*fractal and fractional*

*Article*

# Existence and Uniqueness of Some Unconventional Fractional Sturm–Liouville Equations

**Leila Gholizadeh Zivlaei and Angelo B. Mingarelli ***

School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada; leilagh@math.carleton.ca
* Correspondence: angelo@math.carleton.ca

**Abstract:** In this paper, we provide existence and uniqueness results for the initial value problems associated with mixed Riemann–Liouville/Caputo differential equations in the real domain. We show that, under appropriate conditions in a fractional order, solutions are always square-integrable on the finite interval under consideration. The results are valid for equations that have sign-indefinite leading terms and measurable coefficients. Existence and uniqueness theorem results are also provided for two-point boundary value problems in a closed interval.

**Keywords:** Riemann–Liouville; Caputo; Sturm–Liouville; fractional; existence; uniqueness

**MSC:** 34B24; 34A12; 26A33

## 1. Introduction

There has been keen interest of late in the area of fractional differential equations that are defined in terms of a combination of a left- and/or right-Riemann and/or Caputo differential operators. The reason for this is that it appears as if that, when the operators are defined appropriately, they may be a complete analog of the Sturm–Liouville theory, which is a fractional theory that generalizes equations of the form

$$(p(x)y')' - q(x)y = 0, \quad x \in [a, b], \tag{1}$$

as well as the eigenvalue problems associated with them such as

$$(p(x)y')' + (\lambda w(x) - q(x))y = 0, \quad x \in [a, b]$$

where, $p, q$, and $w$ are real, or are complex-valued and continuous (although these conditions can be relaxed tremendously (see below and e.g., [1])).

In this paper, we consider the basic existence and uniqueness questions for equations of the form

$$\mathbf{D}_b^\alpha (pD_a^\alpha y)(x) + q(x)\, y(x) = 0, \tag{2}$$

where $0 < \alpha < 1$, $\mathbf{D}_b^\alpha$ is a right-Caputo differential operator and $D_a^\alpha$ is a left-Riemann–Liouville differential operator (see Section 2). The advantage of this formulation is that (2) includes (1) upon taking the limit as $\alpha \to 1$.

The recent results dealing with the existence and uniqueness of solutions of some fractional differential equations (but not including those considered here) can be found in [2]. Equations of the form (2) have been considered previously in recent papers such as [3–5] (and the references therein) under the assumption that these solutions actually exist and are unique in some suitable spaces. In [6], the question of the existence of eigenvalues and an expansion theorem was considered, whereas the variational characterization of the eigenvalues was given in the papers [7,8]. In [9], the new idea of Fuzzy-Graph-Kannan contractions were used to estimate the solutions of fractional equations.

**Citation:** Gholizadeh Zivlaei, L.; Mingarelli, A.B. Existence and Uniqueness of Some Unconventional Fractional Sturm–Liouville Equations. *Fractal Fract.* **2024**, *8*, 148. https://doi.org/10.3390/fractalfract8030148

Academic Editor: Ricardo Almeida

Received: 28 January 2024
Revised: 29 February 2024
Accepted: 29 February 2024
Published: 3 March 2024

Applications of fractional differential equations are now widespread. Among them, we cite some current ones such as [10–13] in a list that is far from exhaustive. We encourage the readers to look at these and the references therein for more insight.

To the best of our knowledge, the question of the actual existence and uniqueness of solutions to initial value problems associated with (2), let alone such problems where $p(x)$ is sign-indefinite, has not yet been considered. This is our main purpose herein.

Indeed, in this paper, we relax the continuity and sign conditions on $p, q$ in (2) to a mere Lebesgue measurability over $[a, b]$, along with other integral conditions. In addition, we show that we retain the existence and uniqueness of continuous (specifically absolutely continuous) solutions over $[a, b]$. This is the main contribution of this paper, i.e., to address the fact that the existence and uniqueness of its solutions in appropriate spaces has been seemingly overlooked by authors who have considered equations of the form (2). In so doing this, we fill the gaps in regarding the presentations of such papers outlined in the references below where solutions are assumed to exist.

Our methods make use of the fixed-point theorem of Banach–Cacciopoli [14,15], (which is sometimes simply called the Banach fixed-point theorem). This latter result is a generalization of the classical sequence of Picard iterations in the study of solutions of differential equations. Its advantage lies in the fact that, in a normed space, the iterates, $T^n$, of the contraction map $T$ itself must satisfy the relation $||T^n x - x_o|| < k^n ||x - x_o||$, where $x_o$ is the fixed point in question (i.e., $Tx_o = x_o$ ) and $k < 1$ is the contraction constant. As a result of this exponential decay in the error as the number of iterations increases, we can obtain excellent approximations to the solutions of (2) themselves. Insofar as there are numerical approximations to the solutions of fractional differential equations, we cite [16,17] among the current ones.

## 2. Preliminaries

For the sake of convenience, we adopt the following notation. In the sequel, Caputo (resp. Riemann–Liouville) derivatives will be denoted by boldface (i.e., upper case) letters, while the ordinary derivative has only superscript in the form of an integer. For the sake of brevity, we shall omit the obvious $\pm$ subscripts in expressions such $I_{a^+}^{1-\alpha} y(x)$, which will be written as $I_a^{1-\alpha} y(x)$, and $D_{b^-}^{\alpha} y(x)$ will be written as $D_b^{\alpha} y(x)$, etc. (this includes expressions involving Caputo derivatives). The following abbreviations will also be used from time to time: $(pD_a^{\alpha} y)(x)$ for $p(x) D_a^{\alpha} y(x)$ if $p$ is continuous but otherwise it has a meaning of its own (as the quantity will still exist even if the coefficients are merely measurable); and $I_b^{\alpha}(qy)(x)$ for $I_b^{\alpha}(q(x)y(x))$. In addition, Caputo derivatives will be written with a bold face $\mathbf{D}$. Thus, $\mathbf{D}_a^{\alpha}$ and $\mathbf{D}_b^{\alpha}$ denote the left- and right-Caputo derivatives, respectively, while $D_a^{\alpha}$ and $D_b^{\alpha}$ will refer to the left- and right-Riemann–Liouville derivatives. Ordinary derivatives of order $n$ and $j$ will be denoted by $D^n$ and $D^j$, respectively, etc.

We recall some of the definitions from fractional calculus and refer the reader to standard texts such as [18–20] for further details.

**Definition 1.** *The left- and the right- Riemann–Liouville fractional integrals $I_a^{\alpha}$ and $I_b^{\alpha}$ of the order $\alpha \in \mathbb{R}^+$ are defined by*

$$I_a^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t \in (a, b], \tag{3}$$

*and*

$$I_b^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(s)}{(s-t)^{1-\alpha}} ds, \quad t \in [a, b), \tag{4}$$

*respectively, where $\Gamma(\alpha)$ is the usual Gamma function and $I_a^0(f) = f$, $I_a^{-n}(f) = f^{(n)}$ is the ordinary nth derivative of $f$ [21]. The following properties may be found in any textbook on fractional calculus, see e.g., [18,20].*

**Definition 2.** *The left- and the right-Caputo fractional derivatives $\mathbf{D}_a^\alpha$ and $\mathbf{D}_b^\alpha$ are defined by*

$$\mathbf{D}_a^\alpha f(t) := I_a^{1-\alpha} \circ Df(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^\alpha} ds, \quad t > a, \tag{5}$$

*and*

$$\mathbf{D}_b^\alpha f(t) := -I_b^{1-\alpha} \circ Df(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{f'(s)}{(s-t)^\alpha} ds, \quad t < b, \tag{6}$$

*respectively, where $f$ is assumed to be differentiable and that the integrals exist.*

**Definition 3.** *Similarly, the left- and the right-Riemann–Liouville fractional derivatives $D_a^\alpha$ and $D_b^\alpha$ are defined by*

$$D_a^\alpha f(t) := D \circ I_a^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^\alpha} ds, \quad t > a, \tag{7}$$

*and*

$$D_b^\alpha f(t) := -D \circ I_b^{1-\alpha} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^\alpha} ds, \quad t < b, \tag{8}$$

*respectively, where $f$ is assumed to be differentiable and that the integrals exist.*

**Property 1.** *If $y(t) \in L^1[a,b]$ and $I_a^{1-\alpha} y, I_b^{1-\alpha} y \in AC[a,b]$, then*

$$I_a^\alpha D_a^\alpha y(t) = y(t) - \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} I_a^{1-\alpha} y(a),$$
$$I_b^\alpha D_b^\alpha y(t) = y(t) - \frac{(b-t)^{\alpha-1}}{\Gamma(\alpha)} I_b^{1-\alpha} y(b).$$

**Property 2** (See [18], p. 71)**.**

$$D_a^\alpha\Big((x-a)^\beta\Big) = \begin{cases} 0, & \text{if } \ \alpha - \beta - 1 \in \mathbf{N} = \{0, 1, \dots\}, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (x-a)^{\beta-\alpha}, & \text{otherwise.} \end{cases}$$

**Property 3.** *If $y(t) \in AC[a,b]$ and $0 < \alpha \leq 1$, then*

$$I_a^\alpha \mathbf{D}_a^\alpha y(t) = y(t) - y(a),$$
$$I_b^\alpha \mathbf{D}_b^\alpha y(t) = y(t) - y(b).$$

**Property 4** ([20], p. 44, [18], p. 77)**.** *For $0 < \alpha < 1$ and $f \in L^1[a,b]$, we have*

$$D_a^\alpha I_a^\alpha f(t) = f(t), \text{ and, } D_b^\alpha I_b^\alpha f(t) = f(t).$$

**Property 5.** *The semi-group property holds, i.e., for any $\alpha > 0, \beta$, we have*

$$I_a^\alpha I_a^\beta f(t) = I_a^{\alpha+\beta} f(t), \qquad D(I^{\alpha+1} f)(t) = I^\alpha f(t),$$

*are the case whenever all quantities are defined.*

**Property 6** ([18], p. 71, Property 2.1)**.** *For $\alpha, \beta > 0$ there holds*

$$I_a^\alpha((t-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1}.$$

## 3. Existence and Uniqueness

In this section, we derive an integral equation that will be used later to prove the existence and uniqueness of solutions to (9) and (10) as follows:

$$\mathbf{D}_b^\alpha(pD_a^\alpha y)(x) + q(x)\,y(x) = 0, \tag{9}$$

which is subject to a set of conditions of the form

$$I_a^{1-\alpha}y(a) = K_1 \text{ and given } (pD_a^\alpha y)(a) = K_2, \tag{10}$$

where the $K_i$ are the constants, either real or complex. This is relevant to the case where $p(x) = 1, q(x) = 0$ on $[a,b]$ was considered, in part, in [3]. The analysis in the remaining pages will show that there are two types of solutions. Specifically, solutions that are continuous in $[a,b]$ if $I_a^{1-\alpha}y(a) = 0$, and are—in actuality—absolutely continuous and so are in $L^2[a,b]$, as well as those solutions that are in $L^2[a,b]$ and are continuous on $(a,b]$, if $I_a^{1-\alpha}y(a) \neq 0$. In either case, the solutions are always in $L^2[a,b]$, and so in $L^1[a,b]$, regardless of the value of the initial condition $I_a^{1-\alpha}y(a)$.

Proceeding formally from (9) and applying $I_b^\alpha$ to both sides (see Property 3), we find

$$(pD_a^\alpha y)(x) - (pD_a^\alpha y)(b) + I_b^\alpha(qy)(x) = 0, \tag{11}$$

i.e.,

$$D_a^\alpha y(x) - \frac{1}{p(x)}(pD_a^\alpha y)(b) + \frac{1}{p(x)}I_b^\alpha(qy)(x) = 0. \tag{12}$$

Now, by applying $I_a^\alpha$ to both sides of (12) and using Property 1 we obtain the general integral equation

$$\begin{aligned} y(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}I_a^{1-\alpha}y(a) + I_a^\alpha\left(\frac{1}{p}\right)(x)(pD_a^\alpha y)(b) \\ -I_a^\alpha\left(\frac{1}{p}I_b^\alpha(qy)\right)(x). \end{aligned} \tag{13}$$

The relationship between $(pD_a^\alpha y)(a)$ and $(pD_a^\alpha y)(b)$ is given by (11), which is evaluated at $x = a$, i.e.,

$$K_2 = (pD_a^\alpha y)(b) - \frac{1}{\Gamma(\alpha)}\int_a^b \frac{q(s)\,y(s)}{(s-a)^{1-\alpha}}ds\,dt.$$

Thus, any solution of the initial value problem (9) and (10) must satisfy the equation

$$\begin{aligned} y(x) = K_1\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} + K_2\,I_a^\alpha\left(\frac{1}{p}\right)(x) + I_b^\alpha(qy)(a)\,I_a^\alpha\left(\frac{1}{p}\right)(x) \\ -I_a^\alpha\left(\frac{1}{p}I_b^\alpha(qy)\right)(x). \end{aligned} \tag{14}$$

When dealing with (14), there will be two separate cases here, namely one where $K_1 = I_a^{1-\alpha}y(a) = 0$, i.e., a homogeneous Dirichlet type condition is set at $x = a$, and the other where $I_a^{1-\alpha}y(a) \neq 0$. Each case leads to different types of solutions (more on this in the following sections).

### *3.1. Solutions in $C[a,b]$*

Let $p, q$ be complex-valued Lebesgue measurable functions on $[a,b]$ and let $0 < \alpha < 1$. Here, we show that continuous solutions exist and are unique under various assumptions.

We will always assume that, for every $\alpha$, $0 < \alpha < 1$, we have

$$c_1 \equiv \sup_{x\in[a,b]} I_a^\alpha\left(\frac{1}{|p|}\right)(x) < \infty, \tag{15}$$

and

$$c_2 \equiv \sup_{x\in[a,b]} I_b^\alpha(|q|)(x) < \infty. \tag{16}$$

Observe that there are *no sign restrictions* on the coefficients $p, q$ other than Lebesgue measurability and the integrability conditions (15) and (16). As a result, we will obtain that solutions of (9) and (10), which are not only continuous, but are also absolutely continuous on $[a, b]$. The condition that $K_1 = 0$ is necessary in order that the solutions be continuous at $x = a$. In the next section, we will review the case where $K_1 \neq 0$.

**Theorem 1.** *Let $p, q$ be complex-valued and satisfy (15) and (16), as well as $|p(x)| < \infty$ a.e. on $[a, b]$. If*

$$2c_1c_2 < 1, \tag{17}$$

*then the initial value problem (9) and (10) with $K_1 = 0$ and $K_2$ is arbitrary, has a unique solution of $y \in AC[a, b]$.*

**Proof.** Consider the complete normed space $(X, ||\cdot||_\infty)$ of the real valued continuous functions that are defined on $[a, b]$. Note that $K_1 = I_a^{1-\alpha}y(a) = 0$ is in force in (14). We can define a map $T$ on $X$ by setting

$$Ty(x) = K_2\, I_a^\alpha\left(\frac{1}{p}\right)(x) + I_b^\alpha(qy)(a)\, I_a^\alpha\left(\frac{1}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x). \tag{18}$$

By (15), the first term in (18) is the integral of an absolutely integrable function and so it is, itself, absolutely continuous. On the other hand, since $y \in X$ and $q$ satisfies (16), the second term is also finite and absolutely continuous. Finally, since $y \in X$ and there holds (16), $|I_b^\alpha(qy)(x)| \leq ||y||_\infty\, c_2$ over $[a, b]$ so that this, combined with (15), shows that the third term is also absolutely continuous in $[a, b]$ and is thus continuous. Therefore, $TX \subset X$.

Next, we show that $T$ is a contraction. Observe that

$$\begin{aligned} |Ty(x) - Tz(x)| &\leq I_b^\alpha(|q(y-z)|)(a) I_a^\alpha\left(\frac{1}{|p|}\right)(x) \\ &\quad + I_a^\alpha\left(\frac{1}{|p|} I_b^\alpha(|q||y-z|)\right)(x) \\ &\equiv A + B. \end{aligned} \tag{19}$$

The first term, $A$, in (19), is estimated using (15) and (16), i.e.,

$$A \leq \sup_{x\in[a,b]} I_b^\alpha(|q|)(x)\, ||y-z||_\infty\, I_a^\alpha\left(\frac{1}{|p|}\right)(x) \leq c_1\, c_2\, ||y-z||_\infty. \tag{20}$$

On the other hand, the second term, $B$, satisfies

$$B \leq ||y-z||_\infty I_a^\alpha\left(\frac{1}{|p|} I_b^\alpha(|q|)\right)(x) \leq c_1\, c_2\, ||y-z||_\infty. \tag{21}$$

Through combining (19) with (20) and (21), we obtain

$$\|Ty - Tz\|_\infty < 2\, c_1\, c_2\, \|y-z\|_\infty, \tag{22}$$

such that $T$ is a contraction on $X$ provided there holds (17). The fixed-point theorem of Banach–Cacciopoli now implies the existence of a unique fixed-point $y \in X$ that satisfies

$$y(x) = K_2\, I_a^\alpha\left(\frac{1}{p}\right)(x) + I_b^\alpha(qy)(a)\, I_a^\alpha\left(\frac{1}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x). \tag{23}$$

As inferred from above, since all integrands appearing in (23) are in $L^1(a,b)$, it follows that, in fact, $y \in AC[a,b]$. Finally, we can observe that both initial conditions in (10) are automatically satisfied (once the various properties in Section 2 are used). $\square$

**Remark 1.** *The condition (17) is not sharp and can be readily verified in the case where $\alpha = 1$ (the theorem is clearly also true in that case). By setting $p \equiv 1$, $q \equiv 1$, and $[a,b] = [0,1]$, we can obtain $c_1 = c_2 = b - a$, such that (17) is violated, yet the classical problem $y'' + y = 0$, $y(a) = 0$, $y'(a) = K_2$ always has a solution that exists and is unique on $[0,1]$. In this example, our theorem only gives the existence and uniqueness of solutions on $[0,b]$, where $b < \sqrt{2}/2$. Closed-form solutions in the case where $\alpha < 1$ are generally difficult to find.*

**Corollary 1.** *Let $p, q \in C[a,b]$ and $p(x) > 0$ on $[a,b]$. If*

$$\frac{2\,(b-a)^{2\alpha}}{\Gamma(\alpha+1)^2}\,||1/p||_\infty\,||q||_\infty < 1, \tag{24}$$

*then the initial value problem (9) and (10) with $K_1 = 0$ and $K_2$ is arbitrary, has a unique solution $y \in AC[a,b]$.*

**Proof.** Note that

$$c_1 \leq ||1/p||_\infty \sup_{x\in[a,b]} I_a^\alpha(1)(x) \leq \frac{1}{\Gamma(\alpha)}||1/p||_\infty \sup_{x\in[a,b]} \frac{(x-a)^\alpha}{\alpha} \leq ||1/p||_\infty \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}.$$

Similarly,

$$c_2 \leq ||q||_\infty \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}.$$

Together, these two inequalities imply (17) on account of (24). The above result then follows. $\square$

**Corollary 2.** *In addition to the conditions on $p, q$ in Theorem 1, let $f$ be measurable, complex-valued, and for every $0 < \alpha < 1$ satisfy*

$$\sup_{x\in[a,b]} I_b^\alpha(|f|)(x) < \infty. \tag{25}$$

*Then, the initial value problem (9) and (10) (with $K_1 = 0$ and $K_2$ being arbitrary) for the forced equation*

$$\mathbf{D}_b^\alpha(pD_a^\alpha y)(x) + q(x)\,y(x) = f(x) \tag{26}$$

*has a unique solution in $AC[a,b]$.*

**Proof.** The map $T$ defined by

$$\begin{aligned} Ty(x) = {} & K_2\, I_a^\alpha\left(\frac{1}{p}\right)(x) + I_b^\alpha(qy)(a)\, I_a^\alpha\left(\frac{1}{p}\right)(x) \\ & - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x) + I_a^\alpha\left(\frac{1}{p} I_b^\alpha(f)\right)(x) \end{aligned} \tag{27}$$

is a contraction on $X$ as it is easily verified by the method of Theorem 1 and $TX \subset X$. The result follows by the contraction mapping principle. $\square$

However, the next result, Theorem 2 below, is classical in the case of ordinary derivatives. It is unusual in the case we consider our differential operators as a composition of left-Riemann–Liouville and right-Caputo derivatives. Thus, initial conditions are normally at either the left- or right-endpoint of the interval under consideration, i.e., not in the interior as they are here. Still, we have a uniqueness result.

**Theorem 2.** *Let $p, q$ satisfy the conditions in Theorem 1. In addition, let $x_0 \in (a, b]$ be*

$$I_a^\alpha\left(\frac{1}{p}\right)(x_0) \neq 0,$$

*as well as assume that (17) is satisfied. Then, the only solution of the initial value problem (9) satisfying*

$$I_a^{1-\alpha} y(x_0) = 0, \quad (pD_a^\alpha y)(x_0) = 0, \tag{28}$$

*that is continuous on $[a, b]$ is the trivial solution.*

**Proof.** From Theorem 1, a solution that is continuous on $[a, b]$ must satisfy $I_a^{1-\alpha} y(a) = 0$. As a result, there holds (18), where $K_2 = (pD_a^\alpha y)(a)$. By substituting the first of (28) and using the semi-group property, i.e., Property 5, we obtain the form

$$y(x) = c\, I_a^\alpha\left(\frac{1}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x), \tag{29}$$

where

$$c = \frac{I_a^1\left(\frac{1}{p} I_b^\alpha(qy)\right)(x_0)}{I_a^\alpha\left(\frac{1}{p}\right)(x_0)}.$$

By applying Property 4 to (29), we obtain $(pD_a^\alpha y)(x) = c - I_b^\alpha(qy)(x)$, such that the second of (28) implies that $c = I_b^\alpha(qy)(x_0)$. Thus, the solution of (9) that satisfies both of (28) must look like the solution of the integral equation

$$y(x) = I_b^\alpha(qy)(x_0) I_a^\alpha\left(\frac{1}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x). \tag{30}$$

We now show that (30) can only have the zero solution as a continuous solution. This, however, is similar to the proof of Theorem 1 above with minor revisions, which we now describe. On the space $(C[a, b], ||\cdot||_\infty)$, we define the map

$$Ty(x) = I_b^\alpha(qy)(x_0) I_a^\alpha\left(\frac{1}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x).$$

As in the proof of Theorem 1, $TX \subset X$, and we also note that

$$Ty(x) - Tz(x) = I_b^\alpha(q(y-z))(x_0) I_a^\alpha\left(\frac{1}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(q(y-z))\right)(x),$$

such that

$$\begin{aligned}
|Ty(x) - Tz(x)| &\leq I_b^\alpha(|q(y-z)|)(x_0) I_a^\alpha\left(\frac{1}{|p|}\right)(x) + I_a^\alpha\left(\frac{1}{|p|} I_b^\alpha(|q(y-z)|\right)(x) \\
&\leq ||y-z||_\infty \left(I_b^\alpha(|q|)(x_0)| I_a^\alpha\left(\frac{1}{|p|}\right)(x) + I_a^\alpha\left(\frac{1}{|p|} I_b^\alpha(|q|)\right)(x)\right) \\
&\leq ||y-z||_\infty \,(c_2\, c_1 + c_2\, c_1) \\
&= 2\, c_1\, c_2\, ||y-z||_\infty.
\end{aligned} \tag{31}$$

Thus, $T$ is a contraction on account of (17). The above result then follows. □

*3.2. Solutions in $L^2[a, b]$*

We now consider the initial value problem for (9) where $K_1 \neq 0$. Of course, in this case, there is a singularity at $x = a$, thus we can only expect continuity on $(a, b]$, but we will show that nevertheless solutions exist and are unique when considered in the Hilbert space, $L^2[a, b]$.

**Theorem 3.** *Let $p, q$ be measurable complex-valued functions on $[a, b]$, $|p(x)| < \infty$ a.e., and let $\frac{1}{2} < \alpha < 1$. Assume further that, for every $\alpha \in (1/2, 1)$, we have*

$$c_4 \equiv \sup_{t\in[a,b]} \int_t^b \frac{q^2(s)}{(s-t)^{2-2\alpha}}\, ds < \infty, \tag{32}$$

$$c_5 \equiv \sup_{x\in[a,b]} I_a^\alpha\left(\frac{1}{|p|}\right)(x) < \infty, \tag{33}$$

*and*

$$\frac{2c_5\sqrt{c_4}\sqrt{b-a}}{\Gamma(\alpha)} < 1. \tag{34}$$

*Then, the initial value problem (9) with*

$$I_a^{1-\alpha}y(a) = K_1 \neq 0 \text{ and } (pD_a^\alpha y)(a) = K_2 \tag{35}$$

*has a unique solution $y \in L^2[a, b]$. In addition, the solutions are locally absolutely continuous.*

**Proof.** Note that since $\frac{1}{2} < \alpha < 1$, the Riemann–Liouville integrals $I_a^\alpha, I_b^\alpha$ of $L^2$-functions exist by the Schwarz inequality; therefore, they are absolutely continuous functions of the variable in question.

On the complete normed vector space, $X = (L^2[a, b], ||\cdot||_2)$, for $K_1 \neq 0$, and where $||\cdot||_2$ is the usual norm, define a map $T$ on $X$ by (see (14))

$$\begin{aligned} Ty(x) = K_1\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} + K_2\, I_a^\alpha\left(\frac{1}{p}\right)(x) + I_b^\alpha(qy)(a)\, I_a^\alpha\left(\frac{1}{p}\right)(x) \\ - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x). \end{aligned} \tag{36}$$

Observe that the first term in (36) is $L^2[a, b]$ since $\alpha > 1/2$. The second term is square-integrable by hypothesis (33), while the third term in (36) is also square-integrable by a combination of (32) and (33). The square integrability of the last term in (36) is a consequence of the hypotheses and the Schwarz inequality. Specifically, for $y \in X$, we have

$$\begin{aligned} \left| I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x)\right| &\leq I_a^\alpha\left(\frac{1}{|p|} I_b^\alpha(|qy|)\right)(x) \\ &= \frac{1}{\Gamma^2(\alpha)} \int_a^x \frac{1/|p(t)|}{(x-t)^{1-\alpha}} \left(\int_t^b \frac{|q(s)|\,|y(s)|}{(s-t)^{1-\alpha}}\, ds\right) dt \\ &\leq \frac{1}{\Gamma^2(\alpha)} \int_a^x \frac{1/|p(t)|}{(x-t)^{1-\alpha}} \left(\int_t^b \frac{|q(s)|^2}{(s-t)^{2-2\alpha}}\, ds\right)^{1/2} \\ &\times \left(\int_t^b |y(s)|^2\, ds\right)^{1/2} dt \\ &\leq \frac{1}{\Gamma(\alpha)}\sqrt{c_4}\, ||y||_2\, I_a^\alpha\left(\frac{1}{|p|}\right)(x) \\ &\leq \frac{c_5}{\Gamma(\alpha)}\sqrt{c_4}\, ||y||_2. \end{aligned} \tag{37}$$

Since the right side of (37) is independent of $x$ and the interval $[a, b]$ is finite, we obtain that the fourth term in (36) is also in $L^2[a, b]$. There follows that $TX \subset X$.

We now show that $T$ is a contraction on $X$. For $y, z \in X$, we have, as before (see (36))

$$\begin{aligned} |Ty(x) - Tz(x)| &\le I_b^\alpha(|q(y-z)|)(a)\, I_a^\alpha\left(\frac{1}{|p|}\right)(x) + I_a^\alpha\left(\frac{1}{|p|} I_b^\alpha(|q(y-z)|)\right)(x) \\ &\equiv A + B. \end{aligned} \tag{38}$$

We estimated $A$ and $B$ separately. (Recall that the norm under consideration is the $L^2[a,b]$-norm.) Thus (see the calculation leading to (37)), we have

$$\begin{aligned} A &= I_a^\alpha\left(\frac{1}{|p|}\right)(x)\, I_b^\alpha(|q||y-z|)(a) \\ &= \left(\frac{1}{\Gamma(\alpha)} \int_a^x \frac{1/|p(s)|}{(x-s)^{1-\alpha}}\, ds\right) \left(\frac{1}{\Gamma(\alpha)} \int_a^b \frac{|q(s)|\,|y(s)-z(s)|}{(s-a)^{1-\alpha}}\, ds\right) \\ &\le \frac{c_5}{\Gamma(\alpha)} \int_a^b \frac{|q(s)|\,|y(s)-z(s)|}{(s-a)^{1-\alpha}}\, ds, \\ &\le \frac{c_5\sqrt{c_4}}{\Gamma(\alpha)}\, ||y-z||_2. \end{aligned} \tag{39}$$

The estimate for $B$ was obtained exactly as in the details leading to (37) with $y$ replaced by $y - z$. Hence,

$$B \le \frac{c_5\sqrt{c_4}}{\Gamma(\alpha)}\, ||y-z||_2. \tag{40}$$

By combining (39) and (40), we obtain

$$|Ty(x) - Tz(x)| \le \frac{2c_5\sqrt{c_4}}{\Gamma(\alpha)}\, ||y-z||_2$$

i.e.,

$$||Ty - Tz||_2 \le \frac{2c_5\sqrt{c_4}\sqrt{b-a}}{\Gamma(\alpha)}\, ||y-z||_2. \tag{41}$$

As such, the result eventually follows from (34) as $T$ is a contraction on $X$. □

**Corollary 3.** *Let $p, q \in C[a,b]$, $p(x) > 0$ for all $x \in [a,b]$, and let $1/2 < \alpha < 1$. If*

$$2\frac{||1/p||_\infty ||q||_\infty}{\Gamma(\alpha)\Gamma(\alpha+1)} \frac{(b-a)^{2\alpha}}{\sqrt{2\alpha-1}} < 1, \tag{42}$$

*then the initial value problem (9) subject to*

$$I_a^{1-\alpha} y(a) = K_1 \ne 0 \text{ and } (pD_a^\alpha y)(a) = K_2, \tag{43}$$

*has a unique solution $y \in L^2[a,b]$. In addition, the solutions are at least absolutely continuous in $(a,b]$.*

**Proof.** This is a straightforward consequence of Theorem 3 once the quantities (32) and (33) are estimated trivially and (34) is applied. □

**Remark 2.** *The constants appearing in both (24), (34), and (42) are not intended to be precise.*

**Theorem 4.** *Let $p, q$ be complex-valued and measurable on $[a,b]$, $|p(x)| < \infty$ a.e. on $[a,b]$, and let $1/p \in L^1[a,b]$. Assume further that, for every $\alpha \in (1/2, 1)$, we have*

$$c_4 \equiv \sup_{t\in[a,b]} \int_t^b \frac{q^2(s)}{(s-t)^{2-2\alpha}}\, ds < \infty, \tag{44}$$

*and, for every $\alpha \in (\frac{1}{2}, 1)$, there holds*

$$c_5 \equiv \sup_{x\in[a,b]} I_a^{\alpha}\left(\frac{1}{|p|}\right)(x) < \infty, \tag{45}$$

*as well as*

$$\kappa < 1$$

*where*

$$\kappa = \frac{2}{\Gamma(\alpha)}\left(\frac{(b-a)^{2\alpha-1}}{2\alpha-1} + \frac{2}{\alpha}(b-a)^{\alpha} + b - a\right)^{1/2} c_5\sqrt{c_4}.$$

*Then, for $\alpha \in (1/2, 1)$ and for $x_0 \in (a, b]$, the only solution of the initial value problem* (9) *that satisfies*

$$I_a^{1-\alpha}y(x_0) = 0, \quad (pD_a^{\alpha}y)(x_0) = 0, \tag{46}$$

*and that is in $L^2[a, b]$, is the (a.e.) trivial solution.*

**Proof.** The case $x_0 = a$ is contained in Corollary 3; as such, we consider $x_0 \in (a, b]$. From (14), we know that every solution of (9) satisfies

$$\begin{aligned} y(x) = K_1 \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} + K_2\, I_a^{\alpha}\left(\frac{1}{p}\right)(x) + I_b^{\alpha}(qy)(a)\, I_a^{\alpha}\left(\frac{1}{p}\right)(x) \\ - I_a^{\alpha}\left(\frac{1}{p} I_b^{\alpha}(qy)\right)(x), \end{aligned} \tag{47}$$

where now $K_1$ and $K_2$ are to be determined such that (46) is satisfied for a given $x_0$. By applying the operator $I_a^{1-\alpha}$ to both sides of (47)—as well as by then using both Properties 5 and 6, and setting everything equal to zero for $x = x_0$—we can obtain the relation

$$\begin{aligned} &I_a^{1-\alpha}y(x_0) \\ &= K_1 + (K_2 + I_b^{\alpha}(qy)(a))\, I_a^{1}\left(\frac{1}{p}\right)(x_0) - I_a^{1}\left(\frac{1}{p} I_b^{\alpha}(qy)\right)(x_0) \\ &= 0. \end{aligned} \tag{48}$$

Next, by applying the operator $D_a^{\alpha}$ to both sides of (47) and using both Properties 2 and 4, we can obtain

$$pD_a^{\alpha}y(x) = K_2 + I_b^{\alpha}(qy)(a) - I_b^{\alpha}(qy)(x).$$

From this, the use of the second condition in (46) gives

$$K_2 = I_b^{\alpha}(qy)(x_0) - I_b^{\alpha}(qy)(a). \tag{49}$$

By substituting (48) and (49) back into (47) and simplifying it, we obtain

$$y(x) = K_1 \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} + I_b^{\alpha}(qy)(x_0)\, I_a^{\alpha}\left(\frac{1}{p}\right)(x) - I_a^{\alpha}\left(\frac{1}{p} I_b^{\alpha}(qy)\right)(x), \tag{50}$$

where

$$K_1 = I_a^{1}\left(\frac{1}{p} I_b^{\alpha}(qy)\right)(x_0) - I_b^{\alpha}(qy)(x_0)\, I_a^{1}\left(\frac{1}{p}\right)(x_0),$$

and $K_1$ is a constant. Thus, (50) represents the form of a solution of (47) that satisfies both conditions (46).

This now allows us to define a map $T$ on $X = L^2[a,b]$ that is endowed with the usual, i.e, the $L^2$-norm by, when $y \in X$,

$$\begin{aligned} Ty(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \left( I_a^1 \left( \frac{1}{p} I_b^\alpha(qy) \right)(x_0) - I_b^\alpha(qy)(x_0)\, I_a^1 \left( \frac{1}{p} \right)(x_0) \right) \\ + I_b^\alpha(qy)(x_0)\, I_a^\alpha \left( \frac{1}{p} \right)(x) - I_a^\alpha \left( \frac{1}{p} I_b^\alpha(qy) \right)(x). \end{aligned} \tag{51}$$

By construction, a fixed point of $T$ will be a solution of (47) that satisfies conditions (46). To this end, we used the contraction mapping principle. For $y \in X$, $\alpha \in (\frac{1}{2}, 1)$—as well as $p, q$ satisfying (44) and (45), and using the proof of Theorem 3—we can now verify that each integral appearing in (51) exists and is finite for all $x \in [a,b]$. As such, we have $TX \subset X$.

Next, we show that $T$ is a contraction. For $y, z \in X$, $x \in [a,b]$, we have

$$\begin{aligned} &Ty(x) - Tz(x) = \\ &\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \left( I_a^1 \left( \frac{1}{p} I_b^\alpha(q(y-z)) \right)(x_0) - I_b^\alpha(q(y-z))(x_0)\, I_a^1 \left( \frac{1}{p} \right)(x_0) \right) \\ &+ I_b^\alpha(q(y-z))(x_0)\, I_a^\alpha \left( \frac{1}{p} \right)(x) - I_a^\alpha \left( \frac{1}{p} I_b^\alpha(q(y-z)) \right)(x), \end{aligned}$$

such that

$$\begin{aligned} &|Ty(x) - Tz(x)| \leq \\ &\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \left( I_a^1 \left( \frac{1}{|p|} I_b^\alpha(|q(y-z)|) \right)(x_0) + I_b^\alpha(|q(y-z)|)(x_0)\, I_a^1 \left( \frac{1}{|p|} \right)(x_0) \right) \\ &+ I_b^\alpha(|q(y-z)|)(x_0)\, I_a^\alpha \left( \frac{1}{|p|} \right)(x) + I_a^\alpha \left( \frac{1}{|p|} I_b^\alpha(|q(y-z|)) \right)(x), \\ &\equiv A + B + C, \end{aligned} \tag{52}$$

where

$$A \equiv \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \left( I_a^1 \left( \frac{1}{|p|} I_b^\alpha(|q(y-z)|) \right)(x_0) + I_b^\alpha(|q(y-z)|)(x_0)\, I_a^1 \left( \frac{1}{|p|} \right)(x_0) \right), \tag{53}$$

$$B \equiv I_b^\alpha(|q(y-z)|)(x_0)\, I_a^\alpha \left( \frac{1}{|p|} \right)(x), \tag{54}$$

and

$$C \equiv I_a^\alpha \left( \frac{1}{|p|} I_b^\alpha(|q(y-z|)) \right)(x). \tag{55}$$

Now, $A = A_1 + A_2$, where

$$A_1 \equiv \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\, I_a^1 \left( \frac{1}{|p|} I_b^\alpha(|q(y-z)|) \right)(x_0)$$

and

$$A_2 \equiv \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\, I_b^\alpha(|q(y-z)|)(x_0)\, I_a^1 \left( \frac{1}{|p|} \right)(x_0).$$

We estimate $A_1$ first using the calculations leading to (37). Thus,

$$\begin{aligned} A_1 &= \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)^2} \int_a^{x_0} \frac{1}{|p(s)|} \int_s^b \frac{|q(y-z)|(t)}{(t-s)^{1-\alpha}}\, dt\, ds \\ &\leq \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\, c_5 \sqrt{c_4} ||y-z||_2. \end{aligned} \tag{56}$$

Similarly,

$$A_2 \le \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\, c_5\sqrt{c_4}||y-z||_2. \tag{57}$$

By combining (56) and (57), we obtain

$$A \le \frac{2(x-a)^{\alpha-1}}{\Gamma(\alpha)}\, c_5\sqrt{c_4}||y-z||_2. \tag{58}$$

The estimate for $B$ is similar to the estimate for $A_2$ but without all the terms involving $\alpha$, i.e.,

$$B \le \frac{c_5\sqrt{c_4}}{\Gamma(\alpha)}||y-z||_2. \tag{59}$$

Finally, $C$ is estimated as in the $B$-term in (38), i.e.,

$$C \le \frac{c_5\sqrt{c_4}}{\Gamma(\alpha)}||y-z||_2. \tag{60}$$

Therefore, (58)–(60) yield

$$|Ty(x) - Tz(x)| \le k(x)\,||y-z||_2 \tag{61}$$

$$k(x) = 2\left(\frac{(x-a)^{\alpha-1}+1}{\Gamma(\alpha)}\right)c_5\sqrt{c_4}.$$

Then, it follows that

$$||Ty - Tz||_2 \le \kappa\,||y-z||_2,$$

where $\kappa = ||k||_2$ is given by

$$\kappa = \frac{2}{\Gamma(\alpha)}\left(\frac{(b-a)^{2\alpha-1}}{2\alpha-1} + \frac{2}{\alpha}(b-a)^{\alpha} + b - a\right)^{1/2} c_5\sqrt{c_4}.$$

Thus, $T$ is a contraction on $X$ provided $\kappa < 1$. The conclusion then follows. □

In the case where $p, q$ are (real-valued) continuous and $p(x) > 0$, a similar though more extensive argument gives a different bound for uniqueness. This is our next result.

**Theorem 5.** *Let $p, q \in C[a,b]$, $p(x) > 0$ for all $x \in [a,b]$, and let $\alpha > 1/2$. Thus, let*

$$c_1\left((b-a)c_2^2 + 2c_2c_3\frac{(b-a)^{\alpha}}{\alpha} + c_3^2\,\frac{(b-a)^{2\alpha-1}}{2\alpha-1}\right)^{1/2} < 1, \tag{62}$$

*where*

$$c_1 = 2\,\frac{||q||_\infty||1/p||_\infty}{\Gamma(\alpha)^2\sqrt{2\alpha-1}},$$

$$c_2 = \frac{(b-a)^{2\alpha-1/2}}{\alpha},$$

*and*

$$c_3 = (b-a)^{\alpha+1/2}.$$

*Then, for $x_0 \in (a,b]$, the only solution of the initial value problem (9) that satisfies*

$$I_a^{1-\alpha}y(x_0) = 0, \quad (pD_a^{\alpha}y)(x_0) = 0, \tag{63}$$

*and which is in $L^2[a,b]$ is the (a.e.) trivial solution.*

**Proof.** The case of $x_0 = a$ is contained in Theorem 3, such that we can consider $x_0 \in (a, b]$. From (14), we know that every solution of (9) satisfies

$$y(x) = K_1 \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} + K_2\, I_a^\alpha \left(\frac{1}{p}\right)(x) + I_b^\alpha(qy)(a)\, I_a^\alpha\left(\frac{1}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x). \tag{64}$$

$$pD_a^\alpha y(x) = K_2 + I_b^\alpha(qy)(a) - I_b^\alpha(qy)(x).$$

By using the proof of Theorem 4, we have

$$\begin{aligned} Ty(x) = {} & \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \left( I_a^1\left(\frac{1}{p} I_b^\alpha(qy)\right)(x_0) - I_b^\alpha(qy)(x_0)\, I_a^1\left(\frac{1}{p}\right)(x_0)\right) \\ & + I_b^\alpha(qy)(x_0)\, I_a^\alpha\left(\frac{1}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x) \end{aligned} \tag{65}$$

and

$$\begin{aligned} & |Ty(x) - Tz(x)| \le \\ & \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \left( I_a^1\left(\frac{1}{p} I_b^\alpha(|q(y-z)|)\right)(x_0) + I_b^\alpha(|q(y-z)|)(x_0)\, I_a^1\left(\frac{1}{p}\right)(x_0)\right) \\ & + I_b^\alpha(|q(y-z)|)(x_0)\, I_a^\alpha\left(\frac{1}{p}\right)(x) + I_a^\alpha\left(\frac{1}{p} I_b^\alpha(|q(y-z|))\right)(x), \\ & \equiv A + B + C, \end{aligned} \tag{66}$$

respectively. We estimated $B$ using the same technique that was used in Theorem 3 and Corollary 3, except that $a$ was replaced by $x_0$ in the latter, thus leading to minor changes in the estimate. This gives

$$\begin{aligned} B & \le \frac{||1/p||_\infty ||q||_\infty (b-a)^\alpha}{\alpha \Gamma(\alpha)^2} \int_{x_0}^b (s - x_0)^{\alpha-1}\, |y(s) - z(s)|\, ds \\ & \le \frac{||1/p||_\infty ||q||_\infty (b-a)^\alpha}{\alpha \Gamma(\alpha)^2} \frac{(b-x_0)^{\alpha-1/2}}{\sqrt{2\alpha-1}}\, ||y-z||_2, \\ & \le \frac{||1/p||_\infty ||q||_\infty (b-a)^{2\alpha-1/2}}{\alpha \Gamma(\alpha)^2 \sqrt{2\alpha-1}}\, ||y-z||_2. \end{aligned} \tag{67}$$

Now, $C$ is estimated as in Theorem 3, i.e.,

$$C \le \frac{||1/p||_\infty ||q||_\infty}{\alpha\, \Gamma(\alpha)^2} \frac{(b-a)^{2\alpha-1/2}}{\sqrt{2\alpha-1}} ||y-z||_2. \tag{68}$$

Finally, $A$ in (66) consists of two terms, and we can write $A = A_1 + A_2$ as before, which is where the associations should be clear. Then, we have

$$\begin{aligned} A_1 & = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\, I_a^1\left(\frac{1}{p} I_b^\alpha(|q(y-z)|)\right)(x_0) \\ & = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)^2} \int_a^{x_0} \frac{1}{p(s)} \int_s^b \frac{|q(y-z)|(t)}{(t-s)^{1-\alpha}}\, dt\, ds \\ & \le \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)^2}\, ||q||_\infty \int_a^{x_0} \frac{1}{p(s)} \int_s^b (t-s)^{\alpha-1}\, |y(t) - z(t)|\, dt\, ds \\ & \le \frac{||1/p||_\infty ||q||_\infty}{\Gamma(\alpha)^2} \frac{(b-a)^{\alpha-1/2}}{\sqrt{2\alpha-1}} (x-a)^{\alpha-1} (x_0 - a)\, ||y-z||_2. \end{aligned} \tag{69}$$

Of course, (69) may be strengthened by a bound that is independent of $x_0$, i.e., one such as

$$A_1 \leq \frac{||1/p||_\infty ||q||_\infty}{\Gamma(\alpha)^2} \frac{(x-a)^{\alpha-1}}{\sqrt{2\alpha-1}} (b-a)^{\alpha+1/2} ||y-z||_2. \tag{70}$$

Similarly,

$$A_2 \leq \frac{||1/p||_\infty ||q||_\infty}{\Gamma(\alpha)^2} \frac{(x-a)^{\alpha-1}}{\sqrt{2\alpha-1}} (b-a)^{\alpha+1/2} ||y-z||_2. \tag{71}$$

By combining (70) and (71), we obtain

$$A \leq 2\frac{||1/p||_\infty ||q||_\infty}{\Gamma(\alpha)^2} \frac{(x-a)^{\alpha-1}}{\sqrt{2\alpha-1}} (b-a)^{\alpha+1/2} ||y-z||_2. \tag{72}$$

Thus, through using (67), (68) and (72) together with (66), we obtained the bound

$$\begin{aligned} &|Ty(x) - Tz(x)| \leq \\ &\frac{2\,||1/p||_\infty ||q||_\infty}{\Gamma(\alpha)^2\sqrt{2\alpha-1}} \left\{ \frac{(b-a)^{2\alpha-1/2}}{\alpha} + (b-a)^{\alpha+1/2}(x-a)^{\alpha-1} \right\} ||y-z||_2, \\ &\equiv c_1 \left\{ c_2 + c_3\,(x-a)^{\alpha-1} \right\} ||y-z||_2, \end{aligned} \tag{73}$$

where the definitions of the various constants $c_1, c_2$, and $c_3$ in (73) should be clear from the display. Using (73), we can now obtain

$$\begin{aligned} &\int_a^b |Ty(x) - Tz(x)|^2 dx \\ &\leq c_1^2 ||y-z||_2^2 \int_a^b \{c_2^2 + 2c_2c_3(x-a)^{\alpha-1} + c_3^2(x-a)^{2\alpha-2}\}\,dx \\ &\leq c_1^2 ||y-z||_2^2 \left( (b-a)c_2^2 + 2c_2c_3\frac{(b-a)^\alpha}{\alpha} + c_3^2\,\frac{(b-a)^{2\alpha-1}}{2\alpha-1} \right) \end{aligned} \tag{74}$$

or

$$||Ty - Tz|| \leq c_1 \left( (b-a)c_2^2 + 2c_2c_3\frac{(b-a)^\alpha}{\alpha} + c_3^2\,\frac{(b-a)^{2\alpha-1}}{2\alpha-1} \right)^{1/2} ||y-z||_2. \tag{75}$$

From (75), we find that $T$ is a contraction on $X$ provided that

$$c_1 \left( (b-a)c_2^2 + 2c_2c_3\frac{(b-a)^\alpha}{\alpha} + c_3^2\,\frac{(b-a)^{2\alpha-1}}{2\alpha-1} \right)^{1/2} < 1. \tag{76}$$

The fixed-point theorem guarantees the existence of a unique fixed point, which—of course—must be the (a.e.) zero solution. $\square$

## 4. Two-Point Boundary Problems

We show that the analysis in the previous sections extends naturally to the study of the so-called *two-point boundary value problems* on an interval $[a, b]$. In other words, the initial conditions are placed at two points (usually the end points a and b of the interval under consideration), and then one seeks solutions to the problem at hand with those conditions imposed. As such, now we consider the problem

$$\mathbf{D}_b^\alpha(pD_a^\alpha y)(x) + q(x)\,y(x) = 0, \tag{77}$$

which is subject to a set of conditions of the form

$$I_a^{1-\alpha}y(a) = K_1 \text{ and given } (pD_a^\alpha y)(b) = K_2, \tag{78}$$

where the $K_i$ are both the given constants, i.e., a Dirichlet-type condition at $x = a$ and a Neumann-type condition at $x = b$. Note that the quantity $(pD_a^\alpha y)(x)$ is now evaluated at $x = b$ in lieu of $x = a$. This change leads to a two-point boundary value problem where solutions of (77) are now sought that satisfy both conditions in (78). The techniques from the previous sections led us to formulate the existence and uniqueness results for the solutions of such two-point boundary value problems, i.e., (77) and (78). As will be noted, the problem in this section is actually a little easier to solve than the initial value problem (9) and (10) that were considered earlier.

As before, we proceed formally from (77), except that we now apply $I_b^\alpha$ to both sides (see Property 3) to find

$$(pD_a^\alpha y)(x) - (pD_a^\alpha y)(b) + I_b^\alpha(qy)(x) = 0, \tag{79}$$

i.e.,

$$D_a^\alpha y(x) - \frac{1}{p(x)}(pD_a^\alpha y)(b) + \frac{1}{p(x)} I_b^\alpha(qy)(x) = 0. \tag{80}$$

This time, by applying $I_a^\alpha$ to both sides of (80) and using Property 1, we can obtain (when compared with (13))

$$\begin{aligned} y(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} I_a^{1-\alpha} y(a) + I_a^\alpha\left(\frac{1}{p}\right)(x)(pD_a^\alpha y)(b) \\ - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(qy)\right)(x). \end{aligned} \tag{81}$$

As before, there are two different cases: the case where $I_a^{1-\alpha} y(a) = 0$, and the one where $I_a^{1-\alpha} y(a) \neq 0$. The conditions leading to the existence and uniqueness of solutions to the problems at hand are identical, however. Once again, we do not assume any sign restrictions on the leading coefficient $p$. The proofs are sketched as they lead to no new methods.

**Theorem 6.** *Let $p, q$ be complex-valued measurable functions on $[a, b]$, $|p(x)| < \infty$ a.e. on $[a, b]$, which also satisfy (15) and (16). If $c_1 c_2 < 1$, then the two-point boundary value problem (77) which is subject to (78) with $K_1 = 0$, and where $K_2$ is arbitrary has a unique solution $y \in AC[a, b]$.*

**Proof.** Once again, we considered the normed space $(X, ||\cdot||_\infty)$ of the real valued continuous functions defined on $[a, b]$. Note that $I_a^{1-\alpha} y(a) = 0$ is in force in (81). We can define a map $T$ on $X$ by setting

$$Ty(x) = I_a^\alpha\left(\frac{K_2}{p}\right)(x) - I_a^\alpha\left(\frac{1}{p} I_b^\alpha(q\,y)\right)(x). \tag{82}$$

Then, any fixed point of $T$ will satisfy both the first and the second of (78). The proof of Theorem 1 shows that all quantities appearing in (82) are continuous such that $TX \subset X$. Next, let $y, z \in X$. Then, we obtain

$$|Ty(x) - Tz(x)| \leq I_a^\alpha\left(\frac{1}{|p|} I_b^\alpha(|q\,(y-z)|)\right)(x).$$

The term on the right above corresponds to the term denoted by $B$ in Theorem 1. Hence, by that discussion, we have $|Ty(x) - Tz(x)| \leq c_1 c_2 ||y - z||_\infty$, from which we can obtain

$$||Ty - Tz||_\infty \leq c_1\, c_2 ||y - z||_\infty.$$

As such, $T$ is a contraction on $X$ if $c_1\, c_2 < 1$. The above result then follows. □

The case of continuous coefficients and $p(x) > 0$ are covered as a special case, as was expected.

**Corollary 4.** *Let $p, q \in C[a, b]$, $p(x) > 0$ for all $x \in [a, b]$. If*

$$\frac{(b-a)^{2\alpha}}{\Gamma(\alpha+1)^2} ||1/p||_\infty ||q||_\infty < 1, \tag{83}$$

*then the two-point boundary value problem (77) that is subject to (78), with $K_1 = 0$ and $K_2$ being arbitrary, has a unique solution $y \in AC[a, b]$.*

**Proof.** Using the definitions, it is easy to show that

$$c_1 \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} ||1/p||_\infty$$

and

$$c_2 \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} ||q||_\infty.$$

Thus, (83) implies that $c_1 c_2 < 1$; thus, the theorem applies and gives the conclusion. □

We will now review the case where $K_1 \neq 0$. It is covered similarly but we also now seek solutions in $L^2[a, b]$.

**Theorem 7.** *Let $p, q$ be complex-valued measurable functions on $[a, b]$, $|p(x)| < \infty$ a.e. on $[a, b]$, which also satisfy (32) and (33). Let $1/2 < \alpha < 1$. If*

$$\frac{c_5 \sqrt{c_4} \sqrt{b-a}}{\Gamma(\alpha)} < 1 \tag{84}$$

*then the two-point boundary value problem (77) that is subject to (78), with $K_1 \neq 0$ and $K_2$ being arbitrary, has a unique solution $y \in L^2[a, b]$.*

**Proof.** Let $X = \left(L^2(a, b), || \cdot ||_2\right)$, and let us define a map $T$ on $X$ by (see (81)). We thus have

$$Ty(x) = \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} K_1 + I_a^\alpha \left(\frac{K_2}{p}\right)(x) - I_a^\alpha \left(\frac{1}{p} I_b^\alpha (qy)\right)(x). \tag{85}$$

$TX \subset X$ is a consequence of the discussion in Theorem 3. Next, we have

$$|Ty(x) - Tz(x)| \leq I_a^\alpha \left(\frac{1}{|p|} I_b^\alpha (|q(y-z)|)\right)(x) \leq \frac{c_5 \sqrt{c_4}}{\Gamma(\alpha)} ||y - z||_2,$$

by the estimate (37). Hence, we have

$$||Ty - Tz||_2 \leq \frac{c_5 \sqrt{c_4} \sqrt{b-a}}{\Gamma(\alpha)} ||y - z||_2,$$

which shows that $T$ is a contraction on $X$ provided that

$$\frac{c_5 \sqrt{c_4} \sqrt{b-a}}{\Gamma(\alpha)} < 1.$$

The result then follows as before. □

**Corollary 5.** *Let $p, q \in C[a,b]$, $p(x) > 0$ for all $x \in [a,b]$, and let $1/2 < \alpha < 1$. If*

$$\frac{k}{\alpha\,\Gamma^2(\alpha)}\,\frac{(b-a)^{2\alpha}}{\sqrt{2\alpha-1}} < 1, \tag{86}$$

*where $k = ||1/p||_\infty ||q||_\infty > 0$, then the two-point boundary value problem (77) that is subject to*

$$I_a^{1-\alpha}y(a) = K_1 \neq 0 \text{ and } (pD_a^\alpha y)(b) = K_2 \tag{87}$$

*has a unique solution $y \in L^2[a,b]$.*

**Proof.** In using the definitions and the continuity assumptions, we obtain

$$c_4 \leq ||q||_\infty^2 \int_t^b (s-t)^{2\alpha-2}\,ds \leq \frac{||q||_\infty^2 (b-a)^{2\alpha-1}}{2\alpha-1}$$

and (see the proof of Corollary 1)

$$c_5 \leq \frac{||1/p||_\infty (b-a)^\alpha}{\Gamma(\alpha+1)}.$$

With these estimates, it is a simple matter to see that (86) implies (84), and that this completes the proof. □

**Remark 3.** *We have shown that, under some mild assumptions, the mixed Riemann–Liouville–Caputo fractional differential equation defined as in (77) and (78) always possesses two types of solutions. Either all the solutions are continuous in $[a,b]$ (if $I_a^{1-\alpha}y(a) = 0$ and $0 < \alpha < 1$ ), or they are continuous in $(a,b]$ but are still in $L^2(a,b)$ (if $I_a^{1-\alpha}y(a) \neq 0$ and $1/2 < \alpha < 1$).*

## 5. Conclusions

In this article, we have stated and proved the existence and uniqueness theorems for fractional differential equations of the form

$$\mathbf{D}_b^\alpha (pD_a^\alpha y)(x) + q(x)\,y(x) = 0,$$

where $0 < \alpha < 1$, $\mathbf{D}_b^\alpha$ is a right-Caputo differential operator and $D_a^\alpha$ is a left-Riemann–Liouville differential operator under very general conditions on the coefficients of $p, q$, which involve measurability and no sign conditions on either $p$ or $q$. The advantage of this formulation is that our equation includes the classical Sturm–Liouville equation upon taking the limit as $\alpha \to 1$. We have shown that the initial value problem, when properly formulated and under suitable conditions on $p, q$, will always have its solutions in $L^2[a,b]$. We have also given conditions under which the two-point boundary problem

$$I_a^{1-\alpha}y(a) = K_1 \text{ and given } (pD_a^\alpha y)(b) = K_2$$

that is associated with the above equation has a unique solution in some suitable spaces depending on whether $K_1$ is or is not zero.

**Author Contributions:** Conceptualization, L.G.Z.; methodology, L.G.Z. and A.B.M.; validation, L.G.Z. and A.B.M.; formal analysis, L.G.Z. and A.B.M.; investigation, L.G.Z. and A.B.M.; writing—original draft preparation, L.G.Z. and A.B.M.; writing—review and editing, L.G.Z. and A.B.M.; supervision, A.B.M.; project administration, A.B.M. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Data Availability Statement:** No new data were created or analyzed in this study. Data sharing is not applicable to this article.

**Acknowledgments:** The authors should like to thank Tahereh Houlari (formerly of Carleton University) for their useful conversations and helpful remarks. The authors are grateful to the referees for their useful comments that have led to improvements in this paper.

**Conflicts of Interest:** The authors declare no conflicts of interest.

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