

Informative g -priors for Mixed Models

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Supplementary Material

Appendix A R Code

A.1 Fitting the linear regression model

Consider the linear regression model:

$$y_i | \mathbf{x}_i, \boldsymbol{\beta}, \sigma^2 \stackrel{ind.}{\sim} N(\mathbf{x}'_i \boldsymbol{\beta}, \sigma^2), \quad i = 1, \dots, n, \quad (\text{A.1})$$

where y_i is the i th response, \mathbf{x}_i is a p -vector of covariates which usually includes an intercept, and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$ is the p -vector of regression coefficients. The proposed hierarchical prior is given by

$$\begin{aligned} \boldsymbol{\beta} | \sigma^2 &\sim N_p \left(\mathbf{e}_1 m, \frac{n}{p} (v - \sigma^2) (\mathbf{X}' \mathbf{X})^{-1} \right), \quad \sigma^2 \sim \text{gb}(a, b, v), \\ m &\sim N(m_0, v/k_m), \quad v^{-1} \sim \Gamma(k_v, v_0 k_v). \end{aligned} \quad (\text{A.2})$$

The following R function are used to fit the model, where \mathbf{y} is the vector of responses $\mathbf{y} = (y_1, \dots, y_n)$, \mathbf{X} is the design matrix for \mathbf{x}_i without the intercept, $(m_0, km, v0, kv)$ is (m_0, k_m, v_0, k_v) , and (a, b) is (a, b) .

```
library(R2jags)
fit_lm_gnew <- function(y, m0, km, v0, kv, X, a1=1, b1=1,
                           n.ITER=22000, n.burnin=2000, n.thin=4){
  # m ~ N(m0, v/km); v^{-1} ~ gamma( kv, kv*v0 )
  n <- length(y)
  X1 <- cbind(rep(1,n), X)
  p <- ncol(X1)
  b.v <- crossprod(X1)
  start <- list(list("beta" = c(m0, rep(0, p-1)),
                     "vinv" = rep(1/v0, 1),
                     "m" = rep(m0, 1),
```

```

"s2_v" = rep(0.5, 1) ) # s2_v is sigma^2/v
model <- function(){ # model for JAGS
  for(i in 1:n){
    y[i]~dnorm(inprod(X1[i,1:p], beta), 1/s2) ## s2 is sigma^2
  }
  v  = 1/vinv
  s2 = s2_v*v
  g.jags = (v-s2)/p
  beta ~ dmnorm(c(m, rep(0, p-1)), b.v/(g.jags*n)) # uses precision not covariance matrix
  s2_v ~ dbeta(a1, b1)
  m ~ dnorm(m0, km/v)
  inv ~ dgamma(kv, kv*v0)
}
chal <- list("y","X1","n","b.v","m0", "km", "v0", "kv", "p", "a1", "b1",
             "n.iter", "n.burnin", "n.thin") # data for JAGS
param <- c("beta", "s2", "v")
f <- jags.parallel(data=chal, inits=start, parameters.to.save=param,
                     n.chains=1, n.iter=n.iter, n.burnin=n.burnin, model.file=model,
                     n.thin=n.thin)
sfit <- f$BUGSoutput$summary
sfit
}

```

A.2 Fitting the linear mixed model

Consider the general linear mixed model

$$y_{ij} = \mathbf{x}'_{ij}\boldsymbol{\beta} + \mathbf{z}'_{ij}\boldsymbol{\gamma}_i + \epsilon_{ij}, \text{ or equivalently, } \mathbf{y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\boldsymbol{\gamma}_i + \boldsymbol{\epsilon}_i, \quad (\text{A.3})$$

where $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \sigma^2)$, $\boldsymbol{\gamma}_i$ is a k -vector of random effects, $\mathbf{X}_i = [\mathbf{x}_{i1} \cdots \mathbf{x}_{in_i}]'$, $\mathbf{Z}_i = [\mathbf{z}_{i1} \cdots \mathbf{z}_{in_i}]'$. In this setting, $i = 1, \dots, c$ denotes the data cluster associated with $\boldsymbol{\gamma}_i$ and $j = 1, \dots, n_i$ are the number of repeated measures within cluster i ; the total sample size is $n = \sum_{i=1}^c n_i$. The proposed hierarchical prior is given by

$$\begin{aligned} \boldsymbol{\beta}|g, \sigma^2 &\sim N_p \left(m\mathbf{e}_1, \frac{n}{p}(v - \sigma^2 - gk)(\mathbf{X}'\mathbf{X})^{-1} \right). \\ \boldsymbol{\gamma}_i|g &\stackrel{iid}{\sim} N_k(\mathbf{0}, gn(\mathbf{Z}'\mathbf{Z})^{-1}) \\ g|\sigma^2 &\sim \text{gb}\left(a_1, b_1, \frac{v-\sigma^2}{k}\right) \\ \sigma^2 &\sim \text{gb}(a_2, b_2, v) \\ m &\sim N(m_0, v/k_m), v^{-1} \sim \Gamma(k_v, v_0 k_v). \end{aligned} \quad (\text{A.4})$$

A uniform prior on g obtains from $a_1 = b_1 = 1$ which is the case we consider throughout, since JAGS does not have the generalized beta distribution. The following R function are used to fit the model, where \mathbf{y} is the vector of responses $\mathbf{y} = \{(y_{ij})\}$, \mathbf{X} is the design matrix \mathbf{X} , \mathbf{Z} is the matrix \mathbf{Z} , k is the dimension of $\boldsymbol{\gamma}_i$, $(m_0, km, v0, kv)$ is (m_0, k_m, v_0, k_v) , and $(a2, b2)$ is (a_2, b_2) .

```

library(R2jags)
fit_lme_gnew <- function(y, m0, km, v0, kv, X, Z, k, a2=1, b2=1,
                           n.iter=22000, n.burnin=2000, n.thin=4){
  # m~N(m0, v/km); v^{-1} ~ gamma( kv, kv*v0 )
  njag <- length(y)
}

```

```

pjag <- ncol(X)
XXjag = crossprod(X)
djag <- ncol(Z)
ncjag <- djag/k
ZZjag = crossprod(Z)
ZiZisum = matrix(0, k, k)
for(i in 1:ncjag){
  ZiZisum = ZiZisum + ZZjag[(1:k)+(i-1)*k, (1:k)+(i-1)*k]
}
gam.m <- rep(0, djag)
start <- list(list("beta" = c(m0, rep(0, pjag-1)),
                    "s2_v" = rep(.7, 1),
                    "g" = rep(0.2*v0/k, 1),
                    "gam" = rep(0, djag),
                    "m" = rep(m0, 1),
                    "vinv" = rep(1/v0, 1)))
model <- function() { # model for JAGS
  for(ii in 1:njag){
    y[ii] ~ dnorm(inprod(X[ii,1:pjag], beta) +
                  inprod(Z[ii,1:djag], gam), 1/s2)
  }
  v = 1/vinv
  s2 = s2_v*v
  s2_v ~ dbeta(a2,b2)
  g ~ dunif(0, (v-s2)/k)
  for(i in 1:ncjag){
    gam[(1:k)+(i-1)*k] ~ dmnorm(gam.m[(1:k)+(i-1)*k], ZiZisum/(g*njag))
  }
  beta ~ dmnorm(c(m, rep(0, pjag-1)), XXjag*pjag/njag/(v-s2-g*k))
  s2r = g*k
  m ~ dnorm(m0, km/v)
  inv ~ dgamma(kv, kv*v0)
}
chal <- list("y", "X", "Z", "njag", "pjag", "djag",
              "gam.m", "ZiZisum", "XXjag", "k", "ncjag",
              "a2", "b2", "m0", "km", "v0", "kv",
              "n.iter", "n.burnin", "n.thin") # data for JAGS
param <- c("beta", "s2", "gam", "s2r")
f <- jags.parallel(data=chal, inits=start, parameters.to.save=param,
                     n.chains=1, n.iter=n.iter, n.burnin=n.burnin, model.file=model,
                     n.thin=n.thin) # keep 5000
sfit <- f$BUGSoutput$summary
sfit
}

```

A.3 Fitting the One-way random effects ANOVA

Consider the one-way random effects ANOVA

$$y_{ij} = \beta + \gamma_i + \epsilon_{ij}, \quad i = 1, \dots, c, \quad j = 1, \dots, n_i, \quad (\text{A.5})$$

where $\epsilon_{ij} \stackrel{iid}{\sim} N(0, \sigma^2)$. The prior (A.4) reduces to

$$\begin{aligned}\gamma_i &\sim N(0, g), \frac{\sigma^2}{v} \sim \text{beta}(a_2, b_2), \\ \beta &\sim N(m_0, v/k_m), v^{-1} \sim \Gamma(k_v, v_0 k_v).\end{aligned}\quad (\text{A.6})$$

The following R function are used to fit the model, where \mathbf{y} is the vector of responses $\mathbf{y} = (\{y_{ij}\})$, \mathbf{X} is the design matrix \mathbf{X} , \mathbf{Z} is the matrix \mathbf{Z} , $(m_0, km, v0, kv)$ is (m_0, k_m, v_0, k_v) , and $(a2, b2)$ is (a_2, b_2) .

```
library(R2jags)
fit_anova_gnew <- function(y, m0, km, v0, kv, Z, a2=1, b2=1,
                             n.iter=22000, n.burnin=2000, n.thin=4){
  # m~N(m0, v/km); v^{-1} ~ gamma( kv, kv*v0 )
  njag <- length(y)
  djag <- ncol(Z)
  start <- list(list("beta" = rep(m0, 1),
                      "s2_v" = rep(0.7, 1),
                      "gam" = rep(0, djag),
                      "vinv" = rep(1/v0, 1)))
  model <- function() { # model for JAGS
    for(ii in 1:njag){
      y[ii] ~ dnorm(beta + inprod(Z[ii,1:djag], gam), 1/s2) # tau is 1/variance
    }
    v = 1/vinv
    s2 = s2_v*v
    s2_v ~ dbeta(a2,b2)
    s2r = v-s2
    for(i in 1:djag){
      gam[i] ~ dnorm(0, 1/s2r)
    }
    beta ~ dnorm(m0, km/v)
    inv ~ dgamma(kv, kv*v0)
  }
  chal <- list("y", "Z", "njag", "djag",
               "a2", "b2", "m0", "km", "v0", "kv",
               "n.iter", "n.burnin", "n.thin") # data for JAGS
  param <- c("beta", "s2", "gam", "s2r")
  f <- jags.parallel(data=chal, inits=start, parameters.to.save=param,
                      n.chains=1, n.iter=n.iter, n.burnin=n.burnin, model.file=model,
                      n.thin=n.thin) # keep 5000
  sfit <- f$BUGSoutput$summary
  sfit
}
}
```

References