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Theoretical Investigation on the Conservation Principles of an Extended Davey–Stewartson System with Riesz Space Fractional Derivatives of Different Orders

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Abstract: In this article, a generalized form of the Davey–Stewartson system, consisting of three nonlinear coupled partial differential equations, will be studied. The system considers the presence of fractional spatial partial derivatives of the Riesz type, and extensions of the classical mass, energy, and momentum operators will be proposed in the fractional-case scenario. In this work, we will prove rigorously that these functionals are conserved throughout time using some functional properties of the Riesz fractional operators. This study is intended to serve as a stepping stone for further exploration of the generalized Davey–Stewartson system and its wide-ranging applications.

Keywords: generalized David–Stewartson system; fractional differential equations; Riesz fractional operators; conservative system

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1. Introduction

Nonlinear partial differential equations arise in many areas of mathematical physics and, in particular, in fluid mechanics. For instance, an important system of nonlinear partial differential equations for a complex field is the Zakharov–Schulman system (ZSS), which describes interactions of small-amplitude and high-frequency waves [1]. In particular, the ZSS can lead to the Davey–Stewartson system (DSS), which is one of the most important models in fluid dynamics. The original DSS is a coupled system of nonlinear partial differential equations introduced by A. Davey and K. Stewartson in 1973. This system describes the evolution of a three-dimensional wave packet in water with finite depth [2]. It is important to recall here that Schrödinger's equation also describes the time evolution of such packets [3]. Moreover, the DSS is the closest $(2+1)$-dimensional completely integrable analog of the nonlinear Schrödinger equation [4,5]. The DSS arises as a result of the analysis at multiple scales of modulated nonlinear surface gravity waves that propagate over a horizontal seabed and possesses free surface waves subject to the effects of both gravity and capillarity. Various blow-up phenomena arise in this system, including blow-up in the defocusing case [6], along with long- and short-wave resonances [7].

There are various reports on some properties of the analytical solutions of the DSS. For example, this system can be solved by expansion methods and using a transformation to obtain a system of nonlinear ordinary differential equations [8]. In such a way, different forms of solutions (such as dark, bright, singular, and periodic singular solutions) are obtained. The generalized soliton solutions of the DSS with complex coefficients are calculated by implementing the extended Weierstrass transformation method [9]. Moreover, exact blow-up solutions to the Cauchy problem for the DSS and some special solutions of these equations are obtained by both a tanh method and a variable separation approach [10,11]. The growing-and-decaying mode solution to the DSS models the long-time evolution of the Benjamin–Feir unstable mode in two dimensions [12]. On the other hand, some exact solutions are obtained via commutative algebra by applying the first integral method [13]. Another way to obtain analytical solutions is through the sine-cosine method, which is used to build periodic and solitary wave solutions [14]. In the same sense, the inverse scattering method for DSS II has been developed. In this way, solitons, continuous spectrum solutions, and the explicit formula for the N-soliton solution are derived [15].

On the other hand, the discretization of the DSS has been studied in various reports. Some works tackle the elliptic–hyperbolic form of the DSS, and present finite-difference schemes in which the energy is conserved throughout time [16]. A numerical solution for the DSS based on Galerkin's method, finite time steps, and an extrapolated Crank–Nicolson scheme is provided in [17,18]. In that work, a decoupled semi-implicit multistep scheme is implemented to improve accuracy. The numerical case of the long-time behavior and potential blow-up of solutions for the DSS II is studied through the perturbations of the lump and the Ozawa exact solutions [19]. A change in dependent variables has been proposed for a space discretization of the DSS and an integrable semi-discretization [20,21]. Finally, the blow-up for the DSS II is numerically analyzed based on initial Gaussian data, the lump perturbations, and the explicit Ozawa blow-up solution [22]. These and more reports on the numerical investigation of the DSS can be readily found in the specialized literature, and they provide evidence on the many directions of research that surround the investigation of this mathematical model.

It is important to point out that there are reports in which extended versions of the DSS have been studied. For instance, a DSS consisting of three coupled partial differential equations has been investigated in the context of two-dimensional wave packets in an elastic solid with coupled stresses [23]. This formulation, achieved through a multi-scale expansion of quasi-monochromatic wave solutions, aimed to model wave propagation in a bulk medium comprising an elastic material with couple stresses. This extension led to the following form of the generalized Davey–Stewartson system (GDSS):

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 u}{\partial y^2} = \gamma|u|^2 u + \xi u\left(\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{1}$$

$$\psi\frac{\partial^2 w}{\partial x^2} + \eta\frac{\partial^2 w}{\partial y^2} + \theta\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial |u|^2}{\partial x}, \tag{2}$$

$$\phi\frac{\partial^2 v}{\partial x^2} + \chi\frac{\partial^2 v}{\partial y^2} + \theta\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial |u|^2}{\partial y}. \tag{3}$$

Here, $u : \mathbb{R}^2 \to \mathbb{C}$ represents the complex amplitude of the short transverse wave mode, while $w : \mathbb{R}^2 \to \mathbb{R}$ and $v : \mathbb{R}^2 \to \mathbb{R}$ define the long longitudinal and long transverse wave modes, respectively. For mathematical reasons, we assume that the three functions vanish at infinity. As expected, we let $\mathrm{i} = \sqrt{-1}$, and the three functions depend on $(x, y, t) \in \mathbb{R}^2 \times [0, \infty)$. The real constants α, β, γ, ξ, ψ, η, θ, ϕ, and χ govern the behavior of the system, embodying critical physical characteristics. This system is subjected to the physically relevant constraint

$$\theta^2 = (\phi - \psi)(\eta - \chi), \tag{4}$$

where $\phi > \psi$ and $\eta > \chi$. The parameters γ and ξ can have any sign, with the stipulation that $\gamma \neq 0$. This system of equations and their corresponding parameters offer a comprehensive framework for studying wave propagation in diverse physical media.

It is worthwhile to recall that the GDSS has physical quantities that are conserved throughout time. Physically, these quantities are associated with the total mass, the total energy, and the momenta in the x and y directions. These quantities are time-dependent variables which are denoted, respectively, by the symbols M, E, J_x, and J_y, and they are given by the following formulas:

$$M(t) = \iint_{\mathbb{R}^2} |u(x,y,t)|^2 dxdy, \tag{5}$$

$$\begin{aligned} E(t) = \iint_{\mathbb{R}^2} \Bigg\{ & \alpha \left| \frac{\partial u(x,y,t)}{\partial x} \right|^2 + \beta \left| \frac{\partial u(x,y,t)}{\partial y} \right|^2 \\ & + \frac{1}{2} \left[\gamma |u(x,y,t)|^4 + \xi \left(\psi \left| \frac{\partial w(x,y,t)}{\partial x} \right|^2 + \eta \left| \frac{\partial w(x,y,t)}{\partial y} \right|^2 \right.\right. \\ & \left.\left. + \phi \left| \frac{\partial v(x,y,t)}{\partial x} \right|^2 + \chi \left| \frac{\partial v(x,y,t)}{\partial y} \right|^2 \right) \right] \Bigg\} dxdy, \end{aligned} \tag{6}$$

$$J_x(t) = \mathrm{i} \iint_{\mathbb{R}^2} \left(\overline{u(x,y,t)} \frac{\partial u(x,y,t)}{\partial x} - u(x,y,t) \frac{\partial \overline{u(x,y,t)}}{\partial x} \right) dxdy, \tag{7}$$

and

$$J_y(t) = \mathrm{i} \iint_{\mathbb{R}^2} \left(\overline{u(x,y,t)} \frac{\partial u(x,y,t)}{\partial y} - u(x,y,t) \frac{\partial \overline{u(x,y,t)}}{\partial y} \right) dxdy. \tag{8}$$

The relation of these parameters plays an essential role in the definition and nature of the GDSS system. For example, setting $\theta = \psi - \phi = \chi - \eta$ would reduce the mathematical model to the classical DSS, which is described by the conservative coupled system of nonlinear partial differential equations:

$$\mathrm{i} \frac{\partial u}{\partial t} + \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial^2 u}{\partial dy} = \gamma |u|^2 u + \xi u \frac{\partial w}{\partial x}, \tag{9}$$

$$\eta \frac{\partial^2 w}{\partial x^2} + \theta \frac{\partial^2 w}{\partial y^2} = \frac{\partial |u|^2}{\partial x}. \tag{10}$$

An additional significant feature of the GDSS system lies in its classification based on the signs of its parameters. This classification scheme distinguishes between elliptic–elliptic–elliptic, elliptic–elliptic–hyperbolic, and elliptic–hyperbolic–hyperbolic configurations, determined by the signs of (α, η, χ) as $(+,+,+)$, $(+,+,-)$, and $(+,-,-)$, respectively [23]. This classification underscores the rich spectrum of behaviors exhibited by the GDSS, illustrating its complexity, diversity, and importance in describing different physical phenomena.

The rest of the article is organized as follows. In Section 2, we will extend the GDSS system to the fractional scenario by considering Riesz-type fractional derivatives of different orders of differentiation. To that end, we will recall the definition of this type of fractional operator and recall some of its properties. In that stage, we will propose some fractional forms of the mass, the energy, and the momentum functionals. Section 3 will be devoted to proving that those functionals are preserved throughout time. Various technical results will be necessary to prove the many results of that section. This manuscript concludes with a synthesis of the findings from these demonstrations, providing an insightful understanding of the conservation laws within the realm of fractional-type GDSSs. After the conclusion of this work, it is important to point out that various avenues of research will remain open as a gateway to the exploration of future applications for the fractional GDSS in diverse scientific and engineering domains.

2. Mathematical Model

There is a growing interest in finding new formulations and applications for DSS to further the understanding of complex physical phenomena. One promising area of investigation is the use of fractional differential equations [24,25]. It is well known that fractional differential equations are a type of differential equations that involve fractional-order derivatives [26]. An example of this type of fractional operator is the Riesz derivative, which provides a mathematical tool that can describe complex diffusion and fractional convection phenomena [27–29]. Fractional operators have been applied in the context of Schrödinger's equation, which is a mathematical model used to describe the dynamics of the state of any quantum mechanics system in terms of its wave function given by the Hamiltonian of the system [30]. Another interesting model is the Klein–Gordon equation, which is a relativistic version of the Schrödinger equation used to describe spinless particles [31]. This model has also been analyzed with fractional differential equations to describe different physical phenomena such as the propagation of fluxion between superconductors, motions of a pendulum, and dislocations in crystals [32]. In the present work, we will investigate a fractional form of the GDSS.

Fractional systems have been the subject of numerous studies, each contributing to the understanding and implementation of these equations. For instance, a study explored the impacts of Brownian motion and fractional derivatives on the solutions of the stochastic fractional DSS. Some authors used two different approaches, namely, the Riccati–Bernoulli sub-ordinary differential equations and sine-cosine methods, to obtain novel elliptic, hyperbolic, trigonometric, and rational stochastic solutions [33]. In another study, a new iterative method was proposed to solve fractional DSS [34]. The convergence of this method was proved, and the results obtained were compared with the exact solutions, demonstrating the effectiveness of this approach. Furthermore, a recent study derived general dark solitons and mixed solutions consisting of dark solitons and breathers for a DSS by employing the bilinear method [35]. This study expanded the range of solutions available for these equations. These reports collectively highlight the versatility and applicability of the GDSS in various scientific and engineering contexts.

In the context of this article, the mathematical framework is established by introducing variables $x_L, x_R, y_L, y_R \in \mathbb{R}$ which satisfy the conditions $x_L < x_R$ and $y_L < y_R$. In this way, we consider the compact spatial domain $B \subseteq \mathbb{R}^2$, defined as $B = [x_L, x_R] \times [y_L, y_R]$. The temporal evolution is defined for the time period $T > 0$, and the overall space-time domain is denoted as $\Omega = B \times [0, T]$. Within this domain and motivated by the GDSS in its integer-order form, three smooth functions are introduced, namely, $u : \Omega \to \mathbb{C}$, representing a complex amplitude short transverse wave mode; while $w, v : \Omega \to \mathbb{R}$ denote long longitudinal and long transverse wave modes, respectively. Under these circumstances, the fractional GDSS system under investigation in this article is the coupled system of three nonlinear partial differential equations:

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\frac{\partial^\lambda u}{\partial |x|^\lambda} + \beta\frac{\partial^\lambda u}{\partial |y|^\lambda} = \gamma|u|^2 u + \xi u\left(\frac{\partial^{\mu/2} w}{\partial |x|^{\mu/2}} + \frac{\partial^{\nu/2} v}{\partial |y|^{\nu/2}}\right), \tag{11}$$

$$\psi\frac{\partial^\kappa w}{\partial |x|^\kappa} + \eta\frac{\partial^\kappa w}{\partial |y|^\kappa} + \theta\frac{\partial^\sigma v}{\partial |x|^{\sigma/2}\,\partial |y|^{\sigma/2}} = \frac{\partial^{\mu/2}|u|^2}{\partial |x|^{\mu/2}}, \tag{12}$$

$$\phi\frac{\partial^\rho v}{\partial |x|^\rho} + \chi\frac{\partial^\rho v}{\partial |y|^\rho} + \theta\frac{\partial^\sigma w}{\partial |x|^{\sigma/2}\,\partial |y|^{\sigma/2}} = \frac{\partial^{\nu/2}|u|^2}{\partial |y|^{\nu/2}}. \tag{13}$$

As before, α, β, γ, ξ, ψ, η, θ, ϕ, and χ are real constants, while κ, λ, μ, ν, ρ, and σ are numbers in the interval $(1,2)$. Moreover, we will impose initial-boundary conditions of the form

$$\begin{aligned} u(x,y,0) &= u_0(x,y), \quad \forall (x,y) \in B, \\ v(x,y,0) &= v_0(x,y), \quad \forall (x,y) \in B, \\ w(x,y,0) &= w_0(x,y), \quad \forall (x,y) \in B, \\ u(x,y,t) = v(x,y,t) = w(x,y,t) &= 0, \quad \forall (x,y,t) \in \partial B \times [0,T], \end{aligned} \tag{14}$$

where ∂B represents the boundary of B, and the fractional derivatives are understood in the Riesz sense. More precisely, we observe the following definition.

Definition 1 (Podlubny [36])**.** *Suppose that $u : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$, and let $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ be such that $n-1 < \lambda < n$. The Riesz fractional partial derivative of u of order λ with respect to x is defined as*

$$\frac{\partial^\lambda u(x,y,t)}{\partial |x|^\lambda} = \frac{-1}{2\cos(\frac{\pi\lambda}{2})\Gamma(n-\lambda)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{u(\zeta,y,t)d\zeta}{|x-\zeta|^{\lambda+1-n}}, \quad \forall (x,y,t) \in \mathbb{R}^2 \times [0,T]. \tag{15}$$

Here, Γ denotes the usual gamma function. Similarly, the Riesz fractional partial derivative of u with respect to the variable y of order λ is defined by

$$\frac{\partial^\lambda u(x,y,t)}{\partial |y|^\lambda} = \frac{-1}{2\cos(\frac{\pi\lambda}{2})\Gamma(n-\lambda)} \frac{\partial^n}{\partial y^n} \int_{-\infty}^{\infty} \frac{u(x,\zeta,t)d\zeta}{|y-\zeta|^{\lambda+1-n}}, \quad \forall (x,y,t) \in \mathbb{R}^2 \times [0,T]. \tag{16}$$

Remark 1. *It is important to mention that one of the reviewers asked whether this fractional form of the GDSS offers some practical applications or is it only from the theoretical aspect. Indeed, the present model is interesting from the theoretical aspect given the nontrivial character of the derivations. However, the mathematical model may be seen as a limit of a system of particles with long-range interactions. More precisely, the classical Davey–Stewartson model was derived in the continuous case assuming classical dynamics mechanisms, in which the dynamics of a particle depends only on mechanisms defined in terms of the dynamics of its close neighbors. Classical partial derivatives are obtained then in the model after assuming these conditions. However, Riesz space fractional derivatives may be obtained in the mathematical model if we assume certain types of global interactions. The details are proved in [24], and we provide Appendix A at the end of this work for the sake of convenience.*

Let $L_q(\Omega) = \{g : \Omega \to G : \text{for each } t \in [0,T],\ g(\cdot,t) \in L_q(B)\}$, with $1 \le q < \infty$ and $G = \mathbb{C}, \mathbb{R}$. If $f \in L_p(\Omega)$, define the time-dependent function

$$\|f\|_p = \left(\iint_B |f(x,y,t)|^p dxdy \right)^{\frac{1}{p}}, \quad \forall t \in [0,T]. \tag{17}$$

Given $z \in \mathbb{C}$, represent its conjugate by $\overline{z}$. If $f, g \in L_2(\Omega)$, let

$$\langle f, g \rangle = \iint_{\overline{B}} f(x,y,t)\overline{g(x,y,t)}dxdy, \quad \forall t \in [0,T]. \tag{18}$$

Obviously, the last two equations coincide with the classical p-norm and the inner product in $L_2(\Omega)$, when we consider the functions f and g as functions of (x,t) and fix t. In that sense, the p-norm and the inner product are ultimately functions of the time variable. With this nomenclature, it is important to point out that the Riesz fractional derivatives of order $\lambda \in (1,2)$ satisfy the following property, for each $u_1, u_2 : \Omega \to \mathbb{C}$ (see [37]):

$$\left\langle u_1, -\frac{\partial^\lambda u_2}{\partial |x|^\lambda} \right\rangle = \left\langle -\frac{\partial^\lambda u_1}{\partial |x|^\lambda}, u_2 \right\rangle = \left\langle \frac{\partial^{\lambda/2} u_1}{\partial |x|^{\lambda/2}}, \frac{\partial^{\lambda/2} u_2}{\partial |x|^{\lambda/2}} \right\rangle, \quad \forall t \in [0,T]. \tag{19}$$

Observe that the mass, energy, and momentum operators are identical or are fractional extensions of the corresponding operators for the integer-order GDSS. Also, we provide the definitions of the mass, energy, and momentum functionals associated with the fractional GDSS. Throughout, we will suppose that u, w, and v satisfy the fractional-order system (11)–(13).

Definition 2. *The mass density of the fractional GDSS is given by* $|u(x,y,t)|^2$*, for all* $(x,y,t) \in \Omega$*. Moreover, the total mass at time t is obtained by integrating the mass density over the spatial domain at that time. More precisely,*

$$M(t) = \|u\|_2^2, \quad \forall t \in [0,T]. \tag{20}$$

In turn, for each $(x,y,t) \in \Omega$*, the Hamiltonian associated with the fractional GDSS is defined as*

$$\begin{aligned} H(x,y,t) = \alpha \left| \frac{\partial^{\lambda/2} u}{\partial |x|^{\lambda/2}} \right|^2 + \beta \left| \frac{\partial^{\lambda/2} u}{\partial |y|^{\lambda/2}} \right|^2 + \frac{1}{2} \Bigg[\gamma |u|^4 + \xi \Bigg(\psi \left(\frac{\partial^{\kappa/2} w}{\partial |x|^{\kappa/2}} \right)^2 \\ + \eta \left(\frac{\partial^{\kappa/2} w}{\partial |y|^{\kappa/2}} \right)^2 + \phi \left(\frac{\partial^{\rho/2} v}{\partial |x|^{\rho/2}} \right)^2 + \chi \left(\frac{\partial^{\rho/2} v}{\partial |y|^{\rho/2}} \right)^2 \Bigg) \Bigg]. \end{aligned} \tag{21}$$

As expected, the total energy of the system at time $t \in [0,T]$ *is given by*

$$\begin{aligned} E(t) &= \iint_B H(x,y,t) dx dy \\ &= \alpha \left\| \frac{\partial^{\lambda/2} u}{\partial |x|^{\lambda/2}} \right\|_2^2 + \beta \left\| \frac{\partial^{\lambda/2} u}{\partial |y|^{\lambda/2}} \right\|_2^2 + \frac{1}{2} \Bigg[\gamma \|u\|_4^4 + \xi \Bigg(\psi \left\| \frac{\partial^{\kappa/2} w}{\partial |x|^{\kappa/2}} \right\|_2^2 \\ &\quad + \eta \left\| \frac{\partial^{\kappa/2} w}{\partial |y|^{\kappa/2}} \right\|_2^2 + \phi \left\| \frac{\partial^{\rho/2} v}{\partial |x|^{\rho/2}} \right\|_2^2 + \chi \left\| \frac{\partial^{\rho/2} v}{\partial |y|^{\rho/2}} \right\|_2^2 \Bigg) \Bigg]. \end{aligned} \tag{22}$$

Finally, we define the momentum functionals with respect to the x and y directions, which are, respectively, the functions of time given by

$$J_x(t) = \mathrm{i} \left[\left\langle \frac{\partial u}{\partial x}, u \right\rangle - \left\langle u, \frac{\partial u}{\partial x} \right\rangle \right], \quad \forall t \in [0,T], \tag{23}$$

$$J_y(t) = \mathrm{i} \left[\left\langle \frac{\partial u}{\partial y}, u \right\rangle - \left\langle u, \frac{\partial u}{\partial y} \right\rangle \right], \quad \forall t \in [0,T]. \tag{24}$$

Before closing this section, we provide some technical results which will be useful to prove the main conservation properties associated with the fractional GDSS. The proofs are straightforward and, for that reason, we only provide short proofs. We will use these results without an explicit reference to them. The first lemma will be employed in the proof for the conservation of energy of the fractional GDSS.

Lemma 1. *Suppose that u, w, and v are solutions of the fractional-order system* (11)–(13)*. Then, the following identities are satisfied:*

(a) $\displaystyle \frac{d}{dt} \left\| \frac{\partial^{\lambda/2} u}{\partial |x|^{\lambda/2}} \right\|_2^2 = -2 \,\mathrm{Re} \left\langle \frac{\partial^{\lambda/2} u}{\partial |x|^{\lambda/2}}, \frac{\partial u}{\partial t} \right\rangle$*, for each* $\in [0,T]$.

(b) $\displaystyle \frac{d}{dt} \left\| \frac{\partial^{\lambda/2} u}{\partial |y|^{\lambda/2}} \right\|_2^2 = -2 \,\mathrm{Re} \left\langle \frac{\partial^{\lambda/2} u}{\partial |y|^{\lambda/2}}, \frac{\partial u}{\partial t} \right\rangle$*, for each* $t \in [0,T]$.

(c) $\displaystyle \frac{1}{2} \frac{d}{dt} \|u\|_4^4 = 2 \,\mathrm{Re} \left\langle |u|^2 u, \frac{\partial u}{\partial t} \right\rangle$*, for each* $t \in [0,T]$.

Proof. To start with, we prove property (a). To that end, we apply the product rule and Equation (19) to verify that

$$\begin{aligned}\frac{d}{dt}\left\|\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}}\right\|_2^2 &= \left\langle\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}},\frac{\partial}{\partial t}\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}}\right\rangle + \left\langle\frac{\partial}{\partial t}\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}},\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}}\right\rangle \\ &= -\left\langle\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}},\frac{\partial u}{\partial t}\right\rangle - \left\langle\frac{\partial u}{\partial t},\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}}\right\rangle \\ &= -\left\langle\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}},\frac{\partial u}{\partial t}\right\rangle - \overline{\left\langle\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}},\frac{\partial u}{\partial t}\right\rangle} = -2\,\mathrm{Re}\left\langle\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}},\frac{\partial u}{\partial t}\right\rangle,\end{aligned} \tag{25}$$

for each $t \in [0,T]$. This establishes property (a), and (b) is proved in a similar fashion. To prove property (c), we proceed as follows. Take the derivative with respect to t of $\|u\|_4^4 = \langle u^2, u^2\rangle$; use the product rule followed by an application of Equation (19) to obtain

$$\begin{aligned}\frac{d}{dt}\|u\|_4^4 &= \left\langle\frac{\partial(u^2)}{\partial t},u^2\right\rangle + \left\langle u^2,\frac{\partial(u^2)}{\partial t}\right\rangle = \left\langle 2u\frac{\partial u}{\partial t},u^2\right\rangle + \left\langle u^2,2u\frac{\partial u}{\partial t}\right\rangle \\ &= \left\langle u^2,2u\frac{\partial u}{\partial t}\right\rangle + \overline{\left\langle u^2,2u\frac{\partial u}{\partial t}\right\rangle} = 2\,\mathrm{Re}\left\langle u^2,2u\frac{\partial u}{\partial t}\right\rangle \\ &= 4\,\mathrm{Re}\left\langle |u|^2u,\frac{\partial u}{\partial t}\right\rangle, \quad \forall t\in[0,T].\end{aligned} \tag{26}$$

This is what we wanted to prove. □

The properties in the next technical results will be employed to prove the conservation of the momentum operators in the x and y directions.

Lemma 2. *Let u, w, and v be solutions of the fractional GDSS (11)–(13). For each $t \in [0,T]$ and $\varphi = x, y$, the following identities hold:*

(a) $\dfrac{dJ_\varphi(t)}{dt} = 2\mathrm{i}\left[\left\langle\dfrac{\partial^2 u}{\partial t\partial\varphi},u\right\rangle + \left\langle\dfrac{\partial u}{\partial\varphi},\dfrac{\partial u}{\partial t}\right\rangle\right].$

(b) $\mathrm{Re}\left\langle\dfrac{\partial^{\lambda/2}u}{\partial|\varphi|^{\lambda/2}},\dfrac{\partial}{\partial\xi}\dfrac{\partial^{\lambda/2}u}{\partial|\varphi|^{\lambda/2}}\right\rangle = 0.$

(c) $\mathrm{Re}\left\langle|u|^2u,\dfrac{\partial u}{\partial\varphi}\right\rangle = 0.$

Proof. Let φ be any of x or y. Take the derivative of J_φ with respect to t; use the product rule followed by Equation (19) to obtain

$$\begin{aligned}\frac{dJ_\varphi(t)}{dt} &= \mathrm{i}\left[\left\langle\frac{\partial^2 u}{\partial t\partial\varphi},u\right\rangle + \left\langle\frac{\partial u}{\partial\varphi},\frac{\partial u}{\partial t}\right\rangle - \left\langle\frac{\partial u}{\partial t},\frac{\partial u}{\partial\varphi}\right\rangle - \left\langle u,\frac{\partial^2 u}{\partial t\partial\varphi}\right\rangle\right] \\ &= 2\mathrm{i}\left[\left\langle\frac{\partial^2 u}{\partial t\partial\varphi},u\right\rangle + \left\langle\frac{\partial u}{\partial\varphi},\frac{\partial u}{\partial t}\right\rangle\right], \quad \forall t\in[0,T].\end{aligned} \tag{27}$$

This proves property (a). In this case, we will prove property (c), since property (b) is established using similar arguments. To that end, observe that

$$\begin{aligned}2\,\mathrm{Re}\left\langle|u|^2u,\frac{\partial u}{\partial\varphi}\right\rangle &= \left\langle|u|^2u,\frac{\partial u}{\partial\varphi}\right\rangle + \left\langle\frac{\partial u}{\partial\varphi},|u|^2u\right\rangle = \left\langle u^2,u\frac{\partial u}{\partial\varphi}\right\rangle + \left\langle u\frac{\partial u}{\partial\varphi},u^2\right\rangle \\ &= \frac{1}{2}\left\langle u^2,\frac{\partial(u^2)}{\partial\varphi}\right\rangle + \frac{1}{2}\left\langle\frac{\partial(u^2)}{\partial\varphi},u^2\right\rangle = \frac{1}{2}\iint_B\frac{\partial|u|^4}{\partial\varphi}dxdy\end{aligned} \tag{28}$$

The last term in this chain of identities is equal to zero in light of the boundary conditions on B, whence the result readily follows. □

3. Main Results

The purpose of the present section is to derive the most important conservation properties of the fractional GDSS. More precisely, we will prove that the mass, the energy, and the momentum operators are conserved. In the following, we will continue assuming that the functions u, w, and v satisfy the fractional-order system (11)–(13).

Theorem 1. *The total mass of the fractional GDSS is conserved throughout time.*

Proof. First, multiply Equation (11) by $\overline{u}$ and rearrange terms algebraically to reach the following equation:

$$\mathrm{i}\frac{\partial u}{\partial t}\overline{u} + \left(\alpha\frac{\partial^\lambda u}{\partial |x|^\lambda} + \beta\frac{\partial^\lambda u}{\partial |y|^\lambda}\right)\overline{u} = \left(\gamma|u|^2 + \xi\left(\frac{\partial^{\mu/2} w}{\partial |x|^{\mu/2}} + \frac{\partial^{\nu/2} v}{\partial |x|^{\nu/2}}\right)\right)|u|^2. \tag{29}$$

Then, take the conjugate of Equation (29) to reach that

$$-\mathrm{i}\frac{\partial \overline{u}}{\partial t}u + \left(\alpha\frac{\partial^\lambda \overline{u}}{\partial |x|^\lambda} + \beta\frac{\partial^\lambda \overline{u}}{\partial |y|^\lambda}\right)u = \left(\gamma|u|^2 + \xi\left(\frac{\partial^{\mu/2} w}{\partial |x|^{\mu/2}} + \frac{\partial^{\nu/2} v}{\partial |x|^{\nu/2}}\right)\right)|u|^2. \tag{30}$$

Now, subtract Equation (30) from Equation (29) and multiply by i on both sides. Then, integrate spatially over the domain B. As a result of this and property (19), we obtain that

$$\begin{aligned} M'(t) = \frac{d}{dt}\langle u,u\rangle &= \iint_B \frac{\partial}{\partial t}(u\overline{u})dxdy \\ &= \mathrm{i}\alpha\left\langle \frac{\partial^\lambda u}{\partial |x|^\lambda}, u\right\rangle + \mathrm{i}\beta\left\langle \frac{\partial^\lambda u}{\partial |y|^\lambda}, u\right\rangle - \mathrm{i}\alpha\left\langle u, \frac{\partial^\lambda u}{\partial |x|^\lambda}\right\rangle - \mathrm{i}\beta\left\langle u, \frac{\partial^\lambda u}{\partial |y|^\lambda}\right\rangle \\ &= \mathrm{i}\alpha\left\langle \frac{\partial^\lambda u}{\partial |x|^\lambda}, u\right\rangle + \mathrm{i}\beta\left\langle \frac{\partial^\lambda u}{\partial |y|^\lambda}, u\right\rangle - \mathrm{i}\alpha\left\langle \frac{\partial^\lambda u}{\partial |x|^\lambda}, u\right\rangle - \mathrm{i}\beta\left\langle \frac{\partial^\lambda u}{\partial |y|^\lambda}, u\right\rangle. \end{aligned} \tag{31}$$

Obviously, all the terms on the right-hand side of this equation cancel out. As a consequence, we obtain that $M'(t) = 0$ for all $t \in [0,T]$ or, equivalently, the total mass is conserved throughout time, as desired. □

Theorem 2. *If $t \in [0,T]$, then the energy $E(t)$ of the fractional GDSS is conserved.*

Proof. To verify that the energy of the fractional GDSS is conservative, we multiply Equation (11) by $\partial\overline{u}/\partial t$ to obtain

$$\mathrm{i}\frac{\partial u}{\partial t}\frac{\partial \overline{u}}{\partial t} + \left(\alpha\frac{\partial^\lambda u}{\partial |x|^\lambda} + \beta\frac{\partial^\lambda u}{\partial |y|^\lambda}\right)\frac{\partial \overline{u}}{\partial t} = \left[\gamma|u|^2 + \xi\left(\frac{\partial^{\mu/2} w}{\partial |x|^{\mu/2}} + \frac{\partial^{\nu/2} v}{\partial |y|^{\nu/2}}\right)\right]u\frac{\partial \overline{u}}{\partial t}. \tag{32}$$

Next, we calculate the conjugate of Equation (32). In such a way, we reach the identity

$$-\mathrm{i}\frac{\partial \overline{u}}{\partial t}\frac{\partial u}{\partial t} + \left(\alpha\frac{\partial^\lambda \overline{u}}{\partial |x|^\lambda} + \beta\frac{\partial^\lambda \overline{u}}{\partial |y|^\lambda}\right)\frac{\partial u}{\partial t} = \left[\gamma|u|^2 + \xi\left(\frac{\partial^{\mu/2} w}{\partial |x|^{\mu/2}} + \frac{\partial^{\nu/2} v}{\partial |y|^{\nu/2}}\right)\right]\overline{u}\frac{\partial u}{\partial t}. \tag{33}$$

Next, let us add Equations (32) and (33). After some straightforward simplifications, we readily reach that

$$\begin{aligned} &\alpha\frac{\partial^\lambda u}{\partial |x|^\lambda}\frac{\partial \overline{u}}{\partial t} + \beta\frac{\partial^\lambda u}{\partial |y|^\lambda}\frac{\partial \overline{u}}{\partial t} + \alpha\frac{\partial^\lambda \overline{u}}{\partial |x|^\lambda}\frac{\partial u}{\partial t} + \beta\frac{\partial^\lambda \overline{u}}{\partial |y|^\lambda}\frac{\partial u}{\partial t} \\ &\qquad = 2\gamma\mathrm{Re}\left(|u|^2\overline{u}\frac{\partial u}{\partial t}\right) + \xi\frac{\partial |u|^2}{\partial t}\frac{\partial^{\mu/2} w}{\partial |x|^{\mu/2}} + \xi\frac{\partial |u|^2}{\partial t}\frac{\partial^{\nu/2} v}{\partial |y|^{\nu/2}}. \end{aligned} \tag{34}$$

On the other hand, differentiate Equation (12) with respect to t and multiply by ξw. At the same time, Equation (13) will be derived with respect to t and then multiplied by ξv. As a result, we obtain the following identities:

$$\psi\xi\frac{\partial}{\partial t}\frac{\partial^\kappa w}{\partial|x|^\kappa}w+\eta\xi\frac{\partial}{\partial t}\frac{\partial^\kappa w}{\partial|y|^\kappa}w+\theta\xi\frac{\partial}{\partial t}\frac{\partial^\sigma v}{\partial|x|^{\sigma/2}\partial|y|^{\sigma/2}}w=\xi\frac{\partial}{\partial t}\frac{\partial^{\mu/2}|u|^2}{\partial|x|^{\mu/2}}w, \tag{35}$$

$$\phi\xi\frac{\partial}{\partial t}\frac{\partial^\rho v}{\partial|x|^\rho}v+\chi\xi\frac{\partial}{\partial t}\frac{\partial^\rho v}{\partial|y|^\rho}v+\theta\xi\frac{\partial}{\partial t}\frac{\partial^\sigma w}{\partial|x|^{\sigma/2}\partial|y|^{\sigma/2}}v=\xi\frac{\partial}{\partial t}\frac{\partial^{\nu/2}|u|^2}{\partial|y|^{\nu/2}}v. \tag{36}$$

Now, add Equations (34)–(36), and integrate over the spatial domain. By using the notation in terms of inner products, Equation (19), rearranging terms and simplifying algebraically, it is possible to reach the following equation:

$$\begin{aligned}2\gamma\mathrm{Re}\left\langle|u|^2\overline{u},\frac{\partial u}{\partial t}\right\rangle=&\,2\alpha\mathrm{Re}\left\langle\frac{\partial^\lambda u}{\partial|x|^\lambda},\frac{\partial u}{\partial t}\right\rangle+2\beta\mathrm{Re}\left\langle\frac{\partial^\lambda u}{\partial|y|^\lambda},\frac{\partial u}{\partial t}\right\rangle+\psi\xi\frac{d}{dt}\left\langle\frac{\partial^\kappa w}{\partial|x|^\kappa},w\right\rangle\\&-\psi\xi\left\langle\frac{\partial^\kappa w}{\partial|x|^\kappa},\frac{\partial w}{\partial t}\right\rangle+\eta\xi\frac{d}{dt}\left\langle\frac{\partial^\kappa w}{\partial|y|^\kappa},w\right\rangle-\eta\xi\left\langle\frac{\partial^\kappa w}{\partial|y|^\kappa},\frac{\partial w}{\partial t}\right\rangle\\&+\phi\xi\frac{d}{dt}\left\langle\frac{\partial^\rho v}{\partial|x|^\rho},v\right\rangle-\phi\xi\left\langle\frac{\partial^\rho v}{\partial|x|^\rho},\frac{\partial v}{\partial t}\right\rangle+\chi\xi\frac{d}{dt}\left\langle\frac{\partial^\rho v}{\partial|y|^\rho},v\right\rangle\\&-\chi\xi\left\langle\frac{\partial^\rho v}{\partial|y|^\rho},\frac{\partial v}{\partial t}\right\rangle.\end{aligned} \tag{37}$$

Using the identities of Lemma 1 along with algebraic simplification yields

$$\begin{aligned}0=&\,2\alpha\frac{d}{dt}\left\|\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}}\right\|_2^2+2\beta\frac{d}{dt}\left\|\frac{\partial^{\lambda/2}u}{\partial|y|^{\lambda/2}}\right\|_2^2+\frac{\psi\xi}{2}\frac{d}{dt}\left\|\frac{\partial^{\kappa/2}w}{\partial|x|^{\kappa/2}}\right\|_2^2\\&+\frac{\eta\xi}{2}\frac{d}{dt}\left\|\frac{\partial^{\kappa/2}w}{\partial|y|^{\kappa/2}}\right\|_2^2+\frac{\phi\xi}{2}\frac{d}{dt}\left\|\frac{\partial^{\rho/2}v}{\partial|x|^{\rho/2}}\right\|_2^2+\frac{\chi\xi}{2}\frac{d}{dt}\left\|\frac{\partial^{\rho/2}v}{\partial|y|^{\rho/2}}\right\|_2^2+\frac{1}{2}\frac{d}{dt}\|u\|_4^4.\end{aligned} \tag{38}$$

Factorizing the time derivative and reorganizing terms, we obtain that $E'(t)=0$. As a consequence, we confirm that the energy is constant with respect to time. □

Theorem 3. *The momentum functionals in the x and y directions are constant with respect to time.*

Proof. We will only provide the proof that J_x is conserved since the proof for the conservation of the momentum in the y direction is similar. To start with, multiply Equation (11) by $\frac{\partial\overline{u}}{\partial x}$. As the second step, multiply the conjugate of that equation by -1, integrate over the spatial domain B, and use Equation (19). As a result, we obtain that

$$\begin{aligned}\mathrm{i}\left\langle\frac{\partial u}{\partial x},\frac{\partial u}{\partial t}\right\rangle=&-\alpha\left\langle\frac{\partial}{\partial x}\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}},\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}}\right\rangle-\beta\left\langle\frac{\partial}{\partial x}\frac{\partial^{\lambda/2}u}{\partial|y|^{\lambda/2}},\frac{\partial^{\lambda/2}u}{\partial|y|^{\lambda/2}}\right\rangle\\&-\gamma\left\langle\frac{\partial u}{\partial x},|u|^2u\right\rangle-\xi\left\langle\frac{\partial u}{\partial x},u\frac{\partial^{\mu/2}w}{\partial|x|^{\mu/2}}\right\rangle-\xi\left\langle\frac{\partial u}{\partial x},u\frac{\partial^{\nu/2}v}{\partial|y|^{\nu/2}}\right\rangle\\=&-\alpha\left\langle\frac{\partial}{\partial x}\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}},\frac{\partial^{\lambda/2}u}{\partial|x|^{\lambda/2}}\right\rangle-\beta\left\langle\frac{\partial}{\partial x}\frac{\partial^{\lambda/2}u}{\partial|y|^{\lambda/2}},\frac{\partial^{\lambda/2}u}{\partial|y|^{\lambda/2}}\right\rangle\\&+\gamma\left\langle u,\frac{\partial}{\partial x}|u|^2u\right\rangle+\xi\left\langle u,\frac{\partial}{\partial x}\left(u\frac{\partial^{\mu/2}w}{\partial|x|^{\mu/2}}\right)\right\rangle+\xi\left\langle u,\frac{\partial}{\partial x}\left(u\frac{\partial^{\nu/2}v}{\partial|y|^{\nu/2}}\right)\right\rangle.\end{aligned} \tag{39}$$

Here, we employed integration by parts in the last step. On the other hand, differentiate both sides of Equation (11) with respect to x and multiply by $\overline{u}$. After rearranging terms algebraically, integrating over the spatial domain and applying Equation (19), we obtain

$$\begin{aligned}\mathrm{i}\left\langle \frac{\partial^2 u}{\partial t \partial x}, u \right\rangle = \alpha \left\langle \frac{\partial}{\partial x} \frac{\partial^{\lambda/2} u}{\partial |x|^{\lambda/2}}, \frac{\partial^{\lambda/2} u}{\partial |x|^{\lambda/2}} \right\rangle + \beta \left\langle \frac{\partial}{\partial x} \frac{\partial^{\lambda/2} u}{\partial |y|^{\lambda/2}}, \frac{\partial^{\lambda/2} u}{\partial |y|^{\lambda/2}} \right\rangle + \gamma \left\langle \frac{\partial}{\partial x} |u|^2 u, u \right\rangle \\ + \xi \left\langle \frac{\partial}{\partial x} \left(u \frac{\partial^{\mu/2} w}{\partial |x|^{\mu/2}} \right), u \right\rangle + \xi \left\langle \frac{\partial}{\partial x} \left(u \frac{\partial^{\nu/2} v}{\partial |y|^{\nu/2}} \right), u \right\rangle.\end{aligned} \tag{40}$$

Adding Equations (39) and (40), multiplying both sides by 2, applying the identities of Lemma 2, using integration by parts, reorganizing the terms, and simplifying algebraically yields the following expression:

$$\begin{aligned}\frac{dJ_x(t)}{dt} &= 2\mathrm{i}\left[\left\langle \frac{\partial^2 u}{\partial t \partial x}, u \right\rangle + \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t} \right\rangle \right] \\ &= -4\xi \operatorname{Re} \left\langle u \frac{\partial^{\mu/2} w}{\partial |x|^{\mu/2}}, \frac{\partial u}{\partial x} \right\rangle - 4\xi \operatorname{Re} \left\langle u \frac{\partial^{\nu/2} v}{\partial |y|^{\nu/2}}, \frac{\partial u}{\partial x} \right\rangle \\ &= -2\xi \left\langle \frac{\partial^{\mu/2} |u|^2}{\partial |x|^{\mu/2}}, \frac{\partial w}{\partial x} \right\rangle - 2\xi \left\langle \frac{\partial^{\nu/2} |u|^2}{\partial |y|^{\nu/2}}, \frac{\partial v}{\partial x} \right\rangle.\end{aligned} \tag{41}$$

Then, substitute Equations (12) and (13) into the right-hand side of the last equation and use the identity (19). As a consequence, we obtain

$$\begin{aligned}\frac{dJ_x(t)}{dt} &= 2\xi\psi \left\langle \frac{\partial^{\kappa/2} w}{\partial |x|^{\kappa/2}}, \frac{\partial}{\partial x} \frac{\partial^{\kappa/2} w}{\partial |x|^{\kappa/2}} \right\rangle + 2\xi\eta \left\langle \frac{\partial^{\kappa/2} w}{\partial |y|^{\kappa/2}}, \frac{\partial}{\partial x} \frac{\partial^{\kappa/2} w}{\partial |y|^{\kappa/2}} \right\rangle \\ &\quad - 2\xi\theta \left\langle \frac{\partial^{\sigma} v}{\partial |x|^{\sigma/2} \partial |y|^{\sigma/2}}, \frac{\partial w}{\partial x} \right\rangle + 2\xi\phi \left\langle \frac{\partial^{\rho/2} v}{\partial |x|^{\rho/2}}, \frac{\partial}{\partial x} \frac{\partial^{\rho/2} v}{\partial |x|^{\rho/2}} \right\rangle \\ &\quad + 2\xi\chi \left\langle \frac{\partial^{\rho/2} v}{\partial |y|^{\rho/2}}, \frac{\partial}{\partial x} \frac{\partial^{\rho/2} v}{\partial |y|^{\rho/2}} \right\rangle - 2\xi\theta \left\langle \frac{\partial^{\sigma} w}{\partial |x|^{\sigma/2} \partial |y|^{\sigma/2}}, \frac{\partial v}{\partial x} \right\rangle \\ &= -2\xi\theta \left\langle \frac{\partial^{\sigma} v}{\partial |x|^{\sigma/2} \partial |y|^{\sigma/2}}, \frac{\partial w}{\partial x} \right\rangle - 2\xi\theta \left\langle \frac{\partial^{\sigma} w}{\partial |x|^{\sigma/2} \partial |y|^{\sigma/2}}, \frac{\partial v}{\partial x} \right\rangle.\end{aligned} \tag{42}$$

We conclude that the temporal derivative of the momentum of the fractional GDSS in the x direction is equal to zero. Equivalently, this means that J_x is conserved with respect to time. In a similar fashion, we can establish the conservation property for the momentum in the y direction, as desired. □

The following is a consequence of the properties of the conservation of mass and energy. We provide a short proof in view that it is straightforward.

Corollary 1. *Suppose that u, w, and v satisfy the fractional GDSS, and assume that the model parameters are all positive or all negative. Then, there exists a constant $C \geq 0$ which depends only on the initial conditions of the system, such that*

$$\|u\|_2^2, \|u\|_4^4, \left\| \frac{\partial^{\lambda/2} u}{\partial |\varphi|^{\lambda/2}} \right\|_2^2, \left\| \frac{\partial^{\kappa/2} w}{\partial |\varphi|^{\kappa/2}} \right\|_2^2, \left\| \frac{\partial^{\rho/2} v}{\partial |\varphi|^{\rho/2}} \right\|_2^2 \leq C, \tag{43}$$

for each $t \in [0, T]$ and $\varphi = x, y$.

Proof. We will only prove the case when all the model parameters are positive constants, the case when they are all negative is similar. Let u, w, and v be solutions for the fractional

GDSS. To start with, notice that the mass conservation guarantees that $\|U\|_2^2$ is a non-negative constant. Observe that the previous results imply that the energy is constant throughout time, that is, $E(t) = E(0)$, for each $t \in [0, T]$, where $E(0)$ is clearly a fixed real constant. In light of the positivity of the parameters, it follows that

$$\|u\|_4^4 = \frac{2}{\gamma}\left(\frac{\gamma}{2}\|u\|_4^4\right) \leq \frac{2}{\gamma}E(0), \quad \forall t \in [0, T]. \tag{44}$$

Similarly, we can bound all the other quantities by a constant which depends on the model parameters and $E(0)$. The constant C of the theorem is the maximum of all these individual bounding constants. □

4. Conclusions

In this article, a fractional generalized Davey–Stewartson system was introduced. The system considers the presence of three partial differential equations with nonlinear coupling. One of the differential equations describes the behavior of a complex field while the other two partial differential equations are related to real-valued functions. Meanwhile, the spatial partial derivatives are fractional and understood in the Riesz sense. Six not-necessarily-identical differentiation orders are considered herein, all of them in the interval $(1, 2)$. The present work demonstrates that the system considered has four functional quantities which are preserved throughout time. Those quantities are extensions of the mass, energy, and momentum of the integer-order system that motivates this work. In that sense, the mathematical model investigated in this manuscript is a robust mathematical framework to describe diverse wave phenomena in systems with anomalous diffusion. It is important to point out that this work not only contributes to the development of the theoretical foundations of fractional calculus. Indeed, this work may set a stage for future applications in diverse fields, such as nonlinear optics, particle analysis and fluid dynamics, among others. Evidently, the development of numerical schemes with similar conservation properties is an open area for research in numerical analysis of fractional partial differential equations and scientific computations [38,39].

Before closing this work, it is important to point out that this work still has many avenues of research to be explored. Indeed, as one of the reviewers pointed out, the derivation of exact analytical solutions for the fractional form of the mathematical work investigated in this work is a task that merits attention. Many other works have derived solutions from mathematical models with physical relevance. For example, abundant solutions to the $(3 + 1)$-dimensional Date–Jimbo–Kashiwara–Miwa equation were derived in [40]; some solutions have been provided for the electroosmotic force on micropolar pulsatile bloodstream through aneurysm and stenosis of carotid [41]; optical soliton solutions of perturbing time-fractional nonlinear Schrödinger equations [42]; the entropy generation and curvature effect on peristaltic thrusting of (Cu-Al_2O_3) hybrid nanofluid in resilient channel; and N1-soliton solutions for Schrödinger's equation with competing weakly nonlocal and parabolic law nonlinearities [43]. In the present study, that task is still a task to be accomplished, though its achievement will evidently be the topic of some future work.

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Appendix A. Systems with Long-Range Interactions

This stage is used to describe the relationship between systems with long-range interactions and fractional derivatives of the Riesz type. To that end, we will abbreviate the derivations and the discussions presented in [24,44]. Let us consider a coupled system of ordinary differential equations of the form

$$\frac{du_n(t)}{dt} = I_n + G(u_n(t)), \quad \forall n \in \mathbb{Z}, \ \forall t \in [0, \infty). \tag{A1}$$

Each of these equations describes the global dynamics of an infinite number of particles on a straight line, and we assume that the distance between consecutive particles is equal to $h > 0$. More precisely, the term I_n is an interaction function that depends on all the functions u_n at the time t, for each $n \in \mathbb{Z}$. In the context of this discussion, we will assume that I_n takes on the following explicit form:

$$I_n = \sum_{\substack{m=-\infty \\ m \neq n}}^{\infty} J(n,m)[u_n(t) - u_m(t)], \quad \forall n \in \mathbb{Z}, \ \forall t \in [0, \infty). \tag{A2}$$

Here, J is an interaction function, and we will assume that it satisfies the following properties:

1. If $m, n \in \mathbb{Z}$, then $J(n,m) = J(m-n) = J(n-m)$.
2. $\sum_{n=1}^{\infty} |J(n)|^2 < \infty$.

In the following, we will assume that these two requirements are satisfied by J.

Definition A1. *Assume that $\alpha > 0$. The function J is an α-interaction whenever*

$$J_\alpha(k) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{-ikn} J(n), \quad \forall k \in \mathbb{R}, \tag{A3}$$

is such that $A_k \in \mathbb{R} \setminus \{0\}$, where

$$A_\alpha = \lim_{k \to 0} \frac{J_\alpha(k) - J_\alpha(0)}{|k|^\alpha}. \tag{A4}$$

The following result by Tarasov establishes that certain interaction functions J lead to Riesz fractional derivatives in the continuous limit. This justifies the use of Riesz fractional derivatives in mathematical models, at least from a thermodynamical point of view [25].

Theorem A1 (Tarasov [24])**.** *Let J be an α-interaction, for some $\alpha > 0$. Let $\mathcal{F}^{-1}U : \widetilde{u}(k,t) \to u(x,t)$ represent the inverse Fourier transform, let $\mathcal{L} : \hat{u}(k,t) \to \tilde{u}(k,t)$ be the limit when the inter-particle distance tends to zero, let $\mathcal{F}_h : u_n(t) \to \hat{u}(k,t)$ be the Fourier series transform, and set $\mathcal{T} = \mathcal{F}^{-1} \circ \mathcal{L} \circ \mathcal{F}_h$. Then, $\mathcal{T}$ transforms the system of ordinary differential Equation (A1) into the Riesz fractional partial differential equation*

$$\frac{\partial u(x,t)}{\partial t} - h^{\alpha} A_{\alpha} \frac{\partial^{\alpha} u(x,t)}{\partial |x|^{\alpha}} - G(u(x,t)) = 0, \quad \forall (x,t) \in \mathbb{R} \times [0,\infty). \tag{A5}$$

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