fractal and fractional

Article

A Finite-Dimensional Control Scheme for Fractional-Order Systems under Denial-of-Service Attacks

Ying Zou [1,†], Xinyao Li [2,*,†], Chao Deng [3] and Xiaowen Wu [1]

1 School of Information and Electrical Engineering, Hunan University of Science and Technology, Xiangtan 411201, China; yingz_2020@126.com (Y.Z.); xwu@hnust.edu.cn (X.W.)
2 School of Automation, Guangdong Polytechnic Normal University, Guangzhou 510450, China
3 Institute of Advanced Technology, Nanjing University of Posts and Telecommunications, Nanjing 210023, China; dengchao_neu@126.com
* Correspondence: lixinyao@gpnu.edu.cn or E180209@e.ntu.edu.sg
† These authors are co-first authors.

Abstract: In this article, the security control problem of discrete-time fractional-order networked systems under denial-of-service (DoS) attacks is considered. A practically applicable finite-dimensional control strategy will be developed for fractional-order systems that possess nonlocal characteristics. By employing the Lyapunov method, it is theoretically proved that under the proposed controller, the obtained closed-loop fractional system is globally input-to-state stable (ISS), even in the presence of DoS attacks. Finally, the effectiveness of the designed control method is demonstrated by the numerical example.

Keywords: security control; discrete-time networked systems; denial-of-service (DoS) attacks; fractional-order

check for updates

Citation: Zou, Y.; Li, X.; Deng, C.; Wu, X. A Finite-Dimensional Control Scheme for Fractional-Order Systems under Denial-of-Service Attacks. *Fractal Fract.* **2023**, *7*, 562. https://doi.org/10.3390/fractalfract7070562

Academic Editors: Da-Yan Liu, Driss Boutat, Xuefeng Zhang, Jean-Claude Trigeassou and Jin-Xi Zhang

Received: 1 July 2023
Revised: 17 July 2023
Accepted: 18 July 2023
Published: 21 July 2023

1. Introduction

The utilization of fractional-order calculus, due to its distinctive nonlocal features, is highly advantageous in precisely representing the dynamic characteristics of a multitude of real-world phenomena or systems that have infinite memory; see, for instance, [1–3].

The research and development of fractional-order systems and their associated controls have recently been garnering increased interest [4–6]. It has been proposed that fractional-order differential equations can more accurately capture the rheological constitutive equation (RCE) for the viscoelasticity of polymer materials, as evidenced in [1,5]. In [7], a full-cell model of fractional order with a distinct physical interpretation was developed. Two studies [8,9], utilize fractional-order methods to simulate lithium-ion batteries. Several recent works regarding the application of fractional systems can be found in, for example, [10–13]. For the continuous-time case, the concept of a proportional integral derivative (PID) controller of fractional order was initially presented in [14]. The linear fractional-order systems' stability is discussed in [15]. Sampled-data control schemes for linear fractional-order systems that take the unique properties of fractional-order calculus into account are investigated in [16–18]. The fractional-order Lyapunov direct stability analysis method employed in [19] yields Mittag–Leffler stable conditions for nonlinear fractional-order systems. In [20–23], the development of adaptive backstepping-based controllers for fractional-order uncertain nonlinear systems subject to unknown disturbances is reported. An adaptive fractional controller is developed for high-order nonlinear integer uncertain systems in [24]. By truncating the fractional operator, which is infinite-dimensional, refs. [25–28] propose the finite-dimensional control approaches for fractional-order systems in the discrete-time domain based on the truncated approximated finite-dimensional systems. Further improvement was then made in [29–31], where the finite-dimensional approximation errors were considered in designing a controller by treating them as the additive uncertainty terms, thus ensuring practical asymptotic stability of the actual fractional-order systems.

Attacks on communication links in a networked control system can be divided into two categories: denial-of-service (DoS) attacks and deception attacks. An interruption of communication between networks leads to a DoS attack [32], whereas deception attacks usually involve altering the data that are sent [33]. This article focuses on DoS attacks. Nowadays, some controllers have been created to mitigate the impact of DoS attacks on integer-order systems [34–38]. In [37,38], event-triggered resilient cooperative control schemes are proposed for continuous-time multiagent systems under DoS attacks, such that the controlled multiagent systems can achieve secure consensus exponentially. As for discrete-time multiagent systems in the presence of aperiodic sampling and random DoS attacks, a distributed output-feedback control scheme is developed to reach output consensus by assuming that the sampling process is nonuniform and the consecutive attack duration is upper-bounded [36]. Despite this, there are a limited number of published security control studies for fractional-order systems in the literature. References [39,40] analyze the control problem for continuous-time fractional-order multiagent systems and complex networks that are vulnerable to DoS attacks, respectively. Reference [41] studies the control of discrete-time fractional-order multiagent systems under DoS attacks, disregarding the nonlocal characteristics of fractional-order calculus when designing the control scheme. As of yet, the topic of DoS attacks and their effects on discrete-time fractional-order networked systems has not been explored in depth, which provides the impetus for this work.

In spite of the aforementioned discussion, in this work, we analyze the discrete-time fractional-order systems in which the plant and controller are connected via the network, while the attacker attempts to disrupt the control system's stability by hindering communication between the sensors and controller (measurement channel). The main contributions of this work are outlined in the following:

1. The development of a safety control protocol for discrete-time fractional-order systems subject to external disturbance and DoS attacks is investigated in this article, with the unique properties of fractional-order calculus being taken into account.
2. The controller proposed is finite-dimensional, which makes it possible to calculate the control input with only a limited number of prior system states, making it suitable for practical use.
3. A sufficient condition is provided to guarantee the global stability of the closed-loop system, resulting in the system output eventually settling at an ultimate bound around the origin.

This article is organized as follows: Section 2 presents the problem statement. The controller design procedure is given in Section 3. In Section 4, the proposed control strategy will be examined by simulation example, and finally, Section 5 provides the conclusions of this work.

2. Problem Formulation

Consider a discrete-time linear fractional system described as follows:

$$\begin{cases} {}_0\Delta^{\alpha}_{k+1} z(k+1) = Az(k) + Bu(k) + B_\omega \omega(k) \\ y(k) = C_2 z(k),\ z(0) = z_0 \end{cases} \tag{1}$$

where $z(k) \in \mathbb{R}^n$ is the state vector at time step $k \in \mathbb{N}_0$, ${}_0\Delta^{\alpha}_{k+1} z(k+1) = \left[{}_0\Delta^{\alpha_1}_{k+1} z_1(k+1), \cdots, {}_0\Delta^{\alpha_n}_{k+1} z_n(k+1)\right]^T$, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^p$ is the measurement output, $\omega(k) \in \mathbb{R}^r$ is the exogenous disturbance signal bounded as $\|\omega(k)\| \leq q_\omega$ with $q_\omega > 0$ and A, B, B_ω and C_2 are known real matrices with appropriate dimensions. In accordance with Remark 4 in [18], the state, control signal, and disturbance before the initial time are considered to be all equal to zero in this work, i.e., $z(q) = 0$, $u(q) = 0$ and $\omega(q) = 0, \forall q < 0$.

Remark 1. *For discrete-time fractional systems, several existing works [25–31] focus on the control problem. Nevertheless, none of these present studies concern the influence of attacks when designing control schemes, which highlights the advantages of our work where a safety control strategy is proposed for discrete-time fractional systems.*

According to [42], the Grünwald–Letnikov (GL) fractional-order difference of a discrete-variable bounded function $f(k) : \mathbb{N}_0 \to \mathbb{R}$ is defined as

$$ {}_0\Delta_k^{\alpha} f(k) = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} f(k-j),\ \alpha \in \mathbb{R}^+,\ k \in \mathbb{N}_0, \tag{2} $$

where

$$ \binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & j > 0 \end{cases} \text{for all } j \in \mathbb{N}_0. \tag{3} $$

Define $c_j(\alpha) = (-1)^j \binom{\alpha}{j}$, and due to the fact that $|c_j(\alpha)| \le \frac{\alpha^j}{j!}$, for any $\alpha \in \mathbb{R}^+$, the sequence $\{c_j(\alpha)\}_{j \in \mathbb{N}_0}$ is absolutely summable.

For the i-th state $(i = 1, 2, \cdots, n)$, we have

$$ {}_0\Delta_{k+1}^{\alpha_i} z_i(k+1) = \sum_{j=0}^{k+1} c_j(\alpha_i) z_i(k+1-j), \tag{4} $$

in which α_i is the fractional-order corresponding to z_i. Equation (4) can be rewritten as

$$ z_i(k+1) = {}_0\Delta_{k+1}^{\alpha_i} z_i(k+1) - \sum_{j=1}^{k+1} c_j(\alpha_i) z_i(k+1-j), \tag{5} $$

and hence, the evolution of $z(k+1)$ can be expressed as

$$ z(k+1) = Az(k) + Bu(k) + B_\omega \omega(k) - \sum_{j=1}^{k+1} F_j(\alpha) z(k+1-j) \tag{6} $$

where $F_j(\alpha) = \mathrm{diag}\big(c_j(\alpha_1), c_j(\alpha_2), \cdots, c_j(\alpha_n)\big) \in \mathbb{R}^n$. Alternatively, (6) can be presented as

$$ z(k+1) = \sum_{j=0}^{k} A_j z(k-j) + Bu(k) + B_\omega \omega(k) \tag{7} $$

in which $A_0 = A - F_1(\alpha)$ and $A_j = -F_{j+1}(\alpha)$ for $j \ge 1$. As a result, the linear discrete-time fractional-order system (1) can be described as

$$ \begin{cases} z(k+1) = \sum_{j=0}^{k} A_j z(k-j) + Bu(k) + B_\omega \omega(k) \\ y(k) = C_2 z(k),\ z(0) = z_0. \end{cases} \tag{8} $$

Notation. I_n indicates an identity matrix with a dimension equal to n. $\mathbb{R}^+$, $\mathbb{R}$, $\mathbb{N}_0$, $\mathbb{N}$, and $\mathbb{Z}$, respectively, represent the set of non-negative reals, reals, non-negative integers, positive integers, and integer numbers. The identity function is denoted by Id, i.e., $Id(z) = z$. $\|z\|\_\infty = \sup_{j\in\mathbb{Z}}\|z(j)\|$ and $\|z\|\_{[j_1,j_2]}$ is defined as:

$$\|z\|\_{[j_1,j_2]} = \begin{cases} \max\limits_{j_1\le j\le j_2} \|z(j)\|, & \text{if } j_1 \le j_2 \\ 0, & \text{if } j_1 > j_2. \end{cases}$$

It can be seen from the expression of the GL fractional-order difference of function $f(k)$ given in (2) that the cumulative term used to implement the fractional-order difference will increase as k increases. Different from the controller proposed in [43], which is a linear weighted combination of all the past states of the observer, which will consequently result in the computational explosion as time goes by, a finite-dimensional controller that contains finite steps of recent states is considered in this work.

Reformulate (8) as

$$\begin{cases} \hat{z}(k+1) = \hat{A}_L\hat{z}(k) + \hat{B}u(k) + \hat{B}_\omega\hat{\omega}(k) \\ y(k) = \hat{C}_2\hat{z}(k), \end{cases} \tag{9}$$

where $L \in \mathbb{N}_0$, $\hat{z}(k) = \left[z^{\mathrm{T}}(k), z^{\mathrm{T}}(k-1), \cdots, z^{\mathrm{T}}(k-L)\right]^{\mathrm{T}} \in \mathbb{R}^{(L+1)n}$ is the recent-finite-steps state vector, $\hat{\omega}(k) = \left[\omega^{\mathrm{T}}(k), z^{\mathrm{T}}(k-L-1), \cdots, z^{\mathrm{T}}(0)\right]^{\mathrm{T}} \in \mathbb{R}^{r+(k-L)n}$, and $\hat{A}_L$, $\hat{B}$, $\hat{B}_\omega$ and $\hat{C}_2$ are defined as

$$\hat{A}_L = \begin{bmatrix} A_0 & A_1 & \cdots & A_{L-1} & A_L \\ I_n & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_n & 0 \end{bmatrix}, \hat{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\hat{B}_\omega = \begin{bmatrix} B_\omega & A_{L+1} & \cdots & A_{k-1} & A_k \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \hat{C}_2^T = \begin{bmatrix} C_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Assumption 1. *The pair $(\hat{A}_L, \hat{B})$ is stabilizable.*

Assumption 2. *No DoS attacks occur at initial time $k = 0$.*

Remark 2. *Assumption 1 is needed for the existence of the positive matrices P and Q that will be applied during the controller design in Section 3. As will be mentioned in Section 3, under the influence of DoS attacks, in this paper, the latest received state will be utilized for control until receiving the next successfully transmitted state, and thus, Assumption 2 is required for avoiding the whole control system becoming open-loop at the beginning of the control stage.*

The openness of network communication in networked control makes it susceptible to cyberattacks. This study investigates the effects of aperiodic DoS attacks on the measurement channel. A graphical illustration of the total networked fractional-order system under our proposed control law is depicted in Figure 1.

Suppose that a DoS attack occurs in the measurement channel at the instant $a_i \in [k-L, k)$, where i denotes the i-th attack event. The control objective of this work is to design a finite-dimensional controller that the corresponding closed-loop fractional-order system (1) is globally input-to-state stable (ISS) under DoS attacks, i.e., for arbitrary initial condition $z_0 \in \mathbb{R}^n$, the closed-loop state satisfies

$$\|z(k)\| \le \beta_a(\|z_0\|, k) + \gamma_a(q_\omega), k \in \mathbb{N}_0, \tag{10}$$

where function β_a is a $\mathcal{KL}$-function and γ_a is a $\mathcal{K}$-function. Detailed definitions of the $\mathcal{KL}$-function, $\mathcal{K}_\infty$-function, and $\mathcal{K}$-function [44] are given below.

A continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{K}$-function if it is strictly increasing and $\gamma(0) = 0$. A continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{KL}$-function if for each fixed $t \geq 0$, function $\beta(\cdot, t)$ is a $\mathcal{K}$-function, and for each fixed $s \geq 0$, function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

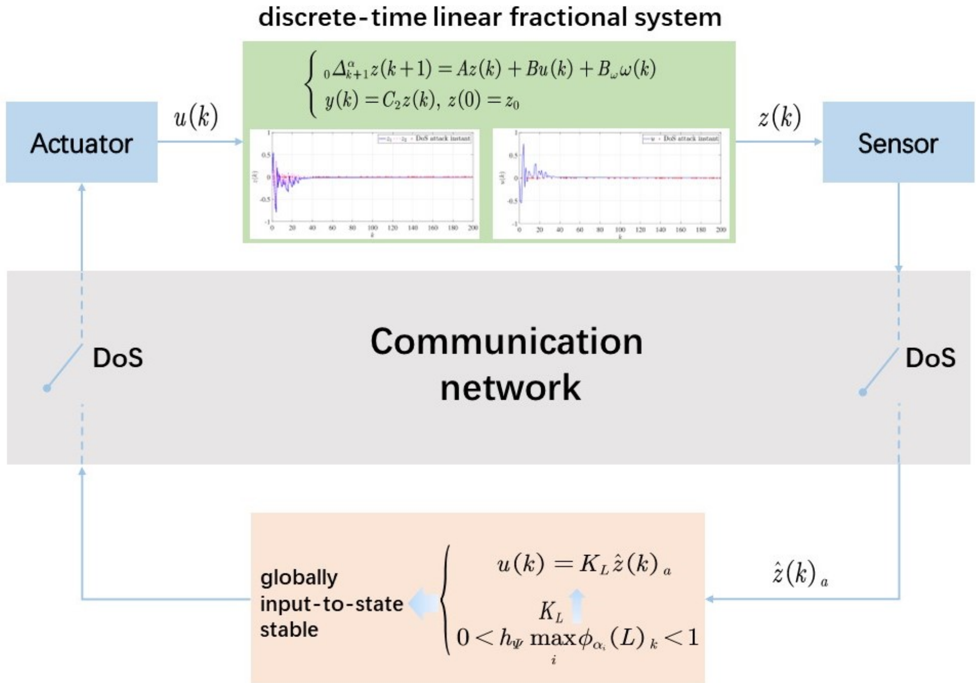

Figure 1. Framework of closed-loop fractional-order networked systems under DoS attacks.

3. Controller Design

Define the difference vector as

$$\hat{e}(k) = \hat{z}(k)_a - \hat{z}(k) \tag{11}$$

where $\hat{z}(k)_a$ stands for the recent-finite-steps state vector under DoS attacks that is utilized for controller design. In this work, if a DoS attack occurs at the current time instant, or in other words, no information is available for the controller currently, then the latest received state will be used for controlling until the next successful state information transmission. The controller is designed as

$$u(k) = K_L \hat{z}(k)_a \tag{12}$$

with control matrix $K_L \in \mathbb{R}^{m \times (L+1)n}$.

Theorem 1. *Consider the closed-loop discrete-time fractional-order system consisting of the system (1) and controller (12). Under Assumptions 1 and 2, for $L \in \mathbb{N}_0$ that satisfies $0 < h_\Psi \max_i \phi_{\alpha_i}(L)_k < 1$ $(i = 1, \cdots, n)$, where h_Ψ and $\phi_{\alpha_i}(L)_k$ will be defined later, then the obtained closed-loop fractional system under DoS attacks is ensured to be globally ISS, with (10) being satisfied.*

Proof. With the controller designed in (12), the evolution of the recent-finite-steps state in the closed-loop system can be expressed as

$$\begin{aligned} \hat{z}(k+1) =& \hat{A}_L \hat{z}(k) + \hat{B} K_L \hat{z}(k)_a + \hat{B}_\omega \hat{\omega}(k)_a \\ =& \hat{A}_L \hat{z}(k) + \hat{B} K_L \hat{z}(k) + \hat{B} K_L \hat{e}(k) + \bar{\omega}(k)_a \end{aligned}$$

$$=\Phi\hat{z}(k)+\hat{B}K_L\hat{e}(k)+\bar{\omega}(k)_a \tag{13}$$

where $\hat{\omega}(k)_a$ is the $\hat{\omega}(k)$ under DoS attacks, $\bar{\omega}(k)_a = \hat{B}_\omega\hat{\omega}(k)_a$ and $\Phi = \hat{A}_L + \hat{B}K_L$. According to Assumption 1, for any positive definite matrix Q, $\Phi^{\mathrm{T}}P\Phi - P + Q = 0$ holds for a positive definite matrix P.

Let the Lyapunov function be $W(k) = \hat{z}^{\mathrm{T}}(k)P\hat{z}(k)$, then

$$\begin{aligned}W(k+1) =&\hat{z}^{\mathrm{T}}(k+1)P\hat{z}(k+1)\\ =&\hat{z}^{\mathrm{T}}(k)(P-Q)\hat{z}(k)+2\hat{z}^{\mathrm{T}}(k)\Phi^{\mathrm{T}}P\hat{B}K_L\hat{e}(k)\\ &+2\hat{z}^{\mathrm{T}}(k)\Phi^{\mathrm{T}}P\bar{\omega}(k)_a+\hat{e}^{\mathrm{T}}(k){K_L}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\hat{B}K_L\hat{e}(k)\\ &+2\hat{e}^{\mathrm{T}}(k){K_L}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\bar{\omega}(k)_a+\bar{\omega}(k)_a^{\mathrm{T}}P\bar{\omega}(k)_a\\ =&W(k)-\hat{z}^{\mathrm{T}}(k)Q\hat{z}(k)+2\hat{z}^{\mathrm{T}}(k)\Phi^{\mathrm{T}}P\hat{B}k_L\hat{e}(k)\\ &+2\hat{z}^{\mathrm{T}}(k)\Phi^{\mathrm{T}}P\bar{\omega}(k)_a+\hat{e}^{\mathrm{T}}(k){K_L}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\hat{B}K_L\hat{e}(k)\\ &+2\hat{e}^{\mathrm{T}}(k){K_L}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\bar{\omega}(k)_a+\bar{\omega}(k)_a^{\mathrm{T}}P\bar{\omega}(k)_a.\end{aligned} \tag{14}$$

From (14), we obtain

$$\begin{aligned}W(k+1)-W(k)\le&-\lambda_{\min}(Q)\|\hat{z}(k)\|^2+\|2\Phi^{\mathrm{T}}P\hat{B}K_L\|\|\hat{z}(k)\|\|\hat{e}(k)\|\\ &+\|2\Phi^{\mathrm{T}}P\|\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|+[{K_L}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\hat{B}K_L]\|\hat{e}(k)\|^2\\ &+\|2P\hat{B}K_L\|\|\hat{e}(k)\|\|\bar{\omega}(k)_a\|+P\|\bar{\omega}(k)_a\|^2.\end{aligned} \tag{15}$$

The following inequality should be guaranteed to ensure closed-loop system stability:

$$\|\hat{e}(k)\|\le\sigma\|\hat{z}(k)\|+\sigma\|\bar{\omega}(k)_a\|, \tag{16}$$

where $\sigma\in\mathbb{R}^+$ is a suitable designed parameter. Substituting (16) into (15), we have

$$\begin{aligned}W(k+1)-W(k)\le&-\lambda_{\min}(Q)\|\hat{z}(k)\|^2+\|2\Phi^{\mathrm{T}}P\hat{B}K_L\|\sigma\|\hat{z}(k)\|^2\\ &+\|2\Phi^{\mathrm{T}}P\hat{B}K_L\|\sigma\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|\\ &+\|2\Phi^{\mathrm{T}}P\|\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|\\ &+[{K_L}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\hat{B}K_L]\sigma^2\big[\|\hat{z}(k)\|^2+\|\bar{\omega}(k)_a\|^2\\ &+2\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|\big]+\|2P\hat{B}K_L\|\sigma\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|\\ &+\|2P\hat{B}K_L\|\sigma\|\bar{\omega}(k)_a\|^2+P\|\bar{\omega}(k)_a\|^2.\end{aligned} \tag{17}$$

Let $\zeta_1=\lambda_{\min}(Q)$, $\zeta_2=\|2\Phi^{\mathrm{T}}P\hat{B}K_L\|$, $\zeta_3=\|2\Phi^{\mathrm{T}}P\|$, $\zeta_4=\lambda_{\max}({K_L}^{\mathrm{T}}\hat{B}^{\mathrm{T}}P\hat{B}K_L)$, $\zeta_5=\|2P\hat{B}K_L\|$, $\eta_2=\lambda_{\max}(P)$, then

$$\begin{aligned}W(k+1)-W(k)\le&-\zeta_1\|\hat{z}(k)\|^2+\zeta_2\sigma\|\hat{z}(k)\|^2+\zeta_2\sigma\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|\\ &+\zeta_3\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|+\zeta_4\sigma^2\|\hat{z}(k)\|^2+\zeta_4\sigma^2\|\bar{\omega}(k)_a\|^2\\ &+\zeta_5\sigma\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|+\zeta_5\sigma\|\bar{\omega}(k)_a\|^2+\eta_2\|\bar{\omega}(k)_a\|^2\\ &+2\zeta_4\sigma^2\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|\\ =&-(\zeta_1-\zeta_2\sigma-\zeta_4\sigma^2)\|\hat{z}(k)\|^2+(\zeta_2\sigma+\zeta_3+\zeta_5\sigma+\\ &2\zeta_4\sigma^2)\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|+(\zeta_4\sigma^2+\zeta_5\sigma+\eta_2)\|\bar{\omega}(k)_a\|^2.\end{aligned} \tag{18}$$

Choosing σ that satisfies $\zeta_1-\zeta_2\sigma-\zeta_4\sigma^2>0$ and letting $\zeta_6=\zeta_1-\zeta_2\sigma-\zeta_4\sigma^2$, $\zeta_7=\zeta_2\sigma+\zeta_3+\zeta_5\sigma+2\zeta_4\sigma^2$, $\zeta_8=\zeta_4\sigma^2+\zeta_5\sigma+\eta_2$, $\zeta_9=\frac{\zeta_7^2}{2\zeta_6}+\zeta_8$, then

$$W(k+1)-W(k)\le-\zeta_6\|\hat{z}(k)\|^2+\zeta_7\|\hat{z}(k)\|\|\bar{\omega}(k)_a\|+\zeta_8\|\bar{\omega}(k)_a\|^2. \tag{19}$$

Since for any positive real scalar δ, the following inequality holds:

$$\|\hat{z}(k)\|\|\bar{\omega}(k)_a\| \leq \frac{\|\hat{z}(k)\|^2}{2\delta} + \frac{\delta\|\bar{\omega}(k)_a\|^2}{2}, \tag{20}$$

Thus, (19) becomes

$$W(k+1) - W(k) \leq -\zeta_6\|\hat{z}(k)\|^2 + \frac{\zeta_7}{2\delta}\|\hat{z}(k)\|^2 + \frac{\zeta_7\delta}{2}\|\bar{\omega}(k)_a\|^2 + \zeta_8\|\bar{\omega}(k)_a\|^2. \tag{21}$$

Let $\delta = \frac{\zeta_7}{\zeta_6}$, then we obtain

$$\begin{aligned} W(k+1) - W(k) \leq& -\zeta_6\|\hat{z}(k)\|^2 + \frac{\zeta_6}{2}\|\hat{z}(k)\|^2 + \frac{\zeta_7^2}{2\zeta_6}\|\bar{\omega}(k)_a\|^2 + \zeta_8\|\bar{\omega}(k)_a\|^2 \\ =& -\frac{\zeta_6}{2}\|\hat{z}(k)\|^2 + \zeta_9\|\bar{\omega}(k)_a\|^2. \end{aligned} \tag{22}$$

Define $\mu_1(\|\hat{z}(k)\|) = \lambda_{\min}(P)\|\hat{z}(k)\|^2$, $\mu_2(\|\hat{z}(k)\|) = \lambda_{\max}(P)\|\hat{z}(k)\|^2$, $\mu_3(\|\hat{z}(k)\|) = \frac{\zeta_6}{2}\|\hat{z}(k)\|^2$, $\mu_4(W(k)) = \mu_3 \circ \mu_2^{-1}(W(k)) = \frac{\zeta_6}{2\eta_2}W(k)$, $\mu_5(\|\bar{\omega}(k)_a\|) = \zeta_9\|\bar{\omega}(k)_a\|^2$; therefore we obtain

$$\mu_1(\|\hat{z}(k)\|) \leq W(k) \leq \mu_2(\|\hat{z}(k)\|), \tag{23}$$

and

$$\begin{aligned} W(k+1) - W(k) &\leq -\mu_3(\|\hat{z}(k)\|) + \mu_5(\|\bar{\omega}(k)_a\|) \\ &\leq -\mu_4(W(k)) + \mu_5(\|\bar{\omega}(k)_a\|). \end{aligned} \tag{24}$$

Given that μ_1, μ_2, μ_3, $\mu_5 \in \mathcal{K}_\infty$, it is evident that the function $W(k)$ is an ISS–Lyapunov function, implying the existence of a $\mathcal{KL}$-function $\beta_a : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ and a $\mathcal{K}$-function $\gamma_a : \mathbb{R}^+ \times \mathbb{R}^+$ such that

$$\|\hat{z}(k_0+k)\| \leq \beta_a(\|\hat{z}(k_0)\|, k) + \gamma_a(\|\bar{\omega}_a\|_{\_[k_0,k_0+k-1]}), \tag{25}$$

where $k_0, k \geq 0$. Let $\hat{\mu}_4(\|\bar{\omega}_a\|)$ be any $\mathcal{K}_\infty$-lower bound of $\mu_4 \in \mathcal{K}_\infty$ such that $Id - \hat{\mu}_4 \in \mathcal{K}$. Hence, we can have $\hat{\mu}_4(\|\bar{\omega}_a\|) = \hat{h}_4\|\bar{\omega}_a\|$, where $\hat{h}_4 = \min(\frac{\zeta_6}{2\eta_2}, \hat{\theta}_a)$ with $\hat{\theta}_a \in (0,1)$. Let $\rho_1(\|\bar{\omega}_a\|) = h_\rho\|\bar{\omega}_a\|$ with $h_\rho \in (0,1)$, then according to [44], (25) holds with $\gamma_a = \mu_1^{-1} \circ \hat{\gamma}_a$ where $\hat{\gamma}_a(\|\bar{\omega}_a\|) = \frac{\zeta_9}{\hat{h}_4 h_\rho}\|\bar{\omega}_a\|^2$ and $\beta_a(s,t) = \mu_1^{-1}(\hat{\beta}_a(\mu_2(s)), t)$ for a $\mathcal{KL}$-function $\hat{\beta}_a : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$; therefore, (25) can be written as

$$\begin{aligned} \|\hat{z}(k_0+k)\| \leq& \beta_a(\|\hat{z}(k_0)\|, k) + \sqrt{\frac{\zeta_9}{\hat{h}_4 h_\rho \lambda_{\min}(P)}}\|\bar{\omega}_a\|_{\_[k_0,k_0+k-1]} \\ =& \beta_a(\|\hat{z}(k_0)\|, k) + h_\Psi\|\bar{\omega}_a\|_{\_[k_0,k_0+k-1]} \end{aligned} \tag{26}$$

with $h_\Psi = \sqrt{\frac{\zeta_9}{\hat{h}_4 h_\rho \lambda_{\min}(P)}}$.

Since $F_{j+1}(\alpha) = \mathrm{diag}\big(c_{j+1}(\alpha_1), c_{j+1}(\alpha_2), \cdots, c_{j+1}(\alpha_n)\big)$, its maximum norm can be obtained as

$$\|F_{j+1}(\alpha)\|_\infty = \max_i |c_{j+1}(\alpha_i)| \leq \max_i \frac{\alpha_i^{j+1}}{(j+1)!}. \tag{27}$$

For simplicity, the maximum norm of matrix F will be expressed as $\|F\|$. Thus, the following inequality can be derived as

$$\begin{aligned}
\|\bar{\omega}(k)_a\| \leq& \|B_\omega\|q_\omega + \sum_{j=L+1}^{k} \|A_j\|\|z(k-j)_a\| \\
\leq& \|B_\omega\|q_\omega + \sum_{j=L+1}^{k} \|A_j\|\|z\|_\infty \\
=& \|B_\omega\|q_\omega + \sum_{j=L+1}^{k} \|F_{j+1}(\alpha)\|\|z\|_\infty \\
\leq& \|B_\omega\|q_\omega + \sum_{j=L+1}^{k} \max_i \frac{\alpha_i^{j+1}}{(j+1)!}\|z\|_\infty \\
=& \|B_\omega\|q_\omega + \max_i \phi_{\alpha_i}(L)_k\|z\|_\infty
\end{aligned} \tag{28}$$

where $\phi_{\alpha_i}(L)_k = \sum_{j=L+1}^{k} \frac{\alpha_i^{j+1}}{(j+1)!}$. Combining (26) with (28) results in

$$\|\hat{z}\|\_\infty \leq \beta_a(\|\hat{z}(0)\|, 0) + h_\Psi\|B_\omega\|q_\omega + h_\Psi \max_i \phi_{\alpha_i}(L)_k\|z\|\_\infty. \tag{29}$$

Due to the fact that $\|z\|\_\infty \leq \|\hat{z}\|\_\infty$, from (29), we obtain

$$\|\hat{z}\|\_\infty \leq \beta_a(\|\hat{z}(0)\|, 0) + h_\Psi\|B_\omega\|q_\omega + h_\Psi \max_i \phi_{\alpha_i}(L)_k\|\hat{z}\|\_\infty. \tag{30}$$

Consider $L \in \mathbb{N}_0$ that satisfies $0 < h_\Psi \max_i \phi_{\alpha_i}(L)_k < 1$ and define $d_z = \left[1 - h_\Psi \max_i \phi_{\alpha_i}(L)_k\right]^{-1}\left[\beta_a(\|\hat{z}(0)\|, 0) + h_\Psi\|B_\omega\|q_\omega\right]$; it then further implies $\|\hat{z}\|\_\infty \leq d_z$. Hence, (28) becomes

$$\|\bar{\omega}(k)_a\| \leq \|B_\omega\|q_\omega + \max_i \phi_{\alpha_i}(L)_k d_z. \tag{31}$$

Furthermore, (26) turns into

$$\begin{aligned}
\|\hat{z}(k)\| \leq& \beta_a(\|\hat{z}(0)\|, k) + h_\Psi\left[\|B_\omega\|q_\omega + \max_i \phi_{\alpha_i}(L)_k d_z\right] \\
=& \beta_a(\|\hat{z}(0)\|, k) + p_z
\end{aligned} \tag{32}$$

where $p_z = h_\Psi\left[\|B_\omega\|q_\omega + \max_i \phi_{\alpha_i}(L)_k d_z\right]$ and $\beta_a : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{KL}$-function.

For any scalar $M \in \mathbb{N}$, the boundedness inequality of $\bar{\omega}_a$ can be written as

$$\begin{aligned}
\|\bar{\omega}(k)_a\| \leq& \|B_\omega\|q_\omega + \sum_{j=L+1}^{L+1+M} \|A_j\|\|z(k-j)_a\| + \sum_{j=L+2+M}^{k} \|A_j\|\|z(k-j)_a\| \\
\leq& \|B_\omega\|q_\omega + \sum_{j=L+1}^{L+1+M} \|A_j\|\|z\|_{[k-L-1-M, k-L-1]} \\
&+ \sum_{j=L+2+M}^{k} \|A_j\|\|z\|_{[0, k-L-2-M]} \\
\leq& \|B_\omega\|q_\omega + \sum_{j=L+1}^{L+1+M} \|A_j\|\|z\|_{[k-L-1-M, k-L]} + \max_i \phi_{\alpha_i}(L+1+M)_k d_z \\
\leq& \|B_\omega\|q_\omega + \sum_{j=L+1}^{L+1+M} \max_i \frac{\alpha_i^{j+1}}{(j+1)!}\|z\|_{[k-L-1-M, k-L]} \\
&+ \max_i \phi_{\alpha_i}(L+1+M)_k d_z
\end{aligned}$$

$$\leq \|B_\omega\| q_\omega + \max_i \psi_{L\alpha_i}(M)\|z\|_{\_[k-L-1-M,k-L]} + \max_i \phi_{\alpha_i}(L+1+M)_k d_z \tag{33}$$

where $\psi_{L\alpha_i}(M) = \sum_{j=L+1}^{L+1+M} \frac{\alpha_i^{j+1}}{(j+1)!}$. Consider for constant $p_l \geq 0$ and function $\beta_{al} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $l \in \mathbb{N}_0$ such that for any $r, s \in \mathbb{R}^+$, $\beta_{al}(r,s)$ is bounded and $\beta_{al}(r,s) \to 0$ as $s \to \infty$, the following inequality holds:

$$\|\hat{z}(k)\| \leq \beta_{al}(\|\hat{z}(0)\|, k) + p_l. \tag{34}$$

Step 1 ($l = 0$): Let $\beta_{a0}(r,s) = \beta_a(r,s)$ and $p_0 = p_z$, from (32) we then obtain

$$\|\hat{z}(k)\| \leq \beta_{a0}(\|\hat{z}(0)\|, k) + p_0. \tag{35}$$

Step 2 ($l > 0$): For a $v_l \in \mathbb{R}$ that satisfies $v_l \in \left(0, \min\{1, \frac{\kappa-\chi}{\chi} p_l\}\right)$, where $\kappa \in (\chi, 1)$ and $\chi = h_\Psi \max_i \phi_{\alpha_i}(L)_k$, it then indicates $\chi(p_l + v_l) \leq \kappa p_l$. Choosing $v_{l+1} \in (0, \kappa v_l)$, there then exists a $k_l = k_l(v_{l+1})$ that for $k \geq k_l$, $\|z\|_{\_[k-L,k]} \leq \|\hat{z}(k)\| \leq p_l + \frac{v_{l+1}}{2}$. This further implies that for any $M_l \in \mathbb{N}$, there exists a $k \geq \bar{k}_l = k_l + M_l + 2$ such that $\|z\|_{\_[k-L-1-M_l,k-L]} \leq p_l + \frac{v_{l+1}}{2}$ holds. After combining with (33), we obtain

$$\begin{aligned} \|\bar{\omega}(k)_a\| \leq& \|B_\omega\| q_\omega + \max_i \psi_{L\alpha_i}(M_l)(p_l + \frac{v_{l+1}}{2}) \\ &+ \max_i \phi_{\alpha_i}(L+1+M_l)_k d_z, \text{ for } k \geq \bar{k}_l. \end{aligned} \tag{36}$$

By selecting M_l that satisfies $h_\Psi \max_i \phi_{\alpha_i}(L+1+M_l)_k d_z \leq \chi \frac{v_{l+1}}{2}$, from (26), we obtain

$$\begin{aligned} \|\hat{z}(\bar{k}_l + k)\| \leq& \beta_a(\|\hat{z}(\bar{k}_l)\|, k) + h_\Psi \Big[\max_i \psi_{L\alpha_i}(M_l)(p_l + \frac{v_{l+1}}{2}) \\ &+ \|B_\omega\| q_\omega + \max_i \phi_{\alpha_i}(L+1+M_l)_k d_z\Big]. \end{aligned} \tag{37}$$

Since for $L, M_l \in \mathbb{N}_0$, $\psi_{L\alpha_i}(M_l) \leq \phi_{\alpha_i}(L)_k$; hence,

$$\begin{aligned} \|\hat{z}(\bar{k}_l + k)\| \leq& \beta_a(\|\hat{z}(\bar{k}_l)\|, k) + h_\Psi \|B_\omega\| q_\omega + h_\Psi \max_i \phi_{\alpha_i}(L)_k \\ &(p_l + \frac{v_{l+1}}{2}) + \chi \frac{v_{l+1}}{2} \\ \leq& \beta_a(\|\hat{z}(\bar{k}_l)\|, k) + h_\Psi \|B_\omega\| q_\omega + \chi(p_l + \frac{v_{l+1}}{2}) \\ &+ \chi \frac{v_{l+1}}{2} \\ \leq& \beta_a(\|\hat{z}(\bar{k}_l)\|, k) + h_\Psi \|B_\omega\| q_\omega + \chi(p_l + v_l) \\ \leq& \beta_a(\|\hat{z}(\bar{k}_l)\|, k) + h_\Psi \|B_\omega\| q_\omega + \kappa p_l. \end{aligned} \tag{38}$$

Thus, combining with (34) gives

$$\|\hat{z}(k)\| \leq \beta_{a(l+1)}(\|\hat{z}(0)\|, k) + \kappa p_l + h_\Psi \|B_\omega\| q_\omega \tag{39}$$

where

$$\begin{aligned} &\beta_{a(l+1)}(r,k) \\ &= \begin{cases} \beta_{al} & (r,k) + (1-\kappa)p_l - h_\Psi \|B_\omega\| q_\omega, \quad k \in [0, \bar{k}_l - 1] \\ \min & \{\beta_{al}(r,k) + (1-\kappa)p_l - \\ & h_\Psi \|B_\omega\| q_\omega, \beta_a(\|\hat{z}(\bar{k}_l)\|, k - \bar{k}_l)\}, \quad k \in [\bar{k}_l, +\infty) \end{cases} \end{aligned} \tag{40}$$

which recursively satisfies that $\beta_{a(l+1)}(r,k) \to 0$ as $k \to \infty$, and it can be noticed from (34) that the state will finally converge to p_l. Also, (39) can be written as $\|\hat{z}(k)\| \leq \beta_{a(l+1)}(\|\hat{z}(0)\|, k) + p_{l+1}$, where

$$\begin{aligned} p_{l+1} &= \kappa p_l + h_\Psi \|B_\omega\| q_\omega \\ &= \kappa\big(\kappa p_{l-1} + h_\Psi \|B_\omega\| q_\omega\big) + h_\Psi \|B_\omega\| q_\omega \\ &= \kappa\big(\kappa(\kappa p_{l-2} + h_\Psi \|B_\omega\| q_\omega) + h_\Psi \|B_\omega\| q_\omega\big) + h_\Psi \|B_\omega\| q_\omega \\ &= \cdots \\ &= \kappa^{l+1} p_0 + h_\Psi \|B_\omega\| q_\omega \Big(\sum_{j=0}^{l} \kappa^j \Big). \end{aligned} \tag{41}$$

As a result, (34) indicates

$$\|\hat{z}(k)\| \leq \beta_{a\infty}(\|\hat{z}(0)\|, k) + p_\infty \tag{42}$$

where $\beta_{a\infty}(\|\hat{z}(0)\|, k) \to 0$ as $k \to \infty$ and

$$p_\infty = \lim_{l \to \infty} p_l = h_\Psi \|B_\omega\| q_\omega \frac{\kappa}{1-\kappa}. \tag{43}$$

In light of Assumption 2, $\|\hat{z}(0)\| = \|z(0)\|$ and $\|z(k)\| \leq \|\hat{z}(k)\|$; thus,

$$\begin{aligned} \|z(k)\| \leq& \beta_a(\|z(0)\|, k) + h_\Psi \|B_\omega\| q_\omega \frac{\kappa}{1-\kappa} \\ =& \beta_a(\|z(0)\|, k) + \gamma_a(q_\omega) \end{aligned} \tag{44}$$

where $\gamma_a(r) = c_{\gamma a} r$ with $c_{\gamma a} = h_\Psi \|B_\omega\| \frac{\kappa}{1-\kappa}$. Therefore, under DoS attacks, the closed-loop fractional-order system (1) with controller (12) is globally ISS, and the control objective is achieved. $\square$

Remark 3. *Although there has been considerable research into secure control for integer-order systems [32,35,37,38], comparatively little attention has been given to systems in fractional-order systems, and even less to those in the discrete-time domain. This research is the first to address the security control of discrete-time fractional-order systems under DoS attacks while taking into account the memory and heredity effects of fractional calculus.*

Remark 4. *References [39,40] respectively investigate the control problem for fractional-order multi-agent systems and complex networks which are vulnerable to DoS attacks in the continuous-time domain. The control issue of discrete-time fractional-order multi-agent systems under DoS attacks is studied in [41], yet the non-local characteristics of fractional-order calculus are ignored when designing the control scheme. Different from such mentioned works, the control for discrete-time fractional systems under the effect of DoS attacks that rigorously consider the unique hereditary and infinite memory properties of fractional calculus is addressed in this work for the first time.*

4. Numerical Example

Consider the fractional-order discrete-time system shown in (1) with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_\omega = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

${}_0\Delta^{\alpha}_{k+1} z(k+1) = \big[{}_0\Delta^{0.1}_{k+1} z_1(k+1), {}_0\Delta^{0.4}_{k+1} z_n(k+1)\big]^T$ $(k \in \mathbb{N}_0)$, $z_0 = [0.1, 0.5]^T$, and the external disturbance $\omega(k)$ is an uniformly distributed random signal with $q_\omega = 0.02$. The numerical example and its visualized results are both implemented in MATLAB. Details of technical implementation can refer to [16–18,45].

The results of the open-loop fractional-order system without control input, depicted in Figure 2, demonstrate its instability, as evidenced by the outputs z_1 and z_2. By choosing $L = 1$, $\sigma = 0.1$, and $K_L = [-1, 0.01, 0.01, 0.05]^T$, we can notice that even under the influence of DoS attacks, the closed-loop system is ISS under our proposed controller, as presented in Figures 3 and 4. Furthermore, it is presented in Figure 3 that under the proposed control input, both system outputs can be driven to zero instead of just being bounded, as illustrated in the proof of Theorem 1, which further indicates an interesting future work to explore the design of security control law for fractional-order systems to achieve asymptotic stability. Moreover, as displayed in Figures 5 and 6 where $K_L = [-1, 0.01]^T$ and $L = 0$, it is worth noting that with a proper selection of design parameters, the global stability of the closed-loop system in the presence of DoS attacks can be ensured under our proposed controller, even if only the current state is considered in designing the control strategy. Consequently, the effectiveness of the investigated control strategy is verified by the simulation results.

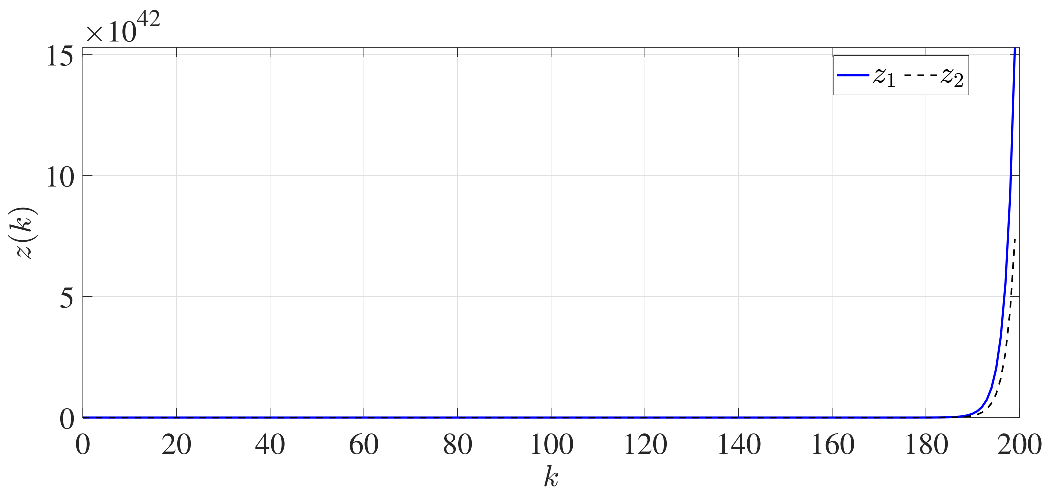

Figure 2. System output z_1 and z_2 without control effort.

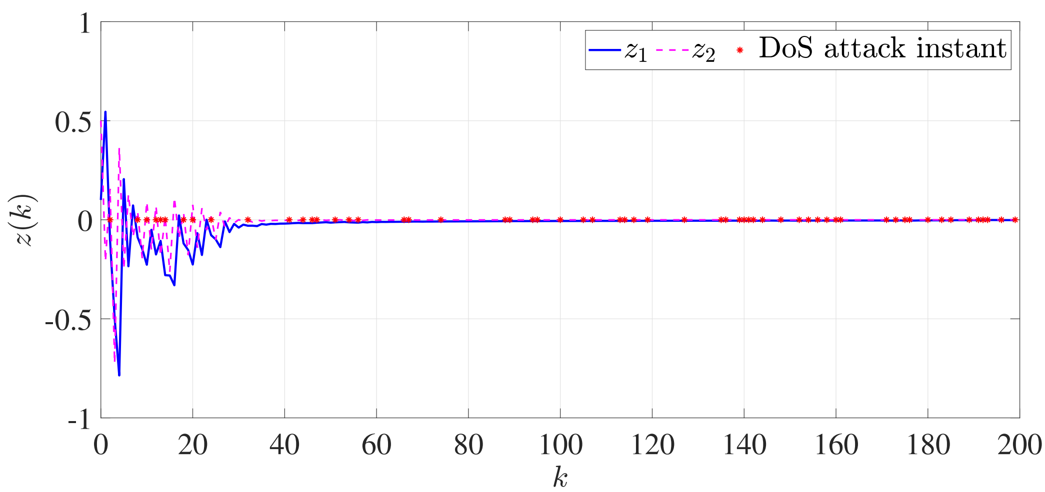

Figure 3. System output z_1 and z_2 with DoS attacks under the proposed control law where $L = 1$.

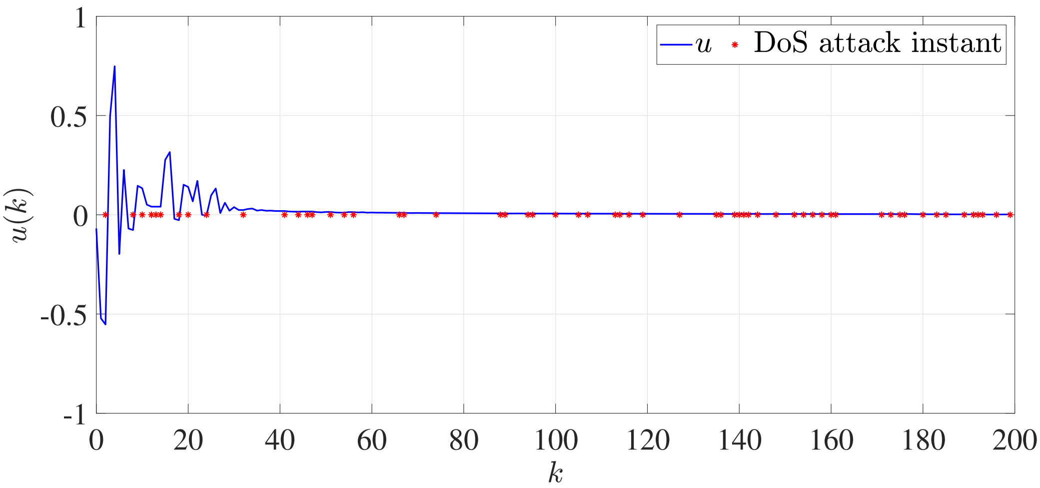

Figure 4. Control input u with $L = 1$.

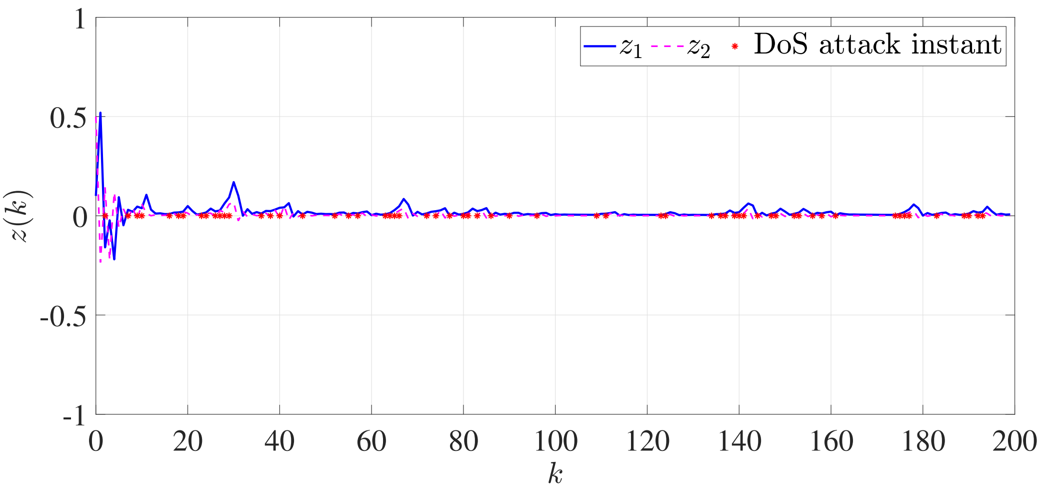

Figure 5. System output z_1 and z_2 with DoS attacks under the proposed control law where $L = 0$.

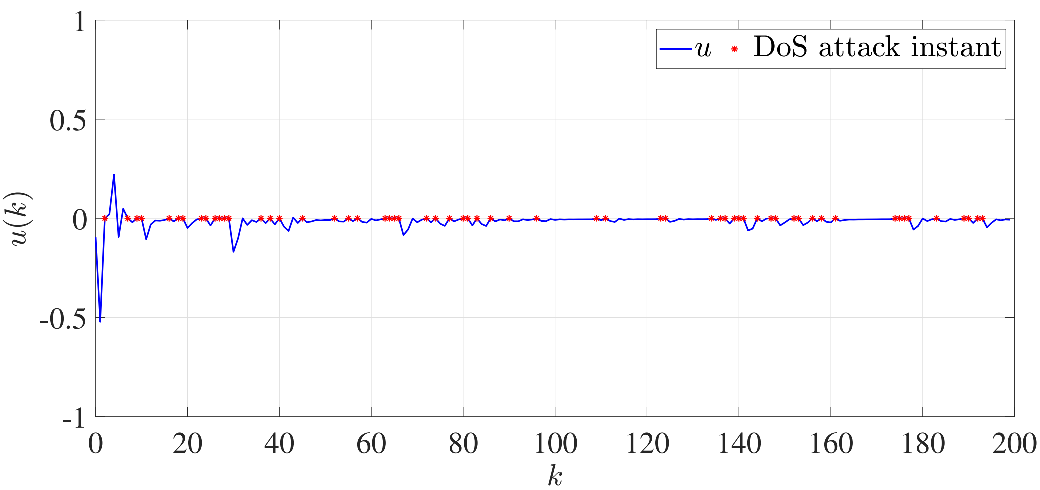

Figure 6. Control input u with $L = 0$.

5. Conclusions

The security control issue for discrete-time fractional-order linear systems under DoS attacks is addressed in this work. A finite-dimensional controller, which requires merely the information of a finite number of the previous state, and hence is practically useful, is proposed in this article. Our proposed control law is verified by numerical simulation, ensuring the global input-to-state stability of the associated closed-loop system. In practice, it is possible that the system matrices could not be known precisely, which consequently implies an interesting future topic for considering the study of adaptive finite-dimensional control for fractional systems under attack by introducing adaptive techniques in our work to estimate system uncertainties.

Author Contributions: Conceptualization, X.L.; Methodology, X.L.; Validation, X.L.; Formal analysis, X.L.; Investigation, X.L.; Writing—original draft, X.L.; Writing—review & editing, Y.Z., X.L. and C.D.; Visualization, Y.Z. and X.L.; Supervision, C.D.; Funding acquisition, Y.Z. and X.W. All authors have read and agreed to the published version of the manuscript.

Funding: This work is supported by the National Natural Science Foundation of China (62203164, 62203165), and the Scientific Research Fund of Hunan Provincial Education Department (Outstanding Young Project) (21B0499).

Data Availability Statement: The data presented in this study may be available on request from the corresponding author.

Conflicts of Interest: The authors declare no potential conflicts of interest with respect to the research, authorship, finance, and/or publication of this article.

References

1. Koeller, R. Applications of fractional calculus to the theory of viscoelasticity. *J. Appl. Mech.* **1984**, *51*, 299–307. [CrossRef]
2. Friedrich, C. Relaxation and retardation functions of the Maxwell model with fractional derivatives. *Rheol. Acta* **1991**, *30*, 151–158. [CrossRef]
3. Cao, H.; Deng, Z.; Li, X.; Yang, J.; Qin, Y. Dynamic modeling of electrical characteristics of solid oxide fuel cells using fractional derivatives. *Int. J. Hydrogen Energy* **2010**, *35*, 1749–1758. [CrossRef]
4. Hammouch, Z.; Mekkaoui, T. Circuit design and simulation for the fractional-order chaotic behavior in a new dynamical system. *Complex Intell. Syst.* **2018**, *4*, 251–260. [CrossRef]
5. Butzer, P.L.; Westphal, U.; Hilfer, R.; West, B.J.; Grigolini, P.; Zaslavsky, G.M.; Douglas, J.F.; Schiessel, H.; Friedrich, C.; Blumen, A.; et al. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, 2000; Volume 35.
6. Liu, L.; Xue, D.; Zhang, S. General type industrial temperature system control based on fuzzy fractional-order PID controller. *Complex Intell. Syst.* **2021**, *9*, 2585–2597. [CrossRef]
7. Guo, D.; Yang, G.; Feng, X.; Han, X.; Lu, L.; Ouyang, M. Physics-based fractional-order model with simplified solid phase diffusion of lithium-ion battery. *J. Energy Storage* **2020**, *30*, 101404. [CrossRef]
8. Wang, Y.; Li, M.; Chen, Z. Experimental study of fractional-order models for lithium-ion battery and ultra-capacitor: Modeling, system identification, and validation. *Appl. Energy* **2020**, *278*, 115736. [CrossRef]
9. Zou, C.; Zhang, L.; Hu, X.; Wang, Z.; Wik, T.; Pecht, M. A review of fractional-order techniques applied to lithium-ion batteries, lead-acid batteries, and supercapacitors. *J. Power Sources* **2018**, *390*, 286–296. [CrossRef]
10. Hakkar, N.; Dhayal, R.; Debbouche, A.; Torres, D.F. Approximate controllability of delayed fractional stochastic differential systems with mixed noise and impulsive effects. *Fractal Fract.* **2023**, *7*, 104. [CrossRef]
11. Johnson, M.; Vijayakumar, V. An analysis on the optimal control for fractional stochastic delay integrodifferential systems of order $1 < \gamma < 2$. *Fractal Fract.* **2023**, *7*, 284. [CrossRef]
12. Kavitha, K.; Vijayakumar, V. An analysis regarding to approximate controllability for Hilfer fractional neutral evolution hemivariational inequality. *Qual. Theory Dyn. Syst.* **2022**, *21*, 80. [CrossRef]
13. Guechi, S.; Dhayal, R.; Debbouche, A.; Malik, M. Analysis and optimal control of φ-Hilfer fractional semilinear equations involving nonlocal impulsive conditions. *Symmetry* **2021**, *13*, 2084. [CrossRef]
14. Podlubny, I. Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers. *IEEE Trans. Autom. Control* **1999**, *44*, 208–214. [CrossRef]
15. Chen, Y.; Petráš, I.; Xue, D. Fractional order control—A tutorial. In Proceedings of the 2009 American Control Conference, St. Louis, MO, USA, 10–12 June 2009; pp. 1397–1411.
16. Li, X.; Wen, C.; Liu, X.K. Sampled-data control based consensus of fractional-order multi-agent systems. *IEEE Control Syst. Lett.* **2021**, *5*, 133–138. [CrossRef]
17. Li, X.; Wen, C.; Liu, X.K. Finite-dimensional sampled-data control of fractional-order systems. *IEEE Control Syst. Lett.* **2022**, *6*, 181–186. [CrossRef]
18. Li, X.; Wen, C.; Li, X.; Deng, C. Stabilization for a general class of fractional-order systems: A sampled-data control method. *IEEE Trans. Circuits Syst. I Regul. Pap.* **2022**, *69*, 4643–4653. [CrossRef]
19. Li, Y.; Chen, Y.; Podlubny, I. Mittag–Leffler stability of fractional order nonlinear dynamic systems. *Automatica* **2009**, *45*, 1965–1969. [CrossRef]

20. Liu, H.; Pan, Y.; Li, S.; Chen, Y. Adaptive fuzzy backstepping control of fractional-order nonlinear systems. *IEEE Trans. Syst. Man Cybern. Syst.* **2017**, *47*, 2209–2217. [CrossRef]
21. Li, X.; Wen, C.; Zou, Y. Adaptive backstepping control for fractional-order nonlinear systems with external disturbance and uncertain parameters using smooth control. *IEEE Trans. Syst. Man Cybern. Syst.* **2021**, *51*, 7860–7869. [CrossRef]
22. Li, X.; He, J.; Wen, C.; Liu, X.K. Backstepping-based adaptive control of a class of uncertain incommensurate fractional-order nonlinear systems With external disturbance. *IEEE Trans. Ind. Electron.* **2022**, *69*, 4087–4095. [CrossRef]
23. Li, X.; Li, X.; Xing, L. Backstepping-based adaptive control for uncertain fractional-order nonlinear systems. In Proceedings of the 2021 IEEE 16th Conference on Industrial Electronics and Applications (ICIEA), Chengdu, China, 1–4 August 2021; pp. 12–17. [CrossRef]
24. Li, X.; Wen, C.; Li, X.; He, J. Adaptive fractional-order backstepping control for a general class of nonlinear uncertain integer-order systems. *IEEE Trans. Ind. Electron.* **2023**, *70*, 7246–7256. [CrossRef]
25. Kaczorek, T. Practical stability of positive fractional discrete-time linear systems. *Bull. Pol. Acad. Sci. Tech. Sci.* **2008**, *56*, 313–317.
26. BusłOwicz, M.; Kaczorek, T. Simple conditions for practical stability of positive fractional discrete-time linear systems. *Int. J. Appl. Math. Comput. Sci.* **2009**, *19*, 263–269. [CrossRef]
27. Guermah, S.; Djennoune, S.; Bettayeb, M. A new approach for stability analysis of linear discrete-time fractional-order systems. In *New Trends in Nanotechnology and Fractional Calculus Applications*; Springer: Dordrecht, The Netherlands, 2010; pp. 151–162. [CrossRef]
28. Guermah, S.; Djennoune, S.; Bettayeb, M. Discrete-time fractional-order systems: Modeling and stability issues. In *Advances in Discrete Time Systems*; InTech: London, UK, 2012; pp. 183–212. [CrossRef]
29. Sopasakis, P.; Ntouskas, S.; Sarimveis, H. Robust model predictive control for discrete-time fractional-order systems. In Proceedings of the 2015 23rd Mediterranean Conference on Control and Automation (MED), Torremolinos, Spain, 16–19 June 2015; pp. 384–389. [CrossRef]
30. Sopasakis, P.; Sarimveis, H. Stabilising model predictive control for discrete-time fractional-order systems. *Automatica* **2017**, *75*, 24–31. [CrossRef]
31. Alessandretti, A.; Pequito, S.; Pappas, G.J.; Aguiar, A.P. Finite-dimensional control of linear discrete-time fractional-order systems. *Automatica* **2020**, *115*, 108512. [CrossRef]
32. Deng, C.; Wen, C.; Zou, Y.; Wang, W.; Li, X. A hierarchical security control framework of nonlinear CPSs against DoS attacks with application to power sharing of AC microgrids. *IEEE Trans. Cybern.* **2022**, *52*, 5255–5266. [CrossRef]
33. Cárdenas, A.A.; Amin, S.; Sastry, S. Research challenges for the security of control systems. In Proceedings of the 3rd Conference on Hot Topics in Security, San Jose, CA, USA, 29 July 2008; pp. 1–6.
34. De Persis, C.; Tesi, P. Input-to-state stabilizing control under denial-of-service. *IEEE Trans. Autom. Control* **2015**, *60*, 2930–2944. [CrossRef]
35. Zhao, L.; Yang, G.H. Adaptive fault-tolerant control for nonlinear multi-agent systems with DoS attacks. *Inf. Sci.* **2020**, *526*, 39–53. [CrossRef]
36. Zhang, D.; Liu, L.; Feng, G. Consensus of heterogeneous linear multiagent systems subject to aperiodic sampled-data and DoS attack. *IEEE Trans. Cybern.* **2019**, *49*, 1501–1511. [CrossRef]
37. Feng, Z.; Hu, G. Secure cooperative event-triggered control of linear multiagent systems Under DoS attacks. *IEEE Trans. Control Syst. Technol.* **2020**, *28*, 741–752. [CrossRef]
38. Chang, B.; Mu, X.; Yang, Z.; Fang, J. Event-based secure consensus of muti-agent systems under asynchronous DoS attacks. *Appl. Math. Comput.* **2021**, *401*, 126120. [CrossRef]
39. Narayanan, G.; Ali, M.S.; Alsulami, H.; Stamov, G.; Stamova, I.; Ahmad, B. Impulsive security control for fractional-order delayed multi-agent systems with uncertain parameters and switching topology under DoS attack. *Inf. Sci.* **2022**, *618*, 169–190. [CrossRef]
40. Bai, J.; Wu, H.; Cao, J. Secure synchronization and identification for fractional complex networks with multiple weight couplings under DoS attacks. *Comput. Appl. Math.* **2022**, *41*, 187. [CrossRef]
41. Narayanan, G.; Ali, M.S.; Ahamad, S. Cyber secure consensus of discrete-time fractional-order multi-agent systems with distributed delayed control against attacks. In Proceedings of the 2021 IEEE International Conference on Systems, Man, and Cybernetics (SMC), Melbourne, Australia, 17–20 October 2021; pp. 2191–2196. [CrossRef]
42. Ostalczyk, P. *Discrete Fractional Calculus: Applications in Control and Image Processing*; World Scientific: Hackensack, NJ, USA, 2016; Volume 4.
43. Chatterjee, S.; Romero, O.; Pequito, S. A separation principle for discrete-time fractional-order dynamical systems and its implications to closed-loop neurotechnology. *IEEE Control Syst. Lett.* **2019**, *3*, 691–696. [CrossRef]
44. Jiang, Z.P.; Wang, Y. Input-to-state stability for discrete-time nonlinear systems. *Automatica* **2001**, *37*, 857–869. [CrossRef]
45. Mendes, E.M.; Salgado, G.H.; Aguirre, L.A. Numerical solution of Caputo fractional differential equations with infinity memory effect at initial condition. *Commun. Nonlinear Sci. Numer. Simul.* **2019**, *69*, 237–247. [CrossRef]