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Existence Result for Coupled Random First-Order Impulsive Differential Equations with Infinite Delay

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Abstract: In this paper, we consider a system of random impulsive differential equations with infinite delay. When utilizing the nonlinear variation of Leray–Schauder's fixed-point principles together with a technique based on separable vector-valued metrics to establish sufficient conditions for the existence of solutions, under suitable assumptions on Υ_1 , Υ_2 and ϖ_1, ϖ_2, which greatly enriched the existence literature on this system, there is, however, no hope to discuss the uniqueness result in a convex case. In the present study, we analyzed the influence of the impulsive and infinite delay on the solutions to our system. In addition, to the best of our acknowledge, there is no result concerning coupled random system in the presence of impulsive and infinite delay.

Keywords: Iterative methods; existence of solutions; impulsive equations; generalized banach space; Schaefer's fixed point theorem; differential equations

MSC: 47H10; 47H30; 54H25; 34K10; 34K40; 34K45

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Citation: Moumen, A.; Ladrani, F.Z.; Ferhat, M.; Benaissa Cherif, A.; Bouye, M.; Bouhali, K. Existence Result for Coupled Random First-Order Impulsive Differential Equations with Infinite Delay. *Fractal Fract.* **2024**, *8*, 10. https://doi.org/10.3390/fractalfract8010010

Academic Editor: Gani Stamov

Received: 4 November 2023
Revised: 9 December 2023
Accepted: 14 December 2023
Published: 21 December 2023

1. Introduction and Position of Problem

1.1. Results and Discussion

Nowadays, mathematics contains many references related to impulsive differential equations. We mention here the development of some of them in this area. Impulsive differential equations is considered in [1], where the authors obtained results related to oscillation and the behaviour of solutions of the system

$$\begin{aligned} &v'(s) + r(s)v(s-\tau) = 0,\ s \neq \phi_k,\ s \geq s_0,\\ &v(\phi_k^+) - v(\phi_k^-) = I_k(v(\phi_k^+)),\ k \in \mathbb{N}. \end{aligned} \tag{1}$$

Impulsive infinite delay differential equations is considered in [2] as a system

$$\begin{aligned} &x'(r) = f(r, x(r), x(r - s(r))),\ r \geq r_0,\ r \neq s_k,\\ &x(r) - x(r^-) = I_k(x(r^-)), r = s_k,\ k = 1, 2... \end{aligned} \tag{2}$$

By using the Lyapunov functions together with the Razumikhin technique, new results related to the existence and behavior of solution were obtained.

In [3], the authors proposed a random impulsive differential equations for $k = 1, ..., m$

$$
\begin{aligned}
& y'(r,t) = f(t, y(t,r), r),\ t \neq t_k, \\
& y'(r, t_k^+) - y'(r, t_k^-) = I_k(y(r, t_k), r), \\
& y(r, a) = \nu(r),
\end{aligned} \tag{3}
$$

here, the function $\nu \in C^0(\Omega \to \mathbb{R}^n)$ is a random variable. Under appropriate conditions in the parameters f, k, I, the existence and uniqueness is established owing to the generalized Schaefer's type random fixed-point theorem. Numerous processes in physics, biology, medicine, population dynamics, and other fields may experience rapid changes like shocks or perturbations (for examples, see [4,5] and the references therein). While this is going on, several models of genuine processes and phenomena explored in physics, chemical technology, population dynamics, biotechnology, and economics are described by delayed impulsive differential systems and evolution differential systems. That is why, in recent years, they have been the object of investigations by many mathematicians [6,7]. We cite the work of Samoilenko and Perestyuk [8], Lakshmikantham et al. [9], and Bainov and Simeonov [10] as sources, where a thorough bibliography is provided and several features of their solutions are investigated. Many studies have been carried out on functional differential equations and inclusions with or without impulses. See the books by Dejabli et al. [11] and Graef et al. [12] for more information on how existence and uniqueness are derived. The boundary value problem on infinite intervals can be found in a variety of real-world models, such as foundation engineering, nonlinear fluid flow problem, and difficulties involving linear elasticity (see [13–19]) and the references therein. Recent years have seen a significant increase in research into impulsive ordinary differential equations and functional differential equations under various conditions; for examples, see the works by Aubin [20] and Benchohra et al. [6] and the references therein. The presence of a delay in the system being studied often turns out to be the cause of phenomena that significantly influence the course of the process. Differential equations with delay argument are differential equations in which an unknown function and its derivatives appear at different values where the time derivatives at the current time depend on the solution and possibly its derivatives at previous times. The most natural methods for solving this type of problem are so-called iterative methods; for more details, please see [21–24]. Motivated by the previous works, in the present paper it is interesting to analyze the influence of the impulsive and infinite delay on the solutions to system (4) under suitable assumptions on Y_1 , Y_2 and ϖ_1, ϖ_2 with the presence of new random properties.

1.2. Position of Problem

To begin with, let Γ be an open domain of $\mathbb{R}^n$, $n > 1$, $\mathcal{J} = [0, \infty)$, $\mathcal{J}_0 = (-\infty, 0]$, $\mathcal{J}_k = (t_{k-1},\ t_k]$, $k = 1, 2, \ldots, \varpi_i = \varpi_i(t,x)$, $\varpi_i' = \varpi_i'(t,x)$, $\varpi_i'' = \varpi_i''(t,x)$, $i = 1, 2, t \in [0, \infty), x \in \Gamma$. The following system of impulsive differential equations by random effects (random parameters) with infinite delay is examined in this paper

$$
\begin{cases}
\varpi_1' = Y_1(t,\ \varpi_{1t}(x),\ \varpi_{2t}(x), x), & a.e \quad t \in \mathcal{J}, t \neq t_k, \\
\varpi_2' = Y_2(t,\ \varpi_{1t}(x),\ \varpi_{2t}(x), x), & a.e \quad t \in \mathcal{J}, t \neq t_k, \\
\Delta\varpi_1(t) = I_k^1(\varpi_1(t_k, x),\ \varpi_2(t_k, x)), & t = t_k, \\
\Delta\varpi_2(t) = I_k^2(\varpi_1(t_k, x),\ \varpi_2(t_k, x)), & t = t_k, \\
A_1\varpi_1 - \varpi_{1,\infty} = \phi_1(t, x), & t \in (-\infty, 0], \\
A_2\varpi_2 - \varpi_{2,\,\infty} = \phi_2(t, x), & t \in (-\infty, 0],
\end{cases} \tag{4}
$$

where $Y_i : \mathcal{J} \times \mathcal{D}_0 \times \mathcal{D}_0 \times \Gamma \to \mathcal{P}(\mathbb{R}^n), i = 1, 2$ and $I_k^1, I_k^2 \in C^0(\mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^n)$ are given functions satisfying some assumptions that will be specified later and

$$
\lim_{t \to \infty} (\varpi_1(t),\ \varpi_2(t)) = (\varpi_{1,\,\infty},\ \varpi_{2,\,\infty}), \tag{5}
$$

here $\phi_1,\ \phi_2 \in \mathcal{D}_0$, and $\mathcal{D}_0$ is called a phase space that will be defined later; the fixed times t_k satisfies

$$
0 < t_1 < t_2 < \ldots < t_m < T,
$$

$\varpi_2(t_k^-)$ and $\varpi_2(t_k^+)$ denotes the left and right limits of $\varpi_2(t)$ at $t = t_k$ and ϕ_1, ϕ_2 are two random maps. The impulse times t_k satisfy

$$0 = t_0 < t_1 < t_2 < \ldots, t_m < T.$$

If $T = \infty$, t_k satisfies

$$0 = t_0 < t_1 < t_2 < \ldots, t_m < \cdots.$$

The functional ϖ_{1t}, represent the infinite delay and as for ϖ_{2t}, we mean the segment solution which is defined in the usual way, that is, if $\varpi_2(.,.) \in C^0((-\infty,\infty) \times \Gamma, \mathbb{R}^n)$, then for any $t \geq 0$, $\varpi_{2t}(.,.) \in C^0((-\infty,0] \times \Gamma, \mathbb{R}^n)$ is given by

$$\varpi_{2t}(\alpha,\omega) = \varpi_2(t+\alpha,\omega), \text{ for } \alpha \in (-\infty,0]. \tag{6}$$

Before going into the characteristics of the operators Υ_1, Υ_2 and I_k^1, I_k^2, we first introduce some notation and define certain spaces.

In this study, we will make use of Hale and Kato's [25] axiomatic description of the phase space $\mathcal{D}_0$.

Definition 1. *By $\mathcal{D}_0$, we mean a linear space containing a family of measurable functions from $(-\infty,0]$ into $\mathbb{R}^n$ and endowed with a norm$\|.\|_{\mathcal{D}_0}$. The following axioms are satisfied:*

(A_1) *If $\varpi_2 \in C^0((-\infty,T), \mathbb{R}^n)$, $T = +\infty$, is such that $\varpi_{2,0} \in \mathcal{D}_0$, then for every $t \in \mathcal{J}$ the following conditions hold*

- (i) $\varpi_{2t} \in \mathcal{D}_0$,
- (ii) $\|\varpi_2(t)\| \leq \mathcal{L}\|\varpi_{2t}\|_{\mathcal{D}_0}$,
- (iii) $\|\varpi_{2t}\|_{\mathcal{D}_0} \leq K(t)\sup\{\|\varpi_2(s)\| : 0 \leq s \leq t\} + N(t)\|\varpi_{2,0}\|_{\mathcal{D}_0}$,
  where $\mathcal{L} > 0$ is a constant; $K, N \in C^0([0,\infty), [0,t))$, K is continuous, N is locally bounded and K, n are independent of $\varpi_2(.)$.

(A_2) *For the function $\varpi_2(.)$ in (A_1), ϖ_{2t} is a $\mathcal{D}_0$-valued function on $[0,t)$.*
(A_3) *The space $\mathcal{D}_0$ is complete.*

Denote

$$\widetilde{K} = \sup_{t\in\mathcal{J}}\{K(t)\} \quad \text{and} \quad \widetilde{N} = \sup_{t\in\mathcal{J}}\{N(t)\}. \tag{7}$$

Remark 1. *In retarded functional differential equations without impulses, the axioms of the abstract phase space $\mathcal{D}_0$ include the continuity of the function $t \to \varpi_{2t}$. Due to the impulsive effect, this property is not satisfied in impulsive delay systems, and, for this reason, it has been eliminated in our abstract description of $\mathcal{D}_0$.*

Let

$$\mathcal{D}_0 = \{\phi_i \in C^0((-\infty,0] \times \Omega, \mathbb{R}^n), \text{ for any }, \sup_{\theta\leq 0}(|\phi_i(\theta)|) < \infty\}. \tag{8}$$

If $\mathcal{D}_0$ is endowed with the norm

$$\|\phi\|_{\mathcal{D}_0} = \sup_{\theta\leq 0}(|\phi_i(\theta)|),$$

then $(\mathcal{D}_0, \|\cdot\|_{\mathcal{D}_0})$ is a Banach space, see [26].

Now, for a given $T = +\infty$, we define

$$\mathcal{D}_\infty = \left\{ \begin{array}{l} \varpi_2 \in C^0((-\infty,\infty) \times \Gamma, \mathbb{R}^n), \ \varpi_{2,k} \in C(\mathcal{J}_k, \mathbb{R}^n), k = 1, \ldots m, \ \varpi_{2,0} \in \mathcal{D}_0, \\ \text{and there exist} \\ \varpi_2(t_k^-) \text{ and } \varpi_2(t_k^+) \text{ with } \varpi_2(t_k) = \varpi_2(t_k^-), \ k \in 1, \cdots, m \text{ and } \sup_{t\in\mathcal{J}}|\varpi_2(t)| < \infty \end{array} \right\},$$

endowed with the norm

$$\|\varpi_2\|_{\mathcal{D}_\infty} = \|\phi_2\|_{\mathcal{D}_0} + \sup_{s\in\mathcal{J}} |\varpi_2(s)|, \tag{9}$$

where $\varpi_{2,k}$ denotes the restriction of ϖ_2 to $\mathcal{J}_k$.

Then we will consider our initial data $\phi_1,\ \phi_2 \in \mathcal{D}_0$. As for the impulse functions, we will assume that $I_k^1, I_k^2 \in C^0(\mathbb{R}^n \times \mathbb{R}^n,\ \mathbb{R}^n)$ and

$$\Delta\varpi_2(t)|_{t=t_k} = \varpi_2(t_k^+) - \varpi_2(t_k^-),$$

$$\varpi_2(t_k^+) = \lim_{h\to 0^+} \varpi_2(t+h),$$

and

$$\varpi_2(t_k^-) = \lim_{h\to 0^-} \varpi_2(t-h).$$

We suppose that the multi-function $\to Y_i(x,\ \varpi_{1t}(x),\ \varpi_{2t}(x))$ is measurable over the entire paper. Applying a novel random fixed point theorem to a system of impulsive random differential equations is the primary objective of this research. Additionally, we provide a random application of the separable vector-valued Banach space Leray–Schauder fixed point theorem.

This article is structured as follows: We provide notations, definitions and introductory information in Section 2 and state some Lemmas and Theorems in Section 3 that will be helpful throughout the proof. Using a nonlinear variant of the Leray–Schauder type theorem on extended Banach spaces in the convex case as in [27], we demonstrate the existence result in Section 4. To finish the work, we give conclusive comments with a discussion of the novelties and some perspictives.

2. Preliminaries and Tools

Here, we make some notes, review some definitions, and talk about some background material that will be used later in the article. In fact, we will use quotes from [28,29]. Although we can only refer to this document when we need it, we prefer to include it here to keep our work as independent as possible and to make it easier to read.

Vector Metric Space

Let

$$\varpi_1 = (\varpi_{1,1},\ \ldots, \varpi_{1,n}) \in \mathbb{R}^n, \tag{10}$$

and

$$\varpi_2 = (\varpi_{2,1},\ \ldots, \varpi_{2,n}) \in \mathbb{R}^n. \tag{11}$$

The interval I be in $\mathbb{R}$ and $c \in \mathbb{R}$, we note that $I_{\mathbb{Z}} = I \cap \mathbb{Z}$,

$$\varpi_1 \le \varpi_2 \text{ implies that } \varpi_{1,j} \le \varpi_{2,j}, j = 1, \ldots, n,$$
$$\varpi_1 \le c \text{ equivalent that } \varpi_{1,j} \le c, j = 1, \ldots, n,$$

$$|\varpi_1| = (|\varpi_{1,1}|,\ \ldots, |\varpi_{1,n}|),$$
$$\max(\varpi_1,\ \varpi_2) = \max_{j=1,\ldots,n} (\max(\varpi_{1,j},\ \varpi_{2,j})).$$

Definition 2. *Let E be a non-empty set and a map $d \in C^0(E \times E,\ \mathbb{R}^n)$, where $d = (d_1, \ldots, d_n)$, we say that the pair (E, d) is said to be a generalized metric space if each pair $((E, d_i))_{i\in[1,n]_{\mathbb{Z}}}$ are metric spaces.*

For

$$a = (a_1,\ \ldots, a_n) \in \mathbb{R}_+^n,$$

we will denote by

$$
\begin{aligned}
B(\varpi_0, a) &= \varpi \in E : d(\varpi_0,\ \varpi) < a\} \\
&= \{\varpi \in E : d_j(\varpi_0,\ \varpi) < a_j,\ \ j = 1, ..., n\},
\end{aligned} \tag{12}
$$

the open ball centered in ϖ_0 with radius a and

$$
\begin{aligned}
\overline{B(\varpi_0, a)} &= \{\varpi \in E : d(\varpi_0,\ \varpi) \leq a\} \\
&= \{\varpi \in E : d_j(\varpi_0,\ \varpi) \leq a_j,\ \ j - 1, ..., n\},
\end{aligned} \tag{13}
$$

the closed ball with radius a, centered in ϖ_0. We point out that the notation of open subset, closed set, convergence, Cauchy sequence, and completion in generalized metric space is comparable to that in conventional metric space.

3. Random Variable and Some Selection Theorems

In this section, symbols, definitions, and introductory information from the multivalued analysis and random variables used throughout this article are presented. Let X be a subset of E, and let (E, d) be a Banach space or a generalized metric space. Let

$$\mathcal{P}_{cl}(E) = \{X \in \mathcal{P}(E) : X \text{ closed }\},$$

$$\mathcal{P}_b(E) = \{X \in \mathcal{P}(E) : X \text{ bounded }\},$$

$$\mathcal{P}_c(E) = \{X \in \mathcal{P}(E) : X \text{ convex }\},$$

$$\mathcal{P}_{cp}(E) = \{X \in \mathcal{P}(E) : X \text{ compact}\}.$$

Definition 3. *Let (Γ, Σ) be a measurable space and $Y \in C^0(\Gamma, \mathcal{P}(E))$ be a multi-valued mapping, Y is called measurable if*

$$Y^+(Q) = \{x \in \Gamma : Y(x) \subset Q\}, \tag{14}$$

for every $Q \in \mathcal{P}_{cl}(E)$, equivalently, for every $\mathcal{U}$ open set of E, the set

$$Y^-(Q) = \{x \in \Gamma : Y(x) \cap \mathcal{U} \neq \emptyset\}, \tag{15}$$

is measurable.

Let E is a metric space, we will use $\mathcal{B}(E)$ to denote the Borel σ-algebra on E. The $\Sigma \times \mathcal{B}(E)$ denotes the smallest σ -algebra on $\Gamma \times E$, which contains all the sets $A \times S$, where $Q \in \Sigma$ and $S \in \mathcal{B}(E)$. Let $Y \in C^0(E,\ \mathcal{P}(X))$ be a multi-valued map. A single-valued map $f \in C^0(E, X)$ is said to be a selection of G, and we write $(f \subset Y)$ whenever $f(\varpi) \in Y(\varpi)$ for every $\varpi \in E$.

Definition 4. *A mapping $Y \in C^0(\Gamma \times E, E)$ is called a random operator if any $\varpi \in E, f(.,\ \varpi)$ is measurable.*

Definition 5. *A random fixed point of f is a measurable function $\varpi \in C^0(\Gamma, E)$ such that*

$$\varpi(x) = f(x,\ \varpi(x)), \quad \forall x \in \Gamma. \tag{16}$$

Equivalently, a measurable selection for the multi-valued map $FixY : \Gamma \to \mathcal{P}(E)$ is defined by

$$FixY_x(\varpi) = \{\varpi \in E : \varpi = f(x,\ \varpi)\}. \tag{17}$$

Theorem 1 ([27])**.** *Let (Γ, Σ), X be a separable metric space and $Y \in C^0(\Gamma,\ \mathcal{P}_{cl}(X))$ be measurable multi-valued. Then Y has a measurable selection.*

The following conclusions can be drawn from Kuratowski–Ryll–Nardzewski and Aumann's selection Theorems.

Theorem 2 ([27])**.** *Let (Γ, Σ), X be a separable generalized metric space and $Y \in C^0(\Gamma,\ \mathcal{P}_{cl}(X))$ be measurable multi-valued. Then Y has a measurable selection.*

Then, in a separable vector Banach space, we propose a few random fixed-point theorems.

Theorem 3 ([27]). *Let E be a separable generalized Banach space, and let $G \in C^0(\Gamma \times E, \mathcal{P}_{cl,cv}(E))$ be an upper semi-continuous and compact map. Then either of the following holds:*

(i) *The random equation $G(x, \varpi) \in \varpi$ has a random solution, i.e., there is a measurable function $\varpi \in C^0(\Gamma, E)$ such that*

$$G(x, \varpi(x)) \in \varpi(x), \ \forall x \in \Gamma.$$

(ii) *The set*

$$\mathcal{M} = \{\varpi : \Gamma \to E : \varpi \text{ is measurable and } \varpi \in \lambda(x)G(x, \varpi)\}, \tag{18}$$

is unbounded for some measurable $\lambda \in C^0(\Gamma, E)$ with $0 < \lambda(x) < 1$ on Γ.

Definition 6. *The function $f \in C^0([0, b] \times \mathbb{R} \times \Gamma, \mathbb{R})$ is called random Carathéodory if*

(i) *The map $(t, x) \to f(t, \varpi, x)$ is jointly measurable $\forall x \in \mathbb{R}$,*

(ii) *The map $\varpi \to f(t, \varpi, x)$ is continuous $\forall t \in [0, b]$ and $x \in \Gamma$.*

Lemma 1 ([27]). *Let E be a separable generalized metric space and $G \in C^0(\Gamma \times E, E)$ be a mapping such that $G(., \varpi)$ is measurable $\forall \varpi \in E$ and $G(x, .)$ is continuous $\forall x \in \Gamma$. Then the map $(x, \varpi) \to G(x, \varpi)$ is jointly measurable.*

Lemma 2 ([12]). *Let E be a Banach space. Let $Y \in C^0(J \times E, \mathcal{P}_{cp,c}(E))$ be an L^1-Carathéodory multi-valued map with $S_{Y,z} \neq \emptyset$ and let R be a linear continuous mapping from $L^1(J, \varpi)$ into $\mathcal{C}(J, \varpi)$. Then the operator*

$$\begin{aligned} R \circ S_Y : \mathcal{C}(J, \varpi) &\to \mathcal{P}_{cp,c}(\mathcal{C}(J, \varpi)) \\ z &\to (R \circ S_Y)(z) = R(S_{Y,z}), \end{aligned}$$

is a closed graph operator in $\mathcal{C}(J, \varpi) \times \mathcal{C}(J, \varpi)$.

4. Main Result: Existence of Solutions

In this section, we provide adequate conditions for the first order of a random system of functional differential Equation (4), to have solutions. We begin by assuming that Y has values that are convex. We define the problem's solution prior to declaring and demonstrating our conclusion for this case.

The Convex Case

Now we first define the concept of the solution to our problem.

Lemma 3. *Given $(\varpi_1, \varpi_2) \in \mathcal{D}_\infty \times \mathcal{D}_\infty$, it is said to be solution of (23) if there exists a functions $f_1(t, x), f_2(t, x)$ such that*

$$(f_1, f_2) \in L^1([0, \infty) \times \Gamma, \mathbb{R}^n) \times L^1([0, \infty) \times \Gamma, \mathbb{R}^n), \tag{19}$$

and

$$(f_1, f_2) \in (Y_1(t, \varpi_{1t}(x), \varpi_{2t}(x), x), Y_2(t, \varpi_{1t}(x), \varpi_{2t}(x), x)), \tag{20}$$

and $(\varpi_1', \varpi_2') = (f_1, f_2)$, with (ϖ_1, ϖ_2) be a solution of the problem (23), for $x \in \Gamma$

$$\varpi_1(t)=\begin{cases}\frac{\phi_1(0,x)}{A(A-1)}+\frac{1}{A-1}\left[\int_0^{\infty}f_1(s,x)ds+\sum_{k=1}^{\infty}I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)\right.\\ \left.+\frac{\phi_1}{A}\right],\quad t\in(-\infty,0],\\ \\ \frac{\phi_1(0,x)}{A-1}+\frac{1}{A-1}\left[\int_0^{\infty}f_1(s,x)ds+\sum_{k=1}^{\infty}I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right]\\ +\int_0^t f_1(s,x)ds+\sum_{k=1}^{\infty}I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)),\quad t\in[0,\ \infty),\end{cases}\tag{21}$$

and

$$\varpi_2(t)=\begin{cases}\frac{\phi_2(0,x)}{A(A-1)}+\frac{1}{A-1}\left[\int_0^{\infty}f_2(s,x)ds+\sum_{k=1}^{\infty}I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x)\right]\\ +\frac{\phi_2}{A},\quad t\in(-\infty,0],\\ \\ \frac{\phi_2(0,x)}{A-1}+\frac{1}{A-1}\left[\int_0^{\infty}f_2(s,x)ds+\sum_{k=1}^{\infty}I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right]\\ +\int_0^t f_2(s,x)ds+\sum_{k=1}^{\infty}I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x)),\quad t\in[0,\ \infty),\end{cases}\tag{22}$$

where

$$\lim_{t\to\infty}(\varpi_1(t),\ \varpi_2(t))=(\varpi_{1,\ \infty},\ \varpi_{2,\ \infty}),$$

if and only if $(\varpi_1,\ \varpi_2)$ *is a solution of the impulsive boundary value problem*

$$\begin{cases}\varpi_1'=f_1(t,x), & a.e\quad t\in J,t\neq t_k,\\ \varpi_2'=f_2(t,x), & a.e\quad t\in J,t\neq t_k,\\ \Delta\varpi_1=I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)), & t=t_k,\\ \Delta\varpi_2=I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x)), & \\ A_1\varpi_1-\varpi_{1,\ \infty}=\phi_1, & \\ A_2\varpi_2-\varpi_{2,\ \infty}=\phi_2. & \end{cases}\tag{23}$$

Proof. Let $(\varpi_1,\ \varpi_2)$ be a solution of the impulsive integral Equations (21) and (22), then for $t\in[0,+\infty)$ and $t\neq t_k$, $k\in[1,\ \infty)_{\mathbb{Z}}$, we have

$$\begin{aligned}\varpi_1 &= \frac{\phi_1(0,x)}{A-1}+\frac{1}{A-1}\left[\int_0^{\infty}f_1(s,x)ds+\sum_{k=1}^{\infty}I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right]\\ &\quad+\int_0^t f_1(s,x)ds+\sum_{k=1}^{\infty}I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)),\end{aligned}$$

and

$$\begin{aligned}\varpi_2 &= \frac{\phi_2(0,x)}{A-1}+\frac{1}{A-1}\left[\int_0^{\infty}f_2(s,x)ds+\sum_{k=1}^{\infty}I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right]\\ &\quad+\int_0^t f_2(s,x)ds+\sum_{k=1}^{\infty}I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x)).\end{aligned}$$

Thus

$$(\varpi_1',\varpi_2')=(f_1(s,x),f_2(s,x)),t\in[0,\ \infty),t\neq t_k,k\in[1,\ \infty)_{\mathbb{Z}}.$$

From the definition of $(\varpi_1,\ \varpi_2)$ we can prove that

$$\begin{cases} \varpi_1(t_k^+, x) - \varpi_1(t_k^-, x) = I_k^1(\varpi_1(t_k, x),\ \varpi_2(t_k, x)), \\ \varpi_2(t_k^+, x) - \varpi_2(t_k^-, x) = I_k^2(\varpi_1(t_k, x),\ \varpi_2(t_k, x)). \end{cases} \tag{24}$$

Finally we prove that

$$(A_1\varpi_1 - \varpi_{1,\,\infty}, A_2\varpi_1 - x_\infty) = (\phi_1, \phi_2).$$

We have

$$\lim_{t\to\infty} \varpi_1 = \frac{\phi_1(0,x)}{A-1} + \frac{A}{A-1}\left(\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right),$$

and

$$\varpi_1 \;=\; \frac{\phi_1(0,x)}{A(A-1)} + \frac{1}{A-1}\left(\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right) + \frac{\phi_1}{A}.$$

Hence

$$\begin{aligned} A\varpi_1 - \lim_{t\to\infty} \varpi_1 \;&=\; \frac{\phi_1(0,x)}{(A-1)} + \phi_1 - \frac{\phi_1(0,x)}{A-1} \\ &\quad \frac{A}{A-1}\left(\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right) \\ &\quad -\left(\frac{A}{A-1}\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right) \\ &= \;\phi_1, \quad t \in [0,+\infty). \end{aligned} \tag{25}$$

Let $(\varpi_1,\ \varpi_2)$ be a solution of the problem (23). Then

$$\varpi_1' = f_1(s,x), \qquad a.e \qquad t \in [0,t_1], t \neq t_k. \tag{26}$$

An integration from 0 to t (here $t \in (0,t_1]$) of both sides of the above equality yields

$$\varpi_1 = \varpi_1(0,x) + \int_0^t f_1(s,x)ds.$$

If $t \in (t_1,t_2]$, then we have

$$\varpi_1 = \varpi_1(0,x) + \int_0^t f_1(s,x)ds + I_1^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)).$$

We obtain for $t \in [0,+\infty)$ that

$$\varpi_1 = \varpi_1(0,x) + \int_0^t f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)). \tag{27}$$

Since

$$(\lim_{t\to\infty}(\varpi_1(t),\ \varpi_2(t)) = (\varpi_{1,\,\infty},\ \varpi_{2,\,\infty}),$$

we obtain

$$\varpi_{1,\,\infty} = \varpi_1(0) + \int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)).$$

Thus
$$\varpi_1(0) = \varpi_{1,\,\infty} - \int_0^\infty f_1(s,x)ds - \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)),$$
and
$$\varpi_{1,\,\infty} = A\varpi_1(0) - \phi_1(0,x),$$
and hence
$$\varpi_1(t) = \frac{\phi_1(0,x)}{A-1} + \frac{1}{A-1}\left(\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right). \tag{28}$$

We replace (28) in (27), to obtain
$$\begin{aligned}\varpi_1 \quad &= \quad \frac{\phi_1(0,x)}{A-1} + \frac{1}{A-1}\left(\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right) \\ &\quad + \int_0^t f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)).\end{aligned}$$

From (28), we have
$$\begin{aligned}\varpi_1 \quad &= \quad \frac{\phi_1}{A} + \frac{1}{A}\left(\varpi_1(0,x) + \int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right) \\ &= \quad \frac{\phi_1(0,x)}{A-1} + \frac{1}{A-1}\left(\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right) \\ &\quad + \frac{\phi_1}{A}.\end{aligned}$$

□

Theorem 4. *Suppose the following hypotheses are satisfied:*

(H_1) *The function*
$$Y \in C^0([0,\ \infty) \times \mathcal{D}_\infty \times \mathcal{D}_\infty,\ \mathcal{P}(\mathbb{R}^n)), \tag{29}$$
is a nonempty, compact, convex, multi-valued map such that:

(a) $(t,.) \to Y(t,.)$ *is measurable;*

(b) $\varpi_1 \to Y(t,\ \varpi_1)$ *is upper semi-continuous for a.e.* $t \in [0,\ \infty)$

(H_2) *There exist bounded measurable functions*
$$P_1, P_2 \in C^0(\Gamma, L^1((0,\ \infty),\ \mathbb{R}^+)),$$
and non-decreasing continuous functions
$$\psi_1,\ \psi_2 \in C^0(\mathbb{R}^+, (0, +\infty)), \tag{30}$$
such that
$$|Y_2(t,\ \varpi_1,\ \varpi, x)| = \sup_{f_1 \in Y_1(t,\ \varpi_1,\ \varpi_2, x)} |f_1(t)| \le p_i, \quad \forall \varpi_1,\ \varpi_2 \in \mathcal{D}_0,$$
and
$$|Y_2(t,\ \varpi_1,\ \varpi_2, x)| = \sup_{f_2 \in Y_2(t,\ \varpi_1,\ \varpi_2, x)} |f_2(t)| \le p_i, \quad \forall \varpi_1,\ \varpi_2 \in \mathcal{D}_0.$$

(H_3) *There exist positive constants* $c_k,\ \ k = 1,\ \ldots$ *such that*
$$|I_k^i(x,\ \varpi_2(t_k,x))| \le c_k^i, \quad \forall \varpi_1,\ \varpi_2 \in \mathcal{D}_0,$$

and

$$\sum_{k=1}^{\infty} c_k^i < \infty,$$

for each $i \in \{1,2\}$*, then problem* (4) *has a unique random solution on* $(-\infty, +\infty)$.

Proof. Consider the operator

$$T \in C^0(\mathcal{D}_\infty \times \mathcal{D}_\infty \times \Gamma,\ \mathcal{P}(\mathcal{D}_\infty \times \mathcal{D}_\infty)),$$

defined by

$$T(x,\ \varpi_1,\ \varpi_2) = (T_1(x,\ \varpi_1,\ \varpi_2), t_2(x,\ \varpi_1,\ \varpi_2)), \quad (\varpi_1,\ \varpi_2) \in \mathcal{D}_\infty \times \mathcal{D}_\infty,$$

and

$$T(x,\ \varpi_1,\ \varpi_2) = \{(h^1, h^2) \in \mathcal{D}_\infty \times \mathcal{D}_\infty\},$$

given by

$$h^1 = \begin{cases} \frac{\phi_1(0,x)}{A(A-1)} + \frac{\phi_1}{A} + \\ \frac{1}{A-1}\left(\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k),x)\right), \quad t \in (-\infty, 0], \\ \\ \frac{\phi_1(0,x)}{A-1} + \frac{1}{A-1}\left(\int_0^\infty f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right) \\ \int_0^t f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x)), \quad t \in [0,\ \infty), \end{cases} \tag{31}$$

and

$$h^2 = \begin{cases} \frac{\phi_2(0,x)}{A(A-1)} + \frac{1}{A-1}\left[\int_0^\infty f_2(s,x)ds \right. \\ \left. + \sum_{k=1}^{\infty} I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k),x)\right] + \frac{\phi_2}{A}, \quad t \in (-\infty, 0], \\ \\ \frac{\phi_2(0,x)}{A-1} + \frac{1}{A-1}\left[\int_0^\infty f_2(s,x)ds + \sum_{k=1}^{\infty} I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x),x))\right] \\ + \int_0^t f_2(s,x)ds + \sum_{k=1}^{\infty} I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x),x)), \quad t \in [0,\ \infty), \end{cases} \tag{32}$$

where

$$f_i \in S_{Y_i,u} = \{f_i \in L^1([0,+\infty) \times \Gamma, \mathbb{R}^n) : \quad f_i \in Y_i(t,\ \varpi_1,\ \varpi_2, x),\ \forall t \in J, \varpi_1,\ \varpi_2 \in \mathcal{D}_\infty\}.$$

Clearly fixed points of the operator T are random solutions of problem (4). For $x \in \Gamma$ fixed $(\varpi_1,\ \varpi_2) \in \mathcal{D}_\infty \times \mathcal{D}_\infty$, consider

$$T_x \in C^0(\mathcal{D}_\infty \times \mathcal{D}_\infty,\ \mathcal{P}(\mathcal{D}_\infty \times \mathcal{D}_\infty)),$$

defined by

$$T_x(\varpi_1, \varpi_2) = (T_1(x,\ \varpi_1, \varpi_2), t_2(x,\ \varpi_1, \varpi_2)).$$

We will prove that T has a fixed point. Let $\alpha_1(.,.)$, $\alpha_2(.,.) \in \mathcal{D}_0$ be a functions defined by

$$\alpha_1 = \begin{cases} \phi_1(0,x), & \text{if} \quad t \in [0, +\infty) \\ \phi_1(t,x), & \text{if} \quad t \in (-\infty, 0], \end{cases}$$

and

$$\alpha_2 = \begin{cases} \phi_2(0,x), & \text{if} \quad t \in [0,+\infty) \\ \phi_2(t,x), & \text{if} \quad t \in (-\infty,0]. \end{cases}$$

Then, it is not difficult to see that $(\alpha_1,\ \alpha_2)$ is an element of $\mathcal{D}_\infty \times \mathcal{D}_\infty$. Set

$$(\varpi_1,\ \varpi_2) = (z_1(t,x) + \alpha_1(t,x), z_2(t,x) + \alpha_2(t,x)), t \in (-\infty,+\infty).$$

It is not hard to see that z_1, z_2 satisfy

$$z_1(t,w) = z_2(t,w) = 0, \quad t \in (-\infty,0].$$

If $(\varpi_1(.,x),\ \varpi_2(.,x))$ satisfies the integral equation

$$\begin{aligned} \varpi_1(x,t) \quad &= \quad \frac{\phi_1(0,x)}{A-1} + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x),x),x)) + \int_0^t f_1(s,\ \varpi_{1s}(x),\ \varpi_{2s}(x),x)ds \\ &+ \frac{1}{A-1}\left(\int_0^\infty f_1(s,\ \varpi_{1s}(x),\ \varpi_{2s}(x),x)ds + \sum_{k=1}^{\infty} I_k^1(\varpi_1(t_k,x),\ \varpi_2(t_k,x),x))\right), \end{aligned}$$

and

$$\begin{aligned} \varpi_2(x,t) \quad &= \quad \frac{\phi_2(0,x)}{A-1} + \frac{1}{A-1}\left(\int_0^\infty f_2(s,\ \varpi_{1s}(x),\ \varpi_{2s}(x),x)ds + \sum_{k=1}^{\infty} I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x))\right) \\ &+ \int_0^t f_2(s,\ \varpi_{1s}(x),\ \varpi_{2s}(x),x)ds + \sum_{k=1}^{\infty} I_k^2(\varpi_1(t_k,x),\ \varpi_2(t_k,x)), \end{aligned}$$

we can decompose $(\varpi_1(.,x),\ \varpi_2(.,x)))$ as

$$(\varpi_1, \varpi_2) = (z_1 + \alpha_1, z_2 + \alpha_2), t \in [0,+\infty),$$

which implies that

$$(\varpi_{1t}(x),\ \varpi_{2t}(x)) = (z_1'(x) + \alpha_1'(x), z_2'(x) + \alpha_2'(x)), t \in [0,+\infty),$$

and the function $z_1(.,x), z_2(.,x)$ satisfies

$$\begin{cases} z_1(x,t) = \displaystyle\int_0^t f_1(s,x)ds \\ + \displaystyle\sum_{k=1}^{\infty} I_k^1(z_1(t_k,x) + \alpha_1(t_k,x), z_2(t_k,x) + \alpha_2(t_k,x),x)), \\ \\ z_2(x,t) = \displaystyle\int_0^t f_2(s,x)ds \\ + \displaystyle\sum_{k=1}^{\infty} I_k^2(z_1(t_k,x) + \alpha_1(t_k,x), z_2(t_k,x) + \alpha_2(t_k,x),x)), \end{cases} \tag{33}$$

where

$$f_i \in Y_i(t, z_1'(x) + \alpha_1'(x), z_2'(x) + \alpha_2'(x), x), \ \ a.e\ t \in [0,+\infty).$$

Set

$$\mathcal{D}''_\infty = \{z_1, z_2 \in \mathcal{D}_\infty : (z_1(0,x), z_2(0,x)) = (0,0)\}.$$

Let the operator

$$P \in C^0(\mathcal{D}''_\infty \times \mathcal{D}''_\infty \times \Gamma,\ \mathcal{P}(\mathcal{D}''_\infty \times \mathcal{D}''_\infty)),$$

we have, then

$$(z_1, z_2) \to (P_1(x, z_1, z_2), P_2(x, z_1, z_2)), \quad (z_1, z_2) \in \mathcal{D}''_\infty \times \mathcal{D}''_\infty,$$

with

$$P(t, z_1, z_2)) = \{(\rho_1, \rho_1) \in \mathcal{D}''_\infty \times \mathcal{D}''_\infty\},$$

where

$$\rho_1 = \begin{cases} 0, & t \in (-\infty, 0] \\ \displaystyle\int_0^t f_1(s,x)ds + \\ \displaystyle + \sum_{k=1}^{\infty} I_k^1(z_1(t_k,x) + \alpha_1(t_k,x), z_2(t_k,x) + \alpha_2(t_k,x)), & t \in [0, +\infty) \end{cases}$$

and

$$\rho_2 = \begin{cases} 0 & t \in (-\infty, 0] \\ \displaystyle\int_0^t f_2(s,x)ds \\ \displaystyle + \sum_{k=1}^{\infty} I_k^2(z_1(t_k,x) + \alpha_1(t_k,x), z_2(t_k,x) + \alpha_2(t_k,x)). & t \in [0, +\infty) \end{cases}$$

Clearly fixed points of the operator P are random solutions of problem (4). For $x \in \Gamma$ fixed, consider

$$P_x \in C^0(\mathcal{D}''_\infty \times \mathcal{D}''_\infty, \mathcal{P}(\mathcal{D}''_\infty \times \mathcal{D}''_\infty)),$$

for

$$(z_1, z_2) \in \mathcal{D}'_\infty \times \mathcal{D}'_\infty,$$

by

$$P_x(z_1, z_2) = (P_1(x, z_1, z_2), P_2(x, z_1, z_2)). \tag{34}$$

Obviously, that the operator T_x has a fixed point is equivalent to P_x has a fixed point. We will prove that T_x verifies the claims of Theorem 3. The proof will be carried out in several steps. First we should prove that P_x is completely continuous.

Claim 1. *$P_x(z_1, z_2)$ is convex for each $(z_1, z_2) \in \mathcal{D}'_0 \times \mathcal{D}'_0$. Indeed, if ρ_1^1, ρ_1^2 belong to $P_1(z_1, z_2)$, then there exist*

$$f_1^1, f_1^2 \in S_{Y_1, z_1+\alpha_1, z_2+\alpha_2},$$

such that, for each $t \in J$, we have

$$\rho_1^i(t) = \int_0^t f_1^i(s,x)ds + \sum_{k=1}^{\infty} I_k^1(z_1(t_k,x) + \alpha_1(t_k,x), z_2(t_k,x) + \alpha_2(t_k,x)).$$

Let $0 \le \delta \le 1$. Then, for each $\in (J, \Gamma)$, we have

$$\begin{aligned} (\delta\rho_1^1 + (1-\delta)\rho_1^1) &= \int_0^t (\delta f_1^i(s,x) + (1-\delta)\delta)ds \\ &+ \sum_{k=1}^{\infty} I_k^1(z_1(t_k,x) + \alpha_1(t_k,x), z_2(t_k,x) + \alpha_2(t_k,x)). \end{aligned}$$

Because $S_{Y_1, z_1+\alpha_1, z_2+\alpha_2}$ is convex ($Y(t, z_1, z_2)$ has convex values), one has

$$(\delta\rho_1^1 + (1-\delta)\rho_1^1) \in P_1(x, z_1, z_2).$$

Similarly, for P_2, we have

$$(\delta\rho_2^1 + (1-\delta)\rho_2^1) \in P_2(x, z_1, z_2).$$

Claim 2. *P_x maps bounded sets into bounded sets in $\mathcal{D}''_\infty \times \mathcal{D}''_\infty$. Indeed, it is enough to show that there exists a positive constant (l_1, l_2) such that for each $(\rho_1, \rho_2) \in P_x$. Let*

$$B_p \times B_q = \{(z_1, z_2) \in \mathcal{D}''_\infty \times \mathcal{D}''_\infty : \|(z_1, z_2)\| \le (p, q)\},$$

where

$$\|(z_1, z_2)\| = \left(\|z_1\|_{\mathcal{D}'_\infty}, \ \|z_2\|_{\mathcal{D}'_\infty}\right).$$

Let $(z_1, z_2) \in B_p \times B_q$*, then for each* $t \in [0, \infty)$*,*

$$\begin{aligned} |\rho_1| &= \left| \int_0^t f_1(s,x)ds + \sum_{k=1}^{\infty} I_k^1(z_1(t_k,x) + \alpha_1(t_k,x), z_2(t_k,x) + \alpha_2(t_k,x)) \right| \\ &\leq \int_0^t p_1(s,x)ds + \sum_{k=1}^{\infty} c_k^1 \\ &= l_1 \\ &< \infty. \end{aligned}$$

Similarly, for ρ_2*, we have*

$$|\rho_2| \leq \int_0^t p_2(s,x)ds + \sum_{k=1}^{\infty} c_k^2 = l_2 < \infty. \tag{35}$$

Claim 3. *P_x maps bounded sets into equi-continuous sets of $\mathcal{D}''_\infty \times \mathcal{D}''_\infty$. Let $B_p \times B_q$ be a bounded set in $\mathcal{D}'_\infty \times \mathcal{D}''_\infty$ as in Step 1 is an equi-continuous set of $\mathcal{D}''_\infty \times \mathcal{D}''_\infty$. Let $\tau_1, \ \tau_2 \in [0, \infty)$* such that $\tau_1 < \tau_2 < \infty$, and $(z_1, z_2) \in B_p \times B_q$*. Then*

$$\begin{aligned} |\rho_i(\tau_2, x) - \rho_i(\tau_1, x)| &\leq \int_{\tau_1}^{\tau_2} |f_i(s, z_{1,s}(x) + \alpha_{1,s}(x), z_{2,s}(x) + \alpha_{2,s}(x), x)| ds \\ &\quad + \sum_{0<t<\tau_2-\tau_1} |I_k^i(z_1(t_k,x) + \alpha_1(t_k,x), z_2(t_k,x) + \alpha_2(t_k,x))| \\ &\leq \int_{\tau_1}^{\tau_2} p_i(s,x)ds + \sum_{0<t<\tau_2-\tau_1} c_k^1. \end{aligned} \tag{36}$$

The RHS tends to 0 as $\tau_2 - \tau_1 \to 0$*. By a similar way we can prove the equi-continuity for* $N_2(B_p, B_q)$*.*

As a consequence of Claim 2 and 3, together with the Arzelà–Ascoli theorem, we conclude that

$$P_x : \mathcal{D}''_\infty \times \mathcal{D}''_\infty \to \mathcal{P}(\mathcal{D}''_\infty \times \mathcal{D}''_\infty),$$

is completely continuous.

Claim 4. *P_x has a closed graph.*

Let (z_1^n, z_2^n) *be a sequence such that*

$$(z_1^n, z_2^n) \to (z_1^*, z_2^*) \text{ in } \mathcal{D}'_\infty \times \mathcal{D}'_\infty \text{ as } n \to \infty,$$

and

$$\rho_i^n \in P_1(x, z_1^n, z_2^n), \ \ \rho_i^n \to \rho_i^* \text{ as } n \to \infty.$$

we shall prove that $\rho_i^* \in P_1(x, z_1^*, z_2^*)$*.*

Because $\rho_i^n \in P_1(x, z_1^n, z_2^n)$*, then there exists* $f_i^n \in S_{\Upsilon_i, z_1^n+\alpha_1, z_2^n+\alpha_2}$ *such that*

$$\rho_i^n = \int_0^t f_1^n(s,x)ds + \sum_{k=1}^{\infty} I_k^2(z_1^n(t_k,x) + \alpha_1(t_k,x), z_2^n(t_k,x) + \alpha_2(t_k,x)), \quad t \in J.$$

We must prove that there exists $f_i^* \in S_{\Upsilon_i, z_1^*+\alpha_1, z_2^*+\alpha_2}$ *such that*

$$\rho_i^* = \int_0^t f_1^*(s,x)ds + \sum_{k=1}^{\infty} I_k^2(z_1^*(t_k,x) + \alpha_1(t_k,x), z_2^*(t_k,x) + \alpha_2(t_k,x)), \quad t \in J.$$

Consider the linear continuous operator

$$R : L^1(J \times \Gamma, \mathbb{R}^n) \to \mathcal{D}''_\infty \times \mathcal{D}''_\infty,$$

defined by

$$R(f)(t)=\left(\int_0^t f_1(s)ds,\ \int_0^t f_1(s)ds\right). \tag{37}$$

From Lemma 2, it follows that $R\circ S_{Y_i}$ is a closed graph operator. Moreover, we have that

$$\rho_i^n-\sum_{k=1}^{\infty} I_k^i(z_1^n(t_k,x)+\alpha_1(t_k,x),z_2^n(t_k,x)+\alpha_2(t_k,x))\in R(S_{Y_i,z_1^*+\alpha_1,z_2^*+\alpha_2}).$$

Because $(z_1^n,z_1^n)\to(z_1^*,z_1^*)$ and $\rho_i^n\to\rho_i^*$ there is $f_i^*\in S_{Y_i,z_1^*+\alpha_1,z_2^*+\alpha_2}$ such that

$$\rho_i^*=\int_0^t f_1^*(s,x)ds+\sum_{k=1}^{\infty} I_k^2(z_1^*(t_k,x)+\alpha_1(t_k,x),z_2^*(t_k,x)+\alpha_2(t_k,x)),\quad t\in J.$$

Therefore, P_x is completely continuous.

Claim 5. *There exist a priori bounds on solutions*

$$\mathcal{M}=\{(z_1,z_2)\in\mathcal{D}''_\infty\times\mathcal{D}''_\infty:\ (z_1,z_2)\in\lambda(x)P_x(z_1,z_2),\ \lambda(x)\in(0,1)\}, \tag{38}$$

is bounded for some measurable function $\lambda:\Gamma\to\mathbb{R}$. Then

$$z_1\in\lambda(x)P_1(x,z_1,z_2),\quad z_2\in\lambda(x)P_2(x,z_1,z_2).$$

For some $0<\lambda(x)<1$, we have

$$\begin{aligned}|z_1| &\le |\lambda(x)|\int_0^t(f_1(s,z_{1,s}(x)+\alpha_{1,s}(x),z_{2,s}(x)+\alpha_{2,s}(x),x))ds\\ &\quad+\sum_{0<t_k<t}\Big(I_k^1(z_1(t_k,x)+\alpha_1(t_k,x),z_2(t_k,x)+\alpha_2(t_k,x))\Big)\\ &\le \int_0^t p_1(s,x)ds+\sum_{k=1}^{\infty}c_k^1.\end{aligned}$$

Similarly

$$|z_2|\le\int_0^t p_2(s,x)ds+\sum_{k=1}^{\infty}c_k^2. \tag{39}$$

By (39)*, we have*

$$|z_1|+|z_2|\le\sum_{i=1}^{2}\int_0^t p_i(s,x)ds+\sum_{i=1}^{2}\sum_{k=1}^{\infty}c_k^i.$$

This implies that for each $t\in[0,\ \infty)$ and there exist positive constants $\beta>0$ such that

$$\|z_1\|_{\mathcal{D}_0}+\|z_2\|_{\mathcal{D}''_\infty}\le\sum_{i=1}^{2}\int_0^\infty p_i(s,x)ds+\sum_{i=1}^{2}\sum_{k=1}^{\infty}c_k^i\le\beta. \tag{40}$$

Finally from (40) there exists a constant $\beta_1,\ \beta_2>0$ such that

$$\|z_1\|_{\mathcal{D}'\infty}\le\beta_1\quad\text{and}\quad\|z_2\|_{\mathcal{D}'\infty}\le\beta_2.$$

Set

$$U=\{(z_1,z_2)\in\mathcal{D}'\infty\times\mathcal{D}'\infty:(\|z_1\|_{\mathcal{D}'\infty},\ \|z_1\|_{\mathcal{D}'\infty})<(\beta_1+1,\ \beta_2+1)\},$$

$P_x:\bar{U}\to\mathcal{P}(\mathcal{D}'_\infty\times\mathcal{D}''_\infty)$ is completely continuous. From the choice of U, there is no $z_1,z_2\in\partial U$ such that $(z_1,z_2)\in\lambda(x)P_x(z_1,z_2)$, for some $\lambda(x)\in(0,1)$. Thus by Theorem 3 the operator P_x has at least one fixed (z_1,z_2) in U. Hence T_x has a fixed point $(\varpi_1,\ \varpi_2)$, which is a random solution to problem (4).

□

5. Conclusions

This work falls within a series of related research carried out by the same authors, and many results were achieved using new recent methods related to iterative theory and developing some techniques to ensure the solutions exist according to different requirements imposed by the random action and delay.

We sought to give as complete and objective studies as possible of the main result in coupled random first-order impulsive differential equations with infinite delay. However, it is surely true that the works that lies in the field of scientific interests of this model can be covered in somewhat more detail. Examples of genuine processes and phenomena explored in physics, chemical technology, population dynamics, biotechnology, and economics are described by delayed impulsive differential systems with the presence of new random properties. The novelties of our contribution are follows:

1. Applying a novel random fixed-point theorem to a system of impulsive random differential equations was our primary objective.
2. We provided a random application of the separable vector-valued Banach space Leray–Schauder fixed-point theorem in nonlinear case.

Extending these results to consider the question of stability (qualitative studies) will make it possible to advance the study in this direction, which will be our next project, see [30–34].

Author Contributions: Conceptualization, A.M., F.Z.L., M.F., A.B.C. and K.B.; Methodology, A.M., F.Z.L., M.F., A.B.C. and K.B.; Writing—original draft, A.M., F.Z.L., M.F., A.B.C. and K.B.; Writing—review & editing, A.M., F.Z.L., M.F., A.B.C., M.B. and K.B.; project administration, A.M. and M.B. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by King Khalid University through large research project under grant number R.G.P.2/252/44.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Acknowledgments: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work.

Conflicts of Interest: Authors declare that they have no conflict of interest.

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