\mathbf{V}_{1\oplus 2}\sqrt{\mathbf{V}_{1\oplus 2}^{-1}} \\
&= 1.
\end{aligned} \tag{60}$$

That is, $\mathbf{R}^{-1} = \mathbf{R}^\dagger$. Now, consider

$$\mathbf{R}^\star = \sqrt{\mathbf{V}_1^*}\sqrt{\mathbf{V}_2^*}\sqrt{\mathbf{V}_{1\oplus 2}}, \tag{61}$$

and compare it to

$$\mathbf{R}^\dagger = \sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_{1\oplus 2}^{-1}}, \tag{62}$$

Actually, $\mathbf{R}^\star = \mathbf{R}^\dagger$, though at first this might not be obvious. Calculate

$$\mathbf{R}^\star\sqrt{\mathbf{V}_{1\oplus 2}} = \sqrt{\mathbf{V}_1^*}\sqrt{\mathbf{V}_2^*}\sqrt{\mathbf{V}_{1\oplus 2}}\sqrt{\mathbf{V}_{1\oplus 2}} = \sqrt{\mathbf{V}_1^*}\sqrt{\mathbf{V}_2^*}\mathbf{V}_{1\oplus 2}. \tag{63}$$

Thence,

$$\mathbf{R}^\star\sqrt{\mathbf{V}_{1\oplus 2}} = \sqrt{\mathbf{V}_1^{-1}}\sqrt{\mathbf{V}_2^{-1}}\left(\sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2}\right) = \sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_2}. \tag{64}$$

Hence,

$$\mathbf{R}^{\star}\sqrt{\mathbf{V}_{1\oplus 2}} = \sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_2} = \mathbf{R}^{\dagger}\sqrt{\mathbf{V}_{1\oplus 2}}. \tag{65}$$

Therefore, $\mathbf{R}^{\star} = \mathbf{R}^{\dagger}$ as claimed. Accordingly, $\mathbf{R}$ is indeed a well-defined rotation. Explicitly one has:

$$\mathbf{R} = \sqrt{\mathbf{V}_{1\oplus 2}^{-1}}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1} = \sqrt{\sqrt{\mathbf{V}_2^{-1}}\,\mathbf{V}_1^{-1}\sqrt{\mathbf{V}_2^{-1}}}\;\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1}. \tag{66}$$

Note that from Equation (65) one has:

$$\mathbf{R}^{\dagger}\sqrt{\mathbf{V}_{1\oplus 2}} = \sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_2}, \tag{67}$$

and so, along with the defining relation Equation (58), one deduces

$$\mathbf{V}_{2\oplus 1} = \sqrt{\mathbf{V}_1}\,\mathbf{V}_2\sqrt{\mathbf{V}_1} = \sqrt{\mathbf{V}_1}\sqrt{\mathbf{V}_2}\;\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1} = \mathbf{R}^{\dagger}\;\sqrt{\mathbf{V}_{1\oplus 2}}\sqrt{\mathbf{V}_{1\oplus 2}}\,\mathbf{R}. \tag{68}$$

That is,

$$\mathbf{V}_{2\oplus 1} = \mathbf{R}^{\dagger}\;\mathbf{V}_{1\oplus 2}\;\mathbf{R}. \tag{69}$$

Therefore, $\mathbf{R}$ is indeed the Wigner rotation as claimed.

6. Non-Associativity of the Combination of Velocities

From the above, one notes that

$$\mathbf{V}_{(1\oplus 2)\oplus 3} = \sqrt{\mathbf{V}_3}\sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_3}, \tag{70}$$

whereas

$$\mathbf{V}_{1\oplus(2\oplus 3)} = \sqrt{\sqrt{\mathbf{V}_3}\mathbf{V}_2\sqrt{\mathbf{V}_3}}\;\mathbf{V}_1\;\sqrt{\sqrt{\mathbf{V}_3}\mathbf{V}_2\sqrt{\mathbf{V}_3}}. \tag{71}$$

This explicitly verifies the general non-associativity of composition of 4-velocities, and furthermore demonstrates why left-composition is much nicer than right-composition. There has in the past been some confusion in this regard [26–28]; see also the recent discussion in reference [34], where an equivalent discussion was presented in terms of quaternionic 3-velocities.

From the above,

$$\begin{aligned}\mathbf{V}_{(1\oplus 2)\oplus 3} \quad = \quad & \sqrt{\mathbf{V}_3}\sqrt{\mathbf{V}_2}\sqrt{\sqrt{\mathbf{V}_3^{-1}}\mathbf{V}_2^{-1}\sqrt{\mathbf{V}_3^{-1}}}\sqrt{\sqrt{\mathbf{V}_3}\mathbf{V}_2\sqrt{\mathbf{V}_3}}\;\;\mathbf{V}_1 \\ & \times\sqrt{\sqrt{\mathbf{V}_3}\mathbf{V}_2\sqrt{\mathbf{V}_3}}\sqrt{\sqrt{\mathbf{V}_3^{-1}}\mathbf{V}_2^{-1}\sqrt{\mathbf{V^{-1}}_3}}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_3}.\end{aligned} \tag{72}$$

That is,

$$\mathbf{V}_{(1\oplus 2)\oplus 3} = \sqrt{\mathbf{V}_3}\sqrt{\mathbf{V}_2}\sqrt{\sqrt{\mathbf{V}_3^{-1}}\mathbf{V}_2^{-1}\sqrt{\mathbf{V}_3^{-1}}}\;\mathbf{V}_{1\oplus(2\oplus 3)}\;\sqrt{\sqrt{\mathbf{V}_3^{-1}}\mathbf{V}_2^{-1}\sqrt{\mathbf{V^{-1}}_3}}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_3}. \tag{73}$$

However, from Equation (66) for the Wigner rotation

$$\mathbf{R}_{1\oplus 2} = \sqrt{\mathbf{V}_{1\oplus 2}^{-1}}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1} = \sqrt{\sqrt{\mathbf{V}_2^{-1}}\,\mathbf{V}_1^{-1}\sqrt{\mathbf{V}_2^{-1}}}\;\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1}, \tag{74}$$

this now implies

$$\mathbf{V}_{(1\oplus 2)\oplus 3} = \mathbf{R}_{2\oplus 3}^{\dagger}\mathbf{V}_{1\oplus(2\oplus 3)}\mathbf{R}_{2\oplus 3}. \tag{75}$$

So, the Wigner rotation is not just relevant for understanding generic non-commutativity when composing two boosts, is also relevant to understanding generic non-associativity when composing three boosts.

Suppose now one considers a specific situation where the composition of 4-velocities is associative, that is, assume:

$$\mathbf{V}_{(1\oplus 2)\oplus 3} = \mathbf{V}_{1\oplus(2\oplus 3)}. \tag{76}$$

Under this condition, one would now have:

$$\sqrt{\mathbf{V}_3}\sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_3} = \sqrt{\sqrt{\mathbf{V}_3}\mathbf{V}_2\sqrt{\mathbf{V}_3}}\;\mathbf{V}_1\;\sqrt{\sqrt{\mathbf{V}_3}\mathbf{V}_2\sqrt{\mathbf{V}_3}}, \tag{77}$$

whence one would need:

$$\sqrt{\sqrt{\mathbf{V}_3^{-1}}\mathbf{V}_2^{-1}\sqrt{\mathbf{V}_3^{-1}}}\;\sqrt{\mathbf{V}_3}\sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_3}\sqrt{\sqrt{\mathbf{V}_3^{-1}}\mathbf{V}_2^{-1}\sqrt{\mathbf{V}_3^{-1}}} = \mathbf{V}_1. \tag{78}$$

One can rewrite this condition in terms of the Wigner rotation as

$$\mathbf{R}_{2\oplus 3}\;\mathbf{V}_1\;\mathbf{R}_{2\oplus 3}^{\dagger} = \mathbf{V}_1. \tag{79}$$

That is,

$$\mathbf{R}_{2\oplus 3}\;\mathbf{V}_1\;\mathbf{R}_{2\oplus 3}^{-1} = \mathbf{V}_1, \tag{80}$$

whence

$$[\mathbf{R}_{2\oplus 3}, \mathbf{V}_1] = 0. \tag{81}$$

Hence, the combination of velocities is associative: $\mathbf{V}_{(1\oplus 2)\oplus 3} = \mathbf{V}_{1\oplus(2\oplus 3)}$, if and only if the boost direction in $\mathbf{V}_1$ is parallel to the rotation axis in $\mathbf{R}_{2\oplus 3}$. However, this holds if and only if

$$[\mathbf{v}_1, [\mathbf{v}_2, \mathbf{v}_3]] = 0 \tag{82}$$

or, in more prosaic language, if and only if

$$\vec{v}_1 \times (\vec{v}_2 \times \vec{v}_3) = 0. \tag{83}$$

The authors had *almost* derived this result in Ref. [34], but only as a sufficient condition; it was never quite established as a necessary and sufficient condition.

7. BCH Approach to the Combination of 4-Velocities

Now consider yet another way of understanding combination of velocities, this time in terms of the (symmetrized) BCH theorem. On has already seen (Equation (44)) that

$$\mathbf{V}_{1\oplus 2} = \exp\left(i\xi_2\hat{\boldsymbol{\xi}}_2/2\right)\exp\left(i\xi_1\hat{\boldsymbol{\xi}}_1\right)\exp\left(i\xi_2\hat{\boldsymbol{\xi}}_2/2\right). \tag{84}$$

Now, differentiate

$$\frac{\partial \mathbf{V}_{1\oplus 2}}{\partial \xi_2} = \frac{i}{2}\left\{\hat{\boldsymbol{\xi}}_2, \exp\left(i\xi_2\hat{\boldsymbol{\xi}}_2/2\right)\exp\left(i\xi_1\hat{\boldsymbol{\xi}}_1\right)\exp\left(i\xi_2\hat{\boldsymbol{\xi}}_2/2\right)\right\}, \tag{85}$$

and rewrite this as

$$\frac{\partial \mathbf{V}_{1\oplus 2}}{\partial \xi_2} = \frac{i}{2}\exp\left(i\xi_2\hat{\boldsymbol{\xi}}_2/2\right)\left\{\hat{\boldsymbol{\xi}}_2, \exp\left(i\xi_1\hat{\boldsymbol{\xi}}_1\right)\right\}\exp\left(i\xi_2\hat{\boldsymbol{\xi}}_2/2\right), \tag{86}$$

However, note that

$$\left\{\hat{\xi}_2, \exp\left(i\xi_1\hat{\xi}_1\right)\right\} = \left\{\hat{\xi}_2, \left(\cosh(\xi_1) + i\sinh(\xi_1)\hat{\xi}_1\right)\right\} = \left(2\cosh(\xi_1)\hat{\xi}_2 + i\sinh(\xi_1)\{\hat{\xi}_2, \hat{\xi}_1\}\right). \tag{87}$$

Furthermore, since $\{\hat{\xi}_2, \hat{\xi}_1\} \in \mathbb{R}$, this implies:

$$\left[\hat{\xi}_2, \left\{\hat{\xi}_2, \exp\left(i\xi_1\hat{\xi}_1\right)\right\}\right] = 0. \tag{88}$$

Consequently, one can pull the factor $\exp\left(i\xi_2\hat{\xi}_2/2\right)$ through the anti-commutator, and rewrite the derivative as

$$\frac{\partial \mathbf{V}_{1\oplus 2}}{\partial \xi_2} = \frac{i}{2}\left\{\hat{\xi}_2, \exp\left(i\xi_1\hat{\xi}_1\right)\right\}\exp\left(i\xi_2\hat{\xi}_2\right). \tag{89}$$

Integrating over ξ_2:

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1 + \frac{i}{2}\int_0^{\xi_2}\left\{\hat{\xi}_2, \exp\left(i\xi_1\hat{\xi}_1\right)\right\}\exp\left(i\xi_2\hat{\xi}_2\right)d\xi_2. \tag{90}$$

Pulling the constant (with respect to ξ_2) anti-commutator outside the integral,

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1 + \frac{i}{2}\left\{\hat{\xi}_2, \exp\left(i\xi_1\hat{\xi}_1\right)\right\}\int_0^{\xi_2}\exp\left(i\xi_2\hat{\xi}_2\right)d\xi_2, \tag{91}$$

and performing the integral, one finds:

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1 + \frac{i}{2}\left\{\hat{\xi}_2, \exp\left(i\xi_1\hat{\xi}_1\right)\right\}\frac{\exp\left(i\xi_2\hat{\xi}_2\right) - 1}{i\hat{\xi}_2}. \tag{92}$$

Noting that $(i\hat{\xi})^2 = 1$ this simplifies to:

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1 - \frac{1}{2}\left\{\hat{\xi}_2, \exp\left(i\xi_1\hat{\xi}_1\right)\right\}\hat{\xi}_2\left(\exp\left(i\xi_2\hat{\xi}_2\right) - 1\right). \tag{93}$$

That is,

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1 - \frac{1}{2}\{\hat{\xi}_2, \mathbf{V}_1\}\hat{\xi}_2(\mathbf{V}_2 - 1). \tag{94}$$

A more tractable result is:

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1 - \frac{1}{2}\left(\hat{\xi}_2\mathbf{V}_1\hat{\xi}_2 - \mathbf{V}_1\right)(\mathbf{V}_2 - 1). \tag{95}$$

Thence,

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1 + \frac{1}{2}\left(\mathbf{V}_1 - \hat{\xi}_2\mathbf{V}_1\hat{\xi}_2\right)(\mathbf{V}_2 - 1). \tag{96}$$

Rearranging:

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1\mathbf{V}_2 - \frac{1}{2}\left(\mathbf{V}_1 + \hat{\xi}_2\mathbf{V}_1\hat{\xi}_2\right)(\mathbf{V}_2 - 1). \tag{97}$$

This gives the composition of 4-velocities $\mathbf{V}_{1\oplus 2}$ *algebraically* in terms of $\mathbf{V}_1$ and $\mathbf{V}_2$ and, at worst, some quaternion multiplication (the need to explicitly evaluate $\sqrt{\mathbf{V}_2}$ has been side-stepped).

Note that this has the right limit for parallel 3-velocities. When $[\mathbf{V}_1, \hat{\xi}_2] = 0$, one sees

$$\mathbf{V}_{1\oplus 2} \to \mathbf{V}_1\mathbf{V}_2, \tag{98}$$

as it should be.

For perpendicular 3-velocities,

$$\mathbf{V}_{1\oplus 2} = \mathbf{V}_1 - \frac{1}{2}\{\hat{\boldsymbol{\xi}}_2, \mathbf{V}_1\}\hat{\boldsymbol{\xi}}_2(\mathbf{V}_2 - 1) \tag{99}$$

reduces to

$$\mathbf{V}_{1\oplus 2} \to \mathbf{V}_1 + \gamma_1(\mathbf{V}_2 - 1). \tag{100}$$

Then (see Equation (29)),

$$\mathbf{V}_{1\oplus 2} \to \gamma_1(1 + \mathbf{v}_1) + \gamma_1\gamma_2(1 + \mathbf{v}_2) - \gamma_1, \tag{101}$$

implying

$$\mathbf{V}_{1\oplus 2} \to \gamma_1\gamma_2\left(1 + \sqrt{1 - v_2^2}\,\mathbf{v}_1 + \mathbf{v}_2\right). \tag{102}$$

That is,

$$\mathbf{v}_{1\oplus 2} = \sqrt{1 - v_2^2}\,\mathbf{v}_1 + \mathbf{v}_2, \tag{103}$$

and

$$|\mathbf{v}_{1\oplus 2}|^2 = v_1^2 + v_2^2 - v_1^2 v_2^2, \tag{104}$$

exactly as expected for perpendicular 3-velocities.

8. Summary

The method of complexified quaternions allows one to prove several interesting results.

- General Lorentz transformations can be factorized into a boost times a rotation:

$$\mathbf{L} = \mathbf{B}\,\mathbf{R} = e^{i\xi\hat{\mathbf{n}}/2}\, e^{\theta\hat{\mathbf{m}}/2}, \tag{105}$$

  with $\hat{\mathbf{n}}$ being the direction of the boost $\mathbf{B}$, while $\hat{\mathbf{m}}$ is the axis of the rotation $\mathbf{R}$.
- Conjugation by the square root of a 4-velocity implements a Lorentz boost:

$$\mathbf{X} \to \sqrt{\mathbf{V}}\,\mathbf{X}\,\sqrt{\mathbf{V}}. \tag{106}$$

- The relativistic combination of 4-velocities has the simple algebraic form:

$$\mathbf{V}_{1\oplus 2} = \sqrt{\mathbf{V}_2}\,\mathbf{V}_1\sqrt{\mathbf{V}_2}, \qquad \mathbf{V}_{2\oplus 1} = \sqrt{\mathbf{V}_1}\,\mathbf{V}_2\sqrt{\mathbf{V}_1}. \tag{107}$$

- The Wigner rotation is given by:

$$\mathbf{R} = \sqrt{\mathbf{V}_{1\oplus 2}^{-1}}\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1} = \sqrt{\sqrt{\mathbf{V}_2^{-1}}\,\mathbf{V}_1^{-1}\sqrt{\mathbf{V}_2^{-1}}}\;\sqrt{\mathbf{V}_2}\sqrt{\mathbf{V}_1}. \tag{108}$$

- The Wigner rotation satisfies, in terms of the generic non-commutativity of two boosts,

$$\mathbf{V}_{2\oplus 1} = \mathbf{R}^\dagger\,\mathbf{V}_{1\oplus 2}\,\mathbf{R}, \tag{109}$$

  and, in terms of the generic non-associativity of three boosts,

$$\mathbf{V}_{(1\oplus 2)\oplus 3} = \mathbf{R}_{2\oplus 3}^\dagger\,\mathbf{V}_{1\oplus(2\oplus 3)}\,\mathbf{R}_{2\oplus 3}. \tag{110}$$

Overall, some rather complicated linear algebra involving 4×4 matrices has been reduced to relatively simple algebra in the $\mathbb{C} \otimes \mathbb{H}$ complexified quaternion number system.

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