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Further Generalizations of Some Fractional Integral Inequalities

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Abstract: This paper aims to establish generalized fractional integral inequalities for operators containing Mittag–Leffler functions. By applying $(\alpha, h-m)-p$-convexity of real valued functions, generalizations of many well-known inequalities are obtained. Hadamard-type inequalities for various classes of functions are given in particular cases.

Keywords: integral operators; fractional integral operators; convex functions

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1. Introduction

Real valued functions with additional constraints provide interesting consequences. For example, a real valued function defined on $\mathbb{R}^n$ satisfying the inequality $f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)(y)$, $\alpha \in [0,1]$, $x, y \in \mathbb{R}^n$ is called a convex function. It was introduced at the start of the nineteenth century and was used very frequently in solving real-world problems of mathematical analysis, functional analysis, optimization theory, etc. In the subject of mathematical inequalities, convex functions are very important, they have fascinating properties, and they provide inequalities that have direct implications to many classical inequalities. In addition, these have been extended and generalized in many ways. We utilized $(\alpha, h-m)-p$-convex functions to establish the results of this paper.

Definition 1 ([1]). *Let $J \subseteq \mathbb{R}$, $I \subset (0,\infty)$ be intervals such that $(0,1) \subset J$, and let $h : J \to \mathbb{R}$ be a non-negative function. A function $\varphi : I \to \mathbb{R}$ is said to be a $(\alpha, h-m)-p$-convex function if*

$$\varphi\left((\tau\varrho^p + m(1-\tau)y^p)^{\frac{1}{p}}\right) \leq h(\tau^\alpha)\varphi(\varrho) + mh(1-\tau^\alpha)\varphi(y), \tag{1}$$

holds for $p \in \mathbb{R}\backslash\{0\}$, provided $(\tau\varrho^p + m(1-\tau)y^p)^{\frac{1}{p}} \in I$, $\tau \in (0,1)$, $(\alpha, m) \in [0,1]^2$.

One can easily find the consequences of inequality (1) by particular substitutions to obtain well-known classes of functions. For example, the (s,m) convex function [2], (α, m) convex function [3], $(h-m)$ convex function [4], (p,h) convex function [5], etc., are all special cases of a $(\alpha, h-m)$-p convex function.

The main goal of this paper is to present certain inequalities for integral operators given in (4) and (5). The consequences of established integral inequalities can be found for several kinds of fractional integral operators and classes of functions linked with convex functions.

In recent past literature, several integral inequalities can be found for different kinds of fractional integral operators. For instance, in [6–10], Hadamard-like inequalities were studied, in [11], Ostrowski-like inequalities were studied, in [12,13], Chebyshev-like inequalities were studied, and Minkowksi-, Hardy- and Grüss-like inequalities were investigated in [14–16].

Next, we define the unified Mittag–Leffler function and associated integral operators as follows:

Definition 2 ([17]). *For $\underline{a} = (a_1, a_2, \dots, a_n)$, $\underline{b} = (b_1, b_2, \dots, b_n)$, $\underline{c} = (c_1, c_2, \dots, c_n)$, where a_i, b_i, $c_i \in \mathbb{C}; i = 1,2,3,\dots,n$ such that $\Re(a_i), \Re(b_i), \Re(c_i) > 0\ \forall\ i$. In addition, let $\alpha, \beta, \gamma, \delta, \mu, \nu, \lambda, \rho, \theta, z \in \mathbb{C}$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta), \Re(\theta)\} > 0$ and $k \in (0,1) \cup \mathbb{N}$ with $k + \Re(\rho) < \Re(\delta + \nu + \alpha)$, $Im(\rho) = Im(\delta + \nu + \alpha)$. Then, the unified Mittag–Leffler function is defined by*

$$M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(z; \underline{a}, \underline{b}, \underline{c}, p') = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{n} B_{p'}(b_i, a_i)(\lambda)_{\rho l}(\theta)_{kl} z^l}{\prod_{i=1}^{n} B(c_i, a_i)(\gamma)_{\delta l}(\mu)_{\nu l}\Gamma(\alpha l + \beta)}, \tag{2}$$

where Γ is a gamma function, and $(\theta)_{lk}$ is the Pochhammer symbol defined by $(\theta)_{kl} = \frac{\Gamma(\theta + lk)}{\Gamma(\theta)}$. The beta function is denoted by B, and $B_{p'}$ is the extension of the beta function defined as follows:

$$B_{p'}(\varrho, y) = \int_0^1 \tau^{\varrho-1}(1-\tau)^{y-1} e^{\frac{-p'}{\tau(1-\tau)}} d\tau. \tag{3}$$

One can easily deduce many kinds of definitions of Mittag–Leffler functions given in recently published papers. For example, the two-parameter Mittag–Leffler function defined in [18], three-parameter Mittag–Leffler function defined in [19] and the extended Mittag–Leffler function defined in [20] can be deduced from the unified Mittag–Leffler function (2). Operators involving the unified Mittag–Leffler function are given in [21] and are defined as follows:

Definition 3. *Let $\phi \in L_1[\xi_1, \xi_2]$, $0 < \xi_1 < \xi_2 < \infty$ be a positive function and let $\Psi : [\xi_1, \xi_2] \to \mathbb{R}$ be a differentiable and strictly increasing function. In addition, let $\frac{\phi}{\varrho}$ be an increasing function on $[\xi_1, \infty)$ and $\varrho \in [\xi_1, \xi_2]$. Then, the unified integral operator in its generalized form satisfying all the convergence conditions stated in Definition 2 is defined by:*

$$({}^{\phi}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_1^+,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varrho; p') = \int_{\xi_1}^{\varrho} \Lambda^{\tau}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Psi; \phi)\Phi(\tau)d(\Psi(\tau)), \tag{4}$$

$$({}^{\phi}_{\Psi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\xi_2^-,\alpha,\beta,\gamma,\delta,\mu,\nu}\Phi)(\varrho; p') = \int_{\varrho}^{\xi_2} \Lambda^{\varrho}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}\Psi; \phi)\Phi(\tau)d(\Psi(\tau)), \tag{5}$$

where

$$\Lambda^{\tau}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}, \Psi; \phi) = \frac{\phi(\Psi(\varrho) - \Psi(\tau))}{\Psi(\varrho) - \Psi(\tau)} M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\delta,\mu,\nu}(\omega(\Psi(\varrho) - \Psi(\tau))^{\mu}; \underline{a}, \underline{b}, \underline{c}, p'). \tag{6}$$

One can note that if Ψ and $\frac{\phi}{\varrho}$ are increasing functions, then for $u < \tau < v, u, v \in [\xi_1, \xi_2]$, the kernel $\Lambda^{u}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}, \Psi; \phi)$ satisfies the following inequality:

$$\Lambda^{u}_{\tau}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}, \Psi; \phi)\Psi'(\tau) \leq \Lambda^{u}_{v}(M^{\lambda,\rho,\theta,k,n}_{\alpha,\beta,\gamma,\mu,\nu}, \Psi; \phi)\Psi'(\tau). \tag{7}$$

There are very interesting implications of the above integral operator involving the unified Mittag–Leffler function. Many fractional integral operators that have been defined by various authors can be obtained by a suitable selection of parameters involved in the kernel. For example, fractional integral operators defined in [17,20,22] can be recovered in particular cases.

This article aims to study some properties of integral operators given in (4) and (5) for $(\alpha, h - m) - p$-convex functions. We establish the bounds of fractional integral operators containing the unified Mittag–Leffler function by utilizing the generalized convexity. A

Hadamard-type inequality is proved that generates plenty of such inequalities in particular cases. The rest of the paper is organized as follows: Section 2 contains some important inequalities for the kernels of integral operators and the $(\alpha, h-m)-p$-convex function. In Section 3, we use the inequalities of Section 2 to obtain desired bounds of the unified integral operators. The established results are generalizations of several inequalities that have been published in the recent past.

2. Some Preliminary Inequalities

From the inequality (7) under its predefined conditions, one can have the following inequalities. These inequalities will be used frequently to prove the main results.

$$\Lambda_{\varrho}^{\tau}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\tau) \leq \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\tau),\ \tau \in (\xi_1,\varrho), \tag{8}$$

$$\Lambda_{\tau}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\tau) \leq \Lambda_{\xi_2}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\tau),\ \tau \in (\varrho,\xi_2), \tag{9}$$

$$\Lambda_{\varrho}^{\xi_1}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varrho) \leq \Lambda_{\xi_2}^{\xi_1}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varrho),\ \ \varrho \in (\xi_1,\xi_2), \tag{10}$$

$$\Lambda_{\xi_2}^{\varrho}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varrho) \leq \Lambda_{\xi_2}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varrho),\ \ \varrho \in (\xi_1,\xi_2). \tag{11}$$

An $(\alpha, h-m)-p$-convex function φ satisfies the inequality (1). From this inequality, one can have the following inequalities, which are also useful in proving the inequalities of the forthcoming section

$$\varphi(\tau^{\frac{1}{p}}) \leq h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\varphi(\xi_1^{\frac{1}{p}}) + mh\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right), \tag{12}$$

$$\varphi(\tau^{\frac{1}{p}}) \leq h\left(\frac{\tau-\varrho}{\xi_2-\varrho}\right)^{\alpha}\varphi(\xi_2^{\frac{1}{p}}) + mh\left(1-\left(\frac{\tau-\varrho}{\xi_2-\varrho}\right)^{\alpha}\right)\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right), \tag{13}$$

$$\varphi(\varrho^{\frac{1}{p}}) \leq h\left(\frac{\varrho-\xi_1}{\xi_2-\xi 1}\right)^{\alpha}\varphi(\xi_2^{\frac{1}{p}}) + mh\left(1-\left(\frac{\varrho-\xi_1}{\xi_2-\xi 1}\right)^{\alpha}\right)\varphi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right). \tag{14}$$

3. Main Results

Theorem 1. *Let $\varphi \in L_1[\xi_1,\xi_2]$ be a positive $(\alpha, h-m)-p$-convex function $m \in (0,1]$, $0<\xi_1<m\xi_2$. In addition, let $\frac{\phi}{\varrho}$ be an increasing function on $[\xi_1,\xi_2]$ and Ψ be a strictly increasing and differentiable function on (ξ_1,ξ_2). Then, we have the following inequality containing the unified Mittag–Leffler function $M_{\alpha,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}(z;\underline{a},\underline{b},\underline{c},p')$ satisfying all the convergence conditions:*

$$\begin{aligned}
&\left({}_{\Lambda}^{\phi}\Upsilon_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}\varphi\circ\chi\right)(\varrho;p') + \left({}_{\Lambda}^{\phi}\Upsilon_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}^{\omega,\lambda,\gamma,\rho,\theta,k,n}\varphi\circ\chi\right)(\varrho;p') \\
&\leq \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)(\varrho-\xi_1)\left(\varphi(\xi_1^{\frac{1}{p}})N_{\varrho}^{\xi_1}(r^{\alpha},h;\Psi') + m\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)N_{\varrho}^{\xi_1}(1-r^{\alpha},h;\Psi')\right) \\
&+ \Lambda_{\xi_2}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)(\xi_2-\varrho)\left(\varphi(\xi_2^{\frac{1}{p}})N_{\varrho}^{\xi_2}(r^{\alpha},h;\Psi') + m\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)N_{\varrho}^{\xi_2}(1-r^{\alpha},h;\Psi')\right),
\end{aligned} \tag{15}$$

while $\chi(\tau) = \tau^{\frac{1}{p}}$, $N_{\varrho}^{\xi_1}(r^{\alpha}, h; \Psi') = \int_0^1 h(r^{\alpha})\Psi'(\varrho - r(\varrho - \xi_1))dr$ *and* $N_{\varrho}^{\xi_1}(1 - r^{\alpha}, h; \Psi') = \int_0^1 h(1 - r^{\alpha})\Psi'(\varrho - r(\varrho - \xi_1))dr$.

Proof. Under the stated conditions, the kernel given in (6) satisfies the inequality (8). In addition, an $(\alpha, h - m) - p$-convex function satisfies the inequality (12). Ultimately, one can have the following inequality:

$$\begin{aligned}\int_{\xi_1}^{\varrho} \Lambda_{\varrho}^{\tau}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)\varphi(\tau^{\frac{1}{p}})d(\Psi(\tau)) &\leq \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)\Bigg(\varphi(\xi_1^{\frac{1}{p}}) \\ &\times \int_{\xi_1}^{\varrho} h\left(\frac{\varrho - \tau}{\varrho - \xi_1}\right)^{\alpha} d(\Psi(\tau)) + m\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\int_{\xi_1}^{\varrho} h\left(1 - \left(\frac{\varrho - \tau}{\varrho - \xi_1}\right)^{\alpha}\right)d(\Psi(\tau))\Bigg).\end{aligned} \tag{16}$$

In the right-hand side, by setting $r = \dfrac{\varrho - \tau}{\varrho - \xi_1}$, while in the left-hand side of the above inequality using Definition 3, the forthcoming inequality is yielded:

$$\begin{aligned}\left({}_{\Lambda}^{\phi}\Upsilon_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}\varphi \circ \chi\right)(\varrho; p') &\leq \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)(\varrho - \xi_1)\Bigg(\varphi(\xi_1^{\frac{1}{p}}) \\ &\times \int_0^1 h(r^{\alpha})\Psi'(\varrho - r(\varrho - \xi_1))dr + m\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\int_0^1 h(1 - r^{\alpha})\Psi'(\varrho - r(\varrho - \xi_1))dr\Bigg).\end{aligned} \tag{17}$$

Inequality (17) is further simplified as follows, which gives an upper bound of the left-sided integral operator:

$$\begin{aligned}\left({}_{\Lambda}^{\phi}\Upsilon_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}\varphi \circ \chi\right)(\varrho; p') &\leq \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)(\varrho - \xi_1) \\ &\times \left(\varphi(\xi_1^{\frac{1}{p}})N_{\varrho}^{\xi_1}(r^{\alpha}, h; \Psi') + m\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)N_{\varrho}^{\xi_1}(1 - r^{\alpha}, h; \Psi')\right).\end{aligned} \tag{18}$$

On the other hand, under stated conditions, kernel (6) also satisfies inequality (9), and φ satisfies inequality (13). Therefore, the following inequality can be yielded:

$$\begin{aligned}\left({}_{\Lambda}^{\phi}\Upsilon_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}^{\omega,\lambda,\rho,\theta,k,n}\varphi \circ \chi\right)(\varrho; p') &\leq \Lambda_{\xi_2}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)(\xi_2 - \varrho)\Bigg(\varphi(\xi_2^{\frac{1}{p}}) \\ &\times \int_0^1 h(r^{\alpha})\Psi'(\varrho - r(\varrho - \xi_2))dr + m\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\int_0^1 h(1 - r^{\alpha})\Psi'(\varrho - r(\varrho - \xi_2))dr\Bigg).\end{aligned} \tag{19}$$

Inequality (19) is further simplified as follows, which gives an upper bound of the right-sided integral operator:

$$\begin{aligned}\left({}_{\Lambda}^{\phi}\Upsilon_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}^{\omega,\lambda,\rho,\theta,k,n}\varphi \circ \chi\right)(\varrho; p') &\leq \Lambda_{\xi_2}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)(\xi_2 - \varrho) \\ &\times \left(\varphi(\xi_2^{\frac{1}{p}})N_{\varrho}^{\xi_2}(r^{\alpha}, h; \Psi') + m\varphi\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)N_{\varrho}^{\xi_2}(1 - r^{\alpha}, h; \Psi')\right).\end{aligned} \tag{20}$$

The required inequality (15) can be composed by adding inequalities (18) and (20). □

The inequality established in the above theorem is linked with many published results. Some of the consequences of inequality (15) are stated in the following remark.

Remark 1. *(i) The inequality stated in [23] (Corollary 3) is followed by setting $p = 1$ in (15). (ii) The inequality stated in [24] (Theorem 2) is followed by setting $p = 1$, $\kappa = \vartheta$ and $h(\tau) = \tau$ in (15).*

For proof of the next theorem, we need the following lemma, which can be easily proved.

Lemma 1. *Let $\varphi : [\xi_1, \xi_2] \to \mathbb{R}$, be a $(\alpha, h-m) - p$-convex function, $m \in (0,1]$, $0 < \xi_1 < m\xi_2$. If $\varphi(\varrho^{\frac{1}{p}}) = \varphi\left(\left(\frac{\xi_1^p + \xi_2^p - \varrho}{m}\right)^{\frac{1}{p}}\right)$, then the following inequality holds:*

$$\varphi\left(\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}}\right) \leq \left(h\left(\frac{1}{2^\alpha}\right) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)\right)\varphi(\varrho^{\frac{1}{p}}). \tag{21}$$

The upcoming theorem provides the Hadamard inequality for the $(\alpha, h-m) - p$-convex function. The special cases of this inequality are specified in the remark given after this theorem.

Theorem 2. *Under the assumptions of Theorem 1, if $\varphi(\varrho^{\frac{1}{p}}) = \varphi\left(\left(\frac{\xi_1^p + \xi_2^p - \varrho}{m}\right)^{\frac{1}{p}}\right)$, then we have*

$$\begin{aligned}
&\frac{\varphi\left(\left(\frac{\xi_1^p + \xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^\alpha}\right) + mh\left(\frac{2^\alpha - 1}{2^\alpha}\right)}\left(\left(\left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma\mu,\nu,\xi_2^-}1\right)(\xi_1;p') + \left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma\mu,\nu,\xi_1^+}1\right)(\xi_2;p')\right)\right) \\
&\leq \left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\varphi \circ \chi\right)(\xi_2;p') + \left({}_{\Lambda}^{\phi}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\varphi \circ \chi\right)(\xi_1;p') \leq (\xi_2 - \xi_1) \\
&\times (\Lambda_{\xi_2}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu}, \Psi;\phi) + \Lambda_{\xi_2}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu}, \Psi;\phi))\left(\varphi(\xi_2^{\frac{1}{p}})N_{\xi_2}^{\xi_1}(r^\alpha, h;\Psi')\right. \\
&\left.+m\varphi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)N_{\xi_2}^{\xi_1}(1 - r^\alpha, h;\Psi')\right).
\end{aligned} \tag{22}$$

Proof. Under stated conditions, kernel (6) satisfies inequality (10). In addition, the $(\alpha, h-m)$-p convex function satisfies inequality (14). Ultimately, one can have the following inequality:

$$\begin{aligned}
&\int_{\xi_1}^{\xi_2} \Lambda_{\varrho}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu}, \Psi;\phi)\varphi((\varrho)^{\frac{1}{p}})d(\Psi(\varrho)) \leq \Lambda_{\xi_2}^{\xi_1}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu}, \Psi;\phi)\left(\varphi(\xi_2)\right. \\
&\left.\times \int_{\xi_1}^{\xi_2} h\left(\frac{\varrho - \xi_1}{\xi_2 - \xi 1}\right)^\alpha d(\Psi(\varrho)) + m\varphi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\int_{\xi_1}^{\xi_2} h\left(1 - \left(\frac{\varrho - \xi_1}{\xi_2 - \xi 1}\right)^\alpha\right)d(\Psi(\varrho))\right).
\end{aligned}$$

In the right-hand side, by setting $r = \frac{\varrho - \xi_1}{\xi_2 - \xi 1}$, while in the left-hand side of the above inequality using Definition 3, the next inequality is yielded

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\varphi\circ\chi\right)(\xi_1;p') \leq \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\xi_2-\xi_1)\left(\varphi(\xi_2^{\frac{1}{p}})\right. \tag{23}$$
$$\left.\times\int_0^1 h(r^\alpha)\Psi'(\xi_1+r(\xi_2-\xi_1))dr + m\varphi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)\int_0^1 h(1-r^\alpha)\Psi'(\xi_1+r(\xi_2-\xi_1))dr\right).$$

Inequality (23) is further simplified as follows, which gives an upper bound of the left-sided integral operator:

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\varphi\circ\chi\right)(\xi_1;p') \leq \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\xi_2-\xi_1) \tag{24}$$
$$\times\left(\varphi(\xi_2^{\frac{1}{p}})N^{\xi_1}_{\xi_2}(r^\alpha,h;\Psi') + m\varphi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)N^{\xi_1}_{\xi_2}(1-r^\alpha,h;\Psi')\right).$$

Adopting the same pattern of simplification as we did for (10) and (14), the following inequality can be observed for (11) and (14):

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\varphi\circ\chi\right)(\xi_1;p') \leq \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)(\xi_2-\xi_1) \tag{25}$$
$$\times\left(\varphi(\xi_2^{\frac{1}{p}})N^{\xi_1}_{\xi_2}(r^\alpha,h;\Psi') + m\varphi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)N^{\xi_1}_{\xi_2}(1-r^\alpha,h;\Psi')\right).$$

By adding (24) and (25), the following inequality can be achieved:

$$\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\varphi\circ\chi\right)(\xi_2;p') + \left({}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\varphi\circ\chi\right)(\xi_1;p') \leq (\xi_2-\xi_1)$$
$$\times\left((\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi) + \Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\right)\left(\varphi(\xi_2^{\frac{1}{p}})N^{\xi_1}_{\xi_2}(r^\alpha,h;\Psi')\right. \tag{26}$$
$$\left. + m\varphi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)N^{\xi_1}_{\xi_2}(1-r^\alpha,h;\Psi')\right).$$

Multiplying both sides of (21) by $\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\Psi'(\varrho)$ and integrating over $[\xi_1,\xi_2]$, one can obtain

$$\varphi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)\int_{\xi_1}^{\xi_2}\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)d(\Psi(\varrho))$$
$$\leq\left(h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)\right)\int_{\xi_1}^{\xi_2}\Lambda^{\xi_1}_{\varrho}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu},\Psi;\phi)\varphi(\varrho^{\frac{1}{p}})d(\Psi(\varrho)).$$

By utilizing Definition 3 in the above inequality one can obtain the following inequality:

$$\frac{\varphi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left({}^{\phi}_{\Lambda}\Upsilon^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}1\right)(\xi_1;p') \tag{27}$$
$$\leq\left({}^{\phi}_{\Lambda}\Upsilon^{\lambda,\rho,\theta,k,n}_{\kappa,\alpha,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\varphi\circ\chi\right)(\xi_1;p').$$

Now, multiplying by $\Lambda_{\xi_2}^{\varrho}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\Psi'(\varrho))$ on both sides of (21), then integrating over $[\xi_1,\xi_2]$, we obtain

$$\frac{\varphi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^\alpha}\right)+mh\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left({}_{\Lambda}^{\phi}\Upsilon_{\vartheta,\alpha,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}1\right)(\xi_2;p') \tag{28}$$
$$\leq\left({}_{\Lambda}^{\phi}\Upsilon_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}\varphi\circ\chi\right)(\xi_2;p').$$

The required inequality (22) can be composed from inequalities (26)–(28). □

Remark 2. *(i) The inequality stated in [23] (Corollary 3) is followed by setting $p=1$ in (22). (ii) The inequality stated in [24] (Theorem 1) is followed by setting $p=1$, $\kappa=\vartheta$ and $h(\tau)=\tau$ in (22).*

Theorem 3. *If $(\alpha,h-m)-p$-convexity of φ is replaced with $(\alpha,h-m)-p$-convexity of $|\varphi'|$, along with same assumptions as in Theorem 1, the following inequality holds for unified integral operators:*

$$\left|\left({}_{\Lambda}^{\phi}\Upsilon_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}(\varphi*\Lambda)\circ\chi\right)(\varrho;p')+\left({}_{\Lambda}^{\phi}\Upsilon_{\mu,\eta,l,\xi_2^-}^{\omega,\lambda,\rho,\theta,k,n}(\varphi*\Lambda)\circ\chi\right)(\varrho;p')\right|$$
$$\leq\Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)(\varrho-\xi_1)\left(|\varphi'(\xi_1^{\frac{1}{p}})|N_{\varrho}^{\xi_1}(r^\alpha,h;\Psi')+m\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|N_{\varrho}^{\xi_1}(1-r^\alpha,h;\Psi')\right) \tag{29}$$
$$+\Lambda_{\xi_2}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n},\Psi;\phi)(\xi_2-\varrho)\left(|\varphi'(\xi_2^{\frac{1}{p}})|N_{\varrho}^{\xi_2}(r^\alpha,h;\Psi')+m\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|N_{\varrho}^{\xi_2}(1-r^\alpha,h;\Psi')\right).$$

where

$$\left({}_{\Lambda}^{\phi}\Upsilon_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}(\varphi*\Lambda)\circ\chi\right)(\varrho;p'):=\int_{\xi_1}^{\varrho}\Lambda_{\varrho}^{\tau}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu,}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\varphi'\left((\tau)^{\frac{1}{p}}\right)d(\Psi(\tau)),$$

$$\left({}_{\Lambda}^{\phi}\Upsilon_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}^{\omega,\lambda,\rho,\theta,k,n}(\varphi*\Lambda)\circ\chi\right)(\varrho;p'):=\int_{\varrho}^{\xi_2}\Lambda_{\tau}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu,}^{\lambda,\rho,\theta,k,n},\Psi;\phi)\varphi'\left((\tau)^{\frac{1}{p}}\right)d(\Psi(\tau)).$$

Proof. Since $|\varphi'|$ is a $(\alpha,h-m)-p$-convex function, one can have

$$|\varphi'\left(\tau^{\frac{1}{p}}\right)|\leq h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}|\varphi'(\xi_1^{\frac{1}{p}})|+mh\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|. \tag{30}$$

Inequality (30) can takes the following form:

$$-\left(h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}|\varphi'(\xi_1^{\frac{1}{p}})|+mh\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right)\leq\varphi'\left((\tau)^{\frac{1}{p}}\right) \tag{31}$$
$$\leq\left(h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}|\varphi'(\xi_1^{\frac{1}{p}})|+mh\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|\right).$$

From inequality (31), we have

$$\varphi'\left((\tau)^{\frac{1}{p}}\right)\leq h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}|\varphi'(\xi_1^{\frac{1}{p}})|+mh\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|. \tag{32}$$

Multiplying (8) with (32) and integrating over $[\xi_1, \varrho]$, we obtain:

$$\int_{\xi_1}^{\varrho} \Lambda_{\varrho}^{\tau}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)\varphi'\left((\tau)^{\frac{1}{p}}\right)d(\Psi(\tau)) \leq \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)\Bigg(|\varphi'(\xi_1^{\frac{1}{p}})|$$
$$\times \int_{\xi_1}^{\varrho} h\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha} d(\Psi(\tau)) + m\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right| \int_{\xi_1}^{\varrho} h\left(1-\left(\frac{\varrho-\tau}{\varrho-\xi_1}\right)^{\alpha}\right)d(\Psi(\tau))\Bigg).$$

which gives

$$\left({}_{\Lambda}^{\phi}Y_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}(\varphi*\Lambda)\circ\chi\right)(\varrho; p') \leq \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)(\varrho-\xi_1) \tag{33}$$
$$\times \left(|\varphi'(\xi_1^{\frac{1}{p}})|N_{\varrho}^{\xi_1}(r^{\alpha}, h; \Psi') + m\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right| N_{\varrho}^{\xi_1}(1-r^{\alpha}, h; \Psi')\right).$$

Using the other inequality of (31) and doing so the same way as adopted for the right-hand inequality, one can obtain

$$\left({}_{\Lambda}^{\phi}Y_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}(\varphi*\Lambda)\circ\chi\right)(\varrho; p') \geq -\Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)(\varrho-\xi_1) \tag{34}$$
$$\times \left(|\varphi'(\xi_1^{\frac{1}{p}})|N_{\varrho}^{\xi_1}(r^{\alpha}, h; \Psi') + m\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right| N_{\varrho}^{\xi_1}(1-r^{\alpha}, h; \Psi')\right).$$

From (33) and (34), the following inequality is observed:

$$\left|\left({}_{\Lambda}^{\phi}Y_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}^{\omega,\lambda,\rho,\theta,k,n}(\varphi*\Lambda)\circ\chi\right)(\varrho; p')\right| \leq \Lambda_{\varrho}^{\xi_1}(M_{\kappa,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)(\varrho-\xi_1) \tag{35}$$
$$\times \left(|\varphi'(\xi_1^{\frac{1}{p}})|N_{\varrho}^{\xi_1}(r^{\alpha}, h; \Psi') + m\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right| N_{\varrho}^{\xi_1}(1-r^{\alpha}, h; \Psi')\right).$$

By applying the $(\alpha, h-m)-p$-convexity of $|\varphi'|$, one can obtain

$$|\varphi'\left((\tau)^{\frac{1}{p}}\right)| \leq h\left(\frac{\tau-\varrho}{\xi_2-\varrho}\right)^{\alpha}|\varphi'(\xi_2^{\frac{1}{p}})| + mh\left(1-\left(\frac{\tau-\varrho}{\xi_2-\varrho}\right)^{\alpha}\right)\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right|. \tag{36}$$

Now, on the same lines as for (8) and (30), from (9) and (36), one can have the following inequality:

$$\left|\left({}_{\Lambda}^{\phi}Y_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}^{\omega,\lambda,\rho,\theta,k,n}(\varphi*\Lambda)\circ\chi\right)(\varrho; p')\right| \leq \Lambda_{\xi_2}^{\varrho}(M_{\vartheta,\beta,\gamma,\delta,\mu,\nu}^{\lambda,\rho,\theta,k,n}, \Psi; \phi)(\xi_2-\varrho) \tag{37}$$
$$\times \left(|\varphi'(\xi_2^{\frac{1}{p}})|N_{\varrho}^{\xi_2}(r^{\alpha}, h; \Psi') + m\left|\varphi'\left(\frac{\varrho^{\frac{1}{p}}}{m}\right)\right| N_{\varrho}^{\xi_2}(1-r^{\alpha}, h; \Psi')\right).$$

The required inequality (29) can be composed by adding inequalities (35) and (37). □

Remark 3. *(i) The inequality stated in [23] (Corollary 3) is followed by setting $p = 1$ in (22).*
(ii) The inequality stated in [24] (Theorem 3) is followed by setting $p = 1$, $\kappa = \vartheta$ and $h(\tau) = \tau$ in (22).

4. Hadamard-Type Inequalities

In this section, we give some Hadamard-type inequalities for $(h,m)-p$-convex functions, $(\alpha,m)-p$-convex functions and $(\alpha,h)-p$-convex functions. First, for $\alpha=1$, (22) gives the result for $(h,m)-p$-convex functions as follows:

Theorem 4. *Under the assumptions of Theorem 2, the following inequality holds:*

$$\begin{aligned}
&\frac{\varphi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2}\right)(1+m)}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}1\right)(\xi_1;p')+\left({}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}1\right)(\xi_2;p')\right)\\
&\leq\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\varphi\right)(\xi_2;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\varphi\right)(\xi_1;p')\\
&\leq(\xi_2-\xi_1)(\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi)+\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi))\\
&\times\left(\varphi(\xi_2^{\frac{1}{p}})N^{\xi_1}_{\xi_2}(r,h;\Psi')+m\varphi\left(\frac{\xi_1^{\frac{1}{p}}}{m}\right)N^{\xi_1}_{\xi_2}(1-r,h;\Psi')\right).
\end{aligned}$$

For $m=1$, (22) gives the result for $(\alpha,h)-p$-convex functions as follows:

Theorem 5. *Under the assumption of Theorem 2, the following inequality holds:*

$$\begin{aligned}
&\frac{\varphi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{h\left(\frac{1}{2^\alpha}\right)+h\left(\frac{2^\alpha-1}{2^\alpha}\right)}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}1\right)(\xi_1;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}1\right)(\xi_2;p')\right)\\
&\leq\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\varphi\right)(\xi_2;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\varphi\right)(\xi_1;p')\\
&\leq(\xi_2-\xi_1)(\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi)+\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi))\\
&\times\left(\varphi(\xi_2^{\frac{1}{p}})N^{\xi_1}_{\xi_2}(r^\alpha,h;\Psi')+\varphi\left(\xi_1^{\frac{1}{p}}\right)N^{\xi_1}_{\xi_2}(1-r^\alpha,h;\Psi')\right).
\end{aligned}$$

For $h(\tau)=\tau$, (22) gives the result for $(\alpha,m)-p$-convex functions as follows:

Theorem 6. *Under the assumption of Theorem 2, the following inequality holds:*

$$\begin{aligned}
&\frac{2^\alpha\varphi\left(\left(\frac{\xi_1^p+\xi_2^p}{2}\right)^{\frac{1}{p}}\right)}{(1+m(2^\alpha-1))}\left(\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}1\right)(\xi_1;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}1\right)(\xi_2;p')\right)\\
&\leq\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,\xi_2^-}\varphi\right)(\xi_1;p')+\left({}^{\phi}_{\Lambda}\Upsilon^{\omega,\lambda,\rho,\theta,k,n}_{\kappa,\beta,\gamma,\delta,\mu,\nu,\xi_1^+}\varphi\right)(\xi_2;p')\\
&\leq(\Lambda^{\xi_1}_{\xi_2}(M^{\lambda,\rho,\theta,k,n}_{\vartheta,\beta,\gamma,\delta,\mu,\nu,},\Psi;\phi))\left(\left(\varphi(\xi_2)\Psi(\xi_2)-m\varphi\left(\frac{\xi_1}{m}\right)\Psi(\xi_1)\right)\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(\xi_2-\xi_1)^\alpha}\left(\varphi(\xi_2)-m\varphi\left(\frac{\xi_1}{m}\right)\right){}^{\alpha}I_{\xi_2^-}\Psi(\xi_1)\right).
\end{aligned}$$

5. Conclusions

We obtained the bounds of fractional integral operators containing the unified Mittag–Leffler function via $(\alpha,h-m)-p$-convex functions. The established results are generalizations of many integral inequalities that have been published in the recent past in articles [23,24] (see Remarks 1–3 and Section 4). The results of this paper are applicable for a wide range of classes of functions linked with $(\alpha, h - m) - p$-convex functions in particular cases.

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