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Brief Report

Poincaré–Chetaev Equations in Dirac's Formalism of Constrained Systems

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Abstract: We single out a class of Lagrangians on a group manifold, for which one can introduce non-canonical coordinates in the phase space, which simplify the construction of the Poisson structure without explicitly calculating the Dirac bracket. In the case of the $SO(3)$ manifold, the application of this formalism leads to the Poincaré–Chetaev equations. The general solution to these equations is written in terms of an exponential of the Hamiltonian vector field.

Keywords: Hamiltonian reduction; Dirac bracket; Poincaré-Chetaev equations; Euler-Poisson equations

Citation: Deriglazov, A.A. Poincaré–Chetaev Equations in Dirac's Formalism of Constrained Systems. *Particles* **2023**, *6*, 913–922. https://doi.org/10.3390/particles6040059

Academic Editor: Armen Sedrakian

Received: 5 September 2023
Revised: 25 September 2023
Accepted: 12 October 2023
Published: 13 October 2023

1. Introduction

The Euler–Poisson equations represent a very interesting example of a Hamiltonian system on the group manifold $SO(3)$, in which the number of conjugate momenta is less than the number of coordinates, and it is endowed with a non-canonical Poisson structure. Generalization of these equations to other group manifolds are known as the Poincaré–Chetaev equations, and represent an active field of research [1–4]. They were suggested by Poincaré in the Lagrangian-like form in [5], while their Hamiltonian character was recognized by Chetaev in [6,7]. For the case of $SO(3)$, the Euler–Poisson equations can be obtained using Dirac's formalism for constrained systems [8–10], by performing the Hamiltonization of the Lagrangian variational problem for a rigid body considered as a system with kinematic constraints [11,12]. Since this works adequately for the $SO(3)$ manifold, it is interesting to see whether it can be applied for other group manifolds and, more generally, for an arbitrary curved manifold. This is the goal of the present work. We show how the Poincaré–Chetaev equations for a certain class of Lagrangians on a curved manifold can be obtained following Dirac's formalism. As compared with [5–7], this work does not assume that the surface is a group manifold. As we will show, Dirac's formalism can be successfully applied to this case. As compared with [11], in the present work we consider an arbitrary k-dimensional surface of the n-dimensional configuration space instead of $SO(3)$.

Let us start from a little more detailed description of the problem.

Consider a mechanical system with non-singular Lagrangian $L(q^A, \dot{q}^A)$, defined on configuration space with the coordinates $q^A(t)$, $A = 1, 2, \ldots n$. Suppose the "particle" q^A was then forced to move on a k-dimensional surface $\mathbb{S}$ given by the algebraic equations $G_\alpha(q^A) = 0$. Then, the equations of motion are known to follow from the modified Lagrangian, where the constraints are taken into account with help of auxiliary variables $\lambda_\alpha(t)$ as follows [10,13]:

$$L(q^A, \dot{q}^A) - \lambda_\alpha G_\alpha(q^A). \tag{1}$$

While we will discuss the case of an arbitrary surface $\mathbb{S}$, the most interesting applications arise when the surface is a group manifold [1–3,14–16]. Then, in tangent space to the manifold $\mathbb{S}$ there is the natural basis composed by vector fields of the Lie algebra of the

group, say, $\mathbf{X}_i$, with the Lie bracket $[\mathbf{X}_i, \mathbf{X}_j] = c_{ij}{}^k \mathbf{X}_k$. Then for any trajectory $q^A(t) \in \mathbb{S}$, we can write $\dot{q}^A = X^A{}_i \eta_i$.

Let us outline and compare three different possibilities to construct a Hamiltonian formulation for the theory (1).

(A) The first possibility is to work with unconstrained variables. Let x^i be local coordinates on $\mathbb{S}$, then (1) is known to be equivalent to the variational problem with the following unconstrained Lagrangian: $\tilde{L}(x^i, \dot{x}^i) \equiv L(q^A(x^i), \dot{q}^A(x^i))$. Denoting the conjugate momenta for x^i by p_i, we immediately obtain the Hamiltonian equations $\dot{x}^i = \{x^i, H\}$, $\dot{p}_i = \{p_i, H\}$, with the canonical Poisson bracket $\{x^i, p_j\} = \delta^i_j$. In the case of the group manifold $\mathbb{S}$, this calculation at first glance completely ignores its group structure. However, Poincaré noticed long ago [5] that this is not the case. In the present context, his observation can be resumed as follows. In the phase space, one can pass to non-canonical variables, $(x^i, p_i) \to (x^i, \eta_i(x, p))$, in which the Poisson structure stores information about the group structure of the manifold. The same observation turns out to be important in the semiclassical description of a spinning electron [17,18] and of a photon [19]. Poincaré performed his calculations in terms of velocities, adjusting infinitesimal variations of $\dot{x}^i$ and η^i, and obtained his equations in the Lagrangian-like form

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \eta_i} = c_{ij}{}^k \frac{\partial \tilde{L}}{\partial \eta_j}\eta_k + f^i. \tag{2}$$

Then, Chetaev in [7] recognized the Hamiltonian character of these equations, pointing out the Hamiltonian and Poisson brackets. In particular, for the $SO(3)$ manifold, the resulting Hamiltonian equations are just the Euler equations of a rigid body [15].

Note that the original problem here is a mechanical system (1) with kinematic (that is, velocity independent) constraints. So, we expect that the Poincaré–Chetaev equations should be obtained by direct application of Dirac's procedure for constrained systems to this theory. This will be one of our tasks in the present work.

We emphasize that the transition to independent variables is not always desirable. For instance, in the case of a rigid body, the q^A variables are nine elements of the matrix R_{ij}, subject to the orthogonality conditions. That is, to describe a rigid body, we need to know the evolution of q^A and not x^i. Furthermore, the description in terms of independent variables often turns out to be local, which can lead to misunderstandings; see [20]. Finally, solving equations for q^A sometimes requires less effort than solving the same equations rewritten through x^i [21].

(B) The second possibility is to work with the original variables using Dirac's formalism [8–10]. We should pass to the Hamiltonian formulation, introducing the conjugate momenta p_A to all original variables q^A. The Hamiltonian equations are then obtained with help of canonical Poisson bracket $\{q^A, p_B\} = \delta^A_B$ and of the Hamiltonian $H(q^A, p_B, \lambda_\alpha)$. Then, the resulting equations depend on the auxiliary variables λ_α. The systematic method to exclude them is as follows. Besides the original constraints $G_\alpha = 0$, in the Hamiltonian formulation arise certain higher-stage constraints $\Phi_\alpha = 0$, and the complete set of constraints form the second-class system, which should be taken into account passing from the Poisson to Dirac bracket, say $\{q^A, p_B\}_D$. When writing equations of motion with the help of a Dirac bracket, it is known that the terms with λ_α in the Hamiltonian can simply be omitted.

Once again, the underlying group structure in this formalism was at first sight ignored. To recognize it, there are two difficulties. First, the Dirac brackets are constructed for the excess number n of momenta, as compared to the dimension k of the Lie algebra. Therefore, some reduction in the number of variables is needed. Second, to construct the Dirac bracket explicitly, it is necessary to invert the matrix composed of Poisson brackets of the constraints. For instance, in the case of $SO(3)$ group this is the 12×12 matrix. In the present work we slightly adjust the Dirac procedure (for the case of kinematic constraints), which allows us

to avoid these two problems. To further clarify this issue, let us discuss the third known possibility to construct the Hamiltonian formulation.

(C) In the case of a rigid body arises a kind of intermediate formalism between (A) and (B). Let us consider the $SO(3)$ manifold with configuration-space variables being nine elements of a 3×3 matrix subjected to six constraints $R^T R = \mathbf{1}$. Then, the Hamiltonian equations are the Euler–Poisson equations, describing a free motion of an asymmetric rigid body [1,11,22,23]

$$\dot{R}_{ij} = -\epsilon_{jkm}\Omega_k R_{im}, \qquad I\dot{\mathbf{\Omega}} = [I\mathbf{\Omega}, \mathbf{\Omega}]. \tag{3}$$

Here, I is the inertia tensor, and the momenta Ω_i are the Hamiltonian counterparts of angular velocity in the body. There are nine redundant coordinates R_{ij}, but only three independent momenta Ω_i. So, if in case (A) we worked with independent set (x^i, p_j), and in case (B) with redundant set (q^A, p_B), then now we have an intermediate situation: (q^A, p_j). As was shown in [11], it is in this formalism that the Poincaré–Chetaev equations can be obtained using Dirac's method. However, in the calculations were used some specific properties of the group $SO(3)$. In Section 2 we show how to construct the intermediate formalism in a more general case of an arbitrary surface, but for a special class of Lagrangians. Then, in Section 3 this formalism will be used to obtain the Poincaré–Chetaev equations on $SO(3)$.

2. Intermediate Formalism for a Special Class of Lagrangians

As we saw above, in the models formulated in n-dimensional configuration space q^A with the dynamics constrained to live in a k-dimensional submanifold $G_\alpha(q^A) = 0$, we can be interested to know dynamics of all original variables $q^A(t)$. In this case, the most economical would be Hamiltonian formulation on the intermediate phase-space submanifold spanned by reducible variables q^A and an irreducible set of momenta p_i, $[i] = [A] - [\alpha]$.

In accordance with this, in the configuration space with the coordinates $q^A(t)$, $A = 1, 2, \ldots n$, consider k-dimensional surface determined by functionally independent functions $G_\alpha(q^A) = 0$. Without loss of generality, we assume that these equations can be resolved with respect to the first $n - k$-coordinates. In accordance to this, the set q^A is divided into two subgroups, q^α and q^i. Greek indices from the beginning of the alphabet run from 1 to $n - k$, while Latin indices from the middle of the alphabet run from 1 to k. So,

$$\mathbb{S}^k = \{q^A = (q^\alpha, q^i),\ G_\alpha(q^A) = 0,\ \det \frac{\partial G_\alpha}{\partial q^\beta}\Big|_{\mathbb{S}} = n - k,\ \alpha = 1, 2,\ \ldots, n - k\}, \tag{4}$$

and our variational problem is (1). Applying Dirac's method to the Lagrangian (1), we introduce conjugate momenta for all dynamical variables. Conjugate momenta for λ_α are the primary constraints: $p_{\lambda\alpha} = 0$. Since the Lagrangian L was assumed non-singular, the expressions for p_A can be resolved with respect to velocities:

$$p_A = \frac{\partial L}{\partial \dot{q}^A} \equiv \tilde{f}_A(q, \dot{q}), \qquad \text{then} \quad \dot{q}^A = f^A(q, p), \quad \det \tilde{f}_{AB} \neq 0, \quad \text{where} \quad \tilde{f}_{AB} \equiv \frac{\partial \tilde{f}_A}{\partial \dot{q}^B}. \tag{5}$$

To find the Hamiltonian, we exclude the velocities from the expression $H = p_A\dot{q}^A - L + \lambda_\alpha G_\alpha + v_\alpha p_{\lambda_\alpha}$, obtaining

$$H = p_A f^A(q, p) - L(q^A, f^B(q, p)) + \lambda_\alpha G_\alpha(q^A) + v_\alpha p_{\lambda_\alpha}. \tag{6}$$

By v_α we denoted the Lagrangian multipliers for the primary constraints. Preservation in time of the primary constraints, $\dot{p}_{\lambda_\alpha} = \{p_{\lambda_\alpha}, H\} = 0$ implies $G_\alpha = 0$ as the secondary

constraints. In turn, the equation $dG_\alpha/dt = \{G_\alpha, H\} = 0$ implies tertiary constraints, that should be satisfied by all true solutions

$$\Phi_\alpha \equiv G_{\alpha B}(q) f^B(q,p) = 0, \qquad \text{where} \quad G_{\alpha B} \equiv \frac{\partial G_\alpha(q)}{\partial q^B}. \tag{7}$$

The Lagrangian counterpart of this constraint is $\dot{q}^A \partial_A G_\alpha = 0$, and means that for true trajectories the velocity vector is tangent to the surface $\mathbb{S}$. Compute

$$\text{rank}\, \frac{\partial \Phi_\alpha}{\partial p_B} = \text{rank}\, (G_{\alpha A} f^{AB}) = n - k, \qquad \text{where} \quad f^{AB} = \frac{\partial f^A(q,p)}{\partial p_B}. \tag{8}$$

This implies that the constraints Φ_α are functionally independent and can be resolved with respect to some $n-k$ momenta of the set p_A. This implies also that the constraints G_β and Φ_α are functionally independent. Computing their Poisson brackets, we obtain the matrix $\{G_\alpha, \Phi_\beta\} = G_{\beta A}(q) f^{AB} G_{\alpha B}$. Rank of this matrix can be analyzed in the coordinates adapted with the surface: $q'^\alpha = G_\alpha(q^\alpha, q^i)$, $q'^i = q^i$. In these coordinates, the surface is just the hyperplane $q'^\alpha = 0$, then $G'_{\alpha A} = \delta_{\alpha A}$, and the matrix of brackets turn into $f'^{\beta\alpha}(q', p')$. It is $(n-k) \times (n-k)$ in the upper left block of the matrix f^{AB}, the latter is the inverse of the Hessian matrix $\tilde{f}_{AB}$ of our theory. We assume that for our Lagrangian this matrix is non-degenerate

$$\det f'^{\beta\alpha}(q', p') \neq 0. \tag{9}$$

This condition is satisfied, in particular, in the theories with a positive-definite Hessian matrix, see Appendix A. Therefore, this may not be a very strong limitation for the applications.

For such Lagrangians, our constraints G_β and Φ_α are of second class. Then preservation in time of the tertiary constraints gives fourth-stage constraints that involve λ_α, and can be used to find them through q^A and p_A. At last, preservation in time of the fourth-stage constraints gives an equation that determines the Lagrangian multipliers v_α. We do not write out these equations; we will not need them.

To proceed further, let us construct non-canonical phase-space coordinates with special properties. The matrix $G_{\alpha B}$ of Equation (7) is composed by $(n-k)$ linearly independent vector fields $\mathbf{G}_\alpha$, which are orthogonal to the surface $\mathbb{S}$. The linear system $G_{\alpha B} x_B = 0$ has a general solution (to avoid a possible confusion, we point out that in the similar Equation (7), representing the tertiary constraints, f^A are given functions of q and p) of the form $x_B = c^i G_{iB}$, where the linearly independent vectors $\mathbf{G}_i$ are fundamental solutions to this system

$$\mathbf{G}_i = (\, G_{i1}(q), G_{i2}(q), \ldots, G_{i,n-k}(q), 0, \ldots, 1, 0 \ldots, 0\,), \qquad G_{\alpha B} G_{iB} = 0. \tag{10}$$

By construction, these vector fields form the basis of space tangent to the surface $\mathbb{S}$. Together with $\mathbf{G}_\alpha$, they form the basis of space tangent to the entire configuration space. Using the rows $\mathbf{G}_\beta$ and $\mathbf{G}_j$, we construct an invertible matrix G_{BA}, and use it to define the new momenta π_B:

$$G_{BA}(q) = \begin{pmatrix} G_{\beta A} \\ G_{jA} \end{pmatrix}, \qquad \pi_B = G_{BA}(q) p_A, \quad \text{then} \quad p_A = G^{-1}_{AB}(q) \pi_B \equiv \tilde{G}_{AB}(q) \pi_B. \tag{11}$$

Let us take q^A and π_A as the new phase-space coordinates. Their special property is that both q^A and π_i have vanishing brackets with the original constraints G_α

$$\{q^A, G_\alpha\} = 0, \qquad \{\pi_i, G_\alpha\} = 0, \tag{12}$$

the latter equality is due to Equation (10).

Let us rewrite our theory using the new variables. Using the canonical brackets $\{q^A, p_B\} = \delta^A{}_B$, we obtain Poisson brackets of the new variables

$$\{q^A, q^B\} = 0, \qquad \{q^A, \pi_B\} = G_{BA}(q), \qquad \{\pi_A, \pi_B\} = -c_{AB}{}^D(q)\tilde{G}_{DE}(q)\pi_E, \tag{13}$$

where appeared the Lie brackets of basic vector fields $\mathbf{G}_A$

$$c_{AB}{}^D = [\mathbf{G}_A, \mathbf{G}_B]^D = G_{AE}\partial_E G_{BD} - G_{BE}\partial_E G_{AD}, \tag{14}$$

$$c_{ij}{}^k = 0. \tag{15}$$

Therefore, the Lie bracket of the vector fields $\mathbf{G}_A$ determines the Poisson structure of our theory in the sector π_A. The structure functions $c_{ij}{}^k$ vanish for our choice of basic vectors $\mathbf{G}_i$ of special form; see Equation (10). The Hamiltonian (6) reads

$$H = \tilde{G}_{AC}\pi_C f^A(q, \tilde{G}\pi) - L(q^A, f^B(q, \tilde{G}\pi)) + \lambda_\alpha G_\alpha(q^A). \tag{16}$$

At last, our second-class constraints in the new coordinates are

$$G_\alpha(q^A) = 0, \qquad \Phi_\alpha \equiv G_{\alpha A}(q) f^A(q, \tilde{G}\pi) = 0. \tag{17}$$

We now show that the variables π_α can be excluded from all these expressions, which gives final Hamiltonian formulation on the intermediate submanifold $\Phi_\alpha = 0$ (all solutions of theory (1) lie in the phase-space submanifold $\Phi_\alpha = 0$, $G_\alpha = 0$, hence the term "intermediate"). To this aim, we construct the Dirac bracket

$$\{A, B\}_D = \{A, B\} - \{A, T^i\}\triangle^{-1}_{ij}\{T^j, B\}. \tag{18}$$

Here, T^i is the set of all constraints: $T^i = (G_\alpha, \Phi_\beta)$. Additionally, denoting symbolically the blocks $b = \{G, \Phi\}$ and $c = \{\Phi, \Phi\}$, the matrices $\triangle$ and $\triangle^{-1}$ are

$$\triangle = \begin{pmatrix} 0 & b \\ -b^T & c \end{pmatrix}, \qquad \triangle^{-1} = \begin{pmatrix} b^{-1T}cb^{-1} & -b^{-1T} \\ b^{-1} & 0 \end{pmatrix}. \tag{19}$$

This implies the following structure of the Dirac bracket:

$$\{A, B\}_D = \{A, B\} - \{A, G\}\triangle'\{G, B\} + \{A, G\}\triangle''\{\Phi, B\}. \tag{20}$$

Taking into account Equation (12), we conclude that in the passage from Poisson bracket (13) to the Dirac bracket, the brackets (13) of the basic variables q^A and π_i will not be modified, retaining their original form. So, fortunately, we do not need to calculate the explicit form of the matrix $\triangle^{-1}$ that appeared in (18). The constraint's functions (17) are Casimir functions of the Dirac bracket (18).

Let us confirm that the tertiary constraints Φ_α from (17) can be resolved with respect to π_α. To this aim we compute $\det(\partial\Phi_\alpha/\partial\pi_\beta)$ in the adapted coordinates, and show that it is not zero:

$$\det\frac{\partial(G'_{\alpha A}f'^A(q', \tilde{G}'\pi'))}{\partial\pi'_\beta} = \det[G'_{\alpha A}f'^{AD}(q', \tilde{G}'\pi')\tilde{G}'_{D\beta}] = \det f'^{\alpha\beta}(q', \pi') \neq 0. \tag{21}$$

Here, we used in adapted coordinates $G'_{\alpha A} = (\delta_{\alpha\beta}, \mathbf{0})$ and $\tilde{G}'_{D\beta} = (\delta_{\alpha\beta}, \mathbf{0})^T$. It is not zero for our class of Lagrangians (9).

The formulation of the theory in terms of Dirac brackets makes it much more transparent. Indeed, according to Dirac's formalism, we now can omit all terms with constraints in the Hamiltonian. Furthermore, we can use the constraints before the calculation of the brackets. Therefore, resolving the constraints (17) for π_α and excluding them from the

formalism, we obtain the desired intermediate formulation of our theory in terms of q^A and π_i.

In particular, excluding π_α from Equation (13) we obtain the Poisson structure of the intermediate formulation as follows:

$$\{q^A, q^B\}_D = 0, \qquad \{q^\alpha, \pi_i\}_D = G_{i\alpha}(q), \qquad \{q^j, \pi_i\}_D = \delta^j{}_i,$$
$$\{\pi_i, \pi_j\}_D = -c_{ij}{}^\alpha[\tilde{G}_{\alpha k}\pi_k + \tilde{G}_{\alpha\beta}\pi_\beta(q^A, \pi_i)], \tag{22}$$

where $\pi_\beta(q^A, \pi_i)$ is the solution to the tertiary constraints $\Phi_\alpha = 0$. In general, the brackets are non-linear for both q^A and π_i. Their dependence on the choice of tangent vector fields $\mathbf{G}_i$ to the surface $\mathbb{S}$ is encoded in three places: in the brackets $\{q^\alpha, \pi_j\}$, in the matrix $\tilde{G}$, and in the structure functions $c_{ij}{}^\alpha$; see Equation (14).

In the Hamiltonian (16), we omit the term containing the constraints G_α, and exclude π_α. Let us denote the resulting expression by $H_0(q^A, \pi_j)$. Hamiltonian equations of intermediate formalism are obtained with use of Dirac brackets:

$$\dot{q}^A = \{q^A, H_0(q^B, \pi_j)\}_D, \qquad \dot{\pi}_i = \{\pi_i, H_0(q^B, \pi_j)\}_D. \tag{23}$$

The matrix G_{BA} can equally be used to construct another coordinates

$$\pi_\alpha = G_{\alpha B}f^B \equiv \Phi_\alpha, \qquad \pi_i = G_{iB}p_B, \tag{24}$$

which contain the constraints Φ_α as a part of new momenta. This change in variables is equivalent to that used above, but turns out to be more convenient in the case of $SO(3)$.

Being one of the classical problems in the theory of integrable systems and classical mechanics, these issues could be of interest in the modern studies of various aspects related with construction and behavior of spinning particles and rotating bodies in external fields beyond the pole–dipole approximation [24–30].

3. Poincaré–Chetaev Equations on $SO(3)$

A rigid body can be defined as a system of n particles, where the distances and angles between them do not change with time

$$(\mathbf{y}_N(t) - \mathbf{y}_K(t), \mathbf{y}_P(t) - \mathbf{y}_M(t)) = \text{const.} \tag{25}$$

Here, $\mathbf{y}_N(t) = (y^1_N(t), y^2_N(t), y^3_N(t))$ are position vectors of the particles with masses m_N, $N = 1, 2, \ldots, n$. Introducing the center-of-mass coordinate $\mathbf{y}_0(t)$, we can work with the body's points using their position vectors with respect to the center of mass, $\mathbf{x_N}(t) = \mathbf{y}_N(t) - \mathbf{y}_0(t)$. Then, according to Euler's theorem, the position of any point at any instant can be written through the initial position $\mathbf{x_N}(0)$ with respect to the center of mass as follows:

$$\mathbf{y}_N(t) = \mathbf{y}_0(t) + R(t)\mathbf{x}_N(0), \tag{26}$$

where $R_{ij}(t)$ is an orthogonal matrix

$$R^T R = \mathbf{1}, \qquad \text{or} \quad R_{ki}R_{kj} - \delta_{ij} = 0. \tag{27}$$

These algebraic equations are analogous of $G_\alpha = 0$ of the general formalism. According to (26), to describe the motion of a rigid body we need to know only the dynamics of this 3×3 orthogonal matrix. Remarkably, equations of motion for R_{ij} follow from its own Lagrangian action [11]

$$S = \int dt \ \frac{1}{2}g_{ij}\dot{R}_{ki}\dot{R}_{kj} - \frac{1}{2}\lambda_{ij}\Big[R_{ki}R_{kj} - \delta_{ij}\Big], \tag{28}$$

with the universal initial conditions $R_{ij}(0) = \delta_{ij}$. We emphasize that solutions to this problem with other initial conditions do not correspond to the movements of a body. Misunderstanding of this point leads to much confusion; see [20] for details. In the expression (28), the mass matrix is taken to be diagonal $g_{ij} = diagonal\ (g_1, g_2, g_3)$, and is related with the inertia tensor as follows: $2g_1 = I_2 + I_3 - I_1$, $2g_2 = I_1 + I_3 - I_2$, $2g_3 = I_1 + I_2 - I_3$.

The variational problem (28) is of the form (1), so we can apply the intermediate formalism of previous section to the present case. The detailed computations of the $SO(3)$ case were presented in [11], so here we only establish the relationship between these calculations and the intermediate formalism developed above.

Denote the conjugate momenta of R_{ij} by $p_{ij} = \partial L/\partial \dot{R}_{ij}$. The Hamiltonian of the theory reads

$$H = \frac{1}{2} g_{ij}^{-1} p_{ki} p_{kj} + \frac{1}{2} \lambda_{ij} [R_{ki} R_{kj} - \delta_{ij}]. \tag{29}$$

Applying the formalism of previous section and comparing it with the calculations of Section XII of the work [11], we have the following table for identification of the basic quantities:

$$q^A \sim R_{ij}, \qquad p_A \sim p_{ij}, \qquad \tilde{f}^A \sim \dot{R}_{ik} g_{ki}, \qquad f_A \sim p_{ik} g_{kj}^{-1}, \tag{30}$$

$$\pi_\alpha = \Phi_\alpha \sim \mathbb{P}_{(ij)} = \frac{1}{2} [R^T p g^{-1} + (R^T p g^{-1})^T]_{ij}, \tag{31}$$

$$\pi_i \sim \hat{\Omega}_{ij} = -\frac{1}{2} [R^T p g^{-1} - (R^T p g^{-1})^T]_{ij} \sim M_n = -I_{nk} \epsilon_{kij} (R^T p)_{ij}. \tag{32}$$

In the case of $SO(3)$, the final results acquire a more simple form if we use the angular momentum in the body $M_n = (I\Omega)_n$ instead of the angular velocity in the body $\hat{\Omega}_{ij}$. Therefore the identification of new coordinates of the general scheme with the $SO(3)$ case is as follows: $(q^A, \pi_\alpha, \pi_i) \sim (R_{ij}, \mathbb{P}_{(ij)}, M_i)$. Making calculations of the previous section in these coordinates, we obtain the following final Hamiltonian and the brackets

$$H_0 = \frac{1}{2} I_{ij}^{-1} M_i M_j; \tag{33}$$

$$\{R_{ij}, R_{ab}\} = 0, \qquad \{M_i, M_j\} = -\epsilon_{ijk} ((R^T R)^{-1} \mathbf{M})_k, \qquad \{M_i, R_{jk}\} = -\epsilon_{ikm} R_{jm}^{-1T}; \tag{34}$$

Using the orthogonality constraint on the right hand side of the brackets (34), we obtain more simple expressions

$$\{R_{ij}, R_{ab}\} = 0, \qquad \{M_i, M_j\} = -\epsilon_{ijk} M_k, \qquad \{M_i, R_{jk}\} = -\epsilon_{ikm} R_{jm}. \tag{35}$$

By direct computations, it can be verified that they still satisfy the Jacobi identity and lead to the same Equation (36). They were suggested by Chetaev [7] as having a possible Poisson structure corresponding to the Euler–Poisson equations.

Denoting the rows of the matrix R_{ij} by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, the bracket (35) reads: $\{M_i, a_j\}_D = -\epsilon_{ijk} a_k$, $\{M_i, b_j\}_D = -\epsilon_{ijk} b_k$, $\{M_i, c_j\}_D = -\epsilon_{ijk} c_k$. Therefore, the Poisson structure of a rigid body can be identified with semidirect sum of the algebra of $s\bar{o}(3)$ with three algebras of translations.

Using the rule $\dot{z} = \{z, H_0\}_D$, we obtain the Euler–Poisson equations

$$\dot{R}_{ij} = -\epsilon_{jkm} (I^{-1} M)_k R_{im}, \tag{36}$$

$$\dot{\mathbf{M}} = [\mathbf{M}, I^{-1} \mathbf{M}]. \tag{37}$$

Here, the bracket $[\ ,\ ]$ means the vector product of $\mathbb{R}^3$. Using the rows $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, Equation (36) can be separated: $\dot{\mathbf{a}} = [\mathbf{a}, I^{-1}\mathbf{M}]$, $\dot{\mathbf{b}} = [\mathbf{b}, I^{-1}\mathbf{M}]$ and $\dot{\mathbf{c}} = [\mathbf{c}, I^{-1}\mathbf{M}]$. Then, the entire system (36), (37) breaks down into three. For instance, in the case of the row $\mathbf{a}$ we have

$$\dot{a}_i = [\mathbf{a}, I^{-1}\mathbf{M}]_i, \qquad \dot{M}_j = [\mathbf{M}, I^{-1}\mathbf{M}]_j, \tag{38}$$

$\{M_i, a_j\}_D = -\epsilon_{ijk}a_k$ in accordance with (35). For this unconstrained Hamiltonian system we can use the known formula of Hamiltonian mechanics to write solutions to the Equation (38) in terms of an exponential of the Hamiltonian vector field [31]. Performing the same function for the vectors $\mathbf{b}$ and $\mathbf{c}$, we obtain the solution to Euler–Poisson Equations (36) and (37) as follows:

$$\begin{aligned}
M_i(t, M_{0k}) &= e^{t[\mathbf{M}_0, I^{-1}\mathbf{M}_0]_j \frac{\partial}{\partial M_{0j}}} M_{0i},\\
a_i(t, M_{0k}) &= e^{t([\mathbf{M}_0, I^{-1}\mathbf{M}_0]_j \frac{\partial}{\partial M_{0j}} + [\mathbf{a}_0, I^{-1}\mathbf{M}_0]_j \frac{\partial}{\partial a_{0j}})} a_{0i},\\
b_i(t, M_{0k}) &= e^{t([\mathbf{M}_0, I^{-1}\mathbf{M}_0]_j \frac{\partial}{\partial M_{0j}} + \mathbf{b}_0, I^{-1}\mathbf{M}_0]_j \frac{\partial}{\partial b_{0j}})} b_{0i},\\
c_i(t, M_{0k}) &= e^{t([\mathbf{M}_0, I^{-1}\mathbf{M}_0]_j \frac{\partial}{\partial M_{0j}} + [\mathbf{c}_0, I^{-1}\mathbf{M}_0]_j \frac{\partial}{\partial c_{0j}})} c_{0i}.
\end{aligned} \tag{39}$$

This depends on three arbitrary constants M_{0k}, and is therefore a general solution to the Euler–Poisson equations (recall that initial conditions for the rotation matrix are universal: $R_{ij}(0) = \delta_{ij}$).

4. Conclusions

The most economical Hamiltonian formulation of the theory (1), in which we are interested in knowing the dynamics of all variables q^A, is achieved on an intermediate submanifold of phase space determined by the constraints (7). We have developed the universal method for Hamiltonian reduction of the dynamics to this submanifold. Roughly speaking, this works as follows. For any theory of the form (1) with a positive-definite Lagrangian L, we present the procedure to find (non-canonical) phase-space coordinates (q^A, π_i, π_α) with special properties. They are constructed with help of the matrix $G_{\alpha A} \equiv \partial G_\alpha / \partial q^A$ and fundamental solutions of the linear system $G_{\alpha A} x_A = 0$. The intermediate formulation of the theory (1) is obtained by first rewriting the Hamiltonian formulation of unconstrained theory L in terms of new coordinates, and then excluding the variables π_α from all resulting expressions with help of the constraint $\Phi_\alpha = 0$. In particular, the Poisson structure on intermediate submanifold turns out to be the canonical Poisson bracket of original variables (q^A, p_B), first rewritten in terms of new coordinates (q^A, π_B), and then restricted to this submanifold.

The final result of the reduction is written out in Equations (22) and (23). As we have shown in the last section, namely the intermediate formalism directly leads to the Euler–Poisson equations of a spinning body.

An intermediate formulation for the theory (1) could be equally obtained by first constructing the Dirac bracket (which is a degenerate Poisson structure on original phase space (q^A, p_B)), and then reducing it on the submanifold $\Phi_\alpha = 0$; see [31]. We hope to compare the two formalisms in a future publication.

Funding: The work has been supported by the Brazilian foundation CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico—Brasil).

Data Availability Statement: No new data were created.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

Properties of a positive-definite matrix. Symmetric real-valued $n \times n$-matrix with the elements M_{ij} is called positive-definite ($M \succ 0$), if for any non-zero column $\mathbf{Y}$ we have $\mathbf{Y}^T M \mathbf{Y} > 0$. The following affirmations turn out to be equivalent [32]:

1A. $M \succ 0$.

1B. There exists $n \times n$ positive-definite matrix B such that $M = B^2 \equiv B^T B$.

1C. All principal minors of M are positive numbers. In particular, $\det M > 0$.

1D. M is the Gram matrix of some set of p-dimensional linearly independent vectors, say $\mathbf{Z}_i$. That is $M_{ij} = (\mathbf{Z}_i, \mathbf{Z}_j)$. If Z_{Ai}, $A = 1, 2, \dots p$ is the matrix formed by the columns $\mathbf{Z}_i$, we can write $M_{ij} = (Z^T)_{iA} Z_{Aj}$.

1E. All eigenvalues of M are positive numbers.

Additionally, there are the following properties:

2A. Diagonal elements of positive-definite matrix are positive numbers: $M_{ii} > 0$ for any i. Then trace $M > 0$.

2B. Positive-definite matrix is invertible, and its inverse is a positive-definite matrix.

Affirmation. Let rank $Q_{Ai} = k$, where $A = 1, 2, \dots p$, $i = 1, 2, \dots k$, $k < p$, and M_{AB} is positive-definite. Then the matrix

$$N_{ij} = (Q^T)_{iA} M_{AB} Q_{Bj}, \tag{A1}$$

is non-degenerate, $\det N \neq 0$.

Proof. Using **1B**, we write $M = B^T B$, then

$$N_{ij} = (BQ)^T_{iA} (BQ)_{Aj}, \tag{A2}$$

where, according to **2B**, the matrix B is non-degenerate. Since the columns of Q_{Aj} are linearly independent, the matrix $(BQ)_{Aj}$ also is composed of linearly independent columns. Then, (A2) means that M_{ij} is the Gram matrix. According to **1D** it is positive-definite. In particular, $\det N > 0$.

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