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On the Relation between Lambert W-Function and Generalized Hypergeometric Functions

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Abstract: In the theory of special functions, finding correlations between different types of functions is of great interest as unifying results, especially when considering issues such as analytic continuation. In the present paper, the relation between Lambert W-function and generalized hypergeometric functions is discussed. It will be shown that it is possible to link these functions by following two different strategies, namely, by means of the direct and inverse Mellin transform of Lambert W-function and by solving the trinomial equation originally studied by Lambert and Euler. The new results can be used both to numerically evaluate Lambert W-function and to study its analytic structure.

Keywords: Lambert W-function; H-function; R-function; Mellin transform; Lagrange's inversion theorem

1. Introduction

The study of special functions and their applications has increasingly grown in the last 50 years. This comes together with the advent of powerful computation techniques and devices which have allowed applied scientists to envision real-world applications to this class of functions.

Even though great work has been carried out in applying the existing theory of special functions to physical and engineering problems, theoretical studies have also been taken into account.

About 25 years ago, the interest in the so-called Lambert W-function tremendously increased as important physical applications and the basic theory of this function were discussed in the seminal work of [1]. The fact that Lambert W-function is linked to exponential-like equations provides a great number of applications in almost every branch of science, as this type of equation is common.

Apparently unrelated to Lambert W-function, the so-called generalized hypergeometric functions have been considered as solutions to ordinary and partial differential equations [2]. This generalized class of functions has as special cases a great number of elementary and special functions, leading researchers to recognize such functions as higher-order special functions.

So far, no relation between Lambert W-function and hypergeometric-type functions has been established. In the theory of special functions, finding such correlations is of great interest, leading to more compact and rigorous representations. This approach is frequent in the literature as seen, for example, in [3–5]. In addition, by connecting functions which apparently have no relation, important analytical continuation strategies can be envisioned.

In the present paper, the relation between Lambert W-function and generalized hypergeometric functions is discussed. It will be shown that this link can be accomplished in two different ways, namely, by solving the trinomial equation originally studied by Euler in [6] and by Lambert; and by means of the direct and inverse Mellin transform of Lambert W-function.

The paper is organized as follows: Section 2 presents some general concepts about generalized hypergeometric functions, while Section 3 introduces the Lambert-W function and its application to solve Lambert's trinomial equation. Section 4 presents the solution of Lambert's trinomial equation both in terms of hypergeometric functions and in terms of the Lambert-W function. Then, Section 5 indicates how general hypergeometric functions and the Lambert-W function can be related while represented as contour integrals. Finally, Section 6 presents the conclusions of the present work.

2. Generalized Hypergeometric Functions

In the present paper, the H-function is the first class of generalized hypergeometric functions which will be explored. This function is defined as a contour complex integral by [2]:

$$
H_{p,q}^{m,n}\left[z \left| \begin{array}{ccccccc} (a_1, A), & \dots, & (a_n, A_n), & (a_{n+1}, A_{n+1}), & \dots, & (a_p, A_p) \\ (b_1, B_1), & \dots, & (b_m, B_m), & (b_{m+1}, B_{m+1}), & \dots, & (b_q, B_q) \end{array} \right. \right]
$$
$$
= \frac{1}{2\pi i} \int_L h(s) z^{-s} ds, \tag{1}
$$

where:

$$
h(s) = \frac{\displaystyle\prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\displaystyle\prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^{p} \Gamma(a_j + A_j s)} \tag{2}
$$

and p, q, m, n are integers such that $0 \le m \le q$ and $0 \le n \le p$, A_j and B_j are positive real quantities, and all the a_j and b_j are complex numbers. The contour L runs from $c - i\infty$ to $c + i\infty$ such that the poles of $\Gamma(b_j + B_j s)$, $j = 1, \dots, m$ lie to the left of L and the poles of $\Gamma(1 - a_j - A_j s)$, $j = 1, \dots, n$ lie to the right of L.

By performing the variable change $s \to -r$ and adjusting the contour L to L^*, where the integral runs from $c^* - i\infty$ to $c^* + i\infty$, the H-function can be alternatively defined as:

$$
H_{p,q}^{m,n}\left[z \left| \begin{array}{ccccccc} (a_1, A), & \dots, & (a_n, A_n), & (a_{n+1}, A_{n+1}), & \dots, & (a_p, A_p) \\ (b_1, B_1), & \dots, & (b_m, B_m), & (b_{m+1}, B_{m+1}), & \dots, & (b_q, B_q) \end{array} \right. \right]
$$
$$
= \frac{1}{2\pi i} \int_{L^*} \frac{\displaystyle\prod_{j=1}^{m} \Gamma(b_j - B_j r) \prod_{j=1}^{n} \Gamma(1 - a_j + A_j r)}{\displaystyle\prod_{j=m+1}^{q} \Gamma(1 - b_j + B_j r) \prod_{j=n+1}^{p} \Gamma(a_j - A_j r)} z^r dr, \tag{3}
$$

for which the same parameter domains' restrictions apply.

By considering the definition in (3), the H-function can be expressed in computable form as [2]:

When the poles of $\displaystyle\prod_{j=1}^{m} \Gamma(b_j - B_j r)$ are simple, we have:

$$H_{p\ q}^{m\ n}(z) = \sum_{h=1}^{m} \sum_{v=0}^{\infty} \frac{\displaystyle\prod_{j=1\neq h}^{m} \Gamma\left(b_j - B_j \frac{b_h + v}{B_h}\right)}{\displaystyle\prod_{j=m+1}^{q} \Gamma\left(1 - b_j + B_j \frac{b_h + v}{B_h}\right)} \times$$

$$\times \frac{\displaystyle\prod_{j=1}^{n} \Gamma\left(1 - a_j + A_j \frac{b_h + v}{B_j}\right)}{\displaystyle\prod_{j=n+1}^{p} \Gamma\left(a_j - A_j \frac{b_h + v}{B_h}\right)} \frac{(-1)^v z^{(b_h+v)/B_h}}{v! B_h} \tag{4}$$

for $z \neq 0$ if $\delta > 0$ and for $0 < |z| < D^{-1}$ if $\delta = 0$, where $\delta = \sum_{j=1}^{p} B_j - \sum_{j=1}^{q} A_j$ and $D = \prod_{j=1}^{p} A_j^{A_j} / \prod_{j=1}^{q} B_j^{B_j}$.

When the poles of $\displaystyle\prod_{j=1}^{n} \Gamma(1 - a_j + A_j r)$ are simple, we have

$$H_{p\ q}^{m\ n}(z) = \sum_{h=1}^{n} \sum_{v=0}^{\infty} \frac{\displaystyle\prod_{j=1\neq h}^{n} \Gamma\left(1 - a_j - A_j \frac{1 - a_h + v}{A_h}\right)}{\displaystyle\prod_{j=n+1}^{p} \Gamma\left(a_j + A_j \frac{1 - a_h + v}{A_h}\right)} \times$$

$$\times \frac{\displaystyle\prod_{j=1}^{m} \Gamma\left(b_j + B_j \frac{1 - a_h + v}{A_h}\right)}{\displaystyle\prod_{j=m+1}^{q} \Gamma\left(1 - b_j - B_j \frac{1 - a_h + v}{A_h}\right)} \frac{(-1)^v (1/z)^{(1-a_h+v)/A_h}}{v! A_h} \tag{5}$$

for $z \neq 0$ if $\delta < 0$ and for $|z| > D^{-1}$ if $\delta = 0$.

Both representations above apply when the poles of the gamma function in the numerator of the quotients are simple. When this simplification does not hold, residue theorem has to be applied. For details about this theorem, one may refer to [7].

Another well-known example of generalized hypergeometric function, which is a special case of the H-function, is the Meijer G-function [2]. More recently, the R-function has also shown its applicability to a number of statistical problems [8].

The R-function (see [8]) is defined as a contour complex integral which contains powers of Gamma functions in their integrands by:

$$_{\gamma,\delta}^{\alpha,\beta} R_{p,q}^{m,n} \left[z \left| \begin{array}{cccccc} (a_1, A), & \ldots, & (a_n, A_n), & (a_{n+1}, A_{n+1}), & \ldots, & (a_p, A_p) \\ (b_1, B_1), & \ldots, & (b_m, B_m), & (b_{m+1}, B_{m+1}), & \ldots, & (b_q, B_q) \end{array} \right. \right]$$

$$= \frac{1}{2\pi i} \int_L \frac{h(s)}{(\alpha + \beta s)^{\gamma + \delta s}} z^{-s} ds, \tag{6}$$

in which $h(s)$ is defined in Equation (2) and α, β, γ, and δ are chosen such that the integral on the right-hand side of Equation (6) exists.

Several functions are special cases of both the H-function and the R-function, for example, generalized hypergeometric series $_pF_q$, MacRobert's E-function, generalized Bessel–Maitland function, Kratzel function, Wright generalized hypergeometric function, Bessel functions, Whittaker function, Mittag–Leffler function, trigonometric functions, exponentials, and so on [2]. In actuality, several authors have studied the properties and generalizations of such hypergeometric functions, such as [9–14].

3. Lambert W-Function and Lambert's Trinomial Equation

Formally, Lambert W-function of a real variable x can be defined as [1]:

$$W(x)e^{W(x)} = x, \ x \in [-1/e, \infty). \tag{7}$$

It is worth noticing that for $x \in [-1/e, 0]$, there are two possible branches of $W(x)$, namely, $W_0(x)$ stands for the branch in which $W(x) \geq -1$, while $W_{-1}(x)$ stands for the branch in which $W(x) \leq -1$. Figure 1 shows the behavior of the function as well as its real branches. The latter are represented by a full line ($W_0(x)$) and a dashed line ($W_{-1}(x)$).
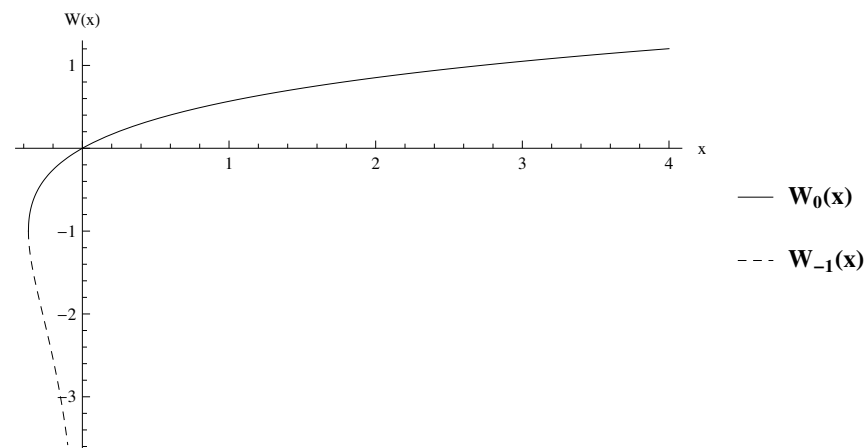
Figure 1. Lambert W-function.

When the Lambert W-function of a complex variable is considered, there are infinitely many branches. This comes from multi-branches of the logarithm function involved in the inversion of Equation (7).

While studying previous works by Lambert, Euler [6] considered the solution of the following trinomial equation, where α, β and $\nu \in \mathbb{R}$:

$$x^\alpha - x^\beta = (\alpha - \beta)\nu x^{\alpha+\beta} \tag{8}$$

By means of taking the limit $\beta \to \alpha$ in Equation (8), the following is obtained [1]:

$$\log x = \nu x^\alpha \tag{9}$$

The solution to Equation (9) is easily obtained in terms of Lambert W-function as $x^\alpha = -W(-\alpha\nu)/\alpha\nu$ [1].

In the present paper, in order to show the relation between Lambert W-function and the H-function, Equation (8) will be solved and the limit will be applied later on.

4. Solution to Lambert's Trinomial Equation

As pointed out in [1], Equation (8) can be brought back to Lambert's original trinomial equation by setting $y = x^{-\beta}$, $q = (\alpha - \beta)\nu$ and $m = \alpha/\beta$. This way, as in the case of Equation (8), α, β and $\nu \in \mathbb{R}$ and we can rearrange (8) using these variable changes to obtain

$$x^{-\beta} = (\alpha - \beta)\nu + x^{-\alpha}, \tag{10}$$

In order to solve equations similar to (8) and (10), Rathie and Ozelim [15] applied Lagrange's inversion theorem [16]. After a series solution was obtained, those authors compared such series to computable representations of the H-function. This way, following the same rationale, we will use the results from [15] to find the solution of Lambert's transcendental equation in terms of the H-function.

Theorem 1. *The transformed version of Lambert's trinomial equation, as presented in (10), can be solved analytically in terms of the H-function as:*
(a) If $\alpha/\beta > 1$:

$$x^\alpha = -\frac{\alpha((\alpha-\beta)\nu)^{-\frac{\alpha}{\beta}}}{\beta} H_{1,2}^{1,1}\left[-((\alpha-\beta)\nu)^{\left(\frac{\alpha}{\beta}-1\right)} \left|\begin{array}{c} \left(\frac{\beta+\alpha}{\beta},\frac{\alpha}{\beta}\right) \\ (0,1),\left(\frac{\alpha}{\beta},\frac{\alpha-\beta}{\beta}\right) \end{array}\right.\right],$$

$$\text{if } |\nu| < \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}}\frac{1}{|\beta|} \tag{11}$$

$$x^\alpha = -\frac{\alpha((\alpha-\beta)\nu)^{-\frac{\alpha}{\beta}}}{\beta} H_{2,1}^{1,1}\left[-((\alpha-\beta)\nu)^{\left(1-\frac{\alpha}{\beta}\right)} \left|\begin{array}{c} (1,1),\left(\frac{\beta-\alpha}{\beta},\frac{\alpha-\beta}{\beta}\right) \\ \left(-\frac{\alpha}{\beta},\frac{\alpha}{\beta}\right) \end{array}\right.\right],$$

$$\text{otherwise} \tag{12}$$

(b) If $0 < \alpha/\beta < 1$:

$$x^\alpha = -\frac{\alpha((\alpha-\beta)\nu)^{-\frac{\alpha}{\beta}}}{\beta} H_{2,1}^{1,1}\left[-((\alpha-\beta)\nu)^{\left(\frac{\alpha}{\beta}-1\right)} \left|\begin{array}{c} \left(\frac{\beta+\alpha}{\beta},\frac{\alpha}{\beta}\right),\left(\frac{\beta-\alpha}{\beta},\frac{\beta-\alpha}{\beta}\right) \\ (0,1) \end{array}\right.\right],$$

$$\text{if } |\nu| < \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha-\beta}}\frac{1}{|\beta|} \tag{13}$$

$$x^\alpha = -\frac{\alpha((\alpha-\beta)\nu)^{-\frac{\alpha}{\beta}}}{\beta} H_{1,2}^{1,1}\left[-((\alpha-\beta)\nu)^{\left(1-\frac{\alpha}{\beta}\right)} \left|\begin{array}{c} (1,1) \\ \left(-\frac{\alpha}{\beta},\frac{\alpha}{\beta}\right),\left(\frac{\alpha}{\beta},\frac{\beta-\alpha}{\beta}\right) \end{array}\right.\right],$$

$$\text{otherwise} \tag{14}$$

In the expressions above, $H_{p,q}^{m,n}$ stands for the H-function (see [2]).

Proof. Rathie and Ozelim [15] obtained general solutions, in terms of the H-function, to trinomial equations. In their work, they considered that $\alpha^*, \beta^*, \gamma^*, and \delta^* \in \mathbb{R}$ and solved the following equation:

$$x^{\alpha^*} = \beta^* + \gamma^* x^{\delta^*}, \tag{15}$$

Thus, by comparing (10) and (15), it is possible to see that when $\alpha^* = -\beta$, $\beta^* = (\alpha-\beta)\nu$, $\gamma^* = 1$ and $\delta^* = -\alpha$, the solutions to the trinomial equation can be given as pointed out in [15]. □

4.1. Particular Case When $\beta \to \alpha$

We are interested in the particular case where $\beta \to \alpha$, in order to match our H-function solutions to the Lambert W-function solution.

Corollary 1. *The W_{-1} branch of the W-function can be related to the H-function solutions in equations from (11) to (14) as:*

$$\overline{W_{-1}}(-\alpha\nu) = \lim_{\beta\to\alpha^-} \frac{\alpha^2\nu((\alpha-\beta)\nu)^{-\frac{\alpha}{\beta}}}{\beta} H_{1,2}^{1,1}\left[-((\alpha-\beta)\nu)^{\left(\frac{\alpha}{\beta}-1\right)} \left|\begin{array}{c} \left(\frac{\beta+\alpha}{\beta},\frac{\alpha}{\beta}\right) \\ (0,1),\left(\frac{\alpha}{\beta},\frac{\alpha-\beta}{\beta}\right) \end{array}\right.\right] \tag{16}$$

if $|\nu| < 1/(e|\alpha|)$ or

$$\overline{W_{-1}}(-\alpha\nu) = \lim_{\beta\to\alpha^-} \frac{\alpha^2\nu((\alpha-\beta)\nu)^{-\frac{\alpha}{\beta}}}{\beta} H_{2,1}^{1,1}\left[-((\alpha-\beta)\nu)^{\left(1-\frac{\alpha}{\beta}\right)} \left|\begin{array}{c} (1,1),\left(\frac{\beta-\alpha}{\beta},\frac{\alpha-\beta}{\beta}\right) \\ \left(-\frac{\alpha}{\beta},\frac{\alpha}{\beta}\right) \end{array}\right.\right] \tag{17}$$

otherwise. Here, $\overline{W_{-1}}(z)$ denotes the complex conjugate of $W_{-1}(z)$. Additionally,

$$\overline{W_{-1}}(-\alpha v) = \lim_{\beta \to \alpha^+} \frac{\alpha^2 v((\alpha-\beta)v)^{-\frac{\alpha}{\beta}}}{\beta} H_{2,1}^{1,1}\left[-((\alpha-\beta)v)^{\left(\frac{\alpha}{\beta}-1\right)} \,\middle|\, \begin{matrix} \left(\frac{\beta+\alpha}{\beta},\frac{\alpha}{\beta}\right),\left(\frac{\beta-\alpha}{\beta},\frac{\beta-\alpha}{\beta}\right) \\ (0,1) \end{matrix}\right] \quad (18)$$

if $|v| < 1/(e|\alpha|)$ *or*

$$\overline{W_{-1}}(-\alpha v) = \lim_{\beta \to \alpha^+} \frac{\alpha^2 v((\alpha-\beta)v)^{-\frac{\alpha}{\beta}}}{\beta} H_{1,2}^{1,1}\left[-((\alpha-\beta)v)^{\left(1-\frac{\alpha}{\beta}\right)} \,\middle|\, \begin{matrix} (1,1) \\ \left(-\frac{\alpha}{\beta},\frac{\alpha}{\beta}\right),\left(\frac{\alpha}{\beta},\frac{\beta-\alpha}{\beta}\right) \end{matrix}\right] \quad (19)$$

otherwise.

Proof. When $\beta \to \alpha$, as pointed out by [1], the trinomial Equation (8) reduces to Equation (9), whose solution can be given in terms of Lambert W-function as $x^\alpha = -W(-\alpha v)/\alpha v$. We can see that the branches satisfying the solutions are $x^\alpha = -W_{-1}(-\alpha v)/\alpha v$ when $0 < \alpha v < 1/e$ and $x^\alpha = -W_0(-\alpha v)/\alpha v$ when $\alpha v < 0$. Finally, numerical checks reveal that the correct branch which matches the equations is $\overline{W_{-1}}$. $\quad\square$

4.2. New Series Expansion Formulas for $\overline{W_{-1}}$

So far, the relation between Lambert W-function and the H-function has been presented in terms of closed-form representations. On the other hand, it is possible to explore numerical alternatives to such representation, as subsequently shown.

Corollary 2. *By considering the alternative representation of $\overline{W_{-1}}$ in terms of the H-function, the following new series representations for $\overline{W_{-1}}$ arise:*

$$\overline{W_{-1}}(-\alpha v) = \lim_{\beta \to \alpha} \frac{\alpha^2 v((\alpha-\beta)v)^{-\frac{\alpha}{\beta}}}{\beta} \sum_{l=0}^{\infty} \frac{((\alpha-\beta)v)^{\left(\frac{\alpha}{\beta}-1\right)l}\Gamma\left(\frac{\alpha}{\beta}l-\frac{\alpha}{\beta}\right)}{\Gamma\left(\frac{\beta-\alpha}{\beta}+\frac{\alpha-\beta}{\beta}l\right)l!}, \quad if \quad |\alpha v| < 1/e \quad (20)$$

$$\overline{W_{-1}}(-\alpha v) = \lim_{\beta \to \alpha} \alpha v((\alpha-\beta)v)^{-\frac{\alpha}{\beta}} \sum_{l=0}^{\infty} (-1)^l \frac{\left(-((\alpha-\beta)v)^{\left(1-\frac{\alpha}{\beta}\right)}\right)^{\frac{\beta l}{\alpha}-1}\Gamma\left(\frac{\beta}{\alpha}l-1\right)}{\Gamma\left(\frac{\beta}{\alpha}l-l\right)l!}, \quad otherwise. \quad (21)$$

Proof. Both the H-functions in Equations (16) and (18) can be converted to a series by directly using the results in (4) and (5). The same is also possible for the H-functions in Equations (17) and (19), which ends the proof. $\quad\square$

The Gamma functions in both Equations (20) and (21) show up after applying Lagrange's inversion theorem, as indicated in [15]. Such ratios may present numerical challenges to some software, especially when the values of the Gamma functions increase.

Corollary 3. *In order to avoid calculating the ratios of Gamma functions in (20) and (21), such equations can be rewritten as:*

$$\overline{W_{-1}}(-\alpha v) = \lim_{\beta \to \alpha}\left[-\alpha v((\alpha-\beta)v)^{-\frac{\alpha}{\beta}} + \right.$$

$$\left. \frac{\alpha^2 v((\alpha-\beta)v)^{-\frac{\alpha}{\beta}}}{\beta} \sum_{l=1}^{\infty} \prod_{k=0}^{l-2}\left(\frac{\alpha}{\beta}l-\frac{\alpha}{\beta}-1-k\right)\frac{((\alpha-\beta)v)^{\left(\frac{\alpha}{\beta}-1\right)l}}{l!}\right], \quad |\alpha v| < 1/e \quad (22)$$

and

$$\overline{W_{-1}}(-\alpha v) = \lim_{\beta \to \alpha} \left[\alpha(\alpha - \beta)^{-1} + \right.$$

$$\left. \alpha v ((\alpha - \beta)v)^{-\frac{\alpha}{\beta}} \sum_{l=1}^{\infty} \prod_{k=0}^{l-2} \left(\frac{\beta}{\alpha} l - 2 - k \right) \frac{(-1)^l}{l!} \left(-((\alpha - \beta)v)^{\left(1 - \frac{\alpha}{\beta}\right)} \right)^{\frac{\beta l}{\alpha} - 1} \right], \quad otherwise \quad (23)$$

Proof. Each ratio of Gamma functions can be further simplified by noticing that [15]:

$$\frac{\Gamma(b+1)}{\Gamma(b-a+1)} = \prod_{k=0}^{a-1}(b-k) \quad (24)$$

By combining (24) with (20) and (21), (22) and (23) follow.   □

It can be seen that, not only it was possible to show that Lambert W-function is a limiting case of the H-function, but also that a new series could be obtained for $\overline{W_{-1}}(z)$. In addition, when $z > -1/e$, $\overline{W_{-1}}(z) = W_{-1}(z)$, since we are dealing with the real values of the branch.

5. Relation to Generalized Hypergeometric Functions of Higher Rank

We can establish another relation between the Lambert W-function and generalized hypergeometric functions by looking at its Mellin transform. As pointed out in [17], the Mellin transform of the W-function can be given as:

$$M(W(z)) = \int_0^\infty x^{s-1} W(x) dx = \frac{(-s)^{-s}\Gamma(s)}{s} \quad (25)$$

for $-1 < \mathbb{R}(s) < 0$.

It is possible to invert the Mellin transform of the W-function to obtain its contour integral representation. This leads to:

$$W(z) = \frac{1}{2\pi i} \int_L \frac{(-s)^{-s}\Gamma(s)}{s} z^{-s} ds = \frac{1}{2\pi i} \int_L \frac{\Gamma(s)}{s^{s+1}} (-z)^{-s} ds = {}_{1,1}^{0,1} R_{0,1}^{1,0} \left[-z \,\middle|\, {(0,1)} \right] \quad (26)$$

By applying the residue Theorem [2] to calculate the contour integral in Equation (26), the series for W_0 is obtained.

Linking the Lambert-W function to such high class hypergeometric function as the R-function can shed light onto problems which could not be addressed in terms of the H-function. These problems can be found in many areas, such as complex multivariate distributions in the area of the inference on multiple time series [18–20], multivariate statistical analysis [21], and distribution of the likelihood ratio criterion for testing specific conditions for the covariance matrix and mean vector of multivariate normal distributions [22–25].

In addition, in physics, this type of special function can be applied to problems involving the energy loss of fast particles by ionization [26,27]. Therefore, there are plenty of future works which may benefit from this relation, both in terms of new analytical results and representations, as well as in new computational calculations made easier by the Lambert-W function.

6. Conclusions

In the present paper we could establish relations between H-functions and R-function and the branches W_{-1} and W_0 of the W-function, respectively. These results were obtained for the first time, establishing a connection between the two classes of functions. Further studies may lead to other relations, allowing interesting new analytic continuation strategies for both classes of functions. The new series are also able to compute $\overline{W_{-1}}(z)$ for arbitrary precision, even when $z < -1/e$.

While the new relations directly follow from previous results presented in the literature, they have not been made available to date. The limiting procedure carried out is novel for these classes of functions, which may shed light on the relation between apparently unrelated functions.

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