fractal and fractional

Article

On Pantograph Problems Involving Weighted Caputo Fractional Operators with Respect to Another Function

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Abstract: In this investigation, weighted *psi*-Caputo fractional derivatives are applied to analyze the solution of fractional pantograph problems with boundary conditions. We establish the existence of solutions to the indicated pantograph equations as well as their uniqueness. The study also takes into account the situation where $\psi(x) = x$. With the aid of Banach's and Krasnoselskii's classic fixed point results, we have established a the qualitative study. Different values of $\psi(x)$ and $w(x)$ are discussed as special cases that are relevant to our current results. Additionally, in light of our findings, we provide a more general fractional system with the weighted ψ-Caputo-type that takes into account both the new problems and certain previously existing, related problems. Finally, we give two illustrations to support and validate the outcomes.

Keywords: fractional pantograph system; Caputo fractional operator; fixed point theorem

MSC: 34A08; 34A12; 47H10

1. Introduction

Fractional calculus (FC) is an extension of the integer order differential and integral. In recent years, a large number of definitions of FC have been developed in response to practical problem modeling requirements, such as Riemann–Liouville, Caputo, Hadamard, Erdelyi–Kober, and Hilfer versions, see [1–4].

In the past few decades, many classes of fractional differential equations (FDEs) have undergone in-depth research and analysis. For instance, concepts concerning the existence and uniqueness of solutions have been mentioned in [5–7] and references therein.

An essential aspect of the theory of FDEs was addressed by the qualitative characteristics of solutions. The area that was previously described has been thoroughly explored for classical differential equations. Nevertheless, there are a number of FDE-related problems and systems that need further research and analysis. Utilizing THE Riemann–Liouville (Caputo, Hilfer) fractional operator, and other fractional operators, the existence and uniqueness have been closely examined; for more information, see [8–13] and the references therein.

Several authors have studied generalized FDs, and their applications. For instance, Kilbas et al. [2] provided some interesting ψ-Riemann–Liouville FD characteristics. Almedia [14] described the ψ-Caputo FD. In the Hilfer sense, Sousa and Oliveira [15] presented another generalization. A singular kernel can be found in the aforementioned derivatives. The substitution of a nonsingular kernel for a singular kernel has resulted in the presentation of new types of FDs by certain authors; for more information, [16–18]. According to [19–22], nonlocal FDs with nonsingular kernels have been shown to be a respectable tool for simulating actual problems in a variety of engineering and scientific fields.

Another significant class of FDEs are the pantograph equations (PEs), which have not been investigated as completely in the context of innovative FDs. PEs are a crucial class of delay equations that provide changes in the dependant worth at a previous time [23],

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and are used in deterministic circumstances. A pantograph is essentially a measuring and drawing tool. Currently, electric trains and electric cells employ this device [24–26]. In 1971, Ockendon and Taylor [27] investigated what is now known as PE, or how electric flow is collected by the pantograph of an electric train. Since then, numerous researchers have investigated it and used it in numerous mathematical and scientific domains, including pharmacology, electrodynamics, probability, number theory, and more (see [27–29] and the references therein). Several authors have thought carefully about the analytical and numerical approaches of the following delay equation

$$\begin{cases} \varpi'(\mathrm{x}) = a\varpi(\mathrm{x}) + b\varpi(\lambda\mathrm{x}), \ \ \mathrm{x} \in [0,T], \ 0 < \lambda < 1, \\ \varpi(0) = \varpi_0, \end{cases} \tag{1}$$

see [30–32]. The PEs were accurately studied in [33,34]. The following nonlinear PE

$$\begin{cases} \varpi'(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda_1\mathrm{x}), ..., \varpi(\lambda_m\mathrm{x})), \ \ \mathrm{x} \in [0,T] \\ \varpi(0) = \varpi_0, \ 0 < \lambda_1 < ... < \lambda_m < 1 \end{cases} \tag{2}$$

has been studied by Liu et al. [35]. In contrast, Sezer et al. [36] considered the nonlinear neutral PE:

$$\begin{cases} \varpi'(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda\mathrm{x}), \varpi'(\lambda\mathrm{x})), \ \ \mathrm{x} > 0, \\ \varpi(0) = \varpi_0, \ \ 0 < \lambda < 1. \end{cases} \tag{3}$$

Because of the importance of fractional PE in many fields, it has been the subject of many studies. For instance, the following Caputo pantograph problem

$$\begin{cases} {}^{C}\mathbb{D}^{\vartheta}_{0+}\varpi(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda\mathrm{x})), \ \ \mathrm{x} \in [0,T], \ 0 < \vartheta < 1 \\ \varpi(0) = \varpi_0 + h(\varpi) \end{cases} \tag{4}$$

was the subject of discussion by Balachandran et al. [37].

In contrast, Agarwal [38], Kolokoltsov [39] and Jarad et al. [40] discussed weighted FDs first, second, and third, respectively. Due to the importance of FDEs in many fields of research, some recent studies addressed the existence of solutions to FDEs; for instance, Abdo et al. [41,42], discussed the following problems ${}^{C}_{0}\mathfrak{D}^{\nu}_{\psi,\mathrm{w}}\Big|\varpi(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x})), \ \ 0 < \mathrm{x} \le 1$, $\varpi(0) = \varpi_0$, and ${}^{PC}_{0}\mathfrak{D}^{\nu}_{\psi}\varpi(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda_1\mathrm{x}), ..., \varpi(\lambda_m\mathrm{x})), 0 \le \mathrm{x} \le b, \varpi(0) = \varpi_0 + h(\varpi)$, where $0 < \nu < 1$, ${}^{C}_{0}\mathfrak{D}^{\nu}_{\psi,\mathrm{w}}\Big|$ and ${}^{PC}_{0}\mathfrak{D}^{\nu}_{\psi}\Big|$ are the ψ-Caputo FD and piecewise Caputo FD, respectively. In this regard, Al-Rafai and Jarrah [43], obtained the uniqueness result of the Cauchy problem involving the $[\mathrm{w}, \psi]$-Caputo–Fabrizio FD with ψ and w which are monotone and weight functions, respectively.

Motivated by the aforementioned works, this paper focuses on novel classes of weighted pantograph FDEs:

$$\begin{cases} {}^{C}_{a}\mathfrak{D}^{\nu}_{\mathrm{x},\mathrm{w}(\mathrm{x})}\varpi(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda_1\mathrm{x}), ..., \varpi(\lambda_m\mathrm{x})), \ \mathrm{x} \in \mho, \\ \varpi(a) = \varpi_a, \ \varpi(b) = \varpi_b, \end{cases} \tag{5}$$

and

$$\begin{cases} {}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\varpi(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda_1\mathrm{x}), ..., \varpi(\lambda_m\mathrm{x})), \ \mathrm{x} \in \mho, \\ \varpi(a) = \varpi_a, \ \varpi(b) = \varpi_b, \end{cases} \tag{6}$$

where $1 < \nu < 2, \mathrm{x} \in \mho := [a,b], 0 < \lambda_i < 1$, for $i = 1,2,...,m$, $m \in \mathbb{N}$, $\mathrm{g} : \mho \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a given function, ${}^{C}\mathfrak{D}^{\nu}_{\mathrm{x},\mathrm{w}(\mathrm{x})}$ and ${}^{C}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}$ are weighted Caputo FD and ψ-weighted Caputo FD, respectively, $\psi(\mathrm{x})$ and $\mathrm{w}(\mathrm{x})$ are monotone and weight functions, respectively, with $\mathrm{w}, \psi \in C^1(\mho)$ with $\mathrm{w}, \mathrm{w}', \psi' > 0$ on $\mho$, and $\varpi_a, \varpi_b \in \mathbb{R}$.

The presented problems have not been addressed yet. By selecting particular kernel functions in the derivative and weight functions, some current results that concentrate on the novel fractional operators, including [40,43–46], are enhanced and supplemented.

Remark 1.

(i) If $\psi(\mathrm{x}) = \mathrm{x}$, then problem (6) is reduced to problem (5).

(ii) Problem (6) with $\varpi(0) = \varpi_0 + \mathrm{h}(\varpi)$ reduces to the problem (4) if $\psi(\mathrm{x}) = \mathrm{x}$, $\mathrm{w}(\mathrm{x}) = 1$ and $m = 1$, see [37].

(iii) Problem (6) with $\psi(\mathrm{x}) = \mathrm{x}$, $\mathrm{w}(\mathrm{x}) = 1$, $\nu = 1$ and $\varpi(0) = \varpi_0$ reduces to the Cauchy problem (2), as shown in [35].

(iv) Our current results for the problem (6) stay available on problem (5).

(v) Our current problems cover a wide range of problems which uses less general derivatives operators by make use of different values of ψ and w.

The following is an outline of the paper's content. Some basic results about weighted FC are presented in Section 2. In Section 3, we present our main results for problems (5) and (6). Section 4 gives a more general pantograph problem. Section 5 offers two examples illustration that demonstrates the validity of the theories. The conclusions of the work are included in the final section.

2. Primitive Results

We first give some notions and definitions of the generalized weighted fractional calculus, and then we state some fundamental results and remarks. Let $\mho := [a, b]$, $a < b < \infty$. The spaces $\mathcal{L}^1_\psi(\mho)$, $\mathcal{C}^1_\psi(\mho)$ and $\mathcal{AC}^n_\psi(\mho)$ are defined as in [46].

Definition 1 ([46])**.** *Let $0 < \nu$, $\rho \in \mathcal{L}^1_\psi(\mho)$ and $\psi, \mathrm{w} \in \mathcal{C}^1_\psi(\mho)$. The weighted ψ-RL fractional integral is defined as*

$$ {}^{RL}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\rho(\mathrm{x}) = \frac{1}{\Gamma(\nu)\mathrm{w}(\mathrm{x})}\int_a^{\mathrm{x}} \psi'(\vartheta)(\psi(\mathrm{x}) - \psi(\vartheta))\mathrm{w}(\vartheta)\rho(\vartheta)d\vartheta. $$

Definition 2 ([46])**.** *Let $0 < \nu$, and $\rho \in \mathcal{AC}^n$. The weighted ψ-Caputo FD is defined as*

$$ {}^{C}_{a}\mathfrak{D}^{\nu;\psi}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\rho(\mathrm{x}) = {}^{RL}_{a}\mathfrak{I}^{n-\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\left(\mathfrak{D}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\right)^n \rho(\mathrm{x}), $$

where $\mathfrak{D}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\rho(\mathrm{x}) := \frac{1}{\mathrm{w}(\mathrm{x})\psi'(\mathrm{x})}\frac{d}{d\mathrm{x}}(\mathrm{w}(\mathrm{x})\rho(\mathrm{x})) = \frac{1}{\psi'(\mathrm{x})}\left[\frac{d}{d\mathrm{x}} + \frac{\mathrm{w}'(\mathrm{x})}{\mathrm{w}(\mathrm{x})}\right]\rho(\mathrm{x})$.

Lemma 1 ([46])**.** *Let $\rho \in \mathcal{C}^n_\psi(\mho)$. Then*

$$ {}^{C}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\rho(\mathrm{x}) = \rho(\mathrm{x}), $$

$$ {}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,{}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\rho(\mathrm{x}) = \rho(\mathrm{x}) - \sum_{k=0}^{n-1}\frac{[\psi(\mathrm{x}) - \psi(a)]^k}{k!}\frac{\mathrm{w}(a^+)}{\mathrm{w}(\mathrm{x})}\lim_{\mathrm{x}\to a^+}\left(\mathfrak{D}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\right)^k\rho(\mathrm{x}). $$

Lemma 2 ([46])**.** *The weighted RL and Caputo operators of ρ with respect to ψ are given as follows:*

$$ {}^{RL}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\frac{[\psi(\mathrm{x}) - \psi(a)]^{\beta}}{\mathrm{w}(\mathrm{x})} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\nu+1)}\frac{[\psi(\mathrm{x}) - \psi(a)]^{\beta-\nu}}{\mathrm{w}(\mathrm{x})}, \quad \nu \in \mathbb{C}, Re(\beta) > -1; $$

$$ {}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\frac{E_\nu\left(\lambda[\psi(\mathrm{x}) - \psi(a)]^{\nu}\right)}{\mathrm{w}(\mathrm{x})} = \lambda\frac{E_\nu\left(\lambda[\psi(\mathrm{x}) - \psi(a)]^{\nu}\right)}{\mathrm{w}(\mathrm{x})}, \quad \lambda \in \mathbb{C}, Re(\beta) > -1, $$

where E_ν is the Mittag–Leffler function.

We will require Krasnoselskii's fixed point theorem [47] and Banach's contraction map [48] for our upcoming analysis.

3. Main Results

Here, we provide some qualitative analyses of pantograph problems (5) and (6).

3.1. Basic Result

Lemma 3. *Let $0 < \nu < 1$, and $\varpi \in \mathcal{AC}$. Then the Cauchy problem*

$$\begin{cases} {}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\varpi(\mathrm{x}) = f(\mathrm{x}), \\ \varpi(a) = \varpi_a \end{cases} \tag{7}$$

has the unique solution

$$\varpi(\mathrm{x}) = \frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\varpi_a + \frac{1}{\Gamma(\nu)\mathrm{w}(\mathrm{x})}\int_a^{\mathrm{x}} \psi'(\vartheta)(\psi(\mathrm{x}) - \psi(\vartheta))\mathrm{w}(\vartheta)f(\vartheta)d\vartheta,\ \mathrm{x} \in \mho. \tag{8}$$

Proof. Assume ϖ satisfies the first equation of (7). From Lemma 1, we have

$${}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\ {}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\varpi(\mathrm{x}) = \varpi(\mathrm{x}) - \frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\varpi(a). \tag{9}$$

Further, from (7), we obtain

$${}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\ {}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\varpi(\mathrm{x}) = {}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,f(\mathrm{x}). \tag{10}$$

By (9) and (10), we find that

$$\varpi(\mathrm{x}) = \frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\varpi(a) + {}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,f(\mathrm{x}).$$

Hence, by initial condition $\varpi(a) = \varpi_a$, we obtain

$$\varpi(\mathrm{x}) = \frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\varpi_a + {}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,f(\mathrm{x}),$$

which is (8).

Conversely, if ϖ satisfies (8), then by Lemmas 2 and 1, we have

$${}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\varpi(\mathrm{x}) = {}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\varpi_a + {}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,f(\mathrm{x}) = {}^{C}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\ {}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,f(\mathrm{x}) = f(\mathrm{x}).$$

Moreover, the condition $\varpi(a) = \varpi_a$ is directly achieved by taking $\mathrm{x} \to a$ of the Equation (8). □

Lemma 4. *Let $1 < \nu < 2$, $0 < \lambda < 1$ and assume that g is a continuous. Then the following ψ-weighted pantograph FDE*

$$\begin{cases} {}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\varpi(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda \mathrm{x})),\ \mathrm{x} \in \mho, \\ \varpi(a) = \varpi_a,\ \varpi(b) = \varpi_b, \end{cases} \tag{11}$$

has the unique solution

$$\begin{aligned} \varpi(\mathrm{x}) \quad = \quad & \frac{1}{\chi}\left(\frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})} + [\psi(\mathrm{x}) - \psi(a)]\frac{\mathrm{w}'(a)}{\psi'(a)\mathrm{w}(\mathrm{x})}\right)\left(\varpi_b - {}_{a}\mathfrak{I}^{\nu}_{\psi(b),\mathrm{w}(b)}\,\mathrm{g}(b, \varpi(b), \varpi(\lambda b))\right) \\ & + {}_{a}\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda \mathrm{x})) \end{aligned} \tag{12}$$

where $\chi := \left(\frac{\mathrm{w}(a)}{\mathrm{w}(b)} + \frac{[\psi(b) - \psi(a)]}{\mathrm{w}(b)}\frac{\mathrm{w}'(a)}{\psi'(a)}\right)$.

Proof. Assume ϖ satisfies the first equation of (11). It follows from Lemma 1 that

$$
\begin{aligned}
{}_a\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,{}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\varpi(\mathrm{x}) &= \varpi(\mathrm{x}) - \frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\varpi(a) - [\psi(\mathrm{x}) - \psi(a)] \\
&\quad \times \frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\lim_{\mathrm{x}\to a}\frac{1}{\psi'(\mathrm{x})}\left[\frac{d}{d\mathrm{x}} + \frac{\mathrm{w}'(\mathrm{x})}{\mathrm{w}(\mathrm{x})}\right]\varpi(\mathrm{x}). \qquad (13)
\end{aligned}
$$

By (11), we have

$$
{}_a\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,{}^{C}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\varpi(\mathrm{x}) = {}_a\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\lambda\mathrm{x})). \qquad (14)
$$

From (13) and (14), we have

$$
\begin{aligned}
\varpi(\mathrm{x}) &= \frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\varpi(a) + [\psi(\mathrm{x}) - \psi(a)]\frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\left(\frac{\varpi'(a)}{\psi'(a)} + \frac{\varpi(a)\mathrm{w}'(a)}{\psi'(a)\mathrm{w}(a)}\right) \\
&\quad + {}_a\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\lambda\mathrm{x}))
\end{aligned}
$$

By boundary conditions $\varpi(a) = \varpi_a$ and $\varpi(b) = \varpi_b$, we get

$$
\begin{aligned}
\varpi_b &= \frac{\mathrm{w}(a)}{\mathrm{w}(b)}\varpi_a + [\psi(b) - \psi(a)]\frac{\varpi_a}{\mathrm{w}(b)}\frac{\mathrm{w}'(a)}{\psi'(a)} \\
&\quad + {}_a\mathfrak{I}^{\nu}_{\psi(b),\mathrm{w}(b)}\,\mathrm{g}(b,\varpi(b),\varpi(\lambda b)),
\end{aligned}
$$

which implies

$$
\varpi_a = \frac{1}{\chi}\left(\varpi_b - \frac{1}{\Gamma(\nu)\mathrm{w}(b)}\int_a^b \psi'(\vartheta)(\psi(b) - \psi(\vartheta))^{\nu-1}\mathrm{w}(\vartheta)\mathrm{g}(\vartheta,\varpi(\vartheta),\varpi(\lambda\vartheta))d\vartheta\right).
$$

Hence,

$$
\begin{aligned}
\varpi(\mathrm{x}) &= \frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})}\frac{1}{\chi}\left(\varpi_b - {}_a\mathfrak{I}^{\nu}_{\psi(b),\mathrm{w}(b)}\,\mathrm{g}(b,\varpi(b),\varpi(\lambda b))\right) \\
&\quad + [\psi(\mathrm{x}) - \psi(a)]\frac{1}{\mathrm{w}(\mathrm{x})}\frac{\frac{1}{\chi}\left(\varpi_b - {}_a\mathfrak{I}^{\nu}_{\psi(b),\mathrm{w}(b)}\,\mathrm{g}(b,\varpi(b),\varpi(\lambda b))\right)\mathrm{w}'(a)}{\psi'(a)} \\
&\quad + {}_a\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\lambda\mathrm{x}))
\end{aligned}
$$

which is (12).

Conversely, if ϖ satisfies (12), then by Lemma 1, we have

$$
\begin{aligned}
{}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\varpi(\mathrm{x}) &= \frac{1}{\chi}\,{}^{C}_{a}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\left(\frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})} + [\psi(\mathrm{x}) - \psi(a)]\frac{\mathrm{w}'(a)}{\psi'(a)\mathrm{w}(\mathrm{x})}\right) \\
&\quad \left(\varpi_b - {}_a\mathfrak{I}^{\nu}_{\psi(b),\mathrm{w}(b)}\,\mathrm{g}(b,\varpi(b),\varpi(\lambda b))\right) \\
&\quad + {}^{C}\mathfrak{D}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,{}_a\mathfrak{I}^{\nu}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\,\mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\lambda\mathrm{x})) \\
&= \mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\lambda\mathrm{x})).
\end{aligned}
$$

Moreover, $\varpi(a) = \varpi_a$ and $\varpi(b) = \varpi_b$. □

Hence, we can deduce the next corollary:

Corollary 1. *Let $1 < \nu < 2$ and $0 < \lambda_i < 1$, for $i = 1, 2, ..., m$. Assume that g is continuous function. The problem (6) is equivalent to*

$$\begin{aligned}\varpi(\mathsf{x}) &= \frac{1}{\chi}\left(\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})} + [\psi(\mathsf{x}) - \psi(a)]\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right) \\ &\quad \times \left(\varpi_b - {}_a\mathfrak{I}^{\nu}_{\psi(b),\mathsf{w}(b)}\,\mathsf{g}(b, \varpi(b), \varpi(\lambda_1 b), ..., \varpi(\lambda_m b))\right) \\ &\quad + {}_a\mathfrak{I}^{\nu}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\,\mathsf{g}(\mathsf{x}, \varpi(\mathsf{x}), \varpi(\lambda_1\mathsf{x}), ..., \varpi(\lambda_m\mathsf{x}))),\end{aligned} \tag{15}$$

where χ is defined as Lemma 4.

Regarding Corollary 1, we define an operator $\mathcal{K} : \mathcal{AC} \to \mathcal{AC}$ by

$$\begin{aligned}&(\mathcal{K}\varpi)(\mathsf{x}) \\ &= \frac{1}{\chi}\left(\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})} + [\psi(\mathsf{x}) - \psi(a)]\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right)\left(\varpi_b - \frac{1}{\Gamma(\nu)\mathsf{w}(b)}\right. \\ &\left.\times \int_a^b \psi'(\vartheta)(\psi(b) - \psi(\vartheta))^{\nu-1}\mathsf{w}(\vartheta)\mathsf{g}(\vartheta, \varpi(\vartheta), \varpi(\lambda_1\vartheta), ..., \varpi(\lambda_m\vartheta)))d\vartheta\right) \\ &+ \frac{1}{\Gamma(\nu)\mathsf{w}(\mathsf{x})}\int_a^{\mathsf{x}} \psi'(\vartheta)(\psi(\mathsf{x}) - \psi(\vartheta))^{\nu-1}\mathsf{w}(\vartheta)\mathsf{g}(\vartheta, \varpi(\vartheta), \varpi(\lambda_1\vartheta), ..., \varpi(\lambda_m\vartheta)))d\vartheta\end{aligned} \tag{16}$$

The following assumptions are necessary in order to prove the main results:

(P_1) $|\mathsf{w}(\mathsf{x})\mathsf{g}(\mathsf{x}, \varpi_1, ..., \varpi_{m+1}) - \mathsf{w}(\mathsf{x})\mathsf{g}(\mathsf{x}, \overline{\varpi}_1, ..., \overline{\varpi}_{m+1})| \leq L_{\mathsf{g}} \sum_{j=1}^{m+1} |\varpi_j - \overline{\varpi}_j|$, $\forall \mathsf{x} \in \mho$, $L_{\mathsf{g}} > 0$, $\varpi_j, \overline{\varpi}_j \in \mathbb{R}$.

(P_2) $|\mathsf{w}(\mathsf{x})\mathsf{g}(\mathsf{x}, \varpi_1, ..., \varpi_{m+1})| \leq n_{\mathsf{g}} + m_{\mathsf{g}} \sum_{j=1}^{m+1} |\varpi_j|$, $\forall \mathsf{x} \in \mho$, $\varpi_j \in \mathbb{R}$, $n_{\mathsf{g}}, m_{\mathsf{g}} > 0$.

For convenience, let us set $\varpi_\lambda(\mathsf{x}) := (\varpi(\lambda_1\mathsf{x}), ..., \varpi(\lambda_m\mathsf{x}))$, $\psi^{\nu}_{\mathsf{x},a} := (\psi(\mathsf{x}) - \psi(a))^{\nu}$, $\psi^{\nu-1}_{\mathsf{x},\vartheta} := \psi'(\vartheta)(\psi(\mathsf{x}) - \psi(\vartheta))^{\nu-1}$,

$$\Lambda := \left|\frac{\mathsf{w}(a)}{\mathsf{w}(b)}\right| + [\psi_{b,a}]\left|\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(b)}\right|,$$

$$\Pi := \frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)} L_{\mathsf{g}}(m+1)\left(\frac{\Lambda}{|\chi|} + 1\right),$$

$$\Delta := \left(\frac{\Lambda}{|\chi|} + 1\right)\mathsf{w}_0\mathsf{g}_0\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)} + \frac{\Lambda}{|\chi|}|\varpi_b|.$$

3.2. Existence Results

Theorem 1. *Assume that* $(\mathsf{wg}) : \mho \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ *is a continuous and satisfies the condition (**P_1**) with*

$$\left[\frac{\Lambda}{\chi} + 1\right]\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}(m+1)L_{\mathsf{g}} < 1, \tag{17}$$

then the ψ-weighted pantograph problem (6) has a unique solution.

Proof. Let $\max_{\mathsf{x}\in\mho}(\mathsf{w}(\mathsf{x})\mathsf{g}(\mathsf{x}, 0, ..., 0)) = \mathsf{w}_0\mathsf{g}_0$ and choosing $r \geq \frac{\Delta}{1-\Pi}$, where $\Pi > 0$. First, we prove that $\mathcal{K}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{\varpi \in \mathcal{AC} : \|\varpi\| \leq r\}$. Indeed, for $\varpi \in \mathcal{B}_r$, we have

$$
\begin{aligned}
&|(\mathcal{K}\varpi)(\mathsf{x})| \\
&\leq \frac{1}{|\chi|}\left(\left|\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})}\right| + [\psi_{\mathsf{x},a}]\left|\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right|\right)\left(|\varpi_b| + \frac{1}{\Gamma(\nu)|\mathsf{w}(b)|}\right. \\
&\times \int_a^b \psi_{b,\vartheta}^{\nu-1}[|\,\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\varpi(\vartheta),\varpi_\lambda(\vartheta)) - \mathsf{w}(\vartheta)\mathsf{g}(\vartheta,0,0)| + |\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,0,0)|]d\vartheta \\
&+ \frac{1}{\Gamma(\nu)|\mathsf{w}(\mathsf{x})|}\int_a^{\mathsf{x}} \psi_{\mathsf{x},\vartheta}^{\nu-1}[|\,\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\varpi(\vartheta),\varpi_\lambda(\vartheta)) - \mathsf{w}(\vartheta)\mathsf{g}(\vartheta,0,0)| \\
&+ |\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,0,0)|]d\vartheta \\
&\leq \frac{\Lambda}{|\chi|}\left(|\varpi_b| + \frac{1}{\Gamma(\nu)|\mathsf{w}(b)|}\int_a^b \psi_{b,\vartheta}^{\nu-1}[L_{\mathsf{g}}(|\varpi| + |\varpi_1| + \ldots + |\varpi_m|) + \mathsf{w}_0\mathsf{g}_0]\,d\vartheta\right) \\
&+ \frac{1}{\Gamma(\nu)|\mathsf{w}(\mathsf{x})|}\int_a^{\mathsf{x}} \psi_{\mathsf{x},\vartheta}^{\nu-1}[L_{\mathsf{g}}(|\varpi| + |\varpi_1| + \ldots + |\varpi_m|) + \mathsf{w}_0\mathsf{g}_0]\,d\vartheta \\
&\leq \frac{\Lambda}{|\chi|}\left(|\varpi_b| + [L_{\mathsf{g}}(m+1)r + \mathsf{w}_0\mathsf{g}_0]\frac{1}{\Gamma(\nu)|\mathsf{w}(b)|}\int_a^b \psi_{b,\vartheta}^{\nu-1}d\vartheta\right) \\
&+ [L_{\mathsf{g}}(m+1)r + \mathsf{w}_0\mathsf{g}_0]\frac{1}{\Gamma(\nu)|\mathsf{w}(\mathsf{x})|}\int_a^{\mathsf{x}} \psi_{\mathsf{x},\vartheta}^{\nu-1}d\vartheta \\
&= \frac{\Lambda}{|\chi|}\left(|\varpi_b| + [L_{\mathsf{g}}(m+1)r + \mathsf{w}_0\mathsf{g}_0]\,{}_a\mathfrak{I}^{\nu}_{\psi(b),\mathsf{w}(b)}\,\mathsf{w}^{-1}(b)\right) \\
&+ [L_{\mathsf{g}}(m+1)r + \mathsf{w}_0\mathsf{g}_0]\,{}_a\mathfrak{I}^{\nu}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\,\mathsf{w}^{-1}(\mathsf{x}) \\
\leq\ & \frac{\Lambda}{|\chi|}\left(|\varpi_b| + [L_{\mathsf{g}}(m+1)r + \mathsf{w}_0\mathsf{g}_0]\frac{\psi_{b,a}^{\nu}}{\Gamma(\nu+1)}\right) \\
&+ [L_{\mathsf{g}}(m+1)r + \mathsf{w}_0\mathsf{g}_0]\frac{\psi_{b,a}^{\nu}}{\Gamma(\nu+1)} \\
=\ & \Pi r + \Delta \leq r.
\end{aligned}
$$

Next, we show that $\mathcal{K}$ is contraction in $\mathcal{AC}$. For $\varpi(\mathsf{x}), \overline{\varpi}(\mathsf{x}) \in \mathcal{AC}$ and for each $\mathsf{x} \in \mho$, we have

$$
\begin{aligned}
&|(\mathcal{K}\varpi)(\mathsf{x}) - (\mathcal{K}\overline{\varpi})(\mathsf{x})| \\
&\leq \frac{1}{\chi}\left(\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})} + \psi_{\mathsf{x},a}\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right) \\
&\times \frac{1}{\Gamma(\nu)\mathsf{w}(b)}\int_a^b \psi_{b,\vartheta}^{\nu-1}|\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\varpi,\varpi_\lambda) - \mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\overline{\varpi},\overline{\varpi}_\lambda)|d\vartheta \\
&+ \frac{1}{\Gamma(\nu)\mathsf{w}(\mathsf{x})}\int_a^{\mathsf{x}} \psi_{\mathsf{x},\vartheta}^{\nu-1}|\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\varpi,\varpi_\lambda) - \mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\overline{\varpi},\overline{\varpi}_\lambda)|d\vartheta \\
&\leq \frac{\Lambda}{\chi}\frac{1}{\Gamma(\nu)\mathsf{w}(b)}\int_a^b \psi_{b,\vartheta}^{\nu-1}L_{\mathsf{g}}\sum_{j=1}^{m+1}|\varpi_j(\vartheta) - \overline{\varpi}_j(\vartheta)|d\vartheta \\
&+ \frac{1}{\Gamma(\nu)\mathsf{w}(\mathsf{x})}\int_a^{\mathsf{x}} \psi_{\mathsf{x},\vartheta}^{\nu-1}L_{\mathsf{g}}\sum_{j=1}^{m+1}|\varpi_j(\vartheta) - \overline{\varpi}_j(\vartheta)|d\vartheta \\
&= \frac{\Lambda}{\chi}\,{}_a\mathfrak{I}^{\nu}_{\psi(b),\mathsf{w}(b)}\,\mathsf{w}^{-1}(b)L_{\mathsf{g}}\sum_{j=1}^{m+1}|\varpi_j(b) - \overline{\varpi}_j(b)| \\
&+ {}_a\mathfrak{I}^{\nu}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\,\mathsf{w}^{-1}(\mathsf{x})L_{\mathsf{g}}\sum_{j=1}^{m+1}|\varpi_j(\mathsf{x}) - \overline{\varpi}_j(\mathsf{x})| \\
&\leq \left[\frac{\Lambda}{\chi} + 1\right]\frac{\psi_{b,a}^{\nu}}{\Gamma(\nu+1)}(m+1)L_{\mathsf{g}}\|\varpi - \overline{\varpi}\|.
\end{aligned}
$$

$\mathcal{K}$ is a contraction in accordance with condition (17), and thanks to the Banach fixed point theorem, $\mathcal{K}$ has a unique fixed point, which is an unique solution to (6). $\square$

Then, in order to prove existence results, we apply Krasnoselskii's fixed point theorem [47].

Theorem 2. *Let* $\mathrm{wg} : \mho \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ *be a continuous satisfying* (P_2). *If*

$$\frac{\Lambda}{\chi}\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}(m+1)L_{\mathrm{g}} < 1, \tag{18}$$

Then the ψ*-weighted pantograph problem* (6) *has a least one solution.*

Proof. From (16), we define the operators $\mathcal{K}_1, \mathcal{K}_2 : \mathcal{AC} \to \mathcal{AC}$ by

$$\begin{aligned}(\mathcal{K}_1\varpi)(\mathrm{x}) &= \frac{1}{\chi}\left(\frac{\mathrm{w}(a)}{\mathrm{w}(\mathrm{x})} + \psi_{\mathrm{x},a}\frac{\mathrm{w}'(a)}{\psi'(a)\mathrm{w}(\mathrm{x})}\right) \\ &\quad \times\left(\varpi_b - \frac{1}{\Gamma(\nu)\mathrm{w}(b)}\int_a^b \psi^{\nu-1}_{b,\vartheta}\mathrm{w}(\vartheta)\mathrm{g}(\vartheta,\varpi(\vartheta),\varpi_\lambda(\vartheta))d\vartheta\right), \ \mathrm{x}\in\mho,\end{aligned}$$

and

$$(\mathcal{K}_2\varpi)(\mathrm{x}) = \tfrac{1}{\Gamma(\nu)\mathrm{w}(\mathrm{x})}\int_a^{\mathrm{x}} \psi^{\nu-1}_{\mathrm{x},\vartheta}\mathrm{w}(\vartheta)\mathrm{g}(\vartheta,\varpi(\vartheta),\varpi_\lambda(\vartheta))d\vartheta, \ \mathrm{x}\in\mho,$$

where $(\mathcal{K}_1\varpi + \mathcal{K}_2\varpi)(\mathrm{x}) = (\mathcal{K}\varpi)(\mathrm{x})$. Let us define $\mathcal{B}_{\bar{r}} = \{\varpi \in \mathcal{AC} : \|\varpi\| \le \bar{r}\}$, we fix

$$\bar{r} \ge \frac{n_{\mathrm{g}}\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}\left(\frac{\Lambda}{\chi}+1\right) + \frac{\Lambda}{\chi}|\varpi_b|}{1 - m_{\mathrm{g}}(m+1)\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}\left(\frac{\Lambda}{\chi}+1\right)}. \tag{19}$$

For $\varpi, \omega \in \mathcal{B}_{\bar{r}}$, we find that

$$\begin{aligned}|(\mathcal{K}_1\varpi + \mathcal{K}_2\omega)(\mathrm{x})| &\le |(\mathcal{K}_1\varpi)(\mathrm{x})| + |(\mathcal{K}_2\omega)(\mathrm{x})| \\ &\le \frac{\Lambda}{\chi}\left(|\varpi_b| + \frac{1}{\Gamma(\nu)\mathrm{w}(b)}\int_a^b \psi^{\nu-1}_{b,\vartheta}|\mathrm{w}(\vartheta)\mathrm{g}(\vartheta,\varpi(\vartheta),\varpi_\lambda(\vartheta))|d\vartheta\right) \\ &\quad + \frac{1}{\Gamma(\nu)\mathrm{w}(\mathrm{x})}\int_a^{\mathrm{x}} \psi^{\nu-1}_{\mathrm{x},\vartheta}|\mathrm{w}(\vartheta)\mathrm{g}(\vartheta,\omega(\vartheta),\omega_\lambda(\vartheta))|d\vartheta \\ &\le \frac{\Lambda}{\chi}\left(|\varpi_b| + \frac{1}{\Gamma(\nu)\mathrm{w}(b)}\int_a^b \psi^{\nu-1}_{b,\vartheta}\left(n_{\mathrm{g}} + m_{\mathrm{g}}\sum_{j=1}^{m+1}|\varpi_j(\vartheta)|\right)d\vartheta\right) \\ &\quad + \frac{1}{\Gamma(\nu)\mathrm{w}(\mathrm{x})}\int_a^{\mathrm{x}} \psi^{\nu-1}_{\mathrm{x},\vartheta}\left(n_{\mathrm{g}} + m_{\mathrm{g}}\sum_{j=1}^{m+1}|\omega_j(\vartheta)|\right)d\vartheta \\ &\le \frac{\Lambda}{\chi}\left(|\varpi_b| + (n_{\mathrm{g}} + m_{\mathrm{g}}(m+1)\|\varpi_j\|)\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}\right) \\ &\quad + (n_{\mathrm{g}} + m_{\mathrm{g}}(m+1)\|\omega_j\|)\frac{\psi^{\nu}_{\mathrm{x},a}}{\Gamma(\nu+1)} \\ &\le \frac{\Lambda}{\chi}\left(|\varpi_b| + (n_{\mathrm{g}} + m_{\mathrm{g}}(m+1)\bar{r})\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}\right) \\ &\quad + (n_{\mathrm{g}} + m_{\mathrm{g}}(m+1)\bar{r})\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)} \\ &\le (n_{\mathrm{g}} + m_{\mathrm{g}}(m+1)\bar{r})\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}\left(\frac{\Lambda}{\chi}+1\right) + \frac{\Lambda}{\chi}|\varpi_b| \\ &= m_{\mathrm{g}}(m+1)\bar{r}\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}\left(\frac{\Lambda}{\chi}+1\right) + n_{\mathrm{g}}\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}\left(\frac{\Lambda}{\chi}+1\right) + \frac{\Lambda}{\chi}|\varpi_b|.\end{aligned}$$

Due to (19), we deduce that $\|\mathcal{K}_1\varpi + \mathcal{K}_2\overline{\varpi}\| \leq \overline{r}$.

Further, $\mathcal{K}_1$ is a contraction operator. Indeed, for each $\varpi(\mathsf{x}), \overline{\varpi}(\mathsf{x}) \in \mathcal{AC}$, and for each $\mathsf{x} \in \mho$, we have

$$\begin{aligned}
&|\mathcal{K}_1\varpi(\mathsf{x}) - \mathcal{K}_1\overline{\varpi}(\mathsf{x})| \\
&\leq \frac{1}{\chi}\left(\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})} + \psi_{\mathsf{x},a}\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right)\frac{1}{\Gamma(\nu)\mathsf{w}(b)} \\
&\times \int_a^b \psi_{b,\vartheta}^{\nu-1}|\,\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\varpi(\vartheta),\varpi_\lambda(\vartheta)) - \mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\overline{\varpi}(\vartheta),\overline{\varpi}_\lambda(\vartheta))|d\vartheta \\
&\leq \frac{\Lambda}{\chi}\frac{1}{\Gamma(\nu)\mathsf{w}(b)}\int_a^b \psi_{b,\vartheta}^{\nu-1} L_{\mathsf{g}}\sum_{j=1}^{m+1}\left|\varpi_j(\vartheta) - \overline{\varpi}_j(\vartheta)\right|d\vartheta \\
&= \frac{\Lambda}{\chi}\,{}_a\mathfrak{I}^{\nu}_{\psi(b),\mathsf{w}(b)}\,\mathsf{w}^{-1}(b)L_{\mathsf{g}}\sum_{j=1}^{m+1}\left|\varpi_j(b) - \overline{\varpi}_j(b)\right| \\
&\leq \frac{\Lambda}{\chi}\frac{\psi_{b,a}^{\nu}}{\Gamma(\nu+1)}(m+1)L_{\mathsf{g}}\|\varpi - \overline{\varpi}\|.
\end{aligned}$$

From (17), $\mathcal{K}_1$ is a contraction.

Continuity of g, w and ψ implies that $\mathcal{K}_2$ is continuous. Further, $\mathcal{K}_2$ is uniformly bounded on $\mathcal{B}_{\overline{r}}$ as

$$|(\mathcal{K}_2\varpi)(\mathsf{x})| \leq (n_{\mathsf{g}} + m_{\mathsf{g}}(m+1)\overline{r})\frac{\psi_{b,a}^{\nu}}{\Gamma(\nu+1)}.$$

Now, we prove that $\mathcal{K}_2$ is compact. In fact

$$\begin{aligned}
\left|(\mathcal{K}_1\varpi)'(\mathsf{x})\right| &\leq D_{\mathsf{w}}\left(\frac{1}{\Gamma(\nu)\mathsf{w}(\mathsf{x})}\int_a^{\mathsf{x}}\psi_{\mathsf{x},\vartheta}^{\nu-1}|\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\varpi(\vartheta),\varpi_\lambda(\vartheta))|d\vartheta\right) \\
&= D_{\mathsf{w}}\,{}_a\mathfrak{I}^{\nu}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\,\mathsf{g}(\mathsf{x},\varpi(\mathsf{x}),\varpi_\lambda(\mathsf{x})) \\
&= {}_a\mathfrak{I}^{\nu-1}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\,\mathsf{g}(\mathsf{x},\varpi(\mathsf{x}),\varpi_\lambda(\mathsf{x})) \\
&= \frac{1}{\Gamma(\nu-1)\mathsf{w}(\mathsf{x})}\int_a^{\mathsf{x}}\psi_{\mathsf{x},\vartheta}^{\nu-2}|\mathsf{w}(\vartheta)\mathsf{g}(\vartheta,\varpi(\vartheta),\varpi_\lambda(\vartheta))|d\vartheta \\
&\leq (n_{\mathsf{g}} + m_{\mathsf{g}}(m+1)\overline{r})\frac{\psi_{b,a}^{\nu-1}}{\Gamma(\nu)}.
\end{aligned}$$

Let $\varpi \in \mathcal{B}_{\overline{r}}$, and $\mathsf{x} \in \mho$ with $\mathsf{x}_\epsilon < \mathsf{x}_\delta \in \mho$. Then

$$|\,(\mathcal{K}_1\varpi)(\mathsf{x}_\delta) - (\mathcal{K}_1\varpi)(\mathsf{x}_\epsilon)| = \int_{\mathsf{x}_\epsilon}^{\mathsf{x}_\delta}\left|(\mathcal{K}_1\varpi)'(\vartheta)\right|d\vartheta \leq (n_{\mathsf{g}} + m_{\mathsf{g}}(m+1)\overline{r})\frac{\psi_{b,a}^{\nu-1}}{\Gamma(\nu)}(\mathsf{x}_\delta - \mathsf{x}_\epsilon).$$

Thus, $|\,(\mathcal{K}_1\varpi)(\mathsf{x}_\delta) - (\mathcal{K}_1\varpi)(\mathsf{x}_\epsilon)| \to 0$ as $\mathsf{x}_\epsilon \to \mathsf{x}_\delta$. Thus, $\mathcal{K}_1$ is equicontinuous on $\mathcal{B}_{\overline{r}}$. Hence, $\mathcal{K}_1$ is relatively compact on $\mathcal{B}_{\overline{r}}$ as a result of the steps that came before, and according to the Arzela–Ascoli theorem, $\mathcal{K}_1$ has at least one fixed point. The Krasnoselskii theorem [47] shows that there is at least one solution to the problem (6). □

Remark 2. *When $\psi(\mathsf{x}) = \mathsf{x}$, the results on the problem (6) still hold true for the problem (5).*

4. Weighted ψ-Caputo Fractional System

Consider a more general problem as

$$\begin{cases} {}^{C}_{a}\mathfrak{D}^{\nu_k}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\varpi_k(\mathsf{x}) = \mathsf{g}_k(\mathsf{x},\varpi_{1,\lambda_i}(\mathsf{x}),\varpi_{2,\lambda_i}(\mathsf{x}),\dots,\varpi_{n,\lambda_i}(\mathsf{x})),\ \mathsf{x} \in \mho := [a,b], \\ \varpi_k(a) = \varpi_a^k,\ \varpi_k(b) = \varpi_b^k,\ ,\ k = 1,\dots,n,\ i = 1,\dots,m \end{cases} \tag{20}$$

where $1 < \nu_k < 2$, ${}^{C}_{a}\mathfrak{D}^{\nu_k}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}$ is the generalized weighted Caputo FD of order ν_i and

$$\begin{aligned}
\varpi_{1,\lambda_i}(\mathsf{x}) &= \varpi_1(\mathsf{x}), \varpi_1(\lambda_1\mathsf{x}), \ldots, \varpi_1(\lambda_m\mathsf{x}),\\
\varpi_{2,\lambda_i}(\mathsf{x}) &= \varpi_2(\mathsf{x}), \varpi_2(\lambda_1\mathsf{x}), \ldots, \varpi_2(\lambda_m\mathsf{x}),\\
&\vdots\\
\varpi_{n,\lambda_i}(\mathsf{x}) &= \varpi_n(\mathsf{x}), \varpi_n(\lambda_1\mathsf{x}), \ldots, \varpi_n(\lambda_m\mathsf{x}),
\end{aligned}$$

It is possible to write the system (20) as

$$\begin{cases} {}^{C}_{a}\mathfrak{D}^{\Upsilon}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\mathcal{V}(\mathsf{x}) = \mathsf{G}(\mathsf{x}, \mathcal{V}_\lambda(\mathsf{x})), \ \mathsf{x} \in \mho,\\ \mathcal{V}(a) = \mathcal{V}_a, \ \mathcal{V}(b) = \mathcal{V}_b, \end{cases} \tag{21}$$

where

$$\mathcal{V}(\mathsf{x}) = \begin{bmatrix} \varpi_1(\mathsf{x})\\ \varpi_2(\mathsf{x})\\ \vdots\\ \varpi_n(\mathsf{x}) \end{bmatrix}, \ \mathsf{G}(\mathsf{x}, \mathcal{V}_\lambda(\mathsf{x})) = \begin{bmatrix} \mathsf{g}_1(\mathsf{x}, \varpi_{1,\lambda_i}(\mathsf{x}))\\ \mathsf{g}_2(\mathsf{x}, \varpi_{2,\lambda_i}(\mathsf{x}))\\ \vdots\\ \mathsf{g}_n(\mathsf{x}, \varpi_{n,\lambda_i}(\mathsf{x})) \end{bmatrix}, \ \text{and}$$

$$\mathcal{V}(a) = \begin{bmatrix} \varpi_1(a)\\ \varpi_2(a)\\ \vdots\\ \varpi_n(a) \end{bmatrix}, \ \mathcal{V}(b) = \begin{bmatrix} \varpi_1(b)\\ \varpi_2(b)\\ \vdots\\ \varpi_n(b) \end{bmatrix}, \ \mathcal{V}_c = \begin{bmatrix} \varpi_c^1\\ \varpi_c^2\\ \vdots\\ \varpi_c^n \end{bmatrix}, \ \Upsilon = \begin{bmatrix} \nu_1\\ \nu_2\\ \vdots\\ \nu_n \end{bmatrix}.$$

By using Corollary 1, the system (21) has the following solution

$$\begin{aligned}
\mathcal{V}(\mathsf{x}) &= \frac{1}{\chi}\left(\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})} + [\psi(\mathsf{x}) - \psi(a)]\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right)\\
&\quad \times\left(\mathcal{V}_b - {}_{a}\mathfrak{I}^{\Upsilon}_{\psi(b),\mathsf{w}(b)}\mathsf{G}(b, \mathcal{V}_\lambda(b))\right)\\
&\quad + {}_{a}\mathfrak{I}^{\Upsilon}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\mathsf{G}(\mathsf{x}, \mathcal{V}_\lambda(\mathsf{x})),
\end{aligned} \tag{22}$$

where χ is defined as Lemma 4. We can write the system (22) as

$$\begin{cases}
\varpi_1(\mathsf{x}) = \begin{cases} \frac{1}{\chi}\left(\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})} + [\psi(\mathsf{x}) - \psi(a)]\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right)\\ \times\left(\varpi_b^1 - {}_{a}\mathfrak{I}^{\nu_1}_{\psi(b),\mathsf{w}(b)}\mathsf{g}_1(b, \varpi_1(b), \varpi_1(\lambda_1 b), \ldots, \varpi_1(\lambda_m b))\right)\\ + {}_{a}\mathfrak{I}^{\nu_1}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\mathsf{g}_1(\mathsf{x}, \varpi_1(\mathsf{x}), \varpi_1(\lambda_1\mathsf{x}), \ldots, \varpi_1(\lambda_m\mathsf{x}), \end{cases}\\
\varpi_2(\mathsf{x}) = \begin{cases} \frac{1}{\chi}\left(\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})} + [\psi(\mathsf{x}) - \psi(a)]\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right)\\ \times\left(\varpi_b^2 - {}_{a}\mathfrak{I}^{\nu_2}_{\psi(b),\mathsf{w}(b)}\mathsf{g}_2(b, \varpi_2(b), \varpi_2(\lambda_1 b), \ldots, \varpi_2(\lambda_m b))\right)\\ + {}_{a}\mathfrak{I}^{\nu_2}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\mathsf{g}_2(\mathsf{x}, \varpi_2(\mathsf{x}), \varpi_2(\lambda_1\mathsf{x}), \ldots, \varpi_2(\lambda_m\mathsf{x}), \end{cases}\\
\qquad\vdots\\
\varpi_n(\mathsf{x}) = \begin{cases} \frac{1}{\chi}\left(\frac{\mathsf{w}(a)}{\mathsf{w}(\mathsf{x})} + [\psi(\mathsf{x}) - \psi(a)]\frac{\mathsf{w}'(a)}{\psi'(a)\mathsf{w}(\mathsf{x})}\right)\\ \times\left(\varpi_b^n - {}_{a}\mathfrak{I}^{\nu_n}_{\psi(b),\mathsf{w}(b)}\mathsf{g}_n(b, \varpi_n(b), \varpi_n(\lambda_1 b), \ldots, \varpi_n(\lambda_m b))\right)\\ + {}_{a}\mathfrak{I}^{\nu_n}_{\psi(\mathsf{x}),\mathsf{w}(\mathsf{x})}\mathsf{g}_n(\mathsf{x}, \varpi_n(\mathsf{x}), \varpi_n(\lambda_1\mathsf{x}), \ldots, \varpi_n(\lambda_m\mathsf{x}). \end{cases}
\end{cases} \tag{23}$$

Banach's and Krasnoselskii's fixed point theorem can be used to present the following theorems without the need for proofs.

Theorem 3. *Assume that* $(\mathsf{wg}_k) : \mho \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ *is a continuous and satisfies*

(P_3) $|\mathrm{w}(\mathrm{x})\mathrm{g}_k(\mathrm{x}, \varpi_1, ..., \varpi_{m+1}) - \mathrm{w}(\mathrm{x})\mathrm{g}_k(\mathrm{x}, \overline{\varpi}_1, ..., \overline{\varpi}_{m+1})| \leq L_{\mathrm{g}_k} \sum_{j=1}^{m+1} |\varpi_j - \overline{\varpi}_j|$, *for* $k = 1, ..., n$, $\mathrm{x} \in \mho$, $L_{\mathrm{g}_k} > 0$, $\varpi_j, \overline{\varpi}_j \in \mathbb{R}$ *with*

$$\left[\frac{\Lambda}{\chi} + 1\right] \frac{\psi_{b,a}^{\nu_k}}{\Gamma(\nu_k + 1)} (m+1) L_{\mathrm{g}_k} < 1.$$

Then the ψ-weighted pantograph system (21) has a unique solution.

Theorem 4. *Let* $\mathrm{wg}_k : \mho \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ *be a continuous satisfying*

(P_4) $|\mathrm{w}(\mathrm{x})\mathrm{g}_k(\mathrm{x}, \varpi_1, ..., \varpi_{m+1})| \leq n_{\mathrm{g}_k} + m_{\mathrm{g}_k} \sum_{j=1}^{m+1} |\varpi_j|$, *for* $k = 1, ..., n$, $\mathrm{x} \in \mho$, $\varpi_j \in \mathbb{R}$, $n_{\mathrm{g}_k}, m_{\mathrm{g}_k} > 0$

$$\frac{\Lambda}{\chi} \frac{\psi_{b,a}^{\nu_k}}{\Gamma(\nu_k + 1)} (m+1) L_{\mathrm{g}_k} < 1.$$

Then the ψ-weighted pantograph system (21) has a least one solution.

Remark 3. *Theorems 3 and 4 for the nonlinear system (21) in light of the formula (22) or (23) can be proved using the same procedure as in the preceding sections.*

5. Examples

To illustrate our acquired results, we provide two examples.

Example 1. *Consider the following weighted ψ-Caputo-type problem*

$$\begin{cases} {}_0^C\mathfrak{D}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}^{\nu} \Big| \varpi(\mathrm{x}) = \mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda_1 \mathrm{x}), \varpi(\lambda_2 \mathrm{x})), \ 0 \leq \mathrm{x} \leq 1, \\ \varpi(0) = 1, \ \varpi(1) = 2, \end{cases} \tag{24}$$

where $m = 2$, $\mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda_1 \mathrm{x}), \varpi(\lambda_2 \mathrm{x})) = \frac{\cos|\varpi(\mathrm{x}) + \varpi(\frac{\mathrm{x}}{4})| + \sin|\varpi(\frac{\mathrm{x}}{6})|}{10 + \mathrm{x}} + \frac{\mathrm{x}+1}{10}$, $\nu = \frac{5}{4}$, $\psi(\mathrm{x}) = e^{\frac{\mathrm{x}}{3}}$, $\mathrm{w}(\mathrm{x}) = e^{-\mathrm{x}}$, $a = 0$, $b = 1$, $\varpi_0 = 1$, $\varpi_1 = 2$, $\lambda_1 = \frac{1}{4}$, *and* $\lambda_2 = \frac{1}{6}$.

(I) Application of Theorem 1: For $\mathrm{x} \in [0,1]$, *and* $\varpi, \omega \in [0, \infty)$, *we have*

$$\begin{aligned} & | \mathrm{w}(\mathrm{x})\mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda_1 \mathrm{x}), \varpi(\lambda_2 \mathrm{x})) - \mathrm{w}(\mathrm{x})\mathrm{g}(\mathrm{x}, \omega(\mathrm{x}), \omega(\lambda_1 \mathrm{x}), \omega(\lambda_2 \mathrm{x}))| \\ & \leq \frac{1}{e^{\mathrm{x}}(10 + \mathrm{x})} | \varpi(\mathrm{x}) - \omega(\mathrm{x})| + \left|\varpi(\frac{\mathrm{x}}{4}) - \omega(\frac{\mathrm{x}}{4})\right| + \left|\varpi(\frac{\mathrm{x}}{6}) - \omega(\frac{\mathrm{x}}{6})\right| \\ & \leq \frac{1}{10} \sum_{j=1}^{3} |\varpi_j - \omega_j| \end{aligned}$$

where $|\varpi_1(\mathrm{x}) - \omega_1(\mathrm{x})| = | \varpi(\mathrm{x}) - \omega(\mathrm{x})|$, $|\varpi_2(\mathrm{x}) - \omega_2(\mathrm{x})| = |\varpi(\frac{\mathrm{x}}{4}) - \omega(\frac{\mathrm{x}}{4})|$ *and* $|\varpi_3(\mathrm{x}) - \omega_3(\mathrm{x})| = |\varpi(\frac{\mathrm{x}}{4}) - \omega(\frac{\mathrm{x}}{4})|$.

Thus, (P_1) *holds with* $L_{\mathrm{g}} = \frac{1}{10}$. *Moreover, the condition (17) holds. Indeed,*

$$\left[\frac{\Lambda}{\chi} + 1\right] \frac{\psi_{b,a}^{\nu}}{\Gamma(\nu + 1)} (m+1) L_{\mathrm{g}} = \left[\frac{3e^{\frac{4}{3}} - 2e}{e - 3e^{\frac{4}{3}}} + 1\right] \frac{\left[e^{\frac{1}{3}} - 1\right]^{\frac{5}{4}}}{\Gamma(\frac{7}{4})} \frac{3}{10} < 1$$

where $\Lambda = 3e^{\frac{4}{3}} - 2e$ *and* $\chi = e - 3e^{\frac{4}{3}}$. *Thus, Theorem 1 shows that (24) has a unique solution on* $[0,1]$.

(II) Application of Theorem 2: For $\mathrm{x} \in [0,1]$, *and* $\varpi \in [0, \infty)$, *we have*

$$\begin{aligned} | \mathrm{w}(\mathrm{x})\mathrm{g}(\mathrm{x}, \varpi(\mathrm{x}), \varpi(\lambda_1 \mathrm{x}), \varpi(\lambda_2 \mathrm{x}))| & \leq \frac{|\varpi(\mathrm{x}) + \varpi(\frac{\mathrm{x}}{4})| + |\varpi(\frac{\mathrm{x}}{6})|}{e^{\mathrm{x}}(10 + \mathrm{x})} + \frac{\mathrm{x}+1}{10} \\ & \leq \frac{1}{5} + \frac{1}{10} \sum_{j=1}^{3} |\varpi_j|, \end{aligned}$$

where $|\varpi_1(\mathrm{x})| = |\,\varpi(\mathrm{x})|$, $|\varpi_2(\mathrm{x})| = |\varpi(\frac{\mathrm{x}}{4})|$ *and* $|\varpi_3(\mathrm{x})| = |\varpi(\frac{\mathrm{x}}{4})|$.

Thus, (P_2) holds with $n_{\mathrm{g}} = \frac{1}{5}$ *and* $m_{\mathrm{g}} = \frac{1}{10}$. *Also,*

$$\frac{\Lambda}{\chi}\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}(m+1)L_{\mathrm{g}} = \left[\frac{3e^{\frac{4}{3}}-2e}{e-3e^{\frac{4}{3}}}\right]\frac{\left[e^{\frac{1}{3}}-1\right]^{\frac{5}{4}}}{\Gamma(\frac{5}{4})}\frac{3}{10} < 1$$

Consequently, (P_2) holds with $n_{\mathrm{g}} = \frac{1}{5}$ *and* $m_{\mathrm{g}} = \frac{1}{10}$. *Thus, Theorem 2's presumptions are all satisfied. As a result, (24) has a solution on* $[0,1]$.

Example 2. *Consider the following weighted ψ-Caputo-type problem*

$$\begin{cases} {}^{C}_{0}\mathfrak{D}^{\frac{4}{3}}_{\psi(\mathrm{x}),\mathrm{w}(\mathrm{x})}\Big|\varpi(\mathrm{x}) = \mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\frac{1}{2}\mathrm{x})),\ 0 \le \mathrm{x} \le \frac{1}{2}, \\ \varpi(0) = 1,\ \varpi(\frac{1}{2}) = 2, \end{cases} \tag{25}$$

where $m = 1$, $\mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\lambda_1\mathrm{x})) = e^{-\mathrm{x}^{\nu}}\varpi(\mathrm{x}) + e^{-\mathrm{x}^{\nu}}\varpi(\frac{\mathrm{x}}{2}) + \frac{1}{8}$, $\nu = \frac{4}{3}$, $\psi(\mathrm{x}) = \frac{\mathrm{x}}{3}$, $\mathrm{w}(\mathrm{x}) = \frac{e^{-\mathrm{x}}}{4}$, $a = 0$, $b = \frac{1}{2}$, $\varpi_0 = 1$, $\varpi_1 = 2$, *and* $\lambda_1 = \frac{1}{2}$.

(I) Application of Theorem 1: For $\mathrm{x} \in [0,1]$, *and* $\varpi, \omega \in [0,\infty)$, *we have*

$$\begin{aligned} &|\,\mathrm{w}(\mathrm{x})\mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\lambda_1\mathrm{x})) - \mathrm{w}(\mathrm{x})\mathrm{g}(\mathrm{x},\omega(\mathrm{x}),\omega(\lambda_1\mathrm{x}))| \\ &\le \frac{1}{4e^{\mathrm{x}}}\left(\frac{1}{e^{\mathrm{x}^{\nu}}}|\,\varpi(\mathrm{x}) - \omega(\mathrm{x})| + \frac{1}{e^{\mathrm{x}^{\nu}}}\left|\varpi(\frac{\mathrm{x}}{2}) - \omega(\frac{\mathrm{x}}{2})\right|\right) \\ &\le \frac{1}{4}\left(\,|\varpi(\mathrm{x}) - \omega(\mathrm{x})| + \left|\varpi(\frac{\mathrm{x}}{2}) - \omega(\frac{\mathrm{x}}{2})\right|\right). \end{aligned}$$

Thus, (P_1) holds with $L_{\mathrm{g}} = \frac{1}{4}$. *Moreover, the condition (17) holds. Indeed,*

$$\left[\frac{\Lambda}{\chi}+1\right]\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}(m+1)L_{\mathrm{g}} = 0.098424 < 1,$$

where $\Lambda = \frac{73}{72\sqrt{e}}$ *and* $\chi = \frac{1}{\sqrt{e}} - \frac{1}{72\sqrt{e}}$. *Thus, Theorem 1 shows that (25) has a unique solution on* $[0,\frac{1}{2}]$.

(II) Application of Theorem 2: For $\mathrm{x} \in [0,\frac{1}{2}]$, *and* $\varpi \in [0,\infty)$, *we have*

$$\begin{aligned} |\,\mathrm{w}(\mathrm{x})\mathrm{g}(\mathrm{x},\varpi(\mathrm{x}),\varpi(\lambda_1\mathrm{x}))| &\le \frac{\left|e^{-\mathrm{x}^{\nu}}\varpi(\mathrm{x})\right| + \left|e^{-\mathrm{x}^{\nu}}\varpi(\frac{\mathrm{x}}{2})\right|}{4e^{\mathrm{x}}} + \frac{1}{8} \\ &\le \frac{1}{8} + \frac{1}{4}\left(|\,\varpi(\mathrm{x})| + \left|\varpi(\frac{\mathrm{x}}{4})\right|\right). \end{aligned}$$

Consequently, (P_2) holds with $n_{\mathrm{g}} = \frac{1}{8}$ *and* $m_{\mathrm{g}} = \frac{1}{4}$. *Also,*

$$\frac{\Lambda}{\chi}\frac{\psi^{\nu}_{b,a}}{\Gamma(\nu+1)}(m+1)L_{\mathrm{g}} = 0.099791 < 1.$$

Thus, Theorem 2's presumptions are all satisfied. As a result, (25) has a solution on $[0,\frac{1}{2}]$.

6. Conclusions

The current paper was epitomized as follows: Sufficient conditions were provided to investigate some qualitative results for the solution of fractional pantograph equations with boundary conditions in scalar real spaces. Weighted ψ-Caputo FDs have been applied; these were based on the weighted Caputo FD which was defined by Jarad et al. [40] and Al-Refai et al. [43]. Recently, it was discovered that the relevant differential operator is a potent tool for spotting crossover behavior in many evolutionary processes. We have established the existence and uniqueness of boundary value problems for pantograph equations as further contributions to this area of study. As a result of Banach and Krasnoselskii's recognized fixed point theorems, we have also established a substantial analysis. Additionally, in light of our most recent discoveries, a more general problem for the fractional pantograph system has been presented, which includes problems comparable to the one being studied. Finally, we have provided two related examples to illustrate potential applications, hence, validating the main results. It would be interesting to study the present problem in the context of the modern operators introduced by Atangana-Baleanu [18] and Al-Refai [44].

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