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Article

Finite-Approximate Controllability of ν-Caputo Fractional Systems

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Abstract: This paper introduces a methodology for examining finite-approximate controllability in Hilbert spaces for linear/semilinear ν-Caputo fractional evolution equations. A novel criterion for achieving finite-approximate controllability in linear ν-Caputo fractional evolution equations is established, utilizing resolvent-like operators. Additionally, we identify a control strategy that not only satisfies the approximative controllability property but also ensures exact finite-dimensional controllability. Leveraging the approximative controllability of the corresponding linear ν-Caputo fractional evolution system, we establish sufficient conditions for achieving finite-approximative controllability in the semilinear ν-Caputo fractional evolution equation. These findings extend and build upon recent advancements in this field. The paper also explores applications to ν-Caputo fractional heat equations.

Keywords: controllability; fixed point theorems; ν-caputo fractional system

MSC: 93B05; 34B15; 34B18

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1. Introduction

In recent developments, there has been significant progress in the field of fractional derivatives, particularly in formulations involving derivatives with respect to another function. A noteworthy contribution is presented in [1], where a novel type of fractional derivative, known as the Ψ-Caputo fractional derivative, is introduced. This derivative incorporates an additional function Ψ and is designed to improve the accuracy of objective modeling. Advantages of the proposed Ψ-Caputo model lie in the flexibility to choose both the classical differential operator and the Ψ function. This implies that, based on the selected Ψ function, the classical differential operator can act on the fractional integral operator or vice versa. In a subsequent study [2], the authors explored the uniqueness/existence/stability of mild solutions for Ψ-Caputo fractional infinite dimensional differential systems.

We will highlight a few selected scientific articles that serve as motivating references for the study of approximate and finite-approximate controllability (reachability), recognizing the enormous amount of relevant literature in this article.

- Several approaches have been employed to establish conditions for approximate reachability in infinite dimensional systems. Zhou, in [3,4], utilized the sequential approach to derive conditions sufficient for the approximate reachability in nonfractional semilinear infinite dimensional systems. Mahmudov, as referenced in [5], employed the resolvent approach, initially introduced by Bashirov and Mahmudov in [6] for linear evolution equations.

Later, this method is successfully applied to fractional semilinear evolution systems in the work of Sakthivel et al. in [7]. Thereafter, several researchers, Bora et al. [8], Kavitha et al. [9], Haq et al. [10], Aimene [11], Bedi [12], Matar [13], Ge et al. [14], Grudzka et al. [15], Ke et al. [16], Kumar et al. [17,18], Liu et al. [19], Sakthivel et al. [20], Wang et al. [21], Yan [22], Yang et al. [23], Rykaczewski [24] have used different methods to study approximate controllability for several fractional differential and integro-differential systems.
- Thereafter, several researchers, Vijayakumar et al. [25], Ding et al. [26], Bose et al. [27] studied the approximate reachability for different kind of ν-fractional systems.
- Variational approach initially employed by Zuazua [28,29] reachability of the heat equation has been adapted and extended by Li et al. [30], Mahmudov [31] to explore these concepts in the context of semilinear evolution systems. Subsequently, this method has been widely applied by various researchers to investigate the finite-approximate reachability of various kinds of evolution systems. Notable contributions include the works of Liu [32], Ding et al. [33], Wang et al. [34], Liu and Yanfang [35].

The potential impact of establishing finite-approximative controllability in the context of ν-Caputo fractional evolution equations could be significant in several areas: Control Design and Engineering Applications, Optimization of Control Systems, Robustness in Control, Advance in Fractional Calculus Theory, Cross-disciplinary Impact, and Technological Innovation. The concept of finite-approximate controllability implies a higher level of robustness in controlling systems with fractional dynamics. This robustness can enhance the stability and performance of controlled systems in the presence of uncertainties or disturbances. It opens up avenues for more effective and reliable control strategies in complex systems with fractional dynamics.

The paper addresses the concept of finite-approximative controllability, a more robust form of approximative controllability that incorporates simultaneous approximative controllability and finite-dimensional complete controllability. Notably, there is currently a gap in the literature regarding finite-approximative controllability, specifically for ν-Caputo fractional evolution equations. This study focuses on investigating the finite-approximative controllability of ν-Caputo fractional evolution equations.

Here are the notations used in the paper.

- $[0,d] \subset \mathbb{R}$.
- $(\mathfrak{Y}, \|\cdot\|)$ -Hilbert space.
- $U-$ Hilbert space.
- $C([0,d],\mathfrak{Y})$ is the Banach space of continuous functions with values in $\mathfrak{Y}$.
- $L^2([0,d],U)-$the Hilbert space measurable and of square integrable functions $u : [0,d] \to U$.
- $\nu \in C^1([0,d],\mathbb{R})$ with ν is strictly increasing with $v(0) \geq 0$ and $\nu'(t) > 0$ for every $t \in [0,d]$, $\nu(t,s) := \nu(t) - \nu(s)$.
- $AC^n_v([0,d],\mathbb{R}) := \left\{ g : [0,d] \to \mathbb{R} \mid \left(\frac{1}{v'(t)} \frac{d}{dt} \right)^{n-1} g \in AC([0,d],\mathbb{R}) \right\}$
- ${}^C D^{\alpha,\nu}_{0^+}$ – ν-Caputo fractional derivative of order α,

We investigate the finite-approximative controllability of the ν-Caputo fractional evolution semilinear system.

$$\begin{cases} {}^C D^{\alpha,\nu}_{0^+} v(t) = \mathcal{A}v(t) + \mathcal{B}u(t) + F(t, v(t)), \;\; t \in [0,d], \\ v(0) = v_0, \;\; \frac{1}{2} < \alpha \leq 1, \end{cases} \tag{1}$$

Here, $A : D(A) \subset \mathfrak{Y} \to \mathfrak{Y}$ is a closed linear operator generating C_0-semigroup $S : [0,d] \to L(\mathfrak{Y})$, $u \in L^2([0,d],U)$, $\mathcal{B} : U \to \mathfrak{Y}$ is a continuous linear map, $F : [0,d] \times \mathfrak{Y} \to \mathfrak{Y}$, and $y_0 \in \mathfrak{Y}$.

We define the approximate reachability and finite-approximate reachability notions for the system (1).

Definition 1. *For the system described by Equation (1), the following concepts are defined:*

(a) *The ν-Caputo fractional system (1) is deemed to be approximately controllable over the interval $[0, d]$, if for any $v_0, v_F \in \mathfrak{Y}$, and for any given $\varpi > 0$, there exists a function $u \in L^2([0, d], U)$ s. t. the mild solution v of the ν-Caputo fractional IVP (1) satisfies the conditions $v(0) = v_0$ and $\|v(d) - v_F\| < \varpi$.*

(b) *Denoting by Π_M the orthogonal projection from $\mathfrak{Y}$ onto a finite-dimensional subspace $M \subset \mathfrak{Y}$, the ν-Caputo fractional system (1) is considered finite-approximatively controllable on $[0, d]$, if $\Pi_M v(d) = \Pi_M v_F$ and it is also approximately controllable.*

In this manuscript, we extend a variational method proposed in [36] to study finite-approximate reachability of the linear ν-Caputo fractional evolution equation. Moreover, we study finite-approximate reachability of semilinear ν-Caputo fractional evolution of systems. Here are the main contributions of this manuscript:

- We extend a variational method from [36] to explore the finite-approximative reachability of linear ν-Caputo fractional evolution equations. Our investigation establishes a criterion for the finite-approximative reachability of linear ν-Caputo fractional evolution systems. This condition is articulated in terms of resolvent-like operators, as specifically outlined in Criteria (iv) of Theorem 1. For $\nu(t) = t$, this condition coincides with that of [36].
- Additionally, we derive a closed explicit form of finite-approximative control that satisfies both the finite-dimensional complete reachability and the approximate reachability criterion, as shown in the Formula (8). This control plays a pivotal role in the proofs of Theorems 2 and 3. By utilizing the closed form of the finite-approximative control (8) and the Schauder Fixed Point Theorem, we investigate the finite-approximate reachability of the semilinear ν-Caputo fractional evolution system. This result is novel even for $\nu(t) = t$. It is important to highlight that in [36], we focused on the case $\nu(t) = t$ and employed a linearization method under the assumptions of continuity and uniform boundedness of the Frechet derivative $F'(t, v)$. Moreover, in [36], Theorem 4 assumes compactness and analyticity of $S(t)$, $t > 0$.

Section 2 provides preliminary remarks, setting the foundation for the subsequent discussions.

In Section 3, we present a variety of results related to parameter-dependent characteristics of positive linear compact operators. Our focus is on introducing resolvent-like operators for the linear fractional ν-Caputo evolution equation. In addition, we define essential conditions for finitely approximated controllability, expressed in the context of resolvent-like operators.

Moving on to Section 4, we establish a control operator, denoted as Ξ_{ϖ}, and prove the existence of fixed points within this framework. Following that, we present the principal outcome related to the finite-approximative controllability in the context of semilinear ν-Caputo fractional evolution systems.

Finally, the article concludes with the presentation of two illustrative examples designed to underscore and elucidate our principal findings. We show that the heat equation is not only approximately controllable, but also finite-approximately controllable

2. Preliminaries

For $F \in AC^n_v([0, d], \mathbb{R})$ the ν-Caputo type fractional derivative of order $\alpha > 0$, $\alpha \notin \mathbb{N}$, is defined as follows

$$\begin{aligned} \left({}^C D^{\alpha,\nu}_{0^+} F\right)(t) &:= \frac{1}{W(n-\alpha)} \int_0^t \nu'(s)(\nu(t,s))^{n-\alpha-1} \left(\frac{1}{\nu'(s)} \frac{d}{ds}\right)^n F(s) ds, \\ &= \frac{1}{W(n-\alpha)} \int_0^t (\nu(t,s))^{n-\alpha-1} \left(\frac{d}{d\nu(s)}\right)^n F(s) d\nu(s), \end{aligned}$$

where $n = [\alpha] + 1$, $[\alpha]$ means the integer part of α , $W(\alpha)$ is Gamma function, see for example [37].

We define family $\{\mathfrak{T}_\alpha(\nu(t,s)) : 0 \leq s \leq t\}$, $\{\mathfrak{V}_\alpha(\nu(t,s)) : 0 \leq s \leq t\}$ of operators by

$$\begin{aligned}
\mathfrak{T}_\alpha(\nu(t,s)) &= \int_0^\infty \eta_\alpha(\theta) S((\nu(t,s))^\alpha \theta) d\theta \\
\mathfrak{V}_\alpha(\nu(t,s)) &= \int_0^\infty \alpha\theta\eta_\alpha(\theta) S((\nu(t,s))^\alpha \theta) d\theta, \\
\eta_\alpha(\theta) &= \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)! W(1-n\alpha)} \sin(n\pi\alpha), \quad \theta \in (0,\infty), \\
\eta_\alpha(\theta) &\geq 0, \quad \int_0^\infty \eta_\alpha(\theta) d\theta = 1, \\
\int_0^\infty \theta_\alpha^\zeta \eta(\theta) d\theta &= \frac{W(1+\zeta)}{W(1+\alpha\zeta)}, \quad \zeta \in (-1,\infty).
\end{aligned}$$

Lemma 1. *Operators $\mathfrak{T}_\alpha$, $\mathfrak{V}_\alpha$ satisfies the following properties:*

(i) *For any $0 \leq s \leq t$, $\mathfrak{T}_\alpha(\nu(t,s))$, $\mathfrak{V}_\alpha(\nu(t,s))$ are linear and bounded operators, and*

$$\|\mathfrak{T}_\alpha(\nu(t,s))x\| \leq M_S\|x\|, \ \ \|\mathfrak{V}_\alpha(\nu(t,s))x\| \leq \frac{M_S}{W(\alpha)}\|x\|, \ \ M_S := \sup_{t\geq 0}\|S(t)\|.$$

(ii) $\{\mathfrak{T}_\alpha(\nu(t,s)) : 0 \leq s \leq t\}$, $\{\mathfrak{V}_\alpha(\nu(t,s)) : 0 \leq s \leq t\}$ *are compact, if* $\{S(t) : t > 0\}$ *is compact.*

(iii) $\{\mathfrak{T}_\alpha(\nu(t,s)) : 0 \leq s \leq t\}$, $\{\mathfrak{V}_\alpha(\nu(t,s)) : 0 \leq s \leq t\}$ *are strongly continuous.*

Now, we present the following definition of mild solutions of ν-Caputo fractional semilinear evolution system (1).

Definition 2. *For $u \in L^2([0,d],U)$, a function $v \in C([0,d],\mathfrak{Y})$ is called a mild solution of (1) if*

$$\begin{aligned}
v(t) &= \mathfrak{T}_\alpha(\nu(t,0))v_0 \\
&+ \int_0^t (\nu(t,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,s))[\mathcal{B}u(s) + F(s,v(s))]\nu'(s)ds, \ \ t \in [0,d].
\end{aligned}$$

3. Linear Systems

Here, our focus is on exploring the finite-approximative controllability of a linear ν-Caputo fractional evolution system:

$$\begin{cases} {}^C D_{0^+}^{\alpha,\nu} v(t) = \mathcal{A}v(t) + \mathcal{B}u(t), \quad t \in [0,d], \\ v(0) = v_0. \end{cases} \tag{2}$$

Finite-approximative controllability property was first studied in [28]. This property is characterized not only by the small distance between $y(d)$ and the target y_F, but also by the coincidence of the projections of $y(d)$ and y_F onto M.

The resolvent operator $\left(\varpi I + W_0^d\right)^{-1}$ has proven utility in investigating the approximative controllability characteristics of both linear/semilinear systems, as referenced in [6]. In light of this, we introduce a criterion for finite-approximative controllability applicable to ν-Caputo fractional linear evolution systems (2), which coincide with that of [36] for the case $v(t) = t$. This criterion is expressed in terms of a resolvent-like map $\left(\varpi(I - \Pi_M) + W_0^d\right)^{-1}$. Our analysis establishes the equivalence between approximate controllability over the interval $[0,d]$ and finite-approximate controllability over the same interval for the ν-Caputo fractional linear evolution system (2). Additionally, we derive a closed explicit form for the finite-approximative control in terms of $\left(\varpi(I - \Pi_M) + W_0^d\right)^{-1}$.

Firstly, we mention some properties of resolvent operators.

Lemma 2 ([36])**.** *Assume that $W(\varpi), W : \mathfrak{Y} \to \mathfrak{Y}$, $\varpi > 0$, are linear positive operators.*

(a) *If*
$$\lim_{\varpi \to 0^+} \|W(\varpi)h - Wh\| = 0,\ h \in \mathfrak{Y},$$
then for a sequence $\{\varpi_m > 0\}$ converging to 0 as $m \to \infty$, we have
$$\lim_{m\to\infty} \left\| \varpi_m (\varpi_m I + W(\varpi_m))^{-1} \Pi_M \right\| = 0.$$

(b) *For any $\varpi > 0$ we have $\left\| \varpi(\varpi I + W(\varpi))^{-1} \Pi_M \right\| < 1$.*

Next lemma shows relationship between the resolvent operator $(\varpi I + W)^{-1}$ and the resolvent-like operator $(\varpi(I - \Pi_M) + W)^{-1}$.

Lemma 3 ([36])**.** *If $W : \mathfrak{Y} \to \mathfrak{Y}$ is a nonnegative linear operator, then $\varpi(I - \Pi_M) + W : \mathfrak{Y} \to \mathfrak{Y}$ is invertible and*
$$\left\| (\varpi(I - \Pi_M) + W)^{-1} h \right\| \le \frac{1}{\min(\varpi, \delta)} \|h\|,\ \ h \in \mathfrak{Y}, \tag{3}$$
where $\delta = \min\{\langle \Pi_M W \Pi_M \varphi, \varphi\rangle : \|\Pi_M \varphi\| = 1\}$. In addition, if $W : \mathfrak{Y} \to \mathfrak{Y}$ is a positive linear operator then
$$(\varpi(I - \Pi_M) + W)^{-1} = \left(I - \varpi(\varpi I + W)^{-1}\Pi_M\right)^{-1} (\varpi I + W)^{-1}. \tag{4}$$

Following that, we introduce a criterion that determines finite-approximate controllability for the ν-Caputo fractional evolution Equation (2).

Bounded linear operator (controllability operator) $L_0^d : L^2([0,d], U) \to \mathfrak{Y}$ defined by
$$L_0^d u := \int_0^d (\nu(d,s))^{\alpha-1} \mathfrak{V}_\alpha(\nu(d,s)) \mathcal{B} u(s) \nu'(s) ds;$$
Controllability Grammian is defined by
$$W_0^d := L_0^d \left(L_0^d\right)^* = \int_0^d (\nu(d,s))^{2(\alpha-1)} \mathfrak{V}_\alpha(\nu(d,s)) \mathcal{B}\mathcal{B}^* \mathfrak{V}_\alpha^*(\nu(d,s)) \nu'(s) ds : \mathfrak{Y} \to \mathfrak{Y}. \tag{5}$$

Theorem 1. *The following conditions are equivalent:*

(a) *System (2) is controllable approximately over $[0,d]$;*

(b) *W_0^d is a positive operator, i.e., $\left\langle W_0^d x, x \right\rangle > 0$ for all $0 \neq x \in \mathfrak{Y}$;*

(c) *$\varpi\left(\varpi I + W_0^d\right)^{-1} h \to 0$ as $\varpi \to 0^+$, $h \in \mathfrak{Y}$;*

(d) *$\varpi\left(\varpi(I - \Pi_M) + W_0^d\right)^{-1} \to 0$ as $\varpi \to 0^+$, $h \in \mathfrak{Y}$;*

(e) *System (2) is finite-approximately controllable over $[0,d]$.*

Proof. The implications (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) are well known, see [6]. To show (c) $\iff$ (e), for any $\varpi > 0$, $h \in \mathfrak{Y}$, introduce the functional $\mathfrak{J}_\varpi(\cdot, h) : \mathfrak{Y} \to R$:
$$\mathfrak{J}_\varpi(\phi, h) = \frac{1}{2}\int_0^d \left\| (\nu(d,s))^{\alpha-1} \mathcal{B}^* \mathfrak{V}_\alpha^*(\nu(d,s)) \phi \right\|^2 ds + \frac{\varpi}{2} \langle (I - \Pi_M)\phi, \phi\rangle - \langle \phi, h - \mathfrak{T}_\alpha(\nu(d,0)) v_0 \rangle.$$

$\mathfrak{J}_\varpi(\cdot, h)$ is differentiable,

$$\mathfrak{J}'_{\varpi}(\phi,h) = W_0^d\phi + \varpi(I-\Pi_M)\phi - h + \mathfrak{T}_\alpha(\nu(d\,0))v_0$$

is monotonic strictly and so $J_\varpi(\cdot,h)$ is convex strictly, as W_0^d is positive. Hence, $J_\varpi(\cdot,h)$ has minimum $\phi_{\min}$, which is unique, and calculated as:

$$W_0^d\phi + \varpi(I-\Pi_M)\phi - h + \mathfrak{T}_\alpha(\nu(d,0))v_0 = 0,$$
$$\phi_{\min} = -\Big(\varpi(I-\Pi_M) + W_0^d\Big)^{-1}(\mathfrak{T}_\alpha(\nu(d,0))v_0 - h).$$

Thus for $u_\varpi(s) = (\nu(d,s))^{\alpha-1}\mathcal{B}^*\mathfrak{V}_\alpha^*(\nu(d,s))\phi_{\min}$, we obtain

$$\begin{aligned}
&v_\varpi(d) - h\\
&= \mathfrak{T}_\alpha(\nu(d,0))v_0 + \int_0^d (\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d)-\nu(s))\mathcal{B}u(s)\nu'(s)ds - h\\
&= \mathfrak{T}_\alpha(\nu(d,0))v_0 - h - W_0^d\Big(\varpi(I-\Pi_M) + W_0^d\Big)^{-1}(\mathfrak{T}_\alpha(\nu(d,0))v_0 - h)\\
&= \mathfrak{T}_\alpha(\nu(d,0))v_0 - h - \Big(W_0^d + \varpi(I-\Pi_M) - \varpi(I-\Pi_M)\Big)\\
&\times\Big(\varpi(I-\Pi_M) + W_0^d\Big)^{-1}(\mathfrak{T}_\alpha(\nu(d,0))v_0 - h)\\
&= \varpi(I-\Pi_M)\Big(\varpi(I-\pi_M) + W_0^d\Big)^{-1}(\mathfrak{T}_\alpha(\nu(d,0))v_0 - h).
\end{aligned}\tag{6}$$

Thus

$$\lim_{\varpi\to 0^+}\|v_\varpi(d) - h\| = \lim_{\varpi\to 0^+}\varpi\left\|(I-\Pi_M)\Big(\varpi(I-\Pi_M) + W_0^d\Big)^{-1}(\mathfrak{T}_\alpha(\nu(d,0))y_0 - h)\right\| = 0,$$
$$\Pi_M(v_\varpi(d) - h) = 0,$$

That is, the system given by Equation (2) is approximately controllable within the finite interval $[0,d]$. Therefore, condition (c) implies condition (e). The reverse implication ((e) $\Rightarrow$ (c)) is straightforward, as finite-approximate controllability naturally implies approximate controllability. Now, to establish (c) $\Rightarrow$ (d), assume that for any $h\in\mathfrak{Y}$

$$\lim_{\varpi\to 0^+}\left\|\Big(\varpi I + W_0^d\Big)^{-1}h\right\| = 0.$$

From (4), for any $h\in\mathfrak{Y}$

$$\begin{aligned}
\left\|\varpi\Big(\varpi(I-\Pi_M) + W_0^d\Big)^{-1}h\right\| &\le \left\|\Big(I - \varpi\Big(\varpi I + W_0^d\Big)^{-1}\Pi_M\Big)^{-1}\right\|\left\|\varpi\Big(\varpi I + W_0^d\Big)^{-1}h\right\|\\
&\le \frac{1}{1 - \left\|\varpi\big(\varpi I + W_0^d\big)^{-1}\Pi_M\right\|}\left\|\varpi\Big(\varpi I + W_0^d\Big)^{-1}h\right\|.
\end{aligned}\tag{7}$$

From

$$\begin{aligned}
&\varpi_1\Big(\varpi_1 I + W_0^d\Big)^{-1}\Pi_M - \varpi\Big(\varpi I + W_0^d\Big)^{-1}\Pi_M\\
&= \varpi_1\Big(\varpi_1 I + W_0^d\Big)^{-1}\Big(I + \varpi^{-1}W_0^d - I - \varpi_1^{-1}W_0^d\Big)\varpi\Big(\varpi I + W_0^d\Big)^{-1}\Pi_M\\
&= \varpi_1\Big(\varpi_1 I + W_0^d\Big)^{-1}\Big(\varpi^{-1}W_0^d - \varpi_1^{-1}W_0^d\Big)\varpi\Big(\varpi I + W_0^d\Big)^{-1}\Pi_M\\
&= \Big(\varpi_1 I + W_0^d\Big)^{-1}\Big(\varpi_1 W_0^d - \varpi W_0^d\Big)\Big(\varpi I + W_0^d\Big)^{-1}\Pi_M\\
&= \Big(\varpi_1 I + W_0^d\Big)^{-1}(\varpi_1 - \varpi)W_0^d W\Big(\varpi I + W_0^d\Big)^{-1}\Pi_M,
\end{aligned}$$

it follows that $\varpi\left(\varpi I+W_0^d\right)^{-1}\Pi_M$ is continuous in as a function of ϖ. Really,

$$\left\|\varpi_1\left(\varpi_1 I+W_0^d\right)^{-1}\pi_M-\varpi\left(\varpi I+W_0^d\right)^{-1}\Pi_M\right\|\leq\frac{|\varpi_1-\varpi|}{\varpi_1}\to 0 \quad \text{as} \quad \varpi_1\to\varpi.$$

By (7), property of $\varpi\left(\varpi I+W_0^d\right)^{-1}\Pi_M$ and Lemma 2, we obtain

$$\gamma=\max_{0\leq\varpi\leq 1}\left\|\varpi\left(\varpi I+W_0^d\right)^{-1}\Pi_M\right\|<1,$$

$$\left\|\varpi\left(\varpi(I-\Pi_M)+W_0^d\right)^{-1}h\right\|\leq\frac{1}{1-\gamma}\left\|\varpi\left(\varpi I+W_0^d\right)^{-1}h\right\|.$$

Therefore, $\varpi\left(\varpi(I-\Pi_M)+W_0^d\right)^{-1}h$ converges to zero as $\varpi\to 0^+$. The implication (c) $\Rightarrow$ (e) follows from (6). $\square$

Remark 1. *The control*

$$u_\varpi(s)=(\nu(d)-\nu(s))^{\alpha-1}\mathcal{B}^*\mathfrak{V}_\alpha^*(\nu(d)-\nu(s))\left(\varpi(I-\Pi_M)+W_0^d\right)^{-1}(h-\mathfrak{T}_\alpha(\nu(d,0))y_0) \tag{8}$$

provides approximate controllability and finite-dimensional exact controllability of (2).

4. Semilinear Systems

The Schauder fixed-point theorem is a result in functional analysis that guarantees the existence of fixed points for certain types of mappings. This theorem is named after the Polish mathematician Juliusz Schauder.

Here is a statement of the Schauder FPT:

Let $\mathfrak{Z}$ be a non-empty convex, closed, and bounded subset of a complete normed vector space. Let $\mathcal{T}:\mathfrak{Z}\to\mathfrak{Z}$ be a compact map. Then, $\exists z\in\mathfrak{Z}$ such that $\mathcal{T}(z)=z$.

This theorem has applications in various areas of mathematics, including differential equations, variational inequalities, and nonlinear functional analysis. It provides a powerful tool for establishing the existence of solutions to certain types of equations and problems involving self-maps on Banach spaces.

We introduce the succeeding assumptions:

Hypothesis 1. $\{S(t):t>0\}$ *is compact* C_0*-semigroup,* $M_S:=\sup_{t\geq 0}\|S(t)\|$.

Hypothesis 2. *Function* $F:[0,d]\times\mathfrak{Y}\to\mathfrak{Y}$ *satisfies the Caratheodory conditions:*

(a) $F(s,\cdot):\mathfrak{Y}\to\mathfrak{Y}$ *is continuous for any* $s\in[0,d]$,
(b) $F(\cdot,v):[0,d]\to\mathfrak{Y}$ *is measurable for any* $v\in\mathfrak{Y}$.

Hypothesis 3. $\exists L_F\in L^\infty([0,d]\times\mathfrak{Y})$ *s. t. for all* $y\in\mathfrak{Y}$

$$\|F(s,v)\|\leq L_F(s), \quad a.e., \quad s\in[0,d].$$

By means of the control

$$u_\varpi(t,v)=\mathcal{B}^*\mathfrak{V}_\alpha^*(\nu(d,t))\left(\varpi(I-\pi_M)+W_0^d\right)^{-1}\times\left(h-\mathfrak{V}_\alpha(\nu(d,0))v_0-\int_0^d(\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,s))F(s,v(s))\nu'(s)ds\right)$$

for $\varpi>0$, define the following operator

$$\Xi_\varpi v(t)=\mathfrak{T}_\alpha(\nu(t,0))v_0$$

$$+\int_0^t (\nu(t,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,s))[\mathcal{B}u_\varpi(s,v)+F(s,v(s))]\nu'(s)ds.$$

Theorem 2. *Assuming that (H1)-(H3) hold, the Equation (1) possesses at least one mild solution belonging to* $C([0,d],\mathfrak{Y})$.

Proof. Set

$$B_\rho := \{y\in C([0,d],\mathfrak{Y}) : \|v\|\le\rho\}.$$

Step 1: For an arbitrary $\varpi>0$, there is a $\rho_0=\rho(\varpi)>0$ s. t. $\Xi_\varpi : B_{\rho_0}\to B_{\rho_0}$. Indeed, the following two inequalities

$$\begin{aligned}
\|\Xi_\varpi v(t)\| \le{}& \|\mathfrak{V}_\alpha(\nu(t,0))v_0\| \\
&+\left\|\int_0^t(\nu(t,r))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,r))\mathcal{B}u_\varpi(r,v)\nu'(r)dr\right\| \\
&+\left\|\int_0^t(\nu(t,r))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,r))F(r,v(r))\nu'(r)dr\right\| \\
&\le\frac{M_S}{W(\alpha)}\|v_0\| \\
&+\frac{M_SM_\mathcal{B}}{W(\alpha)}\int_0^t(\nu(t,r))^{\alpha-1}\|u_\varpi(r,v)\|\nu'(r)dr \\
&+\frac{M_SM_\mathcal{B}}{W(\alpha)}\int_0^t(\nu(t,r))^{\alpha-1}\|F(r,v(r))\|\nu'(r)dr
\end{aligned}$$

and

$$\|u_\varpi(s,v)\|\le\frac{1}{\varpi}M_\mathcal{B}M_S\left[\|h\|+M_S\|v_0\|+\frac{M_S}{W(\alpha)}\|L_F\|_{L^\infty}\frac{(\nu(d,0))^\alpha}{\alpha}\right].$$

These imply that for sufficiently large $\rho_0>0$, $\rho_0>0$

$$\|\Xi_\varpi v(t)\|\le\rho_0,\quad y\in B_{\rho_0}.$$

Step 2: Ξ_ϖ is continuous.

Consider a sequence $v_k\in B_\rho$ such that $v_k\to v$ in B_ρ as $k\to\infty$. Given the hypotheses (H2) and (H3), we can express, for every $s\in[0,d]$,

$$F(r,v_k(r))\to F(r,v(r)) \text{ as } k\to\infty,$$

and

$$\|F(r,v_k(r))-F(r,v(r))\|\le 2L_F(r).$$

We have

$$\begin{aligned}
&\|\Xi_\varpi v_k(r)-\Xi_\varpi v(r)\| \\
&\le\left\|\int_0^t(\nu(t,r))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,r))(F(r,v_k(r))-F(r,v(r)))\nu'(r)dr\right\| \\
&+\left\|\int_0^t(\nu(t,r))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,r))\mathcal{B}\mathcal{B}^*\mathfrak{V}_\alpha^*(\nu(d,t))\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}\nu'(r)dr\right. \\
&\times\left.\int_0^d(\nu(d,r))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,r))(F(r,v_k(r))-F(r,v(r)))\nu'(r)dr\right\| \\
&\le\frac{M_S}{W(\alpha)}\int_0^t(\nu(t,r))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,r))\|F(r,v_k(r))-F(r,v(r))\|\nu'(r)dr \\
&+\frac{1}{\varpi}\frac{M_S^2M_\mathcal{B}^2}{W^2(\alpha)}\frac{(\nu(d,0))^\alpha}{\alpha}\int_0^d(\nu(d,r))^{\alpha-1}\|F(r,v_k(r))-F(r,v(r))\|\nu'(r)dr.
\end{aligned}$$

We take limit as $k \to \infty$ and use the Lebesgue dominated convergence theorem, to obtain continuity

$$\lim_{k\to\infty} \|\Xi_{\varpi} v_k(s) - \Xi_{\varpi} v(s)\| = 0.$$

Step 3: Now, we prove $\{\Xi_{\varpi} y(\cdot) : y \in B_{\rho_0}\}$ is an equicontinuous family on $[0, d]$.

First we show that $\{\Xi_{\varpi} y(s) : y \in B_{\rho_0}\}$ is equicontinuous in $\mathfrak{Y}$. For any $y \in B_{\rho_0}$ and $0 \le s_1 \le s_2 \le d$, we have

$$\begin{aligned}
&\|\Xi_{\varpi} v(s_2) - \Xi_{\varpi} v(s_1)\| \\
&\le \|\mathfrak{T}_\alpha(\nu(s_2,0))y_0 - \mathfrak{T}_\alpha(\nu(s_1,0))v_0\| \\
&+ \left\| \int_{s_1}^{s_2} (\nu(s_2,s))^{\alpha-1} \mathfrak{V}_\alpha(\nu(s_2,s)) F(s, v(s)) \nu'(s) ds \right\| \\
&+ \left\| \int_0^{s_1} \left[(\nu(s_2,s))^{\alpha-1} - (\nu(s_1,s))^{\alpha-1} \right] \mathfrak{V}_\alpha(\nu(s_2,s)) F(s, v(s)) \nu'(s) ds \right\| \\
&+ \left\| \int_0^{s_1} (\nu(s_1,s))^{\alpha-1} [\mathfrak{V}_\alpha(\nu(s_2,s)) - \mathfrak{V}_\alpha(\nu(s_1,s))] F(s, v(s)) \nu'(s) ds \right\| \\
&+ \left\| \int_{s_1}^{s_2} (\nu(s_2,s))^{\alpha-1} \mathfrak{V}_\alpha(\nu(s_2,s)) \mathcal{B}u(s,v) \nu'(s) ds \right\| \\
&+ \left\| \int_0^{s_1} \left[(\nu(s_2,s))^{\alpha-1} - (\nu(s_1,s))^{\alpha-1} \right] \mathfrak{V}_\alpha(\nu(s_2,s)) \mathcal{B}u(s,v) \nu'(s) ds \right\| \\
&+ \left\| \int_0^{s_1} (\nu(s_1,s))^{\alpha-1} [\mathfrak{V}_\alpha(\nu(s_2,s)) - \mathfrak{V}_\alpha(\nu(s_1,s))] \mathcal{B}u(s,v) \nu'(s) ds \right\| \\
&=: I_0 + I_1 + I_2 + \ldots + I_6.
\end{aligned}$$

By Lemma 1, we have

$$\begin{aligned}
I_0 &\le \|\mathfrak{T}_\alpha(\nu(s_2,0))y_0 - \mathfrak{T}_\alpha(\nu(s_1,0))y_0\| \\
I_1 &\le \frac{M_S \|L_F\|_\infty}{W(\alpha)} (\nu(s_2,s_1))^{\alpha-1} \\
I_2 &\le \frac{M_S \|L_F\|_\infty}{\alpha W(\alpha)} \int_0^{s_1} \left| (\nu(s_2,s))^{\alpha-1} - (\nu(s_1,s))^{\alpha-1} \right| \nu'(s) ds.
\end{aligned}$$

Therefore, $I_1 \to 0$, $I_2 \to 0$ as $s_2 \to s_1$. Let η be the arbitrary small positive, we write:

$$\begin{aligned}
I_3 \le &\int_0^{s_1-\eta} (\nu(s_1,s))^{\alpha-1} \|\mathfrak{V}_\alpha(\nu(s_2,s)) - \mathfrak{V}_\alpha(\nu(s_1,s))\| \|F(s,y(s))\| \nu'(s) ds \\
&+ \int_{s_1-\eta}^{s_1} (\nu(s_1,s))^{\alpha-1} \|\mathfrak{V}_\alpha(\nu(s_2,s)) - \mathfrak{V}_\alpha(\nu(s_1,s))\| \|F(s,y(s))\| \nu'(s) ds \\
&\le \|L_F\|_\infty \int_0^{s_1-\eta} (\nu(s_1,s))^{\alpha-1} \nu'(s) ds \sup_{0 \le s \le s_1-\eta} \|\mathfrak{V}_\alpha(\nu(s_2,s)) - \mathfrak{V}_\alpha(\nu(s_1,s))\| \\
&+ \frac{2M_S \|L_F\|_\infty}{W(\alpha)} \int_{s_1-\eta}^{s_1} (\nu(s_1,s))^{\alpha-1} \nu'(s) ds.
\end{aligned}$$

It follows that, $I_3 \to 0$ as $s_2 \to s_1$ and $\eta \to 0$. Using the similar procedure, we obtain that I_4, I_5 and I_6 are tend to zero.

Step 4: We show that $\{\Xi_{\varpi} y(\cdot) : y \in B_{\rho_0}\}$ is relatively compact in $\mathfrak{Y}$.

Take $0 \le s \le d$ then for any $\eta > 0$ and $\delta > 0$ define an operator $\Xi_{\varpi,\eta,\delta}$ on B_{ρ_0} as follows

$$\begin{aligned}
&\Xi_{\varpi,\eta,\delta}(s) \\
&= \int_\delta^\infty \eta_\alpha(\theta) S((\nu(s,0))^\alpha \theta) d\theta v_0 \\
&+ \alpha \int_0^{s-\eta} \int_\delta^\infty \theta (\nu(s,r))^{\alpha-1} \eta_\alpha(\theta) S((\nu(s,r))^\alpha \theta) F(r, v(r)) \nu'(r) d\theta dr
\end{aligned}$$

$$
\begin{aligned}
&+\alpha\int_0^{s-\eta}\int_\delta^\infty \theta(\nu(s,r))^{\alpha-1}\eta_\alpha(\theta)S((\nu(s,r))^\alpha\theta)\mathcal{B}u(r,v)\nu'(r)d\theta dr\\
&=S(\eta^\alpha\delta)\int_\delta^\infty \eta_\alpha(\theta)S((\nu(s,0))^\alpha\theta-\eta^\alpha\delta)d\theta y_0\\
&+\alpha S(\eta^\alpha\delta)\int_0^{s-\eta}\int_\delta^\infty \theta(\nu(s,r))^{\alpha-1}\eta_\alpha(\theta)S((\nu(s,r))^\alpha\theta-\eta^\alpha\delta)F(r,v(r))\nu'(r)d\theta dr\\
&+\alpha S(\eta^\alpha\delta)\int_0^{s-\eta}\int_\delta^\infty \theta(\nu(s,r))^{\alpha-1}\eta_\alpha(\theta)S((\nu(s,r))^\alpha\theta-\eta^\alpha\delta)\mathcal{B}u(r,v)\nu'(r)d\theta dr.
\end{aligned}
$$

Now by compactness of $S(\eta^\alpha\delta)$, $\eta^\alpha\delta>0$, we have relatively compactness of

$$\{\Xi v(s) : v\in B_{\rho_0}\}$$

in $\mathfrak{Y}$. Moreover, for any $y\in B_{\rho_0}$ we obtain

$$
\begin{aligned}
&\left\|\Xi_\varpi v(s)-\Xi_{\varpi,\eta,\delta}v(s)\right\|\\
&\le \int_0^\delta \eta_\alpha(\theta)S((\nu(s,0))^\alpha\theta)d\theta v_0\\
&+\alpha\left\|\int_0^s\int_0^\delta \theta(\nu(s,r))^{\alpha-1}\eta_\alpha(\theta)S((\nu(s,r))^\alpha\theta)[F(r,v(r))+\mathcal{B}u(r,v)]\nu'(r)d\theta dr\right\|\\
&+\alpha\left\|\int_{s-\eta}^s\int_\delta^\infty \theta(\nu(s,r))^{\alpha-1}\eta_\alpha(\theta)S((\nu(s,r))^\alpha\theta)[F(r,v(r))+\mathcal{B}u(r,v)]\nu'(r)d\theta dr\right\|\\
&\le M_S\int_0^\delta \eta_\alpha(\theta)d\theta\\
&+M_S[\|L_F\|_\infty+M_{\mathcal{B}}\|u\|](\nu(s,0))^\alpha\int_0^\delta \theta\eta_\alpha(\theta)d\theta\\
&+M_S[\|L_F\|_\infty+M_{\mathcal{B}}\|u\|](\nu(s,s-\eta))^\alpha\int_0^\infty \theta\eta_\alpha(\theta)d\theta.
\end{aligned}
$$

From $\int_0^\infty\theta\eta_\alpha(\theta)d\theta=\frac{1}{W(\alpha)}$ and the absolute continuity of the integral, we obtain

$$\left\|\Xi_\varpi v(s)-\Xi_{\varpi,\eta,\delta}v(s)\right\|\to 0 \text{ as } \eta,\delta\to 0.$$

A relatively compact set $\Xi_{\varpi,\eta,\delta}(s)$ is arbitrarily close to the set $\Xi_\varpi(s)$ for $s>0$, implying, by the Arzelà–Ascoli theorem, that Ξ_ϖ is itself relatively compact in $C([0,d],\mathfrak{Y})$. Hence, for all $\varpi>0$, Ξ_ϖ is completely continuous operator on $C([0,d],\mathfrak{Y})$. Consequently, according to the Schauder fixed point theorem, Ξ_ϖ possesses a fixed point within B_{ρ_0}, representing the mild solution to the system referenced as (1). □

Now, we focus on the approximate controllability of the Equation (1).

Theorem 3. *Assuming that conditions (H1) to (H3) are satisfied and the function F is uniformly bounded, with the additional condition that the associated linear Equation (1) exhibits approximate controllability over the interval $[0,d]$, it follows that (1) is finite-approximately controllable within the same interval $[0,d]$.*

Proof. By Theorem 2 a fixed point y^ϖ of Ξ_ϖ exists in B_ρ and y^ϖ is a mild solution of (1) with

$$
\begin{aligned}
u_\varpi(t,y^\varpi)=\mathcal{B}^*\mathfrak{V}_\alpha^*(\nu(d,t))\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}&\\
\times\Big(h-\mathfrak{T}_\alpha(\nu(d,0))-\int_0^d(\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,s))F\big(s,v^\varpi(s)\big)\nu'(s)ds\Big)&.
\end{aligned}
$$

Thus

$$v^\varpi(t)=\mathfrak{T}_\alpha(\nu(t,0))v_0$$

$$+\int_0^t (\nu(t,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,s))F\big(s,v^\varpi(s)\big)\nu'(s)ds$$
$$+\int_0^t (\nu(t,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(t,s))\mathcal{B}^*\mathfrak{V}_\alpha^*(\nu(d,s))\nu'(s)ds\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}p(v^\varpi)$$

where

$$p(v^\varpi) = h - \mathfrak{T}_\alpha(\nu(d,0))v_0 - \int_0^d (\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,s))F\big(s,v^\varpi(s)\big)\nu'(s)ds.$$

Using the identity $W_0^d\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1} = I-\varpi(I-\Pi_M)\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}$, we have

$$\begin{aligned}
v^\varpi(d) = {} & \mathfrak{T}_\alpha(\nu(d,0))v_0 \\
& + \int_0^d (\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,s))F\big(s,v^\varpi(s)\big)\nu'(s)ds \\
& + \int_0^d (\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,s))\mathcal{B}^*\mathfrak{V}_\alpha^*(\nu(d,s))\nu'(s)ds\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}p(v^\varpi) \\
& = \mathfrak{T}_\alpha(\nu(d,0))v_0 + \int_0^d (\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,s))F\big(s,v^\varpi(s)\big)\nu'(s)ds \\
& + W_0^d\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}p(v^\varpi) \\
& = \mathfrak{T}_\alpha(\nu(d,0))y_0 + \int_0^d (\nu(d)-\nu(s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d)-\nu(s))F\big(s,v^\varpi(s)\big)\nu'(s)ds \\
& + p(y^\varpi) - \varpi(I-\Pi_M)\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}p(v^\varpi) \\
& = h - \varpi(I-\Pi_M)\Big(\varpi(I-\pi_M)+W_0^d\Big)^{-1}p(v^\varpi).
\end{aligned} \tag{9}$$

By the Dunford–Pettis theorem, $\exists$ a subsequence $\{F(s,v^\varpi(s))\}$ that convergent weakly to $\{F(s,v(s))\}$ in $L^1([0,d],\mathfrak{Y})$. Consider

$$p(y) = h - \mathfrak{T}_\alpha(\nu(d,0))v_0 - \int_0^d (\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,s))F(s,v(s))\nu'(s)ds.$$

We have

$$\begin{aligned}
& \|p(v^\varpi) - p(v)\| \\
& \leq \frac{M_S}{W(\alpha)}\int_0^d (\nu(d,s))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d,s))\big\|F\big(s,v^\varpi(s)\big) - F(s,v(s))\big\|\nu'(s)ds
\end{aligned}$$

Furthermore, approximate controllability of the system (2) implies

$$\begin{aligned}
\|v^\varpi(d) - h\| \leq {} & \varpi\Big\|\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}p(v)\Big\| \\
& + \varpi\Big\|\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}\big[p(v^\varpi)-p(v)\big]\Big\| \\
& \leq \varpi\Big\|\Big(\varpi(I-\Pi_M)+W_0^d\Big)^{-1}p(v)\Big\| + \big\|\big[p(v^\varpi)-p(v)\big]\big\| \\
& \to 0 \text{ as } \varpi \to 0^+.
\end{aligned}$$

Hence, the system (1) is approximately controllable. On the other hand, applying Π_M to the both sides of (9) we obtain finite dimensional exact controllability:

$$\Pi_M v^\varpi(d) = \Pi_M h.$$

□

5. Applications

Example 1. *We consider the ν-Caputo heat equation*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha,\nu}v(t,\eta) = \dfrac{\partial^2 v(t,\eta)}{\partial\eta^2} + \chi_{(\alpha_1,\alpha_2)}(\eta)u(t) + F(v(t,\eta)), \\ v(t,0) = v(t,\pi) = 0, \quad 0 < t < d, \\ I_{0^+}^{1-\alpha}v(0,\eta) = v_0(\eta), \quad 0 \le \eta \le \pi, \end{cases} \tag{10}$$

where $\chi_{(\alpha_1,\alpha_2)}(\eta)$ is the characteristic function of $(\alpha_1,\alpha_2) \subset (0,\pi)$. Let $\mathfrak{Y} = L^2[0,\pi]$, $U = R$, and $A = d^2/d\eta^2$ with $D(\mathcal{A}) = H_0^1[0,\pi] \cap H^2[0,\pi]$. We define the bounded linear operator $\mathcal{B} : R \to L^2[0,\pi]$ by $(\mathcal{B}u)(t) = \chi_{(\alpha_1,\alpha_2)}(\eta)u(t)$, and the operator F is bounded.

Set $M = L_K^2[0,\pi] := \left\{\varphi : \varphi(\eta) = \sum_{i=1}^{K} \alpha_i e_i(\eta),\ \alpha_i \in R\right\}$ and denote by Π_M the operator of the orthogonal projection $L^2[0,\pi]$ onto $L_K^2[0,\pi]$. Define

$$\begin{aligned} \mathfrak{V}_\alpha(t)h &= \sum_{n=1}^{\infty} E_{\alpha,\alpha}\left(-n^2\pi^2(\nu(t))^{\alpha n}\right)\langle h, e_n\rangle e_n, \quad E_{\alpha,\alpha}(t) = \sum_{n=0}^{\infty} \frac{t^n}{W(n\alpha+\alpha)} \quad \text{(Mittag–Leffler series)}, \\ L_0^d u &= \int_0^d (\nu(d)-\nu(t))^{\alpha-1}\mathfrak{V}_\alpha(\nu(d)-\nu(t))(\mathcal{B}u)(t)\nu'(t)dt \\ &= \sum_{n=1}^{\infty} \int_0^d (\nu(d)-\nu(t))^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(\nu(d)-\nu(t))^\alpha\right)\left\langle \chi_{(\alpha_1,\alpha_2)}(\eta), e_n\right\rangle u(t)\nu'(t)dte_n, \\ \left(L_0^d\right)^* h &= \sum_{n=1}^{\infty} \int_0^d (\nu(d)-\nu(t))^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda_n(\nu(d)-\nu(t))^\alpha\right)\left\langle \chi_{(\alpha_1,\alpha_2)}(\eta), e_n\right\rangle\langle h, e_n\rangle \nu'(t)dt, \\ W_0^d h &= L_0^d\left(L_0^d\right)^* h \\ &= \sum_{n=1}^{\infty} \int_0^d (\nu(d)-\nu(t))^{2\alpha-2}E_{\alpha,\alpha}^2(-\lambda_n^\alpha(\nu(d)-\nu(t)))\nu'(t)dt\left\langle \chi_{(\alpha_1,\alpha_2)}(\eta), e_n\right\rangle^2\langle h, e_n\rangle e_n. \end{aligned}$$

Subsequently, we attain

$$\begin{aligned} &\left(\varpi(I-\Pi_M) + W_0^d\right)^{-1} g \\ &= \sum_{n=1}^{\infty} \frac{1}{\left(\varpi(I-\Pi_M) + \int_0^d (\nu(d)-\nu(t))^{2\alpha-2}E_{\alpha,\alpha}^2(-\lambda_n^\alpha(\nu(d)-\nu(t)))\nu'(t)dt\left\langle \chi_{(\alpha_1,\alpha_2)}(\eta), e_n\right\rangle^2\right)}\langle g, e_n\rangle e_n \\ &= \sum_{n=1}^{K} \frac{1}{\int_0^d (\nu(d)-\nu(t))^{2\alpha-2}E_{\alpha,\alpha}^2(-\lambda_n^\alpha(\nu(d)-\nu(t)))\nu'(t)dt\left\langle \chi_{(\alpha_1,\alpha_2)}(\eta), e_n\right\rangle^2}\langle g, e_n\rangle e_n \\ &+ \sum_{n=K+1}^{\infty} \frac{1}{\left(\varpi + \int_0^d (\nu(d)-\nu(t))^{2\alpha-2}E_{\alpha,\alpha}^2(-\lambda_n^\alpha(\nu(d)-\nu(t)))\nu'(t)dt\left\langle \chi_{(\alpha_1,\alpha_2)}(\eta), e_n\right\rangle^2\right)}\langle g, e_n\rangle e_n \end{aligned}$$

It is clear that $\varpi\left(\varpi(I-\Pi_M) + W_0^d\right)^{-1} g \to 0$ as $\varpi \to 0^+$ if

$$\left\langle \chi_{(\alpha_1,\alpha_2)}(\eta), e_n\right\rangle = \int_{\alpha_1}^{\alpha_2} \sqrt{2}\sin(n\pi\eta)d\eta = -\frac{\sqrt{2}}{n\pi}\cos(n\pi\eta)\,\Big|_{\alpha_1}^{\alpha_2} \neq 0,$$

which holds whenever $\alpha_1 \pm \alpha_2$ is an irrational number.

If the sum or difference of α_1 and α_2 is an irrational, then the linear ν-Caputo system associated with (10) exhibits finite-approximate controllability. According to Theorem 2, this implies that the system (10) itself is finite-approximately controllable over the interval $[0,d]$.

Example 2. *Define the differential operator $\mathfrak{L}$ by*

$$\mathfrak{L}v(\eta) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial \eta_i}\left(a_i(\eta)\frac{\partial v}{\partial \eta_j}(\eta)\right) + c(\eta)v(\eta), \;\; \eta \in \Omega,$$

where $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $1 \leq i,j \leq d$, $c \in C(\overline{\Omega})$, $c(\eta) \geq 0$, $\eta \in \overline{\Omega}$, *and*

$$\sum_{i,j=1}^{d} a_{ij}(\eta)\eta_i\eta_j \geq l|\eta|^2, \;\; l > 0, \;\; \eta \in \overline{\Omega}, \; \eta \in R^d.$$

Hence $\mathfrak{L} : D(\mathfrak{L}) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega) \to L^2(\Omega)$.

Consider the following initial value/boundary value problem for a ν*-Caputo fractional system in a bounded domain* $\Omega \subset \mathbb{R}^d$ *with a smooth boundary* $\partial\Omega$.

$$\begin{aligned} {}^{C}D_{0^+}^{\alpha,\nu} v(t,\eta) + \mathfrak{L}v(t,\eta) &= \chi_\omega v(t,\eta) + F(v(t,\eta)) \quad (\eta,t) \in \Omega \times (0,d), \\ v(t,\eta) &= 0, \qquad (\eta,t) \in \partial\Omega \times (0,d), \\ D_{0^+}^{\alpha-1} v(0,\eta) &= v_0, \quad v \in C_0^\infty(\omega \times (0,d)), \;\; \omega \subset \Omega. \end{aligned} \tag{11}$$

For any $v_0 \in L^2(\Omega)$ *the Equation (11) has a unique solution* v *given by*

$$\begin{aligned} v(t,\eta) &= \sum_{n=1}^{\infty} (v_0, e_n) E_{\alpha,\alpha}\left(-\lambda_n(\nu(t) - \nu(0))^\alpha\right) e_n(\eta) \\ &\quad + \sum_{n=1}^{\infty} \int_0^t (F(\cdot, t-r), e_n) r^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n(\nu(t) - \nu(r))^\alpha\right) dr e_n(\eta), \\ L_0^d F &= \sum_{n=1}^{\infty} \int_0^d (F(\cdot, d-r), e_n) r^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_n(\nu(\cdot) - \nu(r))^\alpha\right) dr e_n(\eta), \\ \left(L_0^d\right)^* g &= \chi_\omega (\nu(d) - \nu(t))^{\alpha-1} \sum_{n=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_n(\nu(d) - \nu(t))^\alpha\right)(g, e_n) e_n(\eta). \end{aligned}$$

Assuming $L^* g = 0$, *we derive the equation*

$$(d-t)^{\alpha-1} \sum_{n=1}^{\infty} E_{\alpha,\alpha}\left(-\lambda_n(\nu(d) - \nu(t))^\alpha\right)(g, e_n) e_n(\eta) = 0$$

on $\omega \times (0,d)$. *According to Proposition 4.2 in [38], this implies that* $g = 0$ *on* $\Omega \times (0,d)$, *which is equivalent to the approximate controllability of the linear system associated with (11). Therefore, with the additional assumption of the uniform boundedness of* F, *the system (11) is finite-approximately controllable over the interval* $[0,d]$.

Example 3. *Let* $\mathfrak{Y} = \mathcal{U} = L^2[0,\pi]$. *Consider the following Sobolev type* ν*-Caputo fractional PDE:*

$$\begin{aligned} {}^{C}D_{\xi}^{\frac{3}{4},\nu}(y(\xi,\theta) - y_{\theta\theta}(\xi,\theta)) &= y_{\theta\theta}(\xi,\theta) + g(\xi, y(\xi,\theta)) + \mathfrak{v}(\xi,\theta), \\ y(\xi,0) &= y(\xi,\pi) = 0, \\ y(0,\theta) &= \phi(\theta), : 0 \leq \xi \leq T, : 0 \leq \theta \leq \pi. \end{aligned} \tag{12}$$

Define $\mathcal{A} : D(\mathcal{A}) \subset \mathfrak{Y} \to \mathfrak{Y}$ *by* $\mathcal{A} := y_{\theta\theta}$ *and* $\mathbf{G} : D(\mathbf{G}) \subset \mathfrak{Y} \to \mathfrak{Y}$ *by* $\mathbf{G}y := y - y_{\theta\theta}$, *where*

$$D(\mathcal{A}) = D(\mathbf{G}) = \{y \in \mathfrak{Y} : y, y_\theta \text{ are absolutely continuous}, y_{\theta\theta} \in \mathfrak{Y},\, y(\xi,0) = y(\xi,\pi) = 0\}.$$

$\mathcal{A}$ *and* $\mathbf{G}$ *are defined as follows*

$$\mathcal{A}y = \sum_{n=1}^{\infty} n^2 \langle y, e_n \rangle e_n, y \in D(\mathcal{A}), \mathbf{G}y = \sum_{n=1}^{\infty} \left(1 + n^2\right) \langle y, e_n \rangle e_n, y \in D(\mathbf{G}),$$

respectively, where $e_n(\theta) := \sqrt{\frac{2}{\pi}} \sin n\theta$, $n = 1, 2, \ldots$ *is the set of eigenvalues. Moreover, for any* $y \in \mathbb{U}$ *we have*

$$\begin{aligned} S(\xi)y &= \frac{3}{4} \int_0^{\infty} \theta \xi_{\frac{3}{4}}(\theta) \sum_{n=1}^{\infty} \exp\left(\frac{-n^2}{1+n^2} (\nu(\xi))^{\frac{3}{4}} \theta \right) \langle y, e_n \rangle e_n d\theta \\ &= \sum_{n=1}^{\infty} \mathbb{E}_{\frac{3}{4}, \frac{3}{4}} \left(\frac{-n^2}{1+n^2} (\nu(\xi))^{\frac{3}{4}} \right) \langle y, e_n \rangle e_n, \end{aligned}$$

where $\mathbb{E}_{\frac{3}{4}}$ *and* $\mathbb{E}_{\frac{3}{4}, \frac{3}{4}}$ *the Mittag–Leffler functions. Thus,*

$$\|\mathfrak{T}_{\alpha}(\xi)\| \leq 1, \quad \|\mathfrak{B}_{\alpha}(\xi)\| \leq 1/\Gamma\left(\frac{3}{4}\right), \ \xi \geq 0.$$

Next, define

$$\begin{aligned} \mathfrak{D}^*(\nu(1), \nu(\eta))y = \sum_{k=1}^{m} a_k \chi_k(\eta) (\nu(1) - \nu(\eta))^{-\frac{1}{4}} \mathfrak{B}_{\alpha}^*(\nu(\xi_k) - \nu(\eta)) D^* \mathfrak{T}_{\alpha}(\nu(1)) \left(\mathbf{G}^{-1}\right)^* y \\ + \chi_1(\eta)(1-\eta)^{-\frac{1}{4}} \mathfrak{B}_{\alpha}^*(\nu(1) - \nu(\eta)) \left(\mathbf{G}^{-1}\right)^* y = 0, \ 0 \leq \eta < 1. \\ \Longrightarrow \frac{1}{1+n^2} \langle y, e_n \rangle (\nu(1) - \nu(\eta))^{-\frac{1}{4}} \mathbb{E}_{\frac{3}{4}, \frac{3}{4}} \left(\frac{-n^2}{1+n^2} (\nu(1) - \nu(\eta))^{\frac{3}{4}} \right) = 0 \end{aligned}$$

for any $n \in \mathbb{N}$ *and* $\xi_m < \eta < 1$. *If* $y \neq 0$ *then there is* $n \in \mathbb{N}$ *s. t.* $\mathbb{E}_{\frac{3}{4}, \frac{3}{4}} \left(\frac{-n^2}{1+n^2} (\nu(1) - \nu(\eta))^{\frac{3}{4}} \right) = 0$ *for any* $\xi_m < \eta < 1$. *This is not possible, since*

$$\lim_{\eta \to 1^-} \mathbb{E}_{\frac{3}{4}, \frac{3}{4}} \left(\frac{-n^2}{1+n^2} (\nu(1) - \nu(\eta))^{\frac{3}{4}} \right) = \mathbb{E}_{\frac{3}{4}, \frac{3}{4}}(0) = 1/\Gamma\left(\frac{3}{4}\right).$$

So, $\mathfrak{D}^*(\nu(1), \nu(\eta))y = 0, 0 \leq \eta < 1$ *implies that* $y = 0$, *which means approximate controllability of associated linear system of the problem (12) on* $[0, 1]$.

Define $f : [0, 1] \times \mathfrak{Y} \to \mathfrak{Y}$ *by* $f(\xi, y)(\theta) = g(\xi, y(\xi, \theta))$ *which is assumed to be continuous. Thus, under uniform boundedness of* f, *the system (12) is finite-approximately controllable on* $[0, d]$.

6. Discussion and Conclusions

Finite-approximate controllability builds upon recent advancements in the study of fractional evolution equations by offering a more comprehensive understanding of the controllability aspects of systems described by Caputo fractional equations. Recent research may have explored various aspects of fractional calculus, but finite-approximate controllability specifically addresses the feasibility and methods for controlling these systems within finite-dimensional spaces. This contributes to the ongoing discourse by bridging the gap between theoretical developments in fractional calculus and practical control strategies for systems with fractional dynamics.

Possible directions for exploring finite-approximate controllability include: control strategies, optimal control approaches, sensitivity analysis, numerical simulations, stability analysis, and application in real-world problems. By exploring these avenues, researchers can deepen their understanding of finite approximate controllability in the context of fractional equations and contribute to the development of effective control strategies for systems exhibiting fractional dynamics.

The implications of finite-approximate controllability for numerical implementations are crucial to ensuring the practical feasibility and reliability of the proposed control strategies for systems governed by Caputo fractional evolution equations. The implication revolves around ensuring numerical stability, the accuracy of approximations, appropriate choice of numerical methods, consideration of discretization effects, efficient use of computational resources, and rigorous validation of the proposed methodology.

The methodology's computational complexity and efficiency are significant considerations, especially for larger or more complex systems of fractional evolution equations. Discuss how the proposed controllability approach scales with the size and complexity of the system. Consider whether there are optimizations or computational techniques that can enhance efficiency. Addressing these concerns is crucial for the practical applicability of the methodology, especially in real-world scenarios where computational resources may be limited.

In our study, we investigated the finite-approximate controllability of ν-Caputo fractional differential equations using the fixed point method. The main results were obtained by applying semigroup theory, ν-Caputo fractional derivatives, and fixed point theorems. The practical relevance of these findings is illustrated through a specific application. In our upcoming research, we will focus on investigating the finite-approximate controllability of ν–Hilfer fractional differential systems. Additionally, we aim to explore the existence of ν–Hilfer fractional differential systems, considering both cases with and without delay. Our approach will involve the application of a fixed point method to address these aspects.

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