



On a Quadratic Nonlinear Fractional Equation

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Abstract: In this paper, we study a quadratic nonlinear equation from the fractional point of view. An explicit solution is given in terms of the Lambert special function. A new phenomenon appears involving the collapsing of the solution and the blow-up of the derivative. The explicit representation of the solution reveals the non-elementary nature of the solution.

Keywords: logistic ordinary differential equation; blow-up; fractional differential equation; Lambert function; non-elementary functions; non-uniqueness

MSC: 34A05; 34A34; 34M50

1. Introduction

In logistic population growth, the concept of environmental carrying capacity is modelled by the classical logistic ordinary differential equation [1]:

$$x'(t) = a \cdot x(t) \cdot \left[1 - \frac{x(t)}{K} \right]. \quad (1)$$

The Richards ordinary differential equation is a generalization of the above logistic differential equation by introducing a new parameter, β :

$$x'(t) = a \cdot x(t) \cdot \left[1 - \left(\frac{x(t)}{K} \right)^\beta \right].$$

The latter equation can be explicitly solved

$$x(t) = \frac{K}{[1 + c \cdot \exp(-a\beta(t - t_0))]^{1/\beta'}}$$

and for $\beta = 1$ it gives the solution of the logistic differential Equation (1). We note that the exponential term in the Richards equation has a one-to-one nonlinear correspondence with the basic reproduction number of the SIR (susceptible–infectious–recovered) compartmental epidemiological model [2]. Recently, some researchers have discovered the connection of the Richards model to the epidemic dynamics of COVID-19 [3,4].

For $0 < \alpha < 1$, we have already considered the logistic fractional differential equation (see e.g., [5])

$$D^\alpha x(t) = a \cdot x(t) \cdot \left[1 - \frac{x(t)}{K} \right],$$



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as well as the fractional Richards differential equation for $\beta > 0$,

$$D^\alpha x(t) = a \cdot x(t) \cdot \left[1 - \left(\frac{x(t)}{K} \right)^\beta \right].$$

The fractional Richards equation has been considered in very few articles. For example, it has been considered for water transport [6], but has only been approximated numerically or using peridynamic theory as in [7]. It therefore seems of interest to study the more general nonlinear fractional differential equation

$$D^\alpha x(t) = f(t, x(t)).$$

This is our main motivation in this brief report. We point out that even simple nonlinear equations such as the Caputo fractional logistic equation have no analytical known solution.

For some general references on fractional differential equations and their applications, refer to [8]. If D^α represents the Caputo–Fabrizio fractional derivative [9], then integrating by using the Losada–Nieto fractional integral we obtain

$$x(t) - x_0 = (1 - \alpha)[f(t, x(t)) - f(0, x(0))] + \alpha \int_0^t f(s, x(s)) ds.$$

Thus,

$$x'(t) = (1 - \alpha) \left[\frac{\partial f}{\partial t}(t, x(t)) + \frac{\partial f}{\partial x}(t, x(t)) x'(t) \right] + \alpha f(t, x(t))$$

or equivalently

$$x'(t) \cdot \left[1 - (1 - \alpha) \frac{\partial f}{\partial x}(t, x(t)) \right] = \alpha f(t, x(t)) - (1 - \alpha) \frac{\partial f}{\partial t}(t, x(t)).$$

If, for an initial condition $x(t_0) = x_0$, one has that

$$(1 - \alpha) \frac{\partial f}{\partial x}(t_0, x_0) \neq 1,$$

then the previous equation is equivalent, in a neighbourhood of (t_0, x_0) , to

$$x' = f_\alpha(t, x)$$

with

$$f_\alpha(t, x) = \frac{\alpha f(t, x) - (1 - \alpha) \frac{\partial f}{\partial t}(t, x(t))}{1 - (1 - \alpha) \frac{\partial f}{\partial x}(t, x)}.$$

In general, it is not possible to obtain an exact solution.

In this paper, we focus on the nonlinear fractional equation

$$D^\alpha x(t) = x^2(t). \quad (2)$$

For some biological applications, see [10]. Even for $\alpha = 1$, the solution of (2) is local. Indeed, for the initial condition $x(0) = x_0 > 0$, the solution is given by

$$x(t) = \frac{x_0}{1 - x_0 t}$$

and the right maximal interval of existence is $(0, 1/x_0)$.

Note that $f(t, x) = x^2$ and then

$$\frac{\partial f}{\partial x}(t, x) = 2x \quad \text{and} \quad f_x(x) = \frac{\alpha x^2}{1 - (1 - \alpha)2x}.$$

If D^α represents the Caputo–Fabrizio fractional derivative, then (2) is equivalent to

$$x' - (1 - \alpha)2xx' = \alpha x^2.$$

For an initial condition $x_0 \neq \frac{1}{2(1-\alpha)}$, we have to solve the equation

$$x' = \frac{\alpha x^2}{1 - (1 - \alpha)2x}.$$

Integrating

$$-\frac{1}{\alpha x} + \frac{2(\alpha - 1)}{\alpha} \cdot \log x = t + c_\alpha. \quad (3)$$

For an initial condition $x(0) = x_0$, we can find the value of c_α as

$$c_\alpha = \frac{-1}{\alpha x_0} + \frac{2(\alpha - 1) \log x_0}{\alpha}. \quad (4)$$

Then, (2) is equivalent to

$$-\frac{1}{2(1 - \alpha)x} + \log \frac{1}{2(1 - \alpha)x} + \log(2(1 - \alpha)) = \frac{-\alpha(t + c_\alpha)}{2(\alpha - 1)}.$$

Taking the exponential in both sides,

$$-\frac{1}{2(1 - \alpha)x} \cdot \exp\left(\frac{-1}{2(1 - \alpha)x}\right) = -\frac{1}{2(1 - \alpha)} \cdot \exp\left(\frac{-\alpha(t + c_\alpha)}{2(1 - \alpha)}\right).$$

Setting

$$z = -\frac{1}{2(1 - \alpha)} \cdot \exp\left(\frac{-\alpha(t + c_\alpha)}{2(\alpha - 1)}\right)$$

and

$$w = -\frac{1}{2(1 - \alpha)x}.$$

We observe that

$$w \cdot \exp(w) = z, \quad (5)$$

that is, $w = \mathcal{W}_k(z)$, being \mathcal{W} the Lambert function or product logarithm [11], for some integer k . This gives the solution x as

$$x(t) = \frac{-1}{2(1 - \alpha) \mathcal{W}\left(\frac{1}{2(1 - \alpha)} \cdot \exp\left(\frac{-\alpha(t + c_\alpha)}{2(\alpha - 1)}\right)\right)}.$$

Using the value of the constant c_α given in (4), one can finally write the solution $x(t)$ to (2) in terms of the initial condition:

$$x(t) = \frac{-1}{2(1 - \alpha) \mathcal{W}\left(\frac{-1}{2(1 - \alpha)x_0} \cdot \exp\left(\frac{\alpha t}{2(1 - \alpha)} - \frac{1}{2(1 - \alpha)x_0}\right)\right)}.$$

Noting that the Lambert function cannot be expressed in terms of elementary functions [12], one can conclude that the solution to (2) just obtained is non-elementary.

2. Example

Let us consider $\alpha = 1/2$, $x_0 = 1/2$. Then, the equation is

$$x' = \frac{\frac{1}{2}x}{1-x} \quad (6)$$

and the particular solution is given by

$$x(t) = \frac{1}{\mathcal{W}(-2 \exp(\frac{t}{2} - 2))}.$$

The solution exists locally. Noting that $x'(t)$ is positive for $t > 0$, it is increasing and blows up when x approaches 1^- (see Figure 1); that is,

$$\lim_{t \rightarrow t_\infty} x'(t) = +\infty.$$

To find the maximal interval of existence, we recall that the Lambert function is defined only for $z < 0$ when $z > -\frac{1}{e}$. Therefore, we have the condition

$$-\exp(-1) \leq -2 \exp\left(\frac{t}{2} - 2\right)$$

or

$$t \geq 2 - 2 \cdot \log(2) := t_\infty \approx 0.613706.$$

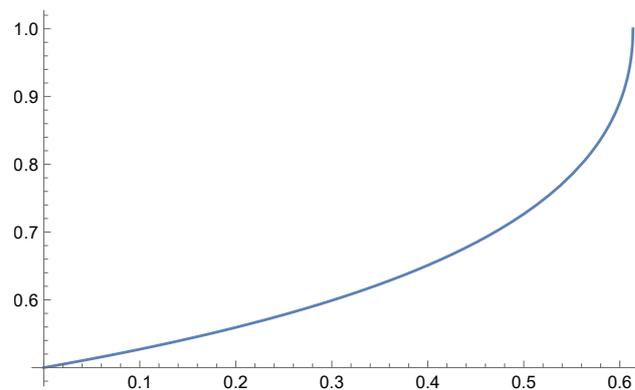


Figure 1. Solution $x(t)$ to (6) in $[0, t_\infty \approx 0.613706]$ with $x_0 = 1/2$.

Observe that $\lim_{t \rightarrow t_\infty} x(t) = +\infty$ for the initial condition $(t_\infty, 1)$, and one cannot solve the differential equation. However, we can ask whether there exists a solution starting at $(0, \zeta)$ that passes through that point $(t_\infty, 1)$. Such a solution is given by

$$x(t) = \frac{-1}{\mathcal{W}\left(\frac{-1}{\zeta} \exp\left(\frac{t}{2} - 2\right)\right)}. \quad (7)$$

Imposing the condition $x(t_\infty) = 1$, we need

$$\mathcal{W}\left(\frac{-1}{\zeta} \exp\left(\frac{t_\infty}{2} - 2\right)\right) = -1,$$

that is,

$$\frac{-1}{\zeta} \exp\left(\frac{t_\infty}{2} - 2\right) = -\exp(-1).$$

Therefore,

$$\frac{-1}{\xi} \exp\left(\frac{-1}{\xi}\right) \exp(1 - \log 2) = -\exp(-1)$$

and (see Figure 2)

$$\xi = \frac{-1}{\mathcal{W}\left(\frac{-2}{\exp(2)}\right)} \approx 2.46078.$$

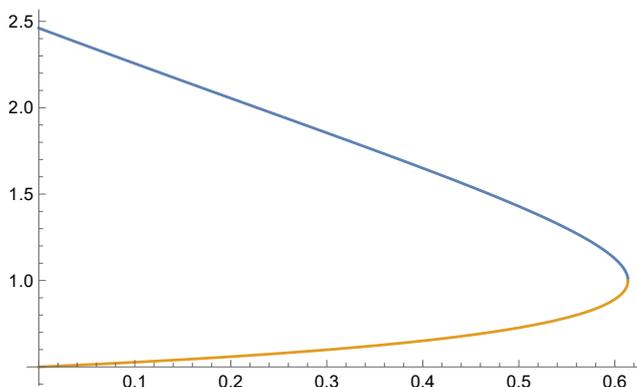


Figure 2. In orange is the solution $x(t)$ to (6) in $[0, t_\infty \approx 0.613706]$ with $x_0 = 1/2$. In blue, $x(t)$ given in (7) for $\xi \approx 2.46078$.

3. Lambert Function

Given a complex number z , we want to solve Equation (5). It has many solutions and hence it is a multivalued function. Thus, (5) is equivalent to

$$w = \mathcal{W}_k(z)$$

for some integer k , where \mathcal{W}_k is a branch of the inverse function $w \rightarrow w \cdot \exp w$. The principal branch is \mathcal{W}_0 and denoted simply by \mathcal{W} .

The Lambert \mathcal{W} relation cannot be expressed in terms of elementary functions, i.e., it is non-elementary. In particular it is non-Liouvillian [12].

For a real number x , the equation

$$w \cdot \exp w = x$$

has a real solution if and only if (see Figure 3)

$$x \geq \frac{-1}{e}.$$

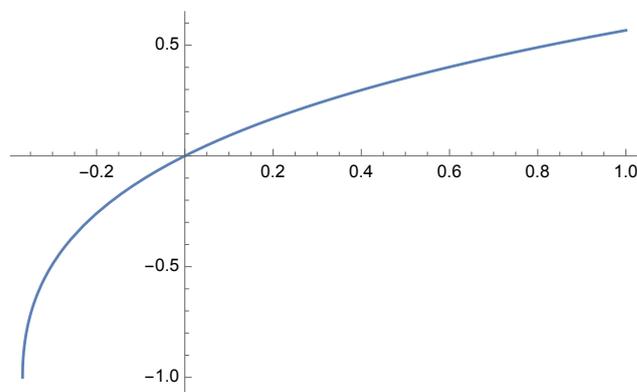


Figure 3. Principal branch of the function $z = we^w$ for $x > -1/e \approx -0.367879$.

For $z \geq 0$, the equation has a single real solution given by $\mathcal{W}(z)$, and for $1/e < z < 0$ we have two solutions: $\mathcal{W}(z)$ and $\mathcal{W}_{-1}(z)$. For $z = -1/e$, $\mathcal{W}(-1/e) = -1$.

The Lambert function satisfies the ordinary differential equation

$$\mathcal{W}'(z) = \frac{1}{z + \exp(\mathcal{W}(z))}, \quad z \neq -1/e.$$

Moreover,

$$\mathcal{W}(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} x^n$$

with a radius of convergence equal to $1/e$.

As another known but very relevant application of the Lambert function, let us compute the final population S_{∞} in the classical compartmental *SIR* epidemic model [13,14]:

$$\begin{aligned} S'(t) &= -\beta S(t)I(t), \\ I'(t) &= \beta S(t)I(t) - \gamma I(t), \\ R'(t) &= \gamma I(t). \end{aligned}$$

Here, as usual, S , I and R represent the susceptible, infectious and recovered population, respectively.

Recall that the basic reproduction number is the initial replacement number when just one infectious individual is introduced into a population of all susceptible individuals:

$$\mathcal{R}_0 = \frac{\beta}{\gamma}.$$

We have

$$S' = -\beta SI = -\beta S \frac{R'}{\gamma}.$$

Then,

$$\frac{S'}{S} = -\mathcal{R}_0 R'$$

and

$$\ln S(t) - \ln S_0 = -\mathcal{R}_0 \cdot [R(t) - R_0].$$

This gives $S(t)$ as the following function of $R(t)$

$$S(t) = S_0 \cdot \exp(-\mathcal{R}_0 \cdot [R(t) - R_0]).$$

Taking the limit as $t \rightarrow \infty$,

$$\begin{aligned} 1 - \mathcal{R}_{\infty} = S_{\infty} &= S_0 \cdot \exp(-\mathcal{R}_0 \cdot [R_{\infty} - R_0]) \\ &= S_0 \cdot \exp(-\mathcal{R}_0 \cdot [R_{\infty} - 1]) \cdot \exp(-\mathcal{R}_0 \cdot [1 - R_0]). \end{aligned}$$

Equivalently,

$$-\mathcal{R}_0 \cdot (R_{\infty} - 1) \cdot \exp(-\mathcal{R}_0 \cdot [R_{\infty} - 1]) = -\mathcal{R}_0 \cdot S_0 \cdot \exp(-\mathcal{R}_0 \cdot [1 - R_0])$$

and

$$\mathcal{R}_0 \cdot (R_{\infty} - 1) = -\mathcal{W}(-\mathcal{R}_0 \cdot S_0 \cdot \exp(-\mathcal{R}_0 \cdot [1 - R_0])).$$

Finally,

$$R_{\infty} = 1 - \frac{1}{\mathcal{R}_0} \mathcal{W}(-\mathcal{R}_0 \cdot S_0 \cdot \exp(-\mathcal{R}_0 \cdot [1 - R_0])),$$

and

$$S_{\infty} = \frac{1}{\mathcal{R}_0} \mathcal{W}(-\mathcal{R}_0 \cdot S_0 \cdot \exp(-\mathcal{R}_0 \cdot [1 - R_0])).$$

It is important to note that the solution I is a *non-elementary* function (see, for example, Theorem 10.3 in [14]).

We recall that an elementary function is defined as a function generated from a finite number of combinations and compositions of algebraic, exponential and logarithm functions under the four algebraic operations. A Liouvillian function is a function lying in some Liouvillian extension of $(C(x),')$ for a constant field C .

Indeed, using the representation of S as a function of I through the Lambert function, it is possible to prove that the solution $I(t)$ is not Liouvillian.

4. Conclusions

We have solved a fractional Richards differential equation.

New aspects of the methodology of solving it and the form of the solution are presented, and further research will be necessary to clearly reveal the complexity of nonlinear differential equations.

It will be of interest to explore fractional Richards differential equations in relation to the dynamics of epidemic compartmental models.

The dependence of the solution on the order of derivation would be of interest. The case when $\alpha > 1$ will be also contemplated.

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References

1. Area, I.; Nieto, J.J. Power series solution of the fractional logistic equation. *Phys. A* **2021**, *573*, 125947. [CrossRef]
2. Wang, X.-S.; Wu, J.; Yang, Y. Richards model revisited: Validation by and application to infection dynamics. *J. Theor. Biol.* **2012**, *313*, 12–19. [CrossRef] [PubMed]
3. Luo, X.; Duan, H.; Xu, K. A novel grey model based on traditional Richards model and its application in COVID-19. *Chaos Solitons Fractals* **2021**, *142*, 110480. [CrossRef] [PubMed]
4. Smirnova, A.; Pidgeon, B.; Chowell, G.; Zhao, Y. The doubling time analysis for modified infectious disease Richards model with applications to COVID-19 pandemic. *Math. Biosci. Eng.* **2022**, *19*, 3242–3268. [CrossRef] [PubMed]
5. Nieto, J.J. Solution of a fractional logistic ordinary differential equation. *Appl. Math. Lett.* **2022**, *123*, 107568. [CrossRef]
6. Romashchenko, M.I.; Bohaienko, V.O.; Matiash, T.V.; Kovalchuk, V.P.; Krucheniuk, A.V. Numerical simulation of irrigation scheduling using fractional Richards equation. *Irrig. Sci.* **2021**, *39*, 385–396. [CrossRef]
7. Berardi, M.; Difonzo, F.V.; Pellegrino, S.F. A numerical method for a nonlocal form of Richards' equation based on peridynamic theory. *Comput. Math. Appl.* **2023**, *143*, 23–32. [CrossRef]
8. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. Theory and Applications of the Fractional Differential Equations. In *North-Holland Mathematics Studies*; Elsevier: Amsterdam, The Netherlands, 2006.
9. Caputo, M.; Fabrizio, M. On the singular kernels for fractional derivatives. Some applications to partial differential equations. *Prog. Fract. Differ. Appl.* **2021**, *7*, 79–82.
10. Slimane, I.; Nazir, G.; Nieto, J.J.; Yaqoob, F. Mathematical analysis of Hepatitis C virus infection model in the framework of non-local and non-singular kernel fractional derivative. *Int. J. Biomath.* **2023**, *16*, 2250064. [CrossRef]
11. Olver, F.W.J.; Daalhuis, A.B.O.; Lozier, D.W.; Schneider, B.I.; Boisvert, R.F.; Clark, C.W.; Mille, B.R.; Saunders, B.V.; Cohl, H.S.; McClain, M.A. NIST Digital Library of Mathematical Functions [Internet]. 2020. Available online: <http://dlmf.nist.gov/> (accessed on 1 April 2023).
12. Bronstein, M.; Corless, R.M.; Davenport, J.H.; Jeffrey, D.J. Algebraic properties of the Lambert W function from a result of Rosenlicht and of Liouville. *Integral Transform. Spec. Funct.* **2008**, *19*, 709–712. [CrossRef]

13. Brauer, F.; Castillo-Chavez, C.; Feng, Z. *Mathematical Models in Epidemiology*; Springer: New York, NY, USA, 2019.
14. Prodanov, D. Chapter 10-Analytical solutions and parameter estimation of the SIR epidemic model. In *Mathematical Analysis of Infectious Diseases*; Agarwal, P., Nieto, J., Torres, D., Eds.; Academic Press: Cambridge, MA, USA, 2022; pp. 163–189. ISBN 978-0-323-90504-6.

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