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Consecutive-k_1 and k_2-out-of-n: F Structures with a Single Change Point

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Abstract: In the present paper, we establish a new consecutive-type reliability model with a single change point. The proposed structure has two common failure criteria and consists of two different types of components. The general framework for constructing the so-called consecutive-k_1 and k_2-out-of-n: F system with a single change point is launched. In addition, the number of path sets of the proposed structure is determined with the aid of a combinatorial approach. Moreover, two crucial performance characteristics of the proposed model are studied. The numerical investigation carried out reveals that the behavior of the new structure is outperforming against its competitors.

Keywords: structures with two common failure criteria; reliability function; mean time to failure

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Citation: Triantafyllou, I.S.; Chalikias, M. Consecutive-k_1 and k_2-out-of-n: F Structures with a Single Change Point. *Stats* **2023**, *6*, 438–449. https://doi.org/10.3390/stats6010027

Academic Editor: Hari Mohan Srivastava

Received: 25 February 2023
Revised: 14 March 2023
Accepted: 15 March 2023
Published: 16 March 2023

1. Introduction

In numerous real-life applications, a captivate concern requests apt structural designs, which can formulate the underlying phenomenon or process. A particular family of reliability models, which has attracted quite a lot of research scrutiny during the last four decades, is known as consecutive-type systems. The wealth of their usage in Reliability Engineering and Statistical Modelling, has turned these systems into a crucial research tool.

Generally speaking, a great collection of consecutive-type systems appears in the existing literature. One of the pioneer members of the aforementioned class is the so-called consecutive-k-out-of-n: F structure consisting of n linearly (or circularly) ordered components. The latter system stops its operation if and only if at least k consecutive components fail (see, e.g., [1–3]). In addition, several generalizations of the consecutive-k-out-of-n: F systems have been introduced and studied in detail. For instance, we refer to the r-within-consecutive-out-of-n: F structure, which fails if and only if there exist k consecutive units which include among them at least r failed ones (see [4–7]). In a slightly different mode, reliability structures with a single failure consecutive-type criterion appear in [8,9].

On the contrary, it is quite often that a practitioner handles problems related to two different failure criteria. For such cases, structures whose operation can stop due to more than one reasoning are more common. The interested reader is referred to the (n,f,k) structures (see, e.g., [10–12]), the $<n,f,k>$ systems ([13,14]), the constrained-(k,d)-out-of-n ones (see [15,16]) or the consecutive-k_1 and k_2-out-of-n: F structure ([17]).

In the present article, we introduce a new reliability system with two common failure criteria having a single change point. More precisely, we propose the consecutive-k_1 and k_2-out-of-n: F structure consisting of two different types of units, namely having a single change point. In Section 2, we describe the general framework of the proposed reliability model. An intriguing result is provided in Section 3 and refers to the determination of the number of path sets of the proposed structure. Moreover, two reliability characteristics of it are also studied. After carrying out a numerical investigation, we provide some evidence for the performance of the consecutive-k_1 and k_2-out-of-n: F structure with a single change point (see Section 4). Finally, the Section 5 summarizes the contribution of the present paper, while some practical concluding remarks are also highlighted.

2. The Constructing Framework for Consecutive-k_1 and k_2-out-of-n: F Systems with a Single Change Point

In the present section, we give an account of the general framework of the proposed reliability structure. Generally speaking, the consecutive-k_1 and k_2-out-of-n structure contains n components in a line and fails if, and only if, there exist at least non-overlapping consecutive k_1 failed components and consecutive k_2 failed components. Kindly note that parameters k_1, k_2 are interchangeable, namely, their role is not distinguished. In that sense, we assume hereafter and without loss of generality that $k_1 < k_2$.

In what follows, we assume that the first n_1 components of the consecutive-k_1 and k_2-out-of-n: F system share a common reliability p_1 (units of Group 1, hereafter), while the remaining ones, namely the remaining $n_2 = n - n_1$ units have reliability p_2 (units of Group 2, hereafter). Note that in our general approach ($1 \leq k_1 < k_2 \leq n$), the aforementioned probabilities p_2, p_1 are not necessarily equal. Therefore, in our framework the location of the n_1-th unit is considered as the change point of the proposed model, which shall be noted hereafter as consecutive-k_1 and k_2-out-of-n: F system with a single change point.

The proposed setup is applicable in some real-life problems from different fields. For instance, in a manufacturing framework we may assume that a production process requires that the components are gradually supplied. In this setup, the first n_1 components could share a common reliability, while the next n_2 components shall have a different reliability from the first ones. Such cases may arise when the production process of the manufacturing plant is improved or modified.

It goes without saying that if the probabilities p_2, p_1 do not differ, the proposed system coincides to the common consecutive-k_1 and k_2-out-of-n: F (see [17] or [18]). Note that the common consecutive-k_1 and k_2-out-of-n: F, which has been introduced in [17] does not contain change points. In that sense, the proposed model generalizes the latter structure, but it also offers a new system which seems to be more flexible and applicable to real-life problems. For studies concerning alternative models with a single change point, the interested reader is referred to [19] or [20].

Figure 1 offers an illustration for the proposed consecutive-k_1 and k_2-out-of-n: F system with a single change point.

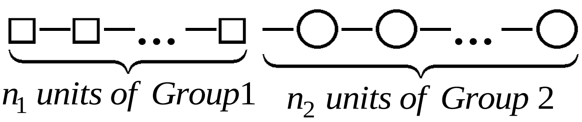

Figure 1. The consecutive-k_1 and k_2-out-of-n: F system with a single change point.

In the above representation, the symbol □ corresponds to units of Group 1 (with reliability p_1), while the symbol ○ is related to units of Group 2 (with reliability p_2). The resulting model fails if and only if there exist k_1 non-overlapping failed units (of either Group) and k_2 non-overlapping failed ones (of either Group). Note that the restriction $k_1 + k_2 \leq n$ is quite obvious, while either k_1 or k_2 could be equal to or larger than $\max(n_1, n_2)$. For example, let us consider the special case where the design parameters of the proposed model are determined as $n_1 = 5, n_2 = 3, k_1 = 2, k_2 = 3$. The resulting consecutive-2 and 3-out-of-8: F structure can be illustrated as

The structure illustrated at Figure 2 fails if and only if there exist two (non-overlapping) consecutive failed units and three (non-overlapping) consecutive failed ones. The whole set of failure scenarios with exactly five failed units are represented below (note that the black boxes correspond to failed components, while the white ones express the operating ones).

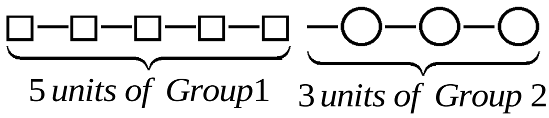

Figure 2. The consecutive-2 and 3-out-of-8*: F* system with a single change point.

As is readily obtained with the aid of Figure 3, the overall failure of the resulting scheme will come whenever one of the following scenarios takes place:

- Two consecutive failed units of Group 1 and three consecutive failed units of Group 2;
- Three consecutive failed units of Group 1 and two consecutive failed units of Group 2;
- Two consecutive failed units of Group 1 and three consecutive failed units of Group 1;
- Four consecutive failed units of Group 1 located at places 2 to 5 (out of 8) and one failed unit of Group 2 located at place 6 (out of 8).

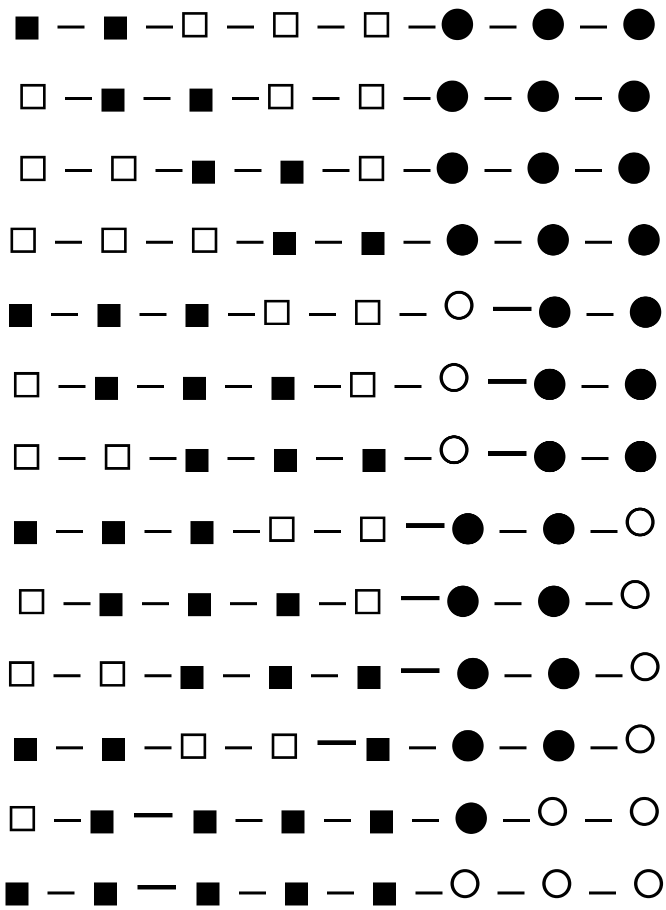

Figure 3. Failure scenarios with exactly 5 failed units for the consecutive-2 and 3-out-of-8*: F* system with a single change point.

3. Main Results

In this section, we establish the main results of the present paper. We first determine the number of path sets of the proposed consecutive-k_1 and k_2-out-of-n*: F* system with a single change point. In particular, we aim at calculating the number of path sets of the structure including i units of Group 1 and j units of Group 2 ($r_{n_1,n_2,k_1,k_2}(i,j)$, hereafter). Based on this outcome, the reliability function and the mean time to failure of the proposed reliability scheme are also studied.

Since there exists a single change point throughout the units of the proposed structure, the first unit of Group 2, namely the first unit after the change point, could be either in a functioning state or failed. In other words, we shall next study the proposed reliability system under two distinct schemes. According to the first scheme (Scheme 1, hereafter) the first component of Group 2, e.g., the first ○ appearing in the structure line, is supposed to be in a working state. Therefore, if we denote by 0 and 1, the failure and functioning state of each unit, respectively, a typical sequence of n binary elements (under Scheme 1), including i working units of Group 1 ($w.u.$G1) and j working units of Group 2 ($w.u.$G2) is illustrated as follows.

In the above figural representation $x_r, r = 2, 3, \ldots, i$ $(i \leq n_1)$ corresponds to the number of 0 s which are located between two successive 1s throughout the units of Group 1, namely x_r is defined as the run of 0 s between the $(r-1) - th$ and the $r - th$ working unit of Group 1. Additionally, x_1 simply denotes the units' failures of Group 1 occurred before the appearance of the first working unit of the same group. In other words, the random variable $x_r, r = 2, 3, \ldots, i$ represents the length of run of 0s in each urn between successive 1s throughout the first n_1 components of the structure. It goes without saying that x_1 expresses the length of the first run of 0s in the same side of the underlying structure. It is easily deduced that quantities $x_r, r = 2, 3, \ldots, i$ obey the next restrictions

$$0 \leq x_r \leq n_1, r = 1,\ 2,\ \ldots,\ i+1 \text{ and } \sum_{r=1}^{i+1} x_r = n_1 - i. \tag{1}$$

Note that Equation (1) could be alternatively written as

$$0 \leq x_r \leq \sum_{m=1}^{r-1} x_m - (r-1),\ r = 1,\ 2,\ \ldots,\ i+1,$$

where $\sum_{r=1}^{i+1} x_r = n_1 - i$.

In a similar manner, we denote by $y_s,\ s = 1, 2, \ldots, j$ $(j \leq n_2)$ the amount of 0s between successive 1s throughout the units of Group 2. In other words, random variables $y_s,\ s = 1,\ 2,\ \ldots,\ j$ express the length of run of 0 s in each urn between successive 1 s throughout the n_2 units of Group 2. The following conditions should be satisfied by the above-mentioned random quantities.

$$0 \leq y_s \leq n_2, s = 1,\ 2,\ \ldots, j \text{ and } \sum_{s=1}^{j} y_s = n_2 - j. \tag{2}$$

On the other hand, there exists a second scheme (Scheme 2, hereafter) for the underlying consecutive-k_1 and k_2-out-of-n: F system with a single change point that we should take into account. Our focus remains to the status of the first unit of Group 2 located in the structure line. Under Scheme 2 we assume that the particular unit has failed. Consequently, a binary sequence of n elements (under Scheme 2), including i working units of Group 1 ($w.u.$G1) and j working units of Group 2 ($w.u.$G2) is illustrated as follows.

In the above figural representation, the random variables $x_r, r = 1,\ 2,\ \ldots,\ i,\ i+1$ and $y_s,\ s = 1,\ 2,\ \ldots,\ j,\ j+1$ are related once again to the length of runs of 0s in the corresponding urn. Clearly, the abovementioned variables satisfy the following set of conditions (under Scheme 2).

$$0 \leq x_r \leq n_1,\ r = 1,\ 2,\ \ldots,\ i,\ i+1,\ \sum_{r=1}^{i+1} x_r = n_1 - i - 1 \text{ and } 0 \leq y_s \leq n_2,\ s = 1,\ 2,\ \ldots,\ j,\ j+1,\ \sum_{s=1}^{j+1} y_s = n_2 - j - 1 \tag{3}$$

It is worth mentioning that the above-mentioned schemes, namely Scheme 1 and Scheme 2, could be presented under an alternative (but in any case, equivalent) way. Indeed, let us define y_1^* as the run of 0s before the first working component of Group 2. Then, under the assumption that $y_1^* > 0$ $(y_1^* = 0)$, the resulting scheme coincides to Scheme 1 (Scheme 2). In what follows, we utilize the notations and schemes illustrated in Figures 4 and 5.

$$\overbrace{\underset{x_1}{0\ldots0}\ \ \underset{\substack{1^{st}\\ w.u.G1}}{1}\ \ \underset{x_2}{0\ldots0}\ \ \underset{\substack{2^{nd}\\ w.u.G1}}{1}\ \ \underset{x_3}{0\ldots0}\ldots\underset{x_i}{0\ldots0}\ \ \underset{\substack{i^{th}\\ w.u.G1}}{1}\ \ \underset{x_{i+1}}{0\ldots0}}^{n_1 \textit{ units of Group } 1}\quad \overbrace{\underset{\substack{1^{st}\\ w.u.G2}}{1}\ \ \underset{y_1}{0\ldots0}\ \ \underset{\substack{2^{nd}\\ w.u.G2}}{1}\ \ \underset{y_2}{0\ldots0}\ \ \underset{\substack{3^{rd}\\ w.u.G2}}{1}\ \ \underset{y_3}{0\ldots0}\ldots\underset{y_{j-1}}{0\ldots0}\ \ \underset{\substack{j^{th}\\ w.u.G2}}{1}\ \ \underset{y_j}{0\ldots0}}^{n_2 \textit{ units of Group } 2}$$

Figure 4. The consecutive-k_1 and k_2-out-of-*n: F* system with a single change point under Scheme 1.

$$\overbrace{\underset{x_1}{0\ldots0}\ \ \underset{\substack{1^{st}\\ w.u.G1}}{1}\ \ \underset{x_2}{0\ldots0}\ \ \underset{\substack{2^{nd}\\ w.u.G1}}{1}\ \ \underset{x_3}{0\ldots0}\ldots\underset{x_i}{0\ldots0}\ \ \underset{\substack{i^{th}\\ w.u.G1}}{1}\ \ \underset{x_{i+1}}{0\ldots0}}^{n_1 \textit{ units of Group } 1}\quad \overbrace{\underset{y_1}{0\ldots0}\ \ \underset{\substack{1^{st}\\ w.u.G2}}{1}\ \ \underset{y_2}{0\ldots0}\ \ \underset{\substack{2^{nd}\\ w.u.G2}}{1}\ \ \underset{y_3}{0\ldots0}\ldots\underset{y_j}{0\ldots0}\ \ \underset{\substack{j^{th}\\ w.u.G2}}{1}\ \ \underset{y_{j+1}}{0\ldots0}}^{n_2 \textit{ units of Group } 2}$$

Figure 5. The consecutive-k_1 and k_2-out-of-*n: F* system with a single change point under Scheme 2.

The next proposition offers an explicit expression for determining the quantities $r_{n_1,n_2,k_1,k_2}(i,j)$, namely the number of path sets including i units of Group 1 and j units of Group 2 for the proposed consecutive-k_1 and k_2-out-of-*n: F* system with a single change point.

Proposition 1. *Let us consider a consecutive-k_1 and k_2-out-of-n: F system with a single change point, consisting of n_1 (n_2) units of Group 1 (2) with common reliability p_1 (p_2). The number of path sets of the structure including i units of Group 1 and j units of Group 2 is determined by the aid of the following*

$$\begin{aligned}
r_{n_1,n_2,k_1,k_2}(i,j) &= C_{k_2}(i+1,n_1-i)\cdot C_{k_2}(j,n_2-j)\\
&+(i+1)\cdot C_{k_1}(j,n_2-j)\cdot\sum_{x=k_2}^{n_1-i}C_{k_1}(i,n_1-i-x)\\
&+j\cdot C_{k_1}(i+1,n_1-i)\cdot\sum_{y=k_2}^{n_2-j+1}C_{k_1}(j-1,n_2-j-y)\\
&+\sum_{x=0}^{k_2-2}\sum_{y=1}^{k_2-x-1}C_{k_2}(i,n_1-i-x)\cdot C_{k_2}(j,n_2-j-y)\\
&+i\cdot\sum_{z=k_2}^{n_1-i}\sum_{x=0}^{k_1-2}\sum_{y=1}^{k_1-x-1}C_{k_1-x-1}(j,n_2-j-y)\cdot C_{k_1}(i-1,n_1-i-x-z)\\
&+j\cdot\sum_{z=k_2}^{n_2-i}\sum_{x=0}^{k_2-2}\sum_{y=1}^{k_2-x-1}C_{k_1}(i,n_1-i-x)\cdot C_{k_1}(j-1,n_2-j-y-z)\\
&+\sum_{x=0}^{n_1-i}\sum_{y=\max(k_2-x,0)}^{n_2-j}C_{k_1}(i,n_1-i-x)\cdot C_{k_1}(j,n_2-j-y)
\end{aligned}\tag{4}$$

where

$$C_h(a,b)=\sum_{g=0}^{\min(a,[b/h])}(-1)^g\binom{a}{g}\binom{a+b-gh-1}{a-1}, a>0, b\geq 0, h>0\tag{5}$$

Proof. The consecutive-k_1 and k_2-out-of-*n: F* system with a single change point operates if and only if

- there is no run of 0s of a length equal to or larger than k_2 (Scenario 1)

  or

- there is exactly one run of 0s of a length equal to or larger than k_2 (and in any case less than $k_1 + k_2$) and simultaneously the length of the remaining runs of 0s is smaller than k_1 (Scenario 2). $\square$

Under Scenario 2, the unique run of 0s having a length equal to or larger than k_2 could be located either at the first part (Choice 1) or at the second part of the structure (Choice 2). Moreover, if the Scheme 2 is under investigation, a third choice is evident. More specifically, under Scheme 2, the occurrence of the unique run of 0s having a length equal to or larger than k_2 can take place partially at the first and partially at the second side of the system (Choice 3). In other words, the specific run could be formulated either exclusively with units of Group 1, or exclusively with units of Group 2 or with a combination of them.

We first assume that the underlying consecutive-k_1 and k_2-out-of-n: F system with a single change point follows Scheme 1 illustrated in Figure 4. In other words, we assume that the first unit of Group 2 appearing in the structure line is working. Then, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 1 shall be determined for each one of the above-mentioned scenarios separately. More specifically,

- under Scenario 1, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 1 equals to the number of integer solutions of the following linear equations
$$x_1 + x_2 + \ldots + x_{i+1} = n_1 - i \tag{6}$$
such that $0 \le x_1 < k_2, 0 \le x_2 < k_2, \ldots, 0 \le x_{i+1} < k_2$,
$$y_1 + y_2 + \ldots + y_j = n_2 - j \tag{7}$$
such that $0 \le y_1 < k_2, 0 \le y_2 < k_2, \ldots, 0 \le y_j < k_2$.
However, the number of integer solutions for Equations (6) and (7) equal to $C_{k_2}(i+1, n_1 - i)$ and $C_{k_2}(j, n_2 - j)$, respectively. Therefore, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 1 and Scenario 1 is determined as
$$r_{1,1} = C_{k_2}(i+1, n_1 - i) \cdot C_{k_2}(j, n_2 - j). \tag{8}$$

- under Scenario 2, there exist two possible choices as mentioned earlier (Choices 1 and 2). Under Choice 1, the unique run of 0s having a length equal to or larger than k_2 contains exclusively failed units of Group 1. Therefore, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 1, Scenario 2 and Choice 1 equals to the number of integer solutions of the following linear equations
$$x_{d_1} + x_{d_2} + \ldots + x_{d_i} = n_1 - i - x_{d_{i+1}} \tag{9}$$
such that $0 \le x_{d_1} < k_1$, $0 \le x_{d_2} < k_1, \ldots, 0 \le x_{d_i} < k_1$ and $x_{d_{i+1}} \ge k_2$, where $\{d_1, d_2, \ldots, d_{i+1}\}$ is a permutation of $\{1, 2, \ldots, i+1\}$,
$$y_1 + y_2 + \ldots + y_j = n_2 - j \tag{10}$$
such that $0 \le y_1 < k_1$, $0 \le y_2 < k_1, \ldots, 0 \le y_j < k_1$. However, the number of integer solutions for Equations (9) and (10) equal to $(i+1) \cdot \sum_{x=k_2}^{n_1-i} C_{k_1}(i, n_1 - i - x)$ and $C_{k_1}(j, n_2 - j)$, respectively. Therefore, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 1, Scenario 2 and Choice 1 is determined as
$$r_{1,2,1} = (i+1) \cdot C_{k_1}(j, n_2 - j) \cdot \sum_{x\,=\,k_2}^{n_1-i} C_{k_1}(i, n_1 - i - x). \tag{11}$$

On the other hand, under Choice 2, the unique run of 0s having a length equal to or larger than k_2 contains exclusively failed units of Group 2. Therefore, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 1, Scenario 2 and Choice 2 equals to the number of integer solutions of the following linear equations

$$x_1 + x_2 + \ldots + x_{i+1} = n_1 - i \tag{12}$$

such that $0 \leq x_1 < k_1, 0 \leq x_2 < k_1, \ldots, 0 \leq x_{i+1} < k_1$,

$$y_{f_1} + y_{f_2} + \ldots + y_{f_{j-1}} = n_2 - j - y_{f_j} \tag{13}$$

such that $0 \leq y_{f_1} < k_1, 0 \leq y_{f_2} < k_1, \ldots, 0 \leq y_{f_{j-1}} < k_1$ and $y_{f_j} \geq k_2$ where $\{f_1, f_2, \ldots, f_j\}$ is a permutation of $\{1, 2, \ldots, j\}$. However, the number of integer solutions for Equations (12) and (13) equal to $C_{k_1}(i+1, n_1 - i)$ and $j \cdot \sum_{y=k_2}^{n_2-j+1} C_{k_1}(j-1, n_2 - j - y)$, respectively. Therefore, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 1, Scenario 2 and Choice 2 is determined as

$$r_{1,2,2} = j \cdot C_{k_1}(i+1, n_1 - i) \cdot \sum_{y\,=\,k_2}^{n_2-j+1} C_{k_1}(j-1, n_2 - j - y). \tag{14}$$

We next assume that the underlying consecutive-k1 and k2-out-of-n: F system with a single change point follows Scheme 2 illustrated in Figure 4. In other words, we assume that the first unit of Group 2 appearing in the structure line is working. Then, the total number of binary sequences of units of both groups under Scheme 2 shall be determined separately for each one of the scenarios and choices mentioned before. More specifically,

- under Scenario 1, the total number of binary sequences of units of both groups under Scheme 2 equals to the number of integer solutions of the following linear equations

$$x_1 + x_2 + \ldots + x_i = n_1 - i - x_{i+1} \tag{15}$$

such that $0 \leq x_1 < k_2, 0 \leq x_2 < k_2, \ldots, 0 \leq x_i < k_2$,

$$y_2 + y_3 + \ldots + y_{j+1} = n_2 - j - y_1 \tag{16}$$

such that $0 \leq y_2 < k_2$, $0 \leq y_3 < k_2, \ldots, 0 \leq y_{j+1} < k_2$ and $x_{i+1} + y_1 < k_2$, where $x_{i+1} \geq 0$, $y_1 > 0$. Therefore, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 2 and Scenario 1 is determined as

$$r_{2,1} = \sum_{x\,=\,0}^{k_2-2} \sum_{y\,=\,1}^{k_2-x-1} C_{k_2}(i, n_1 - i - x) \cdot C_{k_2}(j, n_2 - j - y). \tag{17}$$

- under Scenario 2, there exist three possible choices as mentioned earlier (Choices 1, 2 and 3). Under Choice 1, the unique run of 0s having a length equal to or larger than k_2 contains exclusively failed units of Group 1, namely it coincides to one of the $x_1, x_2, \ldots, x_i$. Therefore, the total number of binary sequences of units of both groups under Scheme 2, Scenario 2 and Choice 1 equals to the number of integer solutions of the following linear equations

$$x_{d_1} + x_{d_2} + \ldots + x_{d_{i-1}} = n_1 - i - x_{d_i} - x_{i+1} \tag{18}$$

such that $0 \leq x_{d_1} < k_1$, $0 \leq x_{d_2} < k_1, \ldots, 0 \leq x_{d_{i-1}} < k_1$ and $x_{d_i} \geq k_2$, where $\{d_1, d_2, \ldots, d_i\}$ is a permutation of $\{1, 2, \ldots, i\}$,

$$y_2 + y_3 + \ldots + y_{j+1} = n_2 - j - y_1 \tag{19}$$

such that $0 \le y_2 < k_1, 0 \le y_3 < k_1, \ldots, 0 \le y_{j+1} < k_1$ and $x_{i+1} + y_1 < k_1$. Therefore, the total number of binary sequences of units of Group 1 and Group 2 under Scheme 2, Scenario 2 and Choice 1 is determined as

$$r_{2,2,1} = i \cdot \sum_{z\,=\,k_2}^{n_1-i} \sum_{x=0}^{k_1-2} \sum_{y=1}^{k_1-x-1} C_{k_1}(j, n_2 - j - y) \cdot C_{k_1}(i-1, n_1 - i - x - z). \quad (20)$$

On the other hand, under Choice 2, the unique run of 0s having a length equal to or larger than k_2 contains exclusively failed units of Group 2, namely it coincides to one of the $y_2, y_3, \ldots, y_{j+1}$. Therefore, the total number of binary sequences of units of both groups under Scheme 2, Scenario 2 and Choice 2 equals to the number of integer solutions of the following linear equations

$$x_1 + x_2 + \ldots + x_i = n_1 - i - x_{i+1} \quad (21)$$

such that $0 \le x_1 < k_1, 0 \le x_2 < k_1, \ldots, 0 \le x_i < k_1$,

$$y_{f_1} + y_{f_2} + \ldots + y_{f_{j-1}} = n_2 - j - y_{f_j} - y_1 \quad (22)$$

such that $0 \le y_{f_1} < k_1$, $0 \le y_{f_2} < k_1$, ..., $0 \le y_{f_{j-1}} < k_1$, $x_{i+1} + y_1 < k_1$ and $y_{f_{j+1}} \ge k_2$, where $\{f_1, f_2, \ldots, f_j\}$ is a permutation of $\{1, 2, \ldots, j\}$.

Therefore, the total number of binary sequences of units of both groups under Scheme 2, Scenario 2 and Choice 2 is determined as

$$r_{2,2,2} = j \cdot \sum_{z\,=\,k_2}^{n_1-i} \sum_{x=0}^{k_1-2} \sum_{y=1}^{k_1-x-1} C_{k_1}(i, n_1 - i - x) \cdot C_{k_1}(j-1, n_2 - j - y - z). \quad (23)$$

Finally, under Choice 3, the unique run of 0s having a length equal to or larger than k_2 coincides to $x_{i+1} + y_1$. Therefore, the total number of binary sequences of units of both group under Scheme 2, Scenario 2 and Choice 3 equals to the number of integer solutions of the following linear equations

$$x_1 + x_2 + \ldots + x_i = n_1 - i - x_{i+1} \quad (24)$$

such that $0 \le x_1 < k_1, 0 \le x_2 < k_1, \ldots, 0 \le x_i < k_1$,

$$y_2 + y_3 + \ldots + y_{j+1} = n_2 - j - y_1 \quad (25)$$

such that $0 \le y_2 < k_1, 0 \le y_3 < k_1, \ldots, 0 \le y_{j+1} < k_1$, $x_{i+1} + y_1 \ge k_2$. Therefore, the total number of binary sequences of units of both groups under Scheme 2, Scenario 2 and Choice 3 is determined as

$$r_{2,2,3} = \sum_{x_{i+1}+y_1 \ge k_2} C_{k_1}(i, n_1 - i - x) \cdot C_{k_1}(j, n_2 - j - y). \quad (26)$$

The desired result is readily deduced by combining Formulas (8), (11), (14), (17), (20), (23) and (26).

Having at hand the expression provided in (4), the reliability and the mean time to failure of the consecutive-k_1 and k_2-out-of-n: F system with a single change point can be readily determined. We next make the common assumption that the number of units of each type in the underlying structure is fixed. That practically means that the design parameters n_1, n_2 are pre-determined. Under this assumption, the reliability of the consecutive-k_1 and k_2-out-of-n: F system with a single change point can be computed via the following formula

$$R_{n_1,n_2,k_1,k_2}(p_1, p_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} r_{n_1,n_2,k_1,k_2}(i,j) p_1^i (1-p_1)^{n_1-i} p_2^j (1-p_2)^{n_2-j}. \quad (27)$$

where $r_{n_1,n_2,k_1,k_2}(i,j)$ can be computed with the aid of (4).

Moreover, let us next denote by $F_1(t)$, $F_2(t)$ the cumulative density functions of the components of Group 1 and Group 2, respectively, while $\overline{F}_1(t) = 1 - F_1(t)$ and $\overline{F}_2(t) = 1 - F_2(t)$ correspond to their reliability function, respectively. Then, the mean time to failure (*MTTF*, hereafter) of the proposed consecutive-k_1 and k_2-out-of-n: F system with a single change point, namely the expected time till the system no longer operates, can be determined as

$$MTTF_{n_1,n_2,m,k}(F_1,F_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} r_{n_1,n_2,m,k}(i,j) \int_0^\infty \overline{F}_1^i(t) F_1{}^{n_1-i}(t) \overline{F}_2^j(t) F_2{}^{n_2-j}(t) dt. \quad (28)$$

Note that Formulas (4), (27) and (28) shall be implemented for providing the numerical results given in the next section of the present work.

4. Numerical Results

In the present section we carry out a numerical investigation to shed light on the behavior of the proposed consecutive-k_1 and k_2-out-of-n: F system with a single change point. The numerical results and figural representations displayed throughout the next lines, are based on the mathematical outcomes which have been presented and proved in the previous section.

Let us first focus on the reliability of the consecutive-k_1 and k_2-out-of-n: F system with a single change point. In particular, we study the performance of the proposed structure under different values of the design parameters n_1, n_2, k_1, k_2. We mainly aim at delivering remarks about how these parameters reflect the performance of the resulting structure.

Figure 6 depicts the reliability of the underlying system in terms of parameter n_1 under specific choices of the remaining parameters.

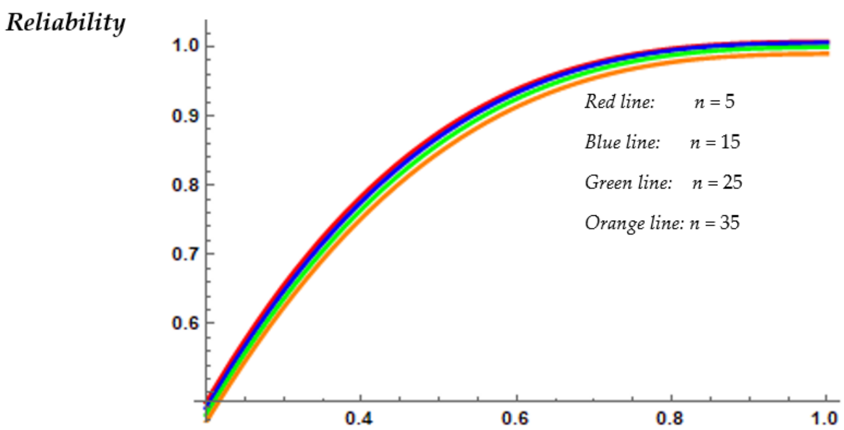

Figure 6. The reliability of the consecutive-k_1 and k_2-out-of-n: F system with a single change point ($n_2 = 4, k_1 = 3, k_2 = 3, p_1 = 0.8$).

As is readily observed, for larger values of parameter n_1, the corresponding reliability becomes smaller. The following figure sheds light on the performance of the proposed structure with respect to the parameter k_1. Figure 7 illustrates the reliability of the consecutive-k_1 and k_2-out-of-n: F system with a single change point for different values of k_1.

Figure 7 reveals that the larger the parameter k_1 is, the better the performance becomes of the corresponding system.

We next investigate the performance of the consecutive-k_1 and k_2-out-of-n: F system with a single change point in comparison with the consecutive-k-out-of-n: F structure with a single change point proposed in [19] and the m-consecutive-k-out-of-n: F structure with a single change point proposed in [20]. In order to provide fair comparisons, we consider for both systems the same design parameters p_1, p_2, n_1, n_2 and we evaluate the corresponding reliability values.

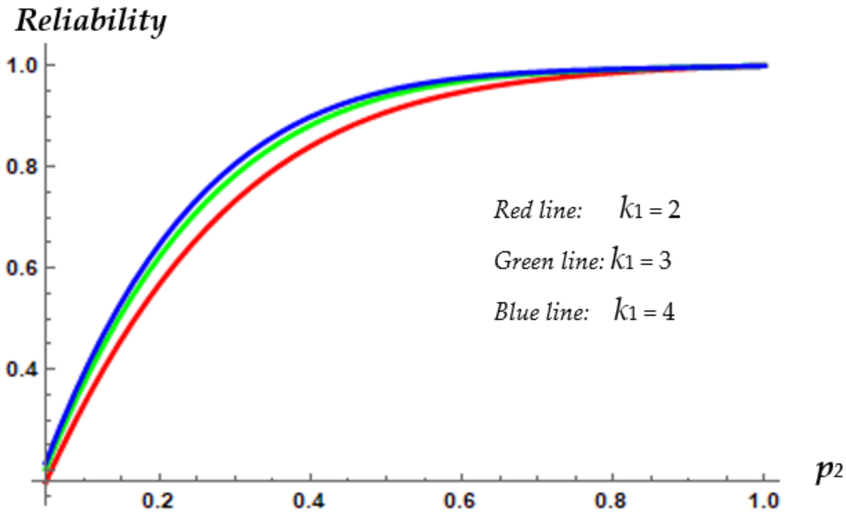

Figure 7. The reliability of the consecutive-k_1 and k_2-out-of-n: F system with a single change point ($n_1 = 6, n_2 = 6, k_2 = 5, p_1 = 0.5$).

Based on the numerical comparisons presented in Table 1, we readily deduce that the consecutive-k_1 and k_2-out-of-n: F system with a single change point achieves larger reliability values and consequently outperforms its competitors, namely the corresponding consecutive-k-out-of-n: F system with a single change point and the m-consecutive-k-out-of-n: F system with a single change point. For instance, for the special case p_1 = 0.7, p_2 = 0.6, n_1 = 10, n_2 = 5 the resulting consecutive-k_1 and k_2-out-of-15: F system for k_1 = 3, k_2 = 4 achieves reliability value equal to 96.8151% and 98.9616%, respectively. At the same time, the consecutive-k-out-of-15: F system with a single change point and k = 3 or k = 4 seems weaker since its corresponding reliability is equal to 69.0048% or 89.5755%, respectively. The same conclusion is drawn if we look at the performance of the corresponding m-consecutive-k-out-of-15: F system with a single change point, whose reliability for (m, k) = (3, 2) or (4, 2) equals to 90.5603% and 88.2080%, respectively.

Table 1. Numerical comparisons between m-consecutive-k-out-of-n: F with a change point systems and consecutive-k-out-of-n: F with a change point systems.

		Consecutive-k-out-of-n: F with a Change Point		**m-Consecutive-k-out-of-n: F with a Change Point**		**Consecutive-k_1 and k_2-out-of-n: F System with a Single Change Point**	
(n1, n2)	**(p1, p2)**	k	**Reliability**	**(m, k)**	**Reliability**	**(k_1, k_2)**	**Reliability**
(10, 5)	(0.8, 0.7)	3	0.868297	(3, 2)	0.958571	(3, 4)	0.994680
		4	0.969613	(4, 2)	0.953963	(4, 3)	0.991787
	(0.7, 0.6)	3	0.690048	(3, 2)	0.905603	(3, 4)	0.968151
		4	0.895755	(4, 2)	0.882080	(4, 3)	0.989616
(12, 8)	(0.8, 0.7)	4	0.950400	(3, 2)	0.916497	(3, 4)	0.991902
		5	0.987742	(3, 3)	0.994343	(3, 5)	0.998053
	(0.7, 0.6)	4	0.842588	(3, 2)	0.837928	(3, 4)	0.954128
		5	0.945907	(3, 3)	0.969490	(3, 5)	0.984154
(15, 10)	(0.8, 0.7)	5	0.983599	(3, 2)	0.888646	(4, 5)	0.997824
		6	0.995840	(3, 3)	0.990176	(4, 6)	0.999482
	(0.7, 0.6)	5	0.929143	(3, 2)	0.826399	(4, 5)	0.985979
		6	0.975524	(3, 3)	0.950575	(4, 6)	0.995288

5. Discussion

In the present work, the consecutive-k_1 and k_2-out-of-n: F system with a single change point was established and studied in some detail. Two reliability characteristics of the proposed system were investigated and the corresponding explicit expressions for determining them were also deduced. The main contribution of the manuscript refers to the determination of the number of path sets of a given size for the proposed reliability structure. Based on the abovementioned result, one may readily reach closed formulae for the computation of the corresponding reliability function and mean time to failure of the system. An intriguing extension of the proposed model emerges under the assumption that the number of components of each type is random. Such a case occurs if the single change point of the underlying structure has not been emplaced at a certain location, but contrariwise the change point is supposed to be random. The latter case simply expresses the dynamic version of the proposed consecutive-k_1 and k_2-out-of-n: F system with a single change point. Finally, an analogous reliability study of structures with two common failure criteria and a single change point and maintenance policy could be an interesting topic for future research.

Author Contributions: All authors contributed equally to this project. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Derman, C.; Lieberman, G.J.; Ross, S.M. On the consecutive-k-out-of-n: F system. *IEEE Trans. Reliab.* **1982**, *31*, 57–63. [CrossRef]
2. Triantafyllou, I.S.; Koutras, M.V. On the signature of coherent systems and applications for consecutive–k–out–of–n: F systems. In *Advances in Mathematical Modeling for Reliability*; Bedford, T., Quigley, J., Walls, L., Alkali, B., Daneshkhah, A., Hardman, G., Eds.; IOS Press: Amsterdam, The Netherlands, 2008; pp. 119–128.
3. Triantafyllou, I.S.; Koutras, M.V. On the signature of coherent systems and applications. *Probab. Eng. Inf. Sci.* **2008**, *22*, 19–35. [CrossRef]
4. Griffith, W.S. On consecutive–k–out–of–n: Failure systems and their generalizations. In *Reliability and Quality Control*; Basu, A.P., Ed.; Elsevier: Amsterdam, The Netherlands, 1986; pp. 157–165.
5. Tong, Y.L. A rearrangement inequality for the longest run, with an application to network reliability. *J. Appl. Probab.* **1985**, *22*, 386–393. [CrossRef]
6. Triantafyllou, I.S.; Koutras, M.V. Signature and IFR preservation of 2-within-consecutive-k-out-of-n:F systems. *IEEE Trans. Reliab.* **2011**, *60*, 315–322. [CrossRef]
7. Triantafyllou, I.S. Signature-based Analysis of the weighted-r-within-consecutive-k-out-of-n: F systems. *Mathematics* **2022**, *11*, 2554. [CrossRef]
8. Eryilmaz, S.; Koutras, M.V.; Triantafyllou, I.S. Signature based analysis of m-consecutive k-out-of-n: F systems with exchangeable components. *Nav. Res. Logist.* **2011**, *58*, 344–354. [CrossRef]
9. Dafnis, S.D.; Makri, F.S.; Philippou, A.N. The reliability of a generalized consecutive system. *Appl. Math. Comput.* **2019**, *359*, 186–193. [CrossRef]
10. Chang, J.G.; Cui, L.; Hwang, F.K. Reliabilities for (n,f,k) systems. *Stat. Probab. Lett.* **1999**, *43*, 237–242. [CrossRef]
11. Zuo, M.J.; Lin, D.; Wu, Y. Reliability evaluation of combined k–out–of–n: F, consecutive–k–out–of–n: F and linear connected–(r,s)–out–of–(m,n):F system structures. *IEEE Trans. Reliab.* **2000**, *49*, 99–104. [CrossRef]
12. Triantafyllou, I.S.; Koutras, M.V. Reliability properties of (n,f,k) systems. *IEEE Trans. Reliab.* **2014**, *63*, 357–366. [CrossRef]
13. Cui, L.; Kuo, W.; Li, J.; Xie, M. On the dual reliability systems of (n,f,k) and $<n,f,k>$. *Stat. Probab. Lett.* **2006**, *76*, 1081–1088.
14. Triantafyllou, I.S. Reliability study of $<n,f,2>$ systems: A generating function approach. *Int. J. Math. Eng. Manag. Sci.* **2021**, *6*, 44–65.
15. Eryilmaz, S.; Zuo, M.J. Constrained (k,d)-out-of-n systems. *Int. J. Syst. Sci.* **2010**, *41*, 679–685. [CrossRef]
16. Triantafyllou, I.S. On the lifetime and signature of the constrained (k,d) out-of-n: F reliability systems. *Int. J. Math. Eng. Manag. Sci.* **2021**, *6*, 66–78. [CrossRef]

17. Zhao, X.; Cui, L.; Yang, X. A consecutive $k1$ and $k2$-out-of-n system and its reliability. In *Advanced Reliability Modeling II*; Yun, W.Y., Dohi, T., Eds.; World Scientific: Singapore, 2006; pp. 97–103.
18. Triantafyllou, I.S. On the consecutive-k_1 and k_2-out-of-n Reliability Systems. *Mathematics* **2020**, *8*, 630. [CrossRef]
19. Eryilmaz, S. Consecutive k-out-of-n: Lines with a change point. *Proc. IMechE Part O J. Risk Reliab.* **2016**, *230*, 545–550. [CrossRef]
20. Triantafyllou, I.S. m-consecutive-k-out-of-n: F structures with a single change point. *Mathematics* **2020**, *8*, 2203. [CrossRef]