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A Temporal Second-Order Difference Scheme for Variable-Order-Time Fractional-Sub-Diffusion Equations of the Fourth Order

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Abstract: In this article, we develop a compact finite difference scheme for a variable-order-time fractional-sub-diffusion equation of a fourth-order derivative term via order reduction. The proposed scheme exhibits fourth-order convergence in space and second-order convergence in time. Additionally, we provide a detailed proof for the existence and uniqueness, as well as the stability of scheme, along with a priori error estimates. Finally, we validate our theoretical results through various numerical computations.

Keywords: variable-order-time fractional-sub-diffusion equation of the fourth-order; compact finite difference scheme; solvable; stability; convergence

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1. Introduction

In recent years, fractional calculus has become a pivotal mathematical tool in various scientific and engineering disciplines. It is particularly noteworthy for its ability to effectively capture the historical memory and holistic relevance of intricate dynamic systems, phenomena, or structures. However, contemporary studies increasingly reveal that these systems' memory or non-local characteristics can evolve over time, space, or under varying conditions [1,2]. Constant-order fractional calculus is not very effective in describing such changes. In contrast, variable-order fractional calculus offers a more nuanced approach to capturing the memory and hereditary properties inherent in many physical phenomena and processes. Therefore, variable-order fractional calculus is a sensible and useful option for accurately describing complex biological systems and processes [3]. Since then, variable-order fractional differential equations have become better known for their ability to simulate various phenomena [4–9].

Due to the intricate nature and analytical difficulty of equations involving variable-order derivatives, developing effective numerical methods stands as a major challenge. Chen et al. [10] investigated a variable-order anomalous subdiffusion equation and developed a numerical scheme characterized by first-order temporal and fourth-order spatial accuracy. Concurrently, they employed Fourier analysis techniques to rigorously analyze their numerical scheme's convergence, stability, and solvability. In [11], Zhao et al. crafted two second-order approximation formulas specifically for the variable-order Caputo fractional-time derivatives. They also provided a comprehensive error analysis to support their formulations. Shivanian [12] introduced a meshless local radial point interpolation method specifically designed for solving two-dimensional fractional-time convection–diffusion–reaction equations. Liu et al. [13] developed optimal piecewise-linear and piecewise-quadratic finite element methods for solving space-time fractional diffusion equations, which have a $2-\gamma$ order temporal accuracy. Liu et al. [14] explored a Galerkin mixed finite element method combined with a time second-order discrete scheme, but its spatial accuracy is less than second-order. Zhao et al. [15] have developed an implicit scheme tailored for time-space fractional diffusion equations. They demonstrate the scheme's convergence in the L2-norm, achieving an order of $\mathcal{O}(\tau^2 + h^2)$. El-Sayed and Agarwal [16] employed shifted Legendre polynomials to construct the numerical solution for multiterm variable-order fractional differential equations. Hajipour et al. [17] described a precise discretization method that can be used to solve variable-order fractional reaction–diffusion problems. Their scheme is characterized by a third-order accuracy in time. However, the equations do not involve a fourth-order derivative term. Du et al. [18] described two discrete difference methods that can solve multidimensional variable-order time fractional-sub-diffusion equations. These methods have second-order accuracy in time and second-order and fourth-order accuracy in space, respectively. Gu et al. [19] put forward an implicit finite difference scheme for a time-fractional diffusion equation with a time-invariant type variable fractional order. This scheme achieves an accuracy order of $\mathcal{O}(\tau + h^2)$. Garrappa et al. [20] contextualized Scarps's concepts within the modern framework of General Fractional Derivatives and Integrals, predominantly based on the Sonine condition. They explore the fundamental characteristics of the resulting variable-order operators. For mobile–immobile variable-order time-fractional diffusion equations, Zhang et al. [21] developed a robust fast method, while Sun et al. [22] introduced a fast and memory-efficient numerical scheme. Both methods are of accuracy-order $\mathcal{O}(\tau + h^2)$. Xu et al. [23] crafted an improved backward substitute method to model variable-order time-fractional advection–diffusion–reaction equation. However, within numerical research focusing on variable-order fractional partial differential equations, studies that attain a second-order temporal accuracy while incorporating higher-order derivative terms remain relatively scarce.

Inspired by this, this article aims to present a high-order, stable numerical scheme for fourth-order variable-order-time fractional-sub-diffusion equations as follows:

$$\begin{cases} {}_0^C D_t^{\alpha(t)} u(x,t) + u_{xxxx}(x,t) + qu(x,t) = f(x,t), & x \in (0,\ L),\ t \in (0,\ T], \\ u(0,t) = 0,\ \ u(L,t) = 0, & t \in [0,\ T], \\ u_{xx}(0,t) = 0,\ u_{xx}(L,t) = 0, & t \in [0,\ T], \\ u(x,0) = \varphi(x), & x \in [0,\ L], \end{cases} \tag{1}$$

where $f(x,t), \varphi(x)$ are given sufficiently smooth functions; q is a positive constant. The expression ${}_0^C D_t^{\alpha(t)} w(t)$ [16] represents the $\alpha(t)$-order time-fractional Caputo derivative of the function $w(t)$, and its definition is

$${}_0^C D_t^{\alpha(t)} w(t) = \begin{cases} w(t) - w(0), \quad \alpha(t) = 0, \\ \dfrac{1}{\Gamma(1-\alpha(t))} \displaystyle\int_0^t w'(s)(t-s)^{-\alpha(t)} ds, \ \ 0 < \alpha(t) < 1, \\ w'(t), \ \ \alpha(t) = 1. \end{cases} \tag{2}$$

The fourth-order fractional diffusion Equation (1) has extensive applications across a diverse array of scientific disciplines, such as wave propagation in complex media, anomalous diffusion, and heat conduction in materials with memory [4]. The intrinsic challenges in numerically solving these equations stem from the non-local properties inherent in fractional derivatives and their overall elevated complexity. To address these challenges, this study adopts an innovative approach. We employ the technique of order reduction to simplify these high-order differential equations into forms that are more amenable to analysis. Furthermore, we approximate the Caputo fractional derivative using a finite difference method, enabling a more manageable and effective computational strategy to tackle these equations.

The structure of the paper is as follows. We present the compact difference scheme, which has temporal second-order precision and spatial fourth-order accuracy, in Section 2, along with various notations. We examine the fully discrete scheme's existence and uniqueness in Section 3. We give the step-by-step convergence and stability analysis in Section 4. We perform some numerical calculations in Section 5 to confirm our theoretical findings. A brief conclusion is included in the next part.

2. The Compact Finite Difference Scheme

This section uses the order reduction method to derive a compact finite difference scheme. Before deriving the difference scheme, we provide some helpful lemmas and notations.

Divide the interval $[0, L]$ into M equal parts and $[0, T]$ into N equal parts. Take the spatial step length $h = \frac{L}{M}$ and the time step length $\tau = \frac{T}{N}$. Let $x_i = ih$ $(0 \leq i \leq M)$ and $t_n = n\tau$ $(0 \leq n \leq N)$; $\Omega_h = \{x_i \mid 0 \leq i \leq M\}$, $\Omega_\tau = \{t_n \mid 0 \leq n \leq N\}$.

Let $\mathcal{V}_h = \{u \mid u = (u_0, u_1, ..., u_M),\ u_0 = u_M = 0\}$ be the grid function on Ω_h. For any grid functions $u,\ v \in \mathcal{V}_h$, we introduce the following symbols:

$$\delta_x u_{i+\frac{1}{2}} = \frac{1}{h}(u_{i+1} - u_i), \qquad \delta_x^2 u_i = \frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}),$$
$$\mathcal{A}u_i = \left(I + \frac{h^2}{12}\delta_x^2\right)u_i = \frac{1}{12}(u_{i-1} + 10u_i + u_{i+1}),$$
$$(u, v) = h\sum_{i=1}^{M-1} u_i v_i, \qquad (\delta_x u, \delta_x v) = h\sum_{i=0}^{M-1}(\delta_x u_{i+\frac{1}{2}})(\delta_x v_{i+\frac{1}{2}}),$$
$$\|u\| = \sqrt{(u, u)}, \qquad |u|_1 = \sqrt{(\delta_x u, \delta_x u)}.$$

Let

$$\sigma_n = \sigma(t_n), \qquad t_{n+\sigma_n} = t_n + \sigma_n\tau, \qquad \alpha_{n+\sigma_n} = \alpha(t_n + \sigma_n).$$

Here, $\sigma(t_n) \in (\frac{1}{2}, 1)$ is the unique root of the equation $\sigma = 1 - \frac{1}{2}\ \alpha(t_n + \sigma\tau)$, $0 \leq n \leq N - 1$ ([18]).

For $u = \{u^k | 0 \leq k \leq N\}$ defined on Ω_τ, we introduce following notation:

$$u^{n+\sigma_n} = \sigma_n u^{n+1} + (1 - \sigma_n)u^n.$$

Lemma 1 ([18])**.** *If $u \in C^3[0, t_{n+1}]$, let $r_n = {}_0^C D_t^{\alpha(t)} u(t)|_{t=t_{n+\sigma_n}} - \mathcal{D}^{\alpha_{n+\sigma_n}} u(t_{n+\sigma_n})$, we have*

$$|r_n| \leq \frac{M\sigma_n^{-\alpha_{n+\sigma_n}}}{\Gamma(1 - \alpha_{n+\sigma_n})}\left[\frac{1}{12} + \frac{\sigma_n}{6(1 - \alpha_{n+\sigma_n})}\right]\tau^{3-\alpha_{n+\sigma_n}},$$

where $M = \max\limits_{0 \leq t \leq t_{n+1}} |u'''(t)|$.

Lemma 2 ([18])**.**

$$\mathcal{D}^{\alpha_{n+\sigma_n}} u(t_{n+\sigma_n}) = \frac{\tau^{-\alpha_{n+\sigma_n}}}{\Gamma(2 - \alpha_{n+\sigma_n})}\sum_{k=0}^{n} c_{n-k}^{(n,\alpha)}[u(t_{k+1}) - u(t_k)]$$
$$= \beta_n \sum_{k=0}^{n} c_k^{(n,\alpha)}[u(t_{n-k+1}) - u(t_{n-k})],$$

where $\beta_n = \frac{\tau^{-\alpha_{n+\sigma_n}}}{\Gamma(2-\alpha_{n+\sigma_n})}$. *When* $n=0$, $c_0^{(n,\alpha)} = \sigma_n^{1-\alpha_{n+\sigma_n}}$; *when* $n \geq 1$,

$$c_l^{(n,\alpha)} = \begin{cases} \frac{1}{2-\alpha_{n+\sigma_n}}\left[(1+\sigma_n)^{2-\alpha_{n+\sigma_n}} - \sigma_n^{2-\alpha_{n+\sigma_n}}\right] - \frac{1}{2}\left[(1+\sigma_n)^{1-\alpha_{n+\sigma_n}} - \sigma_n^{1-\alpha_{n+\sigma_n}}\right], \\ l=0, \\ \frac{1}{2-\alpha_{n+\sigma_n}}\left[(l+\sigma_n+1)^{2-\alpha_{n+\sigma_n}} - 2(l+\sigma_n)^{2-\alpha_{n+\sigma_n}} + (l+\sigma_n-1)^{2-\alpha_{n+\sigma_n}}\right] \\ -\frac{1}{2}\left[(l+\sigma_n+1)^{1-\alpha_{n+\sigma_n}} - 2(l+\sigma_n)^{1-\alpha_{n+\sigma_n}} + (l+\sigma_n-1)^{1-\alpha_{n+\sigma_n}}\right], \\ 1 \leq l \leq n-1, \\ -\frac{1}{2-\alpha_{n+\sigma_n}}\left[(l+\sigma_n)^{2-\alpha_{n+\sigma_n}} - (l+\sigma_n-1)^{2-\alpha_{n+\sigma_n}}\right] \\ +\frac{1}{2}\left[3(l+\sigma_n)^{1-\alpha_{n+\sigma_n}} - (l+\sigma_n-1)^{1-\alpha_{n+\sigma_n}}\right], l=n. \end{cases}$$

Lemma 3 ([24]). *Suppose* $g \in C^6[x_{i-1}, x_{i+1}]$; *then, we have*

$$\frac{1}{12}\left[g''(x_{i-1}) + 10g''(x_i) + g''(x_{i+1})\right] = \frac{1}{h^2}[g(x_{i-1}) - 2g(x_i) + g(x_{i+1})] + \mathcal{O}(h^4).$$

Let

$$v(x) = u_{xx}(x,t);$$

then, we obtain an equivalent form of (1) as follows:

$$\begin{aligned} &{}_0^C D_t^{\alpha(t)} u(x,t) + v_{xx}(x,t) + qu(x,t) = f(x,t), && x \in (0, L), \ t \in (0, T], && (3)\\ &v(x,t) = u_{xx}(x,t), && x \in (0, L), \ t \in (0, T], && (4)\\ &u(0,t) = \alpha_1(t), \ u(L,t) = \alpha_2(t), && t \in [0, T], && (5)\\ &v(0,t) = \gamma_1(t), \ v(L,t) = \gamma_2(t), && t \in [0, T], && (6)\\ &u(x,0) = \varphi(x), && x \in [0, L]. && (7) \end{aligned}$$

Define the grid functions U and V on $\Omega_h \times \Omega_h$:

$$U_i^n = u(x_i, t_n), \ \ V_i^n = v(x_i, t_n), \ \ 0 \leq i \leq M, \ 0 \leq n \leq N.$$

Now, considering Equations (3) and (4) at the points $(x_i, t_{n+\sigma_n})$, we have

$$\begin{aligned} &{}_0^C D_t^{\alpha(t_{n+\sigma_n})} u(x_i, t_{n+\sigma_n}) + v_{xx}(x_i, t_{n+\sigma_n}) + qu(x_i, t_{n+\sigma_n}) = f(x_i, t_{n+\sigma_n}), \\ &\qquad\qquad 1 \leq i \leq M-1, 0 \leq n \leq N-1, \\ &v(x_i, t_{n+\sigma_n}) = u_{xx}(x_i, t_{n+\sigma_n}), \qquad 1 \leq i \leq M-1, 0 \leq n \leq N-1. \end{aligned}$$

Applying the compact operator $\mathcal{A}$ to both sides of the above equations, we obtain

$$\begin{aligned} &\mathcal{A}{}_0^C D_t^{\alpha(t_{n+\sigma_n})} u(x_i, t_{n+\sigma_n}) + \mathcal{A}v_{xx}(x_i, t_{n+\sigma_n}) + q\mathcal{A}u(x_i, t_{n+\sigma_n}) = \mathcal{A}f(x_i, t_{n+\sigma_n}), \\ &\qquad\qquad 1 \leq i \leq M-1, 0 \leq n \leq N-1, && (8)\\ &\mathcal{A}v(x_i, t_{n+\sigma_n}) = \mathcal{A}u_{xx}(x_i, t_{n+\sigma_n}), \qquad 1 \leq i \leq M-1, 0 \leq n \leq N-1. && (9) \end{aligned}$$

Using the Lemmas 1 and 2, we obtain

$$\mathcal{A}{}_0^C D_t^{\alpha(t_{n+\sigma_n})} u(x_i, t_{n+\sigma_n}) = \beta_n \sum_{k=0}^{n} c_k^{(n,\alpha)} \mathcal{A}(U_i^{n-k+1} - U_i^{n-k}) + \mathcal{O}(\tau^{3-\alpha_{n+\sigma_n}}). \tag{10}$$

By the technique in [18] and Lemma 3, we obtain

$$\begin{aligned}\mathcal{A}v_{xx}(x_i,t_{n+\sigma_n}) =&\mathcal{A}(\sigma_n v_{xx}(x_i,t_{n+1})+(1-\sigma_n)v_{xx}(x_i,t_n)+\mathcal{O}(\tau^2))\\ =&\sigma_n\delta_x^2V_i^{n+1}+(1-\sigma_n)\delta_x^2V_i^n+\mathcal{O}(\tau^2+h^4)\\ =&\delta_x^2(\sigma_nV_i^{n+1}+(1-\sigma_n)V_i^n)+\mathcal{O}(\tau^2+h^4)\\ =&\delta_x^2V_i^{n+\sigma_n}+\mathcal{O}(\tau^2+h^4).\end{aligned} \tag{11}$$

Similarly, we have

$$\mathcal{A}u(x_i,t_{n+\sigma_n})=\mathcal{A}U_i^{n+\sigma_n}+\mathcal{O}(\tau^2), \tag{12}$$

$$\mathcal{A}v(x_i,t_{n+\sigma_n})=\mathcal{A}V_i^{n+\sigma_n}+\mathcal{O}(\tau^2), \tag{13}$$

$$\mathcal{A}u_{xx}(x_i,t_{n+\sigma_n})=\delta_x^2U_i^{n+\sigma_n}+\mathcal{O}(\tau^2+h^4). \tag{14}$$

Substituting (10)–(14) into (8) and (9) arrives at

$$\beta_n\sum_{k=0}^{n}c_k^{(n,\alpha)}\mathcal{A}(U_i^{n-k+1}-U_i^{n-k})+\delta_x^2V_i^{n+\sigma_n}+q\mathcal{A}U_i^{n+\sigma_n}=\mathcal{A}f_i^{n+\sigma_n}+R_i^{n+\sigma_n},$$
$$1\le i\le M-1,\ 0\le n\le N-1, \tag{15}$$

$$\mathcal{A}V_i^{n+\sigma_n}=\delta_x^2U_i^{n+\sigma_n}+S_i^{n+\sigma_n},\qquad 1\le i\le M-1,\ 0\le n\le N-1, \tag{16}$$

where there exists a constant c such that

$$|R_i^{n+\sigma_n}|\le c(\tau^2+h^4),\qquad 1\le i\le M-1,\ \ 0\le n\le N-1, \tag{17}$$

$$|S_i^{n+\sigma_n}|\le c(\tau^2+h^4),\qquad 1\le i\le M-1,\ \ 0\le n\le N-1. \tag{18}$$

$$|\delta_xR_i^{n+\sigma_n}|\le c(\tau^2+h^4),\qquad 1\le i\le M-1,\ \ 0\le n\le N-1, \tag{19}$$

$$|\delta_xS_i^{n+\sigma_n}|\le c(\tau^2+h^4),\qquad 1\le i\le M-1,\ \ 0\le n\le N-1. \tag{20}$$

Omitting the small terms $R_i^{n+\sigma_n}$ and $S_i^{n+\sigma_n}$, substituting U_i^n with u_i^n, and noticing boundary conditions

$$U_0^n=0,\ \ U_M^n=0,\qquad 0\le n\le N, \tag{21}$$

$$V_0^n=0,\ \ V_M^n=0,\qquad 0\le n\le N, \tag{22}$$

$$U_i^0=\varphi(x_i),\qquad 1\le i\le M-1, \tag{23}$$

we arrive at the compact finite difference scheme as follows:

$$\beta_n\sum_{k=0}^{n}c_k^{(n,\alpha)}\mathcal{A}(u_i^{n-k+1}-u_i^{n-k})+\delta_x^2v_i^{n+\sigma_n}+q\mathcal{A}u_i^{n+\sigma_n}=\mathcal{A}f_i^{n+\sigma_n},$$
$$1\le i\le M-1,\ \ 0\le n\le N-1, \tag{24}$$

$$\mathcal{A}v_i^{n+\sigma_n}=\delta_x^2u_i^{n+\sigma_n},\qquad 1\le i\le M-1,\ \ 0\le n\le N-1, \tag{25}$$

$$u_0^n=0,\ \ u_M^n=0,\qquad 0\le n\le N, \tag{26}$$

$$v_0^n=0,\ \ v_M^n=0,\qquad 0\le n\le N, \tag{27}$$

$$u_i^0=\varphi(x_i),\qquad 1\le i\le M-1. \tag{28}$$

3. Existence and Uniqueness

The solvability of the difference scheme (24)–(28) is covered in this section.
First, we present some useful lemmas.

Lemma 4 ([25])**.** *For any $\alpha(t)(0 < \alpha(t) < 1)$ and $\{c_l^{(n,\alpha)}(0 \le l \le n, n \ge 1)\}$, it holds that*

$$c_0^{(n,\alpha)} > c_1^{(n,\alpha)} > \cdots > c_{n-1}^{(n,\alpha)} > c_n^{(n,\alpha)} > 0, \tag{29}$$

$$c_n^{(n,\alpha)} > \frac{1-\alpha_{n+\sigma_n}}{2}(n+\sigma_n)^{-\alpha_{n+\sigma_n}} > 0. \tag{30}$$

Lemma 5 ([25])**.** *Let $\mathcal{V}_h$ be an inner product space, and $\langle\cdot,\cdot\rangle_*$ is the inner product with the induced norm $\|\cdot\|_*$. For any grid function $v^n \in \mathcal{V}_h$, $0 \le n \le N$, suppose $\{c_n^{(n,\alpha)}\}$ satisfies* (29)*; we have*

$$\sum_{k=0}^{n} c_k^{(n,\alpha)} \langle v^{n-k+1} - v^{n-k}, \sigma_n v^{n+1} + (1-\sigma_n)v^n\rangle_* \ge \frac{1}{2}\sum_{k=0}^{n} c_k^{(n,\alpha)}(\|v^{n-k+1}\|_*^2 - \|v^{n-k}\|_*^2).$$

Lemma 6 ([24])**.** *For any $u,\ v \in \mathcal{V}_h$, we have*

$$(u,\ \delta_x^2 v) = -(\delta_x u,\ \delta_x v).$$

For any $u,\ v \in \mathcal{V}_h$, we define

$$(u,v)_{\mathcal{A}} = (\mathcal{A}u,\ u), \quad \langle u,v\rangle_{\mathcal{A}} = (\mathcal{A}u,\ -\delta_x^2 u);$$

then, $\langle u,v\rangle_{\mathcal{A}}$, $(u,v)_{\mathcal{A}}$ are the inner product on $\mathcal{V}_h$. So, we denote

$$\|u\|_{\mathcal{A}}^2 = (u,v)_{\mathcal{A}}, \quad |u|_{1,\mathcal{A}}^2 = \langle u,v\rangle_{\mathcal{A}}.$$

Theorem 1. *The difference scheme* (24)–(28) *is uniquely solvable.*

Proof. We use mathematical induction to prove it. By (26)–(28), the value of u^0 and v^0 are determined. If $\{u^k|\ 0 \le k \le n\}$ and $\{v^k|\ 0 \le k \le n\}$ have already been given, then we consider the corresponding homogeneous systems about u^{n+1} and v^{n+1}:

$$\beta_n c_0^{(n,\alpha)} \mathcal{A}u_i^{n+1} + \delta_x^2 v_i^{n+1} + q\mathcal{A}u_i^{n+1} = 0, \quad 1 \le i \le M-1, \tag{31}$$

$$\mathcal{A}v_i^{n+1} = \delta_x^2 u_i^{n+1}, \qquad 1 \le i \le M-1. \tag{32}$$

Making an inner product with u^{n+1} on both sides of (31), we obtain

$$\beta_n c_0^{(n,\alpha)}\|u^{n+1}\|_{\mathcal{A}}^2 + (\delta_x^2 v^{n+1}, u^{n+1}) + q\|u^{n+1}\|_{\mathcal{A}}^2 = 0. \tag{33}$$

Taking an inner product of (32) with v^{n+1}, we obtain

$$\|v^{n+1}\|_{\mathcal{A}}^2 = (\delta_x^2 u^{n+1}, v^{n+1}). \tag{34}$$

Using Lemma 6, we have

$$(\delta_x^2 v^{n+1}, u^{n+1}) = (\delta_x^2 u^{n+1}, v^{n+1}).$$

Combining Equation (33) with Equation (34) arrives at

$$(\beta_n c_0^{(n,\alpha)} + q)\|u^{n+1}\|_{\mathcal{A}}^2 + \|v^{n+1}\|_{\mathcal{A}}^2 = 0.$$

It yields $\|u^{n+1}\|_{\mathcal{A}}^2 = \|v^{n+1}\|_{\mathcal{A}}^2 = 0$, which follows $u^{n+1} = v^{n+1} = 0$. This completes the proof. □

4. Convergence and Stability Analysis

The convergence of the difference scheme (24)–(28) is first explored in this section, and stability is assessed using the Lax Equivalency Theorem.

Lemma 7. *Denote*

$$c_0 = \max_{0\le t\le T}\left[t^{\alpha(t)}\Gamma(1-\alpha(t))\right];$$

we have

$$\frac{1}{c_n^{(n,\alpha)}\beta_n} \le 2c_0.$$

Proof. By Equation (30) in Lemma 4 and the definition of β_n, we have

$$\begin{aligned}\frac{1}{c_n^{(n,\alpha)}\beta_n} <& \frac{2(n+\sigma_n)^{\alpha_{n+\sigma_n}}}{1-\alpha_{n+\sigma_n}}\Gamma(2-\alpha_{n+\sigma_n})\tau^{\alpha_{n+\sigma_n}}\\ =&\Gamma(2-\alpha_{n+\sigma_n})\frac{2(n+\sigma_n)^{\alpha_{n+\sigma_n}}}{1-\alpha_{n+\sigma_n}}\\ =&2t^{\alpha(t)}\Gamma(1-\alpha(t))|_{t_{n+\sigma_n}}\\ \le&2c_0.\end{aligned}$$

□

Lemma 8. *For any $u\in\mathcal{V}_h$, we obtain*

$$\frac{2}{3}|u|_1^2 \le |u|_{1,\mathcal{A}}^2 \le |u|_1^2;$$

it results in the equivalence of the norms $|u|_{1,\mathcal{A}}^2$ and $|u|_1^2$ on $\mathcal{V}_h$.

Proof.

$$|u|_{1,\mathcal{A}}^2 = (\mathcal{A}u,\ -\delta_x^2u) = \left((1+\frac{h^2}{12}\delta_x^2)u, -\delta_x^2u\right) = (u,-\delta_x^2u) - \frac{h^2}{12}(\delta_x^2u,\delta_x^2u).$$

By the inverse estimate $|u|_1^2 \le \frac{4}{h^2}\|u\|^2$, we have $|\delta_xu|_1^2 \le \frac{4}{h^2}\|\delta_xu\|^2$, so

$$\begin{aligned}|u|_{1,\mathcal{A}}^2 =&|u|_1^2 - \frac{h^2}{12}|\delta_xu|_1^2\\ \ge&|u|_1^2 - \frac{1}{3}\|\delta_xu\|^2\\ =&\frac{2}{3}|u|_1^2.\end{aligned}$$

This completes the proof. □

Theorem 2. *Suppose $\{u_i^n \mid 0\le i\le M, 0\le n\le N\}$ and $\{v_i^n \mid 0\le i\le M, 0\le n\le N\}$ are the solutions of the difference scheme below*

$$\beta_n\sum_{k=0}^{n}c_k^{(n,\alpha)}\mathcal{A}(u_i^{n-k+1}-u_i^{n-k}) + \delta_x^2v_i^{n+\sigma_n} + q\mathcal{A}u_i^{n+\sigma_n} = F_i^{n+\sigma_n},$$
$$1\le i\le M-1, 0\le n\le N-1, \qquad (35)$$
$$\mathcal{A}v_i^{n+\sigma_n} = \delta_x^2u_i^{n+\sigma_n} + G_i^{n+\sigma_n}, \qquad 1\le i\le M-1, 0\le n\le N-1, \qquad (36)$$
$$u_0^n = 0,\ u_M^n = 0, \qquad 0\le n\le N, \qquad (37)$$
$$v_0^n = 0,\ v_M^n = 0, \qquad 0\le n\le N, \qquad (38)$$
$$u_i^0 = \varphi(x), \qquad 1\le i\le M-1. \qquad (39)$$

Then, we obtain

$$|u^n|_{1,\mathcal{A}}^2 \le |u^0|_{1,\mathcal{A}}^2 + 2c_0 \max_{0\le s\le n-1}(\frac{3}{4q}\|\delta_x F^{s+\sigma_s}\|^2 + \frac{3}{4}\|\delta_x G^{s+\sigma_s}\|^2), \quad 0\le n\le N. \tag{40}$$

Proof. Taking an inner product of (35) with $-\delta_x^2 u^{n+\sigma_n}$, we obtain

$$\begin{aligned}&(\beta_n\sum_{k=0}^{n} c_k^{(n,\alpha)}\mathcal{A}(u^{n-k+1}-u_i^{n-k}), -\delta_x^2 u^{n+\sigma_n}) + (\delta_x^2 v^{n+\sigma_n}, -\delta_x^2 u^{n+\sigma_n})\\ &+ (q\mathcal{A}u^{n+\sigma_n}, -\delta_x^2 u^{n+\sigma_n}) = (F^{n+\sigma_n}, -\delta_x^2 u^{n+\sigma_n}).\end{aligned} \tag{41}$$

Taking an inner product of (36) with $-\delta_x^2 v^{n+\sigma_n}$, we have

$$(\mathcal{A}v^{n+\sigma_n}, -\delta_x^2 v^{n+\sigma_n}) = (\delta_x^2 u^{n+\sigma_n}, -\delta_x^2 v^{n+\sigma_n}) + (G^{n+\sigma_n}, -\delta_x^2 v^{n+\sigma_n}). \tag{42}$$

Combine Equation (41) with Equation (42), and we obtain

$$\begin{aligned}&(\beta_n\sum_{k=0}^{n} c_k^{(n,\alpha)}\mathcal{A}(u^{n-k+1}-u^{n-k}), -\delta_x^2 u^{n+\sigma_n}) + (q\mathcal{A}u^{n+\sigma_n}, -\delta_x^2 u^{n+\sigma_n})\\ &+ (\mathcal{A}v^{n+\sigma_n}, -\delta_x^2 v^{n+\sigma_n}) = (F^{n+\sigma_n}, -\delta_x^2 u^{n+\sigma_n}) + (G^{n+\sigma_n}, -\delta_x^2 v^{n+\sigma_n}).\end{aligned} \tag{43}$$

For the first item on the left-hand side of (43), using Lemma 5, we have

$$(\beta_n\sum_{k=0}^{n} c_k^{(n,\alpha)}\mathcal{A}(u^{n-k+1}-u_i^{n-k}), -\delta_x^2 u^{n+\sigma_n}) \ge \frac{\beta_n}{2}\sum_{k=0}^{n} c_k^{(n,\alpha)}(|u^{n-k+1}|_{1,\mathcal{A}}^2 - |u^{n-k}|_{1,\mathcal{A}}^2). \tag{44}$$

By Lemma 8, the other items on the left-hand side of (43) arrive at

$$(q\mathcal{A}u^{n+\sigma_n}, -\delta_x^2 u^{n+\sigma_n}) = q|u^{n+\sigma_n}|_{1,\mathcal{A}}^2 \ge \frac{2q}{3}|u^{n+\sigma_n}|_1^2, \tag{45}$$

$$(\mathcal{A}v^{n+\sigma_n}, -\delta_x^2 v^{n+\sigma_n}) = |v^{n+\sigma_n}|_{1,\mathcal{A}}^2 \ge \frac{2}{3}|v^{n+\sigma_n}|_1^2. \tag{46}$$

Substituting (44)–(46) into (43), and using a Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}&\frac{\beta_n}{2}\sum_{k=0}^{n} c_k^{(n,\alpha)}(|u^{n-k+1}|_{1,\mathcal{A}}^2 - |u^{n-k}|_{1,\mathcal{A}}^2) + \frac{2q}{3}|u^{n+\sigma_n}|_1^2 + \frac{2}{3}|v^{n+\sigma_n}|_1^2\\ \le&\|\delta_x F^{n+\sigma_n}\|\|\delta_x u^{n+\sigma_n}\| + \|\delta_x G^{n+\sigma_n}\|\|\delta_x v^{n+\sigma_n}\|\\ \le&\frac{3}{8q}|F^{n+\sigma_n}|_1^2 + \frac{2q}{3}|u^{n+\sigma_n}|_1^2 + \frac{3}{8}|G^{n+\sigma_n}|_1^2 + \frac{2}{3}|v^{n+\sigma_n}|_1^2;\end{aligned}$$

that is,

$$\frac{\beta_n}{2}\sum_{k=0}^{n} c_k^{(n,\alpha)}(|u^{n-k+1}|_{1,\mathcal{A}}^2 - |u^{n-k}|_{1,\mathcal{A}}^2) \le \frac{3}{8q}|F^{n+\sigma_n}|_1^2 + \frac{3}{8}|G^{n+\sigma_n}|_1^2.$$

Transform the above equation and apply Lemma 7; we have

$$\begin{aligned}c_0^{(n,\alpha)}|u^{n+1}|_{1,\mathcal{A}}^2 \le&\sum_{k=0}^{n-1}(c_k^{(n,\alpha)} - c_{k+1}^{(n,\alpha)})|u^{n-k}|_{1,\mathcal{A}}^2 + c_n^{(n,\alpha)}|u^0|_{1,\mathcal{A}}^2 + \frac{1}{\beta_n}(\frac{3}{4q}|F^{n+\sigma_n}|_1^2 + \frac{3}{4}|G^{n+\sigma_n}|_1^2).\\ \le&\sum_{k=0}^{n-1}(c_k^{(n,\alpha)} - c_{k+1}^{(n,\alpha)})|u^{n-k}|_{1,\mathcal{A}}^2 + c_n^{(n,\alpha)}\left[|u^0|_{1,\mathcal{A}}^2 + 2c_0(\frac{3}{4q}|F^{n+\sigma_n}|_1^2 + \frac{3}{4}|G^{n+\sigma_n}|_1^2)\right].\end{aligned}$$

It is easy to know that (40) is true when $n = 0$, so we use mathematical induction to prove (40). Assume (40) is valid for $0 \le n \le l$; now we proved that (40) is valid for $n = l+1$.

$$\begin{aligned}
c_0^{(l,\alpha)}|u^{l+1}|_{1,\mathcal{A}}^2 \le & \sum_{k=0}^{l-1}(c_k^{(l,\alpha)} - c_{k+1}^{(l,\alpha)})|u^{l-k}|_{1,\mathcal{A}}^2 + c_l^{(l,\alpha)}\left[|u^0|_{1,\mathcal{A}}^2 + 2c_0(\frac{3}{4q}|F^{l+\sigma_l}|_1^2 + \frac{3}{4}|G^{l+\sigma_l}|_1^2)\right] \\
\le & \sum_{k=0}^{l-1}(c_k^{(l,\alpha)} - c_{k+1}^{(l,\alpha)})\left(|u^0|_{1,\mathcal{A}}^2 + 2c_0 \max_{0\le s\le l}(\frac{3}{4q}|F^{s+\sigma_s}|_1^2 + \frac{3}{4}|G^{s+\sigma_s}|_1^2)\right) \\
& + c_l^{(l,\alpha)}\left[|u^0|_{1,\mathcal{A}}^2 + 2c_0(\frac{3}{4q}|F^{l+\sigma_l}|_1^2 + \frac{3}{4}|G^{l+\sigma_l}|_1^2)\right] \\
\le & \left[\sum_{k=0}^{l-1}(c_k^{(l,\alpha)} - c_{k+1}^{(l,\alpha)}) + c_l^{(l,\alpha)}\right]\left[|u^0|_{1,\mathcal{A}}^2 + 2c_0 \max_{0\le s\le l}(\frac{3}{4q}|F^{s+\sigma_s}|_1^2 + \frac{3}{4}|G^{s+\sigma_s}|_1^2)\right] \\
\le & c_0^{(l,\alpha)}\left(|u^0|_{1,\mathcal{A}}^2 + 2c_0 \max_{0\le s\le l}(\frac{3}{4q}|F^{s+\sigma_s}|_1^2 + \frac{3}{4}|G^{s+\sigma_s}|_1^2)\right).
\end{aligned}$$

From the above inequality, we have

$$|u^{l+1}|_{1,\mathcal{A}}^2 \le |u^0|_{1,\mathcal{A}}^2 + 2c_0 \max_{0\le s\le l}(\frac{3}{4q}|F^{s+\sigma_s}|_1^2 + \frac{3}{4}|G^{s+\sigma_s}|_1^2).$$

This completes the proof. □

Theorem 3. *Suppose $u(x,t)$, $v(x,t)$ is the solution of (1) and $\{u_i^n,\ v_i^n \mid 0 \le i \le M, 0 \le n \le N\}$ is the solution of the difference scheme (24)–(28). Denote*

$$e_i^n = U_i^n - u_i^n,\ \tilde{e}_i^n = V_i^n - v_i^n, \quad 0 \le i \le M, \quad 0 \le n \le N; \tag{47}$$

then, there exists a constant C such that

$$\|e^n\|_\infty \le C(\tau^2 + h^4), \quad 0 \le n \le N.$$

Proof. Subtracting (24)–(28) from (15)–(16) and (21)–(23), we obtain the system of error equations:

$$\begin{cases}
\beta_n \sum_{k=0}^{n} c_k^{(n,\alpha)}\mathcal{A}(e_i^{n-k+1} - e_i^{n-k}) + \delta_x^2\tilde{e}_i^{n+\sigma_n} + q\mathcal{A}e_i^{n+\sigma_n} = R_i^{n+\sigma_n}, & \\
 & 1 \le i \le M-1, 0 \le n \le N-1, \\
\mathcal{A}\tilde{e}_i^{n+\sigma_n} = \delta_x^2 e_i^{n+\sigma_n} + S_i^{n+\sigma_n}, & 1 \le i \le M-1, 0 \le n \le N-1, \\
e_0^n = 0,\ e_M^n = 0, & 0 \le n \le N, \\
\tilde{e}_0^n = 0,\ \tilde{e}_M^n = 0, & 0 \le n \le N, \\
e_i^0 = 0, & 1 \le i \le M-1.
\end{cases}$$

Applying Theorem 2, we know

$$|e^n|_{1,\mathcal{A}}^2 \le |e^0|_{1,\mathcal{A}}^2 + 2c_0 \max_{0\le s\le n-1}(\frac{3}{4q}\|\delta_x R^{s+\sigma_s}\|^2 + \frac{3}{4}\|\delta_x S^{s+\sigma_s}\|^2), \quad 0 \le n \le N.$$

Noticing (19) and (20), we know

$$|e^n|_{1,\mathcal{A}} \le c_0(\frac{3}{2q} + \frac{3}{2})\sqrt{L}c(\tau^2 + h^4), \quad 0 \le n \le N.$$

According to Lemma 8, we obtain

$$|e^n|_1 \le \sqrt{\frac{3}{2}}|e^n|_{1,\mathcal{A}} \le (\frac{3}{2})^{\frac{3}{2}} c_0(\frac{1}{q}+1)\sqrt{L}c(\tau^2+h^4), \quad 0 \le n \le N.$$

By the inverse estimate $\|u\|_\infty \le \frac{\sqrt{L}}{2}|u|_1$, we obtain

$$\|e^n\|_\infty \le C(\tau^2+h^4), \quad 0 \le n \le N,$$

where

$$C = (\frac{3}{2})^{\frac{3}{2}} c_0(\frac{1}{q}+1)\frac{L}{2}c.$$

It completes the proof. □

Theorem 4 (The Lax Equivalence Theorem [26])**.** *For a consistent finite difference scheme, stability is equivalent to convergence.*

Theorem 5. *In accordance with Theorem 3 and Theorem 4, the solution of the compact finite difference scheme* (24)–(28) *is stable with respect to initial value u^0 and the source term f, and we have*

$$|u^n|_{1,\mathcal{A}}^2 \le |u^0|_{1,\mathcal{A}}^2 + 2c_0 \max_{0\le s\le n-1}(\frac{3}{4q}|\mathcal{A}f^{s+\sigma_s}|_1^2), \quad 0 \le n \le N.$$

5. Numerical Results

In this section, the compact scheme (24)–(28) will be utilized to address problem (1). On a computer with Intel(R) Core(TM) i5-8265U CPU@1.60GHz 1.80 GHz and 8.00GB RAM, we offer two numerical instances to confirm that the theoretical analysis is accurate via Python 3.9.0.

Denote

$$E(h,\tau) = \max_{0\le k\le N}\|U^k - u^k\|_\infty,$$

$$order_\tau = \log_2\frac{E(h,2\tau)}{E(h,\tau)}. \quad order_h = \log_2\frac{E(2h,4\tau)}{E(h,\tau)}.$$

Example 1. *For Problem* (1)*, we consider the initial condition $u^0(x) = \sin x$, and the source term is*

$$f(x,t) = \left(\frac{6t^{3-\alpha(t)}}{\Gamma(4-\alpha(t))} + \frac{6t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + 2(t^3+3t^2+1)\right)\sin x.$$

Take $L = \pi$, $T = 1$, $q = 1$. The exact solution is given by $u(x,t) = (t^3+3t^2+1)\sin x$.

In Table 1, we take the fixed spatial step size $h = \frac{1}{500\pi}$ and verify temporal step τ from $1/10$ to $/160$. Table 1 displays the maximum error and corresponding convergence order of the compact difference scheme for different values $\alpha(t) = 1 - \frac{1}{2}t^2$, e^{-t}, $\frac{1-2sint}{4}$. The numerical results in Table 1 show that the difference scheme is second-order convergent in time.

The maximum error and accompanying spatial convergence order of the compact difference scheme with varying step sizes are presented in Table 2. According to Table 2, the spatial convergence order of the scheme varies about around order 4. The convergence orders in space and time that have been observed align with the theoretical outcomes found in Theorem 3.

Table 1. The maximum error and convergence order in time for $h = \pi/500$ for Example 1.

$\alpha(t)$	τ	$E(h,\tau)$	$order_\tau$
$1-\frac{1}{2}t^2$	1/10	8.794433×10^{-3}	
	1/20	2.184325×10^{-3}	2.0094
	1/40	5.435953×10^{-4}	2.0066
	1/80	1.354184×10^{-4}	2.0051
	1/160	3.361582×10^{-5}	2.0102
e^{-t}	1/10	6.611430×10^{-3}	
	1/20	1.647638×10^{-3}	2.0046
	1/40	4.103318×10^{-4}	2.0055
	1/80	1.019858×10^{-4}	2.0084
	1/160	2.554354×10^{-5}	1.9973
$\frac{1+2\sin t}{4}$	1/10	9.074528×10^{-3}	
	1/20	2.261408×10^{-3}	2.0046
	1/40	5.626816×10^{-4}	2.0068
	1/80	1.400681×10^{-4}	2.0062
	1/160	3.467035×10^{-5}	2.0144

Table 2. The maximum error and convergence order in space for Example 1.

$\alpha(t)$	h	τ	$E(h,\tau)$	$order_h$
$1-\frac{1}{2}t^2$	$\pi/5$	1/10	6.585513×10^{-3}	
	$\pi/10$	1/40	4.278681×10^{-4}	3.9441
	$\pi/20$	1/160	2.650930×10^{-5}	4.0126
	$\pi/40$	1/640	1.646259×10^{-6}	4.0092
	$\pi/80$	1/2560	1.024713×10^{-7}	4.0059
e^{-t}	$\pi/5$	1/10	4.405910×10^{-3}	
	$\pi/10$	1/40	2.884598×10^{-4}	3.9330
	$\pi/20$	1/160	1.791444×10^{-5}	4.0092
	$\pi/40$	1/640	1.114702×10^{-6}	4.0064
	$\pi/80$	1/2560	6.960226×10^{-8}	4.0014
$\frac{1+2\sin t}{4}$	$\pi/5$	1/10	6.852871×10^{-3}	
	$\pi/10$	1/40	4.471970×10^{-4}	3.9377
	$\pi/20$	1/160	2.768960×10^{-5}	4.0135
	$\pi/40$	1/640	1.718657×10^{-6}	4.0100
	$\pi/80$	1/2560	1.069261×10^{-7}	4.0066

Example 2. *Take $L = \pi$, $T = 1$, $q = 1$. The exact solution is given as*

$$u(x,t) = (t^5 + 4t^3 + 1)\sin x.$$

Based on the exact solution, we know the source term and the initial value

$$f(x,t) = \left(\frac{120t^{5-\alpha(t)}}{\Gamma(6-\alpha(t))} + \frac{24t^{3-\alpha(t)}}{\Gamma(4-\alpha(t))} + 2(t^5 + 4t^3 + 1)\right)\sin x,$$

$$u_0(x) = \sin x.$$

In Tables 3 and 4, we show the error and convergence for various $\alpha(t)$ for Example 2. Additionally, it shows a near consistency between the theoretical and numerical results.

Table 3. The maximum error and convergence order in space for Example 2.

$\alpha(t)$	h	τ	$E(h,\tau)$	$order_h$
$1-\frac{1}{2}t^2$	$\pi/5$	$1/10$	2.409320×10^{-2}	
	$\pi/10$	$1/40$	1.549210×10^{-3}	3.9590
	$\pi/20$	$1/160$	9.445871×10^{-5}	4.0357
	$\pi/40$	$1/640$	5.794117×10^{-6}	4.0270
	$\pi/80$	$1/2560$	3.576473×10^{-7}	4.0180
e^{-t}	$\pi/5$	$1/10$	1.816586×10^{-2}	
	$\pi/10$	$1/40$	1.186513×10^{-3}	3.9364
	$\pi/20$	$1/160$	7.299520×10^{-5}	4.0228
	$\pi/40$	$1/640$	4.512097×10^{-6}	4.0159
	$\pi/80$	$1/2560$	2.803956×10^{-7}	4.0083
$\frac{1+2\sin t}{4}$	$\pi/5$	$1/10$	2.560157×10^{-2}	
	$\pi/10$	$1/40$	1.647386×10^{-3}	3.9580
	$\pi/20$	$1/160$	1.000595×10^{-4}	4.0412
	$\pi/40$	$1/640$	6.117342×10^{-6}	4.0318
	$\pi/80$	$1/2560$	3.767160×10^{-7}	4.0214

Table 4. The maximum error and convergence order in time for $h=\pi/500$ for Example 2.

$\alpha(t)$	τ	$E(h,\tau)$	$order_\tau$
$1-\frac{1}{2}t^2$	$1/10$	2.740388×10^{-2}	
	$1/20$	6.789080×10^{-3}	2.0131
	$1/40$	1.677644×10^{-3}	2.0168
	$1/80$	4.144795×10^{-4}	2.0171
	$1/160$	1.023590×10^{-4}	2.0177
e^{-t}	$1/10$	2.135416×10^{-2}	
	$1/20$	5.336100×10^{-3}	2.0007
	$1/40$	1.325432×10^{-3}	2.0093
	$1/80$	3.285335×10^{-4}	2.0124
	$1/160$	8.168648×10^{-5}	2.0079
$\frac{1+2\sin t}{4}$	$1/10$	2.893380×10^{-2}	
	$1/20$	7.180678×10^{-3}	2.0106
	$1/40$	1.771918×10^{-3}	2.0188
	$1/80$	4.368764×10^{-4}	2.0200
	$1/160$	1.075720×10^{-4}	2.0219

6. Conclusions

In this article, we proposed a novel and efficient compact finite difference scheme for variable-order-time fractional-sub-diffusion equations of the fourth order. The core of our method is to use the reduced-order method to simplify the equation and implement a second-order temporal discretization formula to develop the difference scheme. The robustness of our proposed scheme is thoroughly validated through rigorous theoretical analysis. We have comprehensively proven the solvability, stability, and convergence of the scheme, ensuring that it is effective and reliable for practical applications. Numerical tests further demonstrate the scheme's efficacy. Looking ahead, integrating cutting-edge high-order temporal methodologies, mainly designed for time-fractional equations as in reference [27], into our framework offers a promising avenue for enhancing accuracy.

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