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Several Quantum Hermite–Hadamard-Type Integral Inequalities for Convex Functions

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Abstract: The aim of this study was to present several improved quantum Hermite–Hadamard-type integral inequalities for convex functions using a parameter. Thus, a new quantum identity is proven to be used as the main tool in the proof of our results. Consequently, in some special cases several new quantum estimations for q-midpoints and q-trapezoidal-type inequalities are derived with an example. The results obtained could be applied in the optimization of several economic geology problems.

Keywords: quantum calculus; convex functions; Hermite–Hadamard-type inequalities

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1. Introduction

The field of mathematical inequalities and applications has seen great advancements in the last three decades. This has a significant impact on areas such as physics, earth sciences, engineering [1], statistics [2], economics [3] and approximation theory [4], information theory [5] and numerical analysis [6,7]. "Mathematical inequalities have streamlined the concept of classical convexity" [8]. It is known that "the scientific observations and calculations rely on convex functions and their relationship to mathematical inequalities" [8]. Integral inequalities represent a fundamental way to establish qualitative or quantitative mathematical results. The strong correlation between different convexities and symmetric functions, as well as between convex functions and integral inequalities, open a broad framework for studying a large category of complex problems.

Convexity is a natural concept for solving many problems in mathematics with numerous uses in industry, business and medicine. Various types of convexities have been investigated, such as, h-convexity defined by Varosanec [9], exponentially convex functions introduced by Bernstein [10] with covariance analysis applications, r-convex functions studied by Avriel [11], convex functions on coordinates introduced by Dragomir [12], h-convexity on coordinates introduced by Alomari et al. [13], exponentially h-convexity defined by Rashid et al. [14], and exponential h-convex functions [15] on coordinates given by W. Iqbal et al. On the other hand, Pal and Wong provided a base for exponential convex functions in the fields of information theory [16] and optimization theory. An important generalization of convexity was given in 1981 [17], by the introduction of invexity, due to its importance in optimization. Studies on convex and pre-invex functions have potential applications [18] in maximizing the likelihood [19] from multiple linear regressions [20] involving Gauss–Laplace distributions.

Quantum calculus has various applications in the interdisciplinary field of quantum information theory. This field contains many subfields, such as computer science, information theory [21], philosophy, and cryptography. Quantum calculus (i.e., q-calculus) is known as the study of calculus without limitations. This topic has become a reliable instrument in many areas of physics [22] and mathematics in recent years. The quantum integral inequalities are more useful and interesting than their classical equivalents. Jackson studied the quantum difference operator [23]. The first use of quantum calculus and the difference equations [23] was in physics and chemistry problems. Siegel investigated string theory [24] involving quantum calculus. Recently, new applications have been established in various branches of physics and mathematics. In [25], in 1969, Agarwal studied q-fractional derivatives for the first time. q-calculus concepts [26] on finite intervals was used to find quantum analogues of known mathematical definitions and results. New quantum analogues [27] of the Ostrowski inequalities [28], using first-order quantum differentiable convex mappings, were presented by Noor et al. Several bounds for the left-hand side (LHS) of quantum H–H inequalities [29] were established. New quantum analogues of the classical Simpson's inequality were presented [30] for pre-invex functions. These new q-analogues of the (s, m) that generalized (s, m) pre-invex functions were given in [31] by Deng et al. The mathematical field [32] of time scale calculus contains quantum calculus as a subfield. "In studying quantum calculus, we are concerned with a specific time scale, called the q-time scale, defined as: $T = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$", see [32].

Many inequalities, especially the Hermite–Hadamard (H–H)-type inequalities, contain a kind of symmetry, an important characteristic as symmetry commonly has a central role [21] in finding the correct way to solve dynamic inequalities. This famous inequality [12] has had many improvements and extensions [13] during recent decades [33], see, for example, [34], and nowadays, for quantum and post-quantum calculus, fractional calculus and fuzzy environment. Recently new generalizations and refinements of H–H-type integral inequalities for quantum [29,31] and post-quantum calculus [35,36] have been obtained, e.g., [37,38] and the references cited therein.

Motivated by [36], we aimed to obtain new parametrized q-H–H-type inequalities for third-order q-differentiable functions using a new quantum identity, concerning the third-order q-left and right derivatives. These results are different from the results of [39] and are similar to the results of [36]. Our work represents the case when the functions accept the third q-derivative instead of the first q-derivative, see [36], and the left term of these inequalities contains two integrals defined as two new intervals, $[a, \lambda b + (1-\lambda)a]$ and $[\lambda a + (1-\lambda)b, b]$, different from intervals $[a, x]$ and $[x, b]$ which appear in [39] and [35]. The values of the parameter λ change these intervals in a different way, which is advantageous. Furthermore, here new terms appear in the left term of these inequalities with different coefficients of the third q-derivatives in the right term.

The paper is structured as four sections as follows. Section 2 provides a brief summary of the fundamental definitions and properties regarding quantum calculus and the H–H integral inequality. In Section 3, we state and prove our main results in Lemma 1, Theorems 4–6. In these theorems and consequent new q-midpoints, trapezoidal and q-H–H-type integral inequalities are established for three times differentiable convex functions. Many consequences are formulated with a given example. Figure and several calculus in Example 1 were performed using the Matlab R2023a software version. Applications to special cases of real numbers are presented in Section 3 using the newly generated results. In Section 4, a discussion and conclusions are drawn.

2. Preliminary on q-Calculus and Inequalities

The classical H–H inequality says that "if $\theta : [a, b] \to \mathbb{R}$ is a convex function, then the following inequality holds:

$$\theta\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b \theta(x)dx \le \frac{\theta(a)+\theta(b)}{2}. \tag{1}$$

When θ is a concave function, then the previous inequality holds but in the opposite direction", see [40].

We assume that $0 < q < 1$. Let $[a, b]$ be a real interval, where $a < b$. We assume that $[n]_q = \frac{1-q^n}{1-q} = 1 + q + \ldots + q^{n-2} + q^{n-1}$, $n \in \mathbb{N}$.

Now we present some important definitions, remarks and lemmas of the q-calculus which will be used throughout this paper.

Definition 1 ([36]). *The right or q^b-derivative of $\theta : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is expressed as:*

$$^{b}D_q\theta(x) = \frac{\theta(qx + (1-q)b) - \theta(x)}{(1-q)(b-x)}, \quad x \neq b.$$

Definition 2 ([36,41]). *The left or q_a-derivative of $\theta : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is expressed as:*

$$_{a}D_q\theta(x) = \frac{\theta(x) - \theta(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a.$$

Definition 3 ([36]). *The right or q^b-integral of $\theta : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is defined as:*

$$\int_x^b \theta(t)^b d_q t = (1-q)(b-x)\sum_{n=0}^{\infty} q^n \theta(q^n x + (1-q^n)b) = (b-a)\int_0^1 \theta(tb + (1-t)x)^1 d_q t.$$

Definition 4 ([36]). *The left or q_a-integral of $\theta : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is defined as:*

$$\int_a^x \theta(t)_a d_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n \theta(q^n x + (1-q^n)a) = (b-a)\int_0^1 \theta(tx + (1-t)a) d_q t.$$

Definition 5 ([36]). *We have the equality for q_a-integrals, defined as*

$$\int_a^b (x-a)^{\alpha}{}_a d_q x = \frac{(b-a)^{\alpha+1}}{[\alpha+1]_q},$$

for $\alpha \in \mathbb{R} - \{-1\}$.

The fundamental properties of these derivatives and integrals can be found in [41,42]. Recently, new refinements and generalizations of q-H–H integral inequalities for q-differentiable functions were given in [36].

Theorem 1 ([36]). *"We assume that the conditions of Lemma 2 ([36]) hold. If $|_aD_q\theta|$ and $|^bD_q\theta|$ are convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \theta(t)_a d_q t + \int_{\lambda a+(1-\lambda)b}^{b} \theta(t)^b d_q t\right) - \frac{\theta(\lambda b + (1-\lambda)a) + \theta(\lambda a + (1-\lambda)b)}{2}\right| \\ &\leq \frac{\lambda q(b-a)}{2[2]_q[3]_q}[([3]_q - \lambda[2]_q)[|^bD_q\theta(b)| + |_aD_q\theta(a)|] + \lambda[2]_q[|^bD_q\theta(a)| + |_aD_q\theta(b)|]]." \end{aligned}$$

Theorem 2 ([36]). *"We assume that the conditions of Lemma 2 ([36]) hold. If $|_aD_q\theta|^s$ and $|^bD_q\theta|^s$, $s > 1$ are convex, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \theta(t)_a d_q t + \int_{\lambda a+(1-\lambda)b}^{b} \theta(t)^b d_q t\right) - \frac{\theta(\lambda b + (1-\lambda)a) + \theta(\lambda a + (1-\lambda)b)}{2}\right| \\ &\leq \frac{\lambda q(b-a)}{2}\left(\frac{1}{[r+1]_q}\right)^{\frac{1}{r}}\left[\left(\frac{[2]_q - \lambda}{[2]_q}|^bD_q\theta(b)|^s + \frac{\lambda}{[2]_q}|^bD_q\theta(a)|^s\right)^{\frac{1}{s}}\right. \\ &\left. + \left(\frac{[2]_q - \lambda}{[2]_q}|_aD_q\theta(a)|^s + \frac{\lambda}{[2]_q}|_aD_q\theta(b)|^s\right)^{\frac{1}{s}}\right], \end{aligned}$$

where $s^{-1} + r^{-1} = 1$."

Theorem 3 ([36]). *"We assume that the conditions of Lemma 2 ([36]) hold. If $|_aD_q\theta|^s$ and $|^bD_q\theta|^s$, $s \geq 1$ are convex, then the following inequality holds:*

$$|\frac{1}{2\lambda(b-a)}\left(\int_a^{\lambda b+(1-\lambda)a} \theta(t)_a d_qt + \int_{\lambda a+(1-\lambda)b}^b \theta(t)^b d_qt\right) - \frac{\theta(\lambda b+(1-\lambda)a)+\theta(\lambda a+(1-\lambda)b)}{2}|$$
$$\leq \frac{\lambda q(b-a)}{2[2]_q}[\left(\frac{([3]_q-\lambda[2]_q)|^bD_q\theta(b)|^s+\lambda[2]_q|^bD_q\theta(a)|^s}{[3]_q}\right)^{\frac{1}{s}}$$
$$+\left(\frac{([3]_q-\lambda[2]_q)|_aD_q\theta(a)|^s+\lambda[2]_q|_aD_q\theta(b)|^s}{[3]_q}\right)^{\frac{1}{s}}]."$$

3. Results

A new quantum identity is given below as the main tool in the proof of our results. New estimates of parametrized q-H–H-type integral inequalities for three time quantum differentiable functions are presented below starting from the results from [36]. In addition, new consequent terms and applications, including an example, are given to illustrate the investigated results.

Lemma 1. *Let $\theta : [a,b] \to \mathbb{R}$ be a third-order q-differentiable function with ${}_aD_q^3\theta$ and ${}^bD_q^3\theta$ continuous and q-integrable functions on $[a,b]$, respectively. Thus, the following equality holds,*

$$ {}_a^bP_q(\lambda) = \frac{\lambda^3(b-a)^3}{2}\int_0^1 q^6t^3[{}^bD_q^3\theta((1-\lambda t)b+\lambda ta) - {}_aD_q^3\theta((1-\lambda t)a+\lambda tb)]d_qt, \quad (2)$$

where

$$\begin{aligned} {}_a^bP_q(\lambda) &= \frac{[2]_q[3]_q}{2\lambda(b-a)}[\int_{\lambda a+(1-\lambda)b}^b \theta(t)^b d_qt + \int_a^{\lambda b+(1-\lambda a)} \theta(t)_a d_qt] \\ &- \frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(\lambda a+(1-\lambda)b)+\theta(\lambda b+(1-\lambda)a)] \\ &- \frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(q\lambda a+(1-q\lambda)b)+\theta(q\lambda b+(1-q\lambda)a)] \\ &- \frac{q^2}{2(1-q)^2}[\theta(q^2\lambda a+(1-q^2\lambda)b)+\theta(q^2\lambda b+(1-q^2\lambda)a)]. \end{aligned}$$

Proof. Using Definition 1, Definition 3 and calculus, we obtain

$$\begin{aligned} I_1 &= \int_0^1 t^{3b}D_q^3\theta((1-\lambda t)b+\lambda ta)d_qt \\ &= \int_0^1 \frac{1}{q^3(1-q)^3\lambda^3(b-a)^3}[\theta(q^3\lambda ta+b(1-q^3\lambda t)) - [3]_q\theta(q^2\lambda ta+b(1-q^2\lambda t)) \\ &+ q[3]_q\theta(q\lambda ta+b(1-q\lambda t)) - q^3\theta(\lambda ta+(1-\lambda t)b)]d_qt \\ &= \frac{1}{(b-a)^3\lambda^3q^3(1-q)^2}[\sum_{n=0}^{\infty} q^n\theta(q^{n+3}\lambda a+b(1-q^{n+3}\lambda)) \\ &- [3]_q\sum_{n=0}^{\infty} q^n\theta(q^{n+2}\lambda a+b(1-q^{n+2}\lambda)) \\ &+ q[3]_q\sum_{n=0}^{\infty} q^n\theta(q^{n+1}\lambda a+b(1-q^{n+1}\lambda)) - q^3\sum_{n=0}^{\infty} q^n\theta(q^n\lambda a+b(1-q^n\lambda))] \\ &= \frac{1-q}{(b-a)^3\lambda^3q^3(1-q)^3}[\frac{1}{q^3}\sum_{n=3}^{\infty} q^n\theta(q^n\lambda a+b(1-q^n\lambda)) - \frac{[3]_q}{q^2}\sum_{n=2}^{\infty} q^n\theta(q^n\lambda a+b(1-q^n\lambda)) \end{aligned}$$

$$
\begin{aligned}
+ \quad & [3]_q \sum_{n=1}^{\infty} q^n \theta(q^n \lambda a + b(1-q^n\lambda)) - q^3 \sum_{n=0}^{\infty} q^n \theta(q^n \lambda a + b(1-q^n\lambda))] \\
= \quad & \frac{1-q}{(b-a)^3\lambda^3 q^3(1-q)^3}\{\frac{1}{q^3}[\sum_{n=0}^{\infty} q^n \theta(q^n \lambda a + b(1-q^n\lambda)) - \theta(\lambda a + b(1-\lambda)) \\
- \quad & q\theta(q\lambda a + b(1-q\lambda)) - q^2\theta(q^2\lambda a + b(1-q^2\lambda))] \\
- \quad & \frac{[3]_q}{q^2}[\sum_{n=0}^{\infty} q^n \theta(q^n \lambda a + b(1-q^n\lambda)) - \theta(\lambda a + b(1-\lambda)) - q\theta(q\lambda a + b(1-q\lambda))] \\
+ \quad & [3]_q[\sum_{n=0}^{\infty} q^n \theta(q^n \lambda a + b(1-q^n\lambda)) - \theta(\lambda a + b(1-\lambda))] - q^3 \sum_{n=0}^{\infty} q^n \theta(q^n \lambda a + b(1-q^n\lambda))]\} \\
= \quad & \frac{1}{(b-a)^4\lambda^4 q^3(1-q)^3}\left(\frac{1}{q^3} - \frac{[3]_q}{q^2} + [3]_q - q^3\right)\int_{\lambda a + b(1-\lambda)}^{b} \theta(t)\, {}^{b}d_q t \\
- \quad & \frac{1}{(b-a)^3\lambda^3 q^3(1-q)^2}\left(\frac{1}{q^3} - \frac{[3]_q}{q^2} + [3]_q\right)\theta(\lambda a + b(1-\lambda)) \\
- \quad & \frac{1}{(b-a)^3\lambda^3 q^5(1-q)^2}(1 - q[3]_q)\theta(q\lambda a + b(1-q\lambda)) \\
- \quad & \frac{1}{(b-a)^3\lambda^3 q^4(1-q)^2}\theta(q^2\lambda a + b(1-q^2\lambda)).
\end{aligned}
$$

In the same way, from Definition 2, Definition 4 and calculus, we obtain

$$
\begin{aligned}
I_2 \quad = \quad & \int_0^1 t^3 {}_aD_q^3 \theta((1-\lambda t)a + \lambda t b) d_q t \\
= \quad & \int_0^1 \frac{1}{q^3(1-q)^3\lambda^3(b-a)^3}[-\theta(q^3\lambda t b + a(1-q^3\lambda t)) + [3]_q\theta(q^2\lambda t b + a(1-q^2\lambda t) \\
- \quad & q[3]_q\theta(q\lambda t b + a(1-q\lambda t)) + q^3\theta(\lambda t b + (1-\lambda t)a)]d_q t \\
= \quad & \frac{1}{(b-a)^3\lambda^3 q^3(1-q)^2}[-\sum_{n=0}^{\infty} q^n \theta(q^{n+3}\lambda b + a(1-q^{n+3}\lambda)) \\
+ \quad & [3]_q \sum_{n=0}^{\infty} q^n \theta(q^{n+2}\lambda b + a(1-q^{n+2}\lambda)) \\
- \quad & q[3]_q \sum_{n=0}^{\infty} q^n \theta(q^{n+1}\lambda b + a(1-q^{n+1}\lambda)) + q^3 \sum_{n=0}^{\infty} q^n \theta(q^n\lambda b + a(1-q^n\lambda))] \\
= \quad & \frac{1-q}{(b-a)^3\lambda^3 q^3(1-q)^3}[-\frac{1}{q^3}\sum_{n=3}^{\infty} q^n \theta(q^n\lambda b + a(1-q^n\lambda)) \\
+ \quad & \frac{[3]_q}{q^2}\sum_{n=2}^{\infty} q^n \theta(q^n\lambda b + a(1-q^n\lambda)) - [3]_q \sum_{n=1}^{\infty} q^n \theta(q^n\lambda b + a(1-q^n\lambda)) \\
+ \quad & q^3 \sum_{n=0}^{\infty} q^n \theta(q^n\lambda b + a(1-q^n\lambda))] \\
= \quad & \frac{1-q}{(b-a)^3\lambda^3 q^3(1-q)^3}\{-\frac{1}{q^3}[\sum_{n=0}^{\infty} q^n \theta(q^n\lambda b + a(1-q^n\lambda)) - \theta(\lambda b + a(1-\lambda)) \\
-q \quad & \theta(q\lambda b + a(1-q\lambda)) - q^2\theta(q^2\lambda b + a(1-q^2\lambda))] + \frac{[3]_q}{q^2}[\sum_{n=0}^{\infty} q^n \theta(q^n\lambda b + a(1-q^n\lambda)) \\
- \quad & \theta(\lambda b + a(1-\lambda)) - q\theta(q\lambda b + a(1-q\lambda))]
\end{aligned}
$$

$$
\begin{aligned}
&- \quad [3]_q[\sum_{n=0}^{\infty} q^n\theta(q^n\lambda b + a(1-q^n\lambda)) - \theta(\lambda b + a(1-\lambda))] + q^3\sum_{n=0}^{\infty} q^n\theta(q^n\lambda b + a(1-q^n\lambda))]\} \\
&= \quad \frac{1}{(b-a)^4\lambda^4 q^3(1-q)^3}\left(-\frac{1}{q^3} + \frac{[3]_q}{q^2} - [3]_q + q^3\right)\int_a^{\lambda b + a(1-\lambda)} \theta(t)_a d_q t \\
&+ \quad \frac{1}{(b-a)^3\lambda^3 q^3(1-q)^2}\left(\frac{1}{q^3} - \frac{[3]_q}{q^2} + [3]_q\right)\theta(\lambda b + a(1-\lambda)) \\
&+ \quad \frac{1}{(b-a)^3\lambda^3 q^5(1-q)^2}(1 - q[3]_q)\theta(q\lambda b + a(1-q\lambda)) \\
&+ \quad \frac{1}{(b-a)^3\lambda^3 q^4(1-q)^2}\theta(q^2\lambda b + a(1-q^2\lambda)).
\end{aligned}
$$

By subtracting I_2 from I_1 and multiplying the result by $\frac{\lambda^3 q^6(b-a)^3}{2}$, it follows that

$$
\begin{aligned}
\frac{\lambda^3 q^6(b-a)^3}{2}(I_1 - I_2) &= \frac{[2]_q[3]_q}{2\lambda(b-a)}[\int_{\lambda a + (1-\lambda)b}^{b} \theta(t)^b d_q t + \int_a^{\lambda b + (1-\lambda)a} \theta(t)_a d_q t] \\
&- \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(\lambda a + b(1-\lambda)) + \theta(\lambda b + a(1-\lambda))] \\
&- \frac{q(1 - q[3]_q)}{2(1-q)^2}[\theta(q\lambda a + b(1-q\lambda)) + \theta(q\lambda b + a(1-q\lambda))] \\
&- \frac{q^2}{2(1-q)^2}[\theta(q^2\lambda a + b(1-q^2\lambda)) + \theta(q^2\lambda b + a(1-q^2\lambda))],
\end{aligned}
$$

and thus, the proof is complete. □

Theorem 4. *We assume that the hypotheses of Lemma 1 are satisfied. If $|{}_aD_q^3\theta|$ and $|{}^bD_q^3\theta|$ are convex on $[a,b]$ then the following inequality holds:*

$$
\begin{aligned}
|{}_a^bP_q(\lambda)| &= |\frac{[2]_q[3]_q}{2\lambda(b-a)}[\int_{\lambda a + (1-\lambda)b}^{b} \theta(t)^b d_q t + \int_a^{\lambda b + (1-\lambda a)} \theta(t)_a d_q t] \\
&- \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(\lambda a + (1-\lambda)b) + \theta(\lambda b + (1-\lambda)a)] \\
&- \frac{q(1 - q[3]_q)}{2(1-q)^2}[\theta(q\lambda a + (1-q\lambda)b) + \theta(q\lambda b + (1-q\lambda)a)] \\
&- \frac{q^2}{2(1-q)^2}[\theta(q^2\lambda a + (1-q^2\lambda)b) + \theta(q^2\lambda b + (1-q^2\lambda)a)]| \\
&\leq \frac{\lambda^3 q^6(b-a)^3}{2[4]_q[5]_q}\{([5]_q - \lambda[4]_q)[|{}^bD_q^3\theta(b)| + |{}_aD_q^3\theta(a)|] \\
&\quad + \lambda[4]_q[|{}^bD_q^3\theta(a)| + |{}_aD_q^3\theta(b)|]\}. \qquad (3)
\end{aligned}
$$

Proof. By using Lemma 1, we obtain

$$
\begin{aligned}
|{}_a^bP_q(\lambda)| &= |\frac{[2]_q[3]_q}{2\lambda(b-a)}[\int_{\lambda a+(1-\lambda)b}^{b}\theta(t)^b d_qt+\int_a^{\lambda b+(1-\lambda a)}\theta(t)_a d_qt] \\
&- \frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(\lambda a+(1-\lambda)b)+\theta(\lambda b+(1-\lambda)a)] \\
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(q\lambda a+(1-q\lambda)b)+\theta(q\lambda b+(1-q\lambda)a)] \\
&- \frac{q^2}{2(1-q)^2}[\theta(q^2\lambda a+(1-q^2\lambda)b)+\theta(q^2\lambda b+(1-q^2\lambda)a)]| \\
&\leq \frac{\lambda^3q^6(b-a)^3}{2}\{\int_0^1 t^3|{}^bD_q^3\theta((1-\lambda t)b+\lambda ta)|d_qt \\
&+ \int_0^1 t^3|{}_aD_q^3\theta((1-\lambda t)a+\lambda tb)|d_qt\}
\end{aligned}
$$

and then with the help of convexity of $|{}_aD_q^3\theta|$ and $|{}^bD_q^3\theta|$, we have

$$
\begin{aligned}
|{}_a^bP_q(\lambda)| &\leq \frac{\lambda^3q^6(b-a)^3}{2}\{\int_0^1 t^3[(1-\lambda t)|{}^bD_q^3\theta(b)|+\lambda t|{}^bD_q^3\theta(a)|]d_qt \\
&+ \int_0^1 t^3[(1-\lambda t)|{}_aD_q^3\theta(a)|+\lambda t|{}_aD_q^3\theta(b)|]d_qt\} \\
&= \frac{\lambda^3q^6(b-a)^3}{2}\{[|{}^bD_q^3\theta(b)|+|{}_aD_q^3\theta(a)|]\int_0^1 t^3(1-\lambda t)d_qt \\
&+ [|{}^bD_q^3\theta(a)|+|{}_aD_q^3\theta(b)|]\int_0^1 t^4\lambda d_qt\} \\
&= \frac{\lambda^3q^6(b-a)^3}{2}\{(\frac{1}{[4]_q}-\frac{\lambda}{[5]_q})[|{}^bD_q^3\theta(b)|+|{}_aD_q^3\theta(a)|] \\
&+ \frac{\lambda}{[5]_q}[|{}^bD_q^3\theta(a)|+|{}_aD_q^3\theta(b)|]\}.
\end{aligned}
$$

Using calculus we obtain the desired inequality. □

Remark 1. *If we take $\lambda = 1$ in Theorem 4, the following trapezoid-type inequality is obtained:*

$$
\begin{aligned}
|{}_a^bP_q(1)| &= |\frac{[2]_q[3]_q}{2(b-a)}[\int_a^b\theta(t)^b d_qt+\int_a^b\theta(t)_a d_qt]-\frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(a)+\theta(b)] \\
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(qa+(1-q)b)+\theta(qb+(1-q)a)] \\
&- \frac{q^2}{2(1-q)^2}[\theta(q^2a+(1-q^2)b)+\theta(q^2b+(1-q^2)a)]| \\
&\leq \frac{q^6(b-a)^3}{2[4]_q[5]_q}\{([5]_q-[4]_q)[|{}^bD_q^3\theta(b)|+|{}_aD_q^3\theta(a)|]+[4]_q[|{}^bD_q^3\theta(a)|+|{}_aD_q^3\theta(b)|]\}.
\end{aligned}
$$

Remark 2. *If we assign $\lambda = \frac{1}{2}$ in Theorem 4, then the following midpoint-type inequality is obtained:*

$$
\begin{aligned}
|{}_{a}^{b}P_q(\frac{1}{2})| &= |\frac{[2]_q[3]_q}{(b-a)}[\int_{\frac{a+b}{2}}^{b}\theta(t)^{b}d_qt+\int_{a}^{\frac{a+b}{2}}\theta(t)_{a}d_qt]-\frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}\theta(\frac{a+b}{2}) \\
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(\frac{q}{2}a+(1-\frac{q}{2})b)+\theta(\frac{q}{2}b+(1-\frac{q}{2})a)] \\
&- \frac{q^2}{2(1-q)^2}[\theta(\frac{q^2}{2}a+(1-\frac{q^2}{2})b)+\theta(\frac{q^2}{2}b+(1-\frac{q^2}{2})a)]| \\
&\leq \frac{q^6(b-a)^3}{32[4]_q[5]_q}\{(2[5]_q-[4]_q)[|{}^{b}D_q^3\theta(b)|+|{}_{a}D_q^3\theta(a)|]+[4]_q[|{}^{b}D_q^3\theta(a)|+|{}_{a}D_q^3\theta(b)|]\}.
\end{aligned}
$$

Remark 3. *If we assign $\lambda=\frac{1}{[2]_q}$ in Theorem 4, then the following inequality is obtained:*

$$
\begin{aligned}
|{}_{a}^{b}P_q(\frac{1}{[2]_q})| &= |\frac{[2]_q^2[3]_q}{2(b-a)}[\int_{\frac{a+qb}{[2]_q}}^{b}\theta(t)^{b}d_qt+\int_{a}^{\frac{qa+b}{[2]_q}}\theta(t)_{a}d_qt]-\frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(\frac{a+qb}{[2]_q}) \\
&+ \theta(\frac{qa+b}{[2]_q})]-\frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(\frac{qa+b}{[2]_q})+\theta(\frac{qb+a}{[2]_q})] \\
&- \frac{q^2}{2(1-q)^2}[\theta(\frac{q^2a+b([2]_q-q^2)}{[2]_q})+\theta(\frac{q^2b+a([2]_q-q^2)}{[2]_q})]| \\
&\leq \frac{q^6(b-a)^3}{2[2]_q^4[4]_q[5]_q}\{([2]_q[5]_q-[4]_q)[|{}^{b}D_q^3\theta(b)|+|{}_{a}D_q^3\theta(a)|] \\
&+ [4]_q[|{}^{b}D_q^3\theta(a)|+|{}_{a}D_q^3\theta(b)|]\}.
\end{aligned}
$$

Theorem 5. *We assume that the conditions from Lemma 1 hold. If $|{}_{a}D_q^3\theta|^s$ and $|{}^{b}D_q^3\theta|^s$ are convex functions when $s>1$ then the following inequality holds:*

$$
\begin{aligned}
|{}_{a}^{b}P_q(\lambda)| &\leq \frac{\lambda^3q^6(b-a)^3}{2}\left(\frac{1}{[3r+1]_q}\right)^{\frac{1}{r}}\{[\frac{[2]_q-\lambda}{[2]_q}|{}^{b}D_q^3\theta(b)|^s+\frac{\lambda}{[2]_q}|{}^{b}D_q^3\theta(a)|^s]^{\frac{1}{s}} \\
&+ [\frac{[2]_q-\lambda}{[2]_q}|{}_{a}D_q^3\theta(a)|^s+\frac{\lambda}{[2]_q}|{}_{a}D_q^3\theta(b)|^s]^{\frac{1}{s}}\},
\end{aligned}
$$

where $\frac{1}{s}+\frac{1}{r}=1$.

Proof. This time, the parameters will be applied to Holder's inequality after being used before the modulus properties in Theorem 4, obtaining:

$$
\begin{aligned}
|{}_{a}^{b}P_q(\lambda)| &\leq \frac{\lambda^3q^6(b-a)^3}{2}\{\int_0^1 t^3|{}^{b}D_q^3\theta((1-\lambda t)b+\lambda ta)|d_qt \\
&+ \int_0^1 t^3|{}_{a}D_q^3\theta((1-\lambda t)a+\lambda tb)|d_qt\} \\
&\leq \frac{\lambda^3q^6(b-a)^3}{2}[\left(\int_0^1 t^{3r}d_qt\right)^{\frac{1}{r}}\left(\int_0^1|{}^{b}D_q^3\theta((1-\lambda t)b+\lambda ta)|^sd_qt\right)^{\frac{1}{s}} \\
&+ \left(\int_0^1 t^{3r}d_qt\right)^{\frac{1}{r}}\left(\int_0^1|{}_{a}D_q^3\theta((1-\lambda t)a+\lambda tb)|^sd_qt\right)^{\frac{1}{s}}].
\end{aligned}
$$

Now we use the convex functions $|{}_{a}D_q^3\theta|^s$ and $|{}^{b}D_q^3\theta|^s$ to obtain

$$
\begin{aligned}
|{}_a^bP_q(\lambda)| &\leq \frac{\lambda^3 q^6(b-a)^3}{2}\left(\frac{1}{[3r+1]_q}\right)^{\frac{1}{r}}\Bigg[\left(\int_0^1(1-\lambda t)|{}^bD_q^3\theta(b)|^s d_qt+\int_0^1\lambda t|{}^bD_q^3\theta(a)|^s d_qt\right)^{\frac{1}{s}}\\
&+ \left(\int_0^1(1-\lambda t)|{}_aD_q^3\theta(a)|^s d_qt+\int_0^1\lambda t|{}_aD_q^3\theta(b)|^s d_qt\right)^{\frac{1}{s}}\Bigg]\\
&= \frac{\lambda^3 q^6(b-a)^3}{2}\left(\frac{1}{[3r+1]_q}\right)^{\frac{1}{r}}\Big\{\Big[\frac{[2]_q-\lambda}{[2]_q}|{}^bD_q^3\theta(b)|^s+\frac{\lambda}{[2]_q}|{}^bD_q^3\theta(a)|^s\Big]^{\frac{1}{s}}\\
&+ \Big[\frac{[2]_q-\lambda}{[2]_q}|{}_aD_q^3\theta(a)|^s+\frac{\lambda}{[2]_q}|{}_aD_q^3\theta(b)|^s\Big]^{\frac{1}{s}}\Big\},
\end{aligned}
$$

which completes the proof. □

Remark 4. *If we assign $\lambda = 1$ in Theorem 5, then the following trapezoid-type inequality is obtained:*

$$
\begin{aligned}
|{}_a^bP_q(1)| &= \Big|\frac{[2]_q[3]_q}{2(b-a)}\Big[\int_a^b\theta(t)\,{}^bd_qt+\int_a^b\theta(t)\,{}_ad_qt\Big]-\frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(a)+\theta(b)]\\
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(qa+(1-q)b)+\theta(qb+(1-q)a)]\\
&- \frac{q^2}{2(1-q)^2}[\theta(q^2a+(1-q^2)b)+\theta(q^2b+(1-q^2)a)]\Big|\\
&\leq \frac{q^6(b-a)^3}{2}\left(\frac{1}{[3r+1]_q}\right)^{\frac{1}{r}}\Big\{\Big[\frac{[2]_q-1}{[2]_q}|{}^bD_q^3\theta(b)|^s+\frac{1}{[2]_q}|{}^bD_q^3\theta(a)|^s\Big]^{\frac{1}{s}}\\
&+ \Big[\frac{[2]_q-1}{[2]_q}|{}_aD_q^3\theta(a)|^s+\frac{1}{[2]_q}|{}_aD_q^3\theta(b)|^s\Big]^{\frac{1}{s}}\Big\}.
\end{aligned}
$$

Remark 5. *If we assign $\lambda = \frac{1}{2}$ in Theorem 5, then the following midpoint-type inequality is obtained:*

$$
\begin{aligned}
|{}_a^bP_q(\tfrac{1}{2})| &= \Big|\frac{[2]_q[3]_q}{(b-a)}\Big[\int_{\frac{a+b}{2}}^b\theta(t)\,{}^bd_qt+\int_a^{\frac{a+b}{2}}\theta(t)\,{}_ad_qt\Big]-\frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}\theta(\frac{a+b}{2})\\
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(\frac{q}{2}a+(1-\frac{q}{2})b)+\theta(\frac{q}{2}b+(1-\frac{q}{2})a)]\\
&- \frac{q^2}{2(1-q)^2}[\theta(\frac{q}{2}^2a+(1-\frac{q}{2}^2)b)+\theta(\frac{q}{2}^2b+(1-\frac{q}{2}^2)a)]\Big|\\
&\leq \frac{q^6(b-a)^3}{16}\left(\frac{1}{[2]_q}\right)^{\frac{1}{s}}\left(\frac{1}{[3r+1]_q}\right)^{\frac{1}{r}}\{[(2[2]_q-1)|{}^bD_q^3\theta(b)|^s+|{}^bD_q^3\theta(a)|^s]^{\frac{1}{s}}\\
&+ [(2[2]_q-1)||{}_aD_q^3\theta(a)|^s+|{}_aD_q^3\theta(b)|^s]^{\frac{1}{s}}\}.
\end{aligned}
$$

Theorem 6. *We assume that the conditions from Lemma 1 hold. If $|{}_aD_q^3\theta|^s$ and $|{}^bD_q^3\theta|^s$ are convex functions for $s \geq 1$, then the following inequality holds:*

$$
\begin{aligned}
|{}_a^bP_q(\lambda)| &\leq \frac{\lambda^3q^6(b-a)^3}{2[4]_q[5]_q^{\frac{1}{s}}}\\
&\times \{[([5]_q-\lambda[4]_q)|{}^bD_q^3\theta(b)|^s+\lambda[4]_q|{}^bD_q^3\theta(a)|^s]^{\frac{1}{s}}\\
&+ [([5]_q-\lambda[4]_q)|{}_aD_q^3\theta(a)|^s+\lambda[4]_q|{}_aD_q^3\theta(b)|^s]^{\frac{1}{s}}\}.
\end{aligned}
$$

Proof. By applying the properties of modulus and the power mean inequality we obtain,

$$|{}_a^bP_q(\lambda)| \leq \frac{\lambda^3 q^6 (b-a)^3}{2} \{ \left(\int_0^1 t^3 d_q t \right)^{1-\frac{1}{s}} \left(\int_0^1 t^3 |{}^bD_q^3 \theta((1-\lambda t)b + \lambda t a)|^s d_q t \right)^{\frac{1}{s}}$$

$$+ \left(\int_0^1 t^3 d_q t \right)^{1-\frac{1}{s}} \left(\int_0^1 t^3 |{}_aD_q^3 \theta((1-\lambda t)a + \lambda t b)|^s d_q t \right)^{\frac{1}{s}} \}.$$

By using the convexity of the functions $|{}_aD_q^3\theta|^s$ and $|{}^bD_q^3\theta|^s$ we have,

$$\begin{aligned}
|{}_a^bP_q(\lambda)| &\leq \frac{\lambda^3 q^6 (b-a)^3}{2} \frac{1}{[4]_q^{1-\frac{1}{s}}} \{ \left(\int_0^1 (t^3(1-\lambda t)) |{}^bD_q^3\theta(b)|^s + \lambda t^4 |{}^bD_q^3\theta(a)|^s) d_q t \right)^{\frac{1}{s}} \\
&+ \left(\int_0^1 (t^3(1-\lambda t)) |{}_aD_q^3\theta(a)|^s + \lambda t^4 |{}_aD_q^3\theta(b)|^s) d_q t \right)^{\frac{1}{s}} \} \\
&= \frac{\lambda^3 q^6 (b-a)^3}{2} \frac{1}{[4]_q^{1-\frac{1}{s}}} \{ [(\frac{1}{[4]_q} - \frac{\lambda}{[5]_q}) |{}^bD_q^3\theta(b)|^s + \frac{\lambda}{[5]_q} |{}^bD_q^3\theta(a)|^s]^{\frac{1}{s}} \\
&+ [(\frac{1}{[4]_q} - \frac{\lambda}{[5]_q}) |{}_aD_q^3\theta(a)|^s + \frac{\lambda}{[5]_q} |{}_aD_q^3\theta(b)|^s)]^{\frac{1}{s}} \} \\
&= \frac{\lambda^3 q^6 (b-a)^3}{2[4]_q [5]_q^{\frac{1}{s}}} \{ [([5]_q - \lambda[4]_q) |{}^bD_q^3\theta(b)|^s + \lambda[4]_q |{}^bD_q^3\theta(a)|^s]^{\frac{1}{s}} \\
&+ [([5]_q - \lambda[4]_q) |{}_aD_q^3\theta(a)|^s + \lambda[4]_q |{}_aD_q^3\theta(b)|^s)]^{\frac{1}{s}} \}.
\end{aligned}$$

Therefore, the proof is complete. □

Remark 6. *If we take $\lambda = 1$ in Theorem 6, then the following trapezoid-type inequality is obtained:*

$$\begin{aligned}
|{}_a^bP_q(1)| &= | \frac{[2]_q[3]_q}{2(b-a)} [\int_a^b \theta(t)\, {}^bd_q t + \int_a^b \theta(t)\, {}_ad_q t] - \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2} [\theta(a) + \theta(b)] \\
&- \frac{q(1-q[3]_q)}{2(1-q)^2} [\theta(qa + (1-q)b) + \theta(qb + (1-q)a)] \\
&- \frac{q^2}{2(1-q)^2} [\theta(q^2 a + (1-q^2)b) + \theta(q^2 b + (1-q^2)a)] | \\
&\leq \frac{q^6(b-a)^3}{2[4]_q[5]_q^{\frac{1}{s}}} \{ [([5]_q - [4]_q) |{}^bD_q^3\theta(b)|^s + [4]_q |{}^bD_q^3\theta(a)|^s]^{\frac{1}{s}} \\
&+ [([5]_q - [4]_q) |{}_aD_q^3\theta(a)|^s + [4]_q |{}_aD_q^3\theta(b)|^s]^{\frac{1}{s}} \}.
\end{aligned}$$

Remark 7. *If we take $\lambda = \frac{1}{2}$ in Theorem 6, then the following midpoint-type inequality holds:*

$$\begin{aligned}
|{}_a^bP_q(\frac{1}{2})| &= | \frac{[2]_q[3]_q}{(b-a)} [\int_{\frac{a+b}{2}}^b \theta(t)\, {}^bd_q t + \int_a^{\frac{a+b}{2}} \theta(t)\, {}_ad_q t] - \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2} \theta(\frac{a+b}{2}) \\
&- \frac{q(1-q[3]_q)}{2(1-q)^2} [\theta(\frac{q}{2}a + (1-\frac{q}{2})b) + \theta(\frac{q}{2}b + (1-\frac{q}{2})a)] \\
&- \frac{q^2}{2(1-q)^2} [\theta(\frac{q^2}{2}a + (1-\frac{q^2}{2})b) + \theta(\frac{q^2}{2}b + (1-\frac{q^2}{2})a)] | \\
&\leq \frac{q^6(b-a)^3}{16[2]_q^{\frac{1}{s}}[4]_q[5]_q^{\frac{1}{s}}} \{ [(2[5]_q - [4]_q) |{}^bD_q^3\theta(b)|^s + [4]_q |{}^bD_q^3\theta(a)|^s]^{\frac{1}{s}} \\
&+ [(2[5]_q - [4]_q) |{}_aD_q^3\theta(a)|^s + [4]_q |{}_aD_q^3\theta(b)|^s]^{\frac{1}{s}} \}.
\end{aligned}$$

Example 1. *Let us consider the convex function $\theta : [0,1] \to \mathbb{R}$ defined by $\theta(x) = x^5$ and $\lambda = 1$, which satisfies the conditions of Theorem 4. Using calculus, under these assumptions, we obtain for the left-hand side of inequality (3), the expression,*

$$|\frac{[2]_q[3]_q}{2}[\int_0^1 \theta(t)^1 d_q t + \int_0^1 \theta(t)_0 d_q t] - \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(0) + \theta(1)]$$

$$\begin{aligned}
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(1-q) + \theta(q)] - \frac{q^2}{2(1-q)^2}[\theta(1-q^2) + \theta(q^2)]| \\
&= |\frac{[2]_q[3]_q}{2}[\int_0^1 t^{5}{}^1 d_q t + \int_0^1 t^5{}_0 d_q t] - \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2} \\
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[(1-q)^5 + q^5] - \frac{q^2}{2(1-q)^2}[(1-q^2)^5 + q^{10}]| \\
&= |\frac{[2]_q[3]_q}{2}(1-q)[\sum_0^\infty q^n(1-q^n)^5 + \sum_0^\infty q^{6n}] - \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2} \\
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[(1-q)^5 + q^5] - \frac{q^2}{2(1-q)^2}[(1-q^2)^5 + q^{10}]| \\
&= |\frac{[2]_q[3]_q}{2}[1 - \frac{5}{[2]_q} + \frac{10}{[3]_q} - \frac{10}{[4]_q} + \frac{5}{[5]_q}] - \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2} \\
&- \frac{q(1-q[3]_q)}{2(1-q)^2}[(1-q)^5 + q^5] - \frac{q^2}{2(1-q)^2}[(1-q^2)^5 + q^{10}]|.
\end{aligned}$$

Then, using calculus, the right-hand side of inequality (3) becomes:

$$\frac{q^6}{2[4]_q[5]_q}\{([5]_q - [4]_q)[|{}^1D_q^3\theta(1)| + |{}_0D_q^3\theta(0)|] + [4]_q[|{}^1D_q^3\theta(0)| + |{}_0D_q^3\theta(1)|]\}.$$

Thus, in this case inequality (3) becomes,

$$\begin{aligned}
&|\frac{[2]_q[3]_q}{2}[1 - \frac{5}{[2]_q} + \frac{10}{[3]_q} - \frac{10}{[4]_q} + \frac{5}{[5]_q}] - \frac{1 - q[2]_q[3]_q(1-q)}{2(1-q)^2} \\
&-\frac{q(1-q[3]_q)}{2(1-q)^2}[(1-q)^5 + q^5] - \frac{q^2}{2(1-q)^2}[(1-q^2)^5 + q^{10}]| \\
&\leq \frac{q^6}{2[4]_q[5]_q}\{([5]_q - [4]_q)[|{}^1D_q^3\theta(1)| + |{}_0D_q^3\theta(0)|] + [4]_q[|{}^1D_q^3\theta(0)| + |{}_0D_q^3\theta(1)|]\}. \quad (4)
\end{aligned}$$

On the other hand we have, ${}_0D_q^3\theta(x) = [5]_q[4]_q[3]_q x^2$,; therefore, ${}_0D_q^3\theta(0) = 0$ and ${}_0D_q^3\theta(1) = [5]_q[4]_q[3]_q$. Using this in our case gives,

$${}^1D_q^3\theta(x) = \frac{(q^3x + 1 - q^3)^5 - [3]_q(q^2x + 1 - q^2)^5 + q[3]_q(qx + 1 - q)^5 - q^3x^5}{q^3(1-q)^3(1-x)^3}$$

finding that ${}^1D_q^3\theta(0) = q^9 + 3q^8 + 6q^7 + 4q^6 - 4q^5 - 14q^4 - 11q^3 + q^2 + 8q + 6$ *and* ${}^1D_q^3\theta(1) = 10(q^3 + 2q^2 + 2q + 1)$.

One can see the validity of inequality (4) in Figure 1, where the red line represents the left term of inequality (4) and the right term is represented by the blue line in the figure.

The results could be used to optimise economic geology analyses; for example, in the study of metal ore resources, fossil fuels, and other materials of commercial value.

Here the Matlab R2023a software version was utilized to create Figure 1 and perform partial calculus operations of last two derivatives.

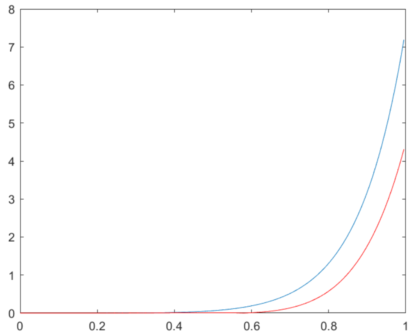

Figure 1. An example for inequality (3) from Theorem 4.

Remark 8. *If we consider the same convex function $\theta : [0,1] \to \mathbb{R}$ defined by $\theta(x) = x^5$ and $\lambda = \frac{1}{2}$, or $\lambda = \frac{1}{[2]_q}$ the analogue inequalities and figures can be analysed as in Example 1.*

Some important applications to the special means of real numbers can be found in [38], where the definitions of the arithmetic mean, harmonic mean and geometric mean are discussed. Similar inequalities can be obtained in our case and we formulate these results for the arithmetic mean.

Remark 9. *If we assume $|{}_aD_q^3\theta| \le M$ and $|{}^bD_q^3\theta| \le M$ in Remark 1, we obtain*

$$
\begin{aligned}
&|\frac{[2]_q[3]_q}{2(b-a)}[\int_a^b \theta(t)^b d_qt + \int_a^b \theta(t)_a d_qt] - \frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}[\theta(a)+\theta(b)] \\
&-\frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(qa+(1-q)b)+\theta(qb+(1-q)a)] \\
&-\frac{q^2}{2(1-q)^2}[\theta(q^2a+(1-q^2)b)+\theta(q^2b+(1-q^2)a)]| \le \frac{Mq^6(b-a)^3}{[4]_q}.
\end{aligned}
$$

Proposition 1. *For $a, b \in \mathbb{R}$, $a < b$ we have,*

$$
\begin{aligned}
&|\frac{[2]_q[3]_q}{b-a}\mathcal{A}(\theta_1,\theta_2) - \frac{1-q[2]_q[3]_q(1-q)}{(1-q)^2}\mathcal{A}(a^5,b^5) \\
&-\frac{q(1-q[3]_q)}{(1-q)^2}\mathcal{A}((qa+(1-q)b)^5,(qb+(1-q)a)^5) \\
&-\frac{q^2}{(1-q)^2}\mathcal{A}((q^2a+(1-q^2)b)^5,(q^2b+(1-q^2)a)^5)| \le \frac{Mq^6(b-a)^3}{[4]_q},
\end{aligned}
$$

where

$$\theta_1 = (1-q)(b-a)\sum_{n=0}^{\infty} q^n(q^na+(1-q^n)b)^5$$

and

$$\theta_2 = (1-q)(b-a)\sum_{n=0}^{\infty} q^n(q^nb+(1-q^n)a)^5.$$

Proof. The inequality from Remark 9 used for the function $\theta(x) = x^5$ leads to the desired result. □

Remark 10. *If we assume* $|_a D_q^3 \theta| \leq M$ *and* $|^b D_q^3 \theta| \leq M$ *in Remark 2, we obtain*

$$\begin{aligned}
&|\frac{[2]_q[3]_q}{(b-a)}[\int_{\frac{a+b}{2}}^{b} \theta(t)^b d_q t + \int_a^{\frac{a+b}{2}} \theta(t)_a d_q t] - \frac{1-q[2]_q[3]_q(1-q)}{2(1-q)^2}\theta(\frac{a+b}{2}) \\
&-\frac{q(1-q[3]_q)}{2(1-q)^2}[\theta(\frac{q}{2}a+(1-\frac{q}{2})b)+\theta(\frac{q}{2}b+(1-\frac{q}{2})a)] \\
&-\frac{q^2}{2(1-q)^2}[\theta(\frac{q}{2}^2 a+(1-\frac{q}{2}^2)b)+\theta(\frac{q}{2}^2 b+(1-\frac{q}{2}^2)a)]| \leq \frac{Mq^6(b-a)^3}{8[4]_q}.
\end{aligned}$$

Proposition 2. *For* $a, b \in \mathbb{R}$, $a < b$ *we have,*

$$\begin{aligned}
&|\frac{2[2]_q[3]_q}{b-a}\mathcal{A}(\theta_1', \theta_2') - \frac{1-q[2]_q[3]_q(1-q)}{(1-q)^2}\mathcal{A}^5(a,b) \\
&-\frac{q(1-q[3]_q)}{(1-q)^2}\mathcal{A}((\frac{q}{2}a+(1-\frac{q}{2})b)^5, (\frac{q}{2}b+(1-\frac{q}{2})a)^5) \\
&-\frac{q^2}{(1-q)^2}\mathcal{A}((\frac{q^2}{2}a+(1-\frac{q^2}{2})b)^5, (\frac{q^2}{2}b+(1-\frac{q^2}{2})a)^5)| \leq \frac{Mq^6(b-a)^3}{8[4]_q},
\end{aligned}$$

where

$$\theta_1' = \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n (q^n \mathcal{A}(a,b) + (1-q^n)b)^5$$

and

$$\theta_2' = \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n (q^n \mathcal{A}(a,b) + (1-q^n)a)^5.$$

Proof. The inequality from Remark 10 used for the function $\theta(x) = x^5$ leads to the desired result. □

Proposition 3. *For* $a, b \in \mathbb{R}$, $a < b$ *we have,*

$$\begin{aligned}
&|\frac{[2]_q[3]_q}{2(b-a)}\mathcal{A}(\theta_1, \theta_2) - \frac{1-q[2]_q[3]_q(1-q)}{(1-q)^2}\mathcal{A}(a^5, b^5) \\
&-\frac{q(1-q[3]_q)}{(1-q)^2}\mathcal{A}((qa+(1-q)b)^5, (qb+(1-q)a)^5) \\
&-\frac{q^2}{(1-q)^2}\mathcal{A}((q^2 a+(1-q^2)b)^5, (q^2 b+(1-q^2)a)^5)| \leq \frac{Mq^6(b-a)^3}{[3r+1]_q^{\frac{1}{r}}},
\end{aligned}$$

where

$$\theta_1 = (1-q)(b-a)\sum_{n=0}^{\infty} q^n (q^n a + (1-q^n)b)^5$$

and

$$\theta_2 = (1-q)(b-a)\sum_{n=0}^{\infty} q^n (q^n b + (1-q^n)a)^5.$$

Proof. The inequality from Remark 4 used for the function $\theta(x) = x^5$ leads to the desired result. □

Proposition 4. *For $a, b \in \mathbb{R}$, $a < b$ we have,*

$$\begin{aligned}
&\left|\frac{2[2]_q[3]_q}{b-a}\mathcal{A}(\theta_1', \theta_2') - \frac{1-q[2]_q[3]_q(1-q)}{(1-q)^2}\mathcal{A}^5(a,b)\right. \\
&-\frac{q(1-q[3]_q)}{(1-q)^2}\mathcal{A}((\frac{q}{2}a + (1-\frac{q}{2})b)^5, (\frac{q}{2}b + (1-\frac{q}{2})a)^5) \\
&\left.-\frac{q^2}{(1-q)^2}\mathcal{A}((\frac{q^2}{2}a + (1-\frac{q^2}{2})b)^5, (\frac{q^2}{2}b + (1-\frac{q^2}{2})a)^5)\right| \leq \frac{Mq^6(b-a)^3 2^{\frac{1}{s}}}{8[3r+1]_q^{\frac{1}{r}}},
\end{aligned}$$

where

$$\theta_1' = \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n (q^n \mathcal{A}(a,b) + (1-q^n)b)^5$$

and

$$\theta_2' = \frac{(1-q)(b-a)}{2}\sum_{n=0}^{\infty} q^n (q^n \mathcal{A}(a,b) + (1-q^n)a)^5.$$

Proof. The inequality from Remark 5 for function $\theta(x) = x^5$ leads to the desired result. □

Analogue inequalities can be obtained if the function $\theta(x) = x^5$ is chosen in Remarks 6 and 7.

4. Discussion and Conclusions

In this paper, several new parametrized q-H–H-type integral inequalities were given for functions whose third left and right q-derivatives are convex. Some basic inequalities, such as quantum Holder's inequality and power mean inequality, were used to obtain new bounds and an auxiliary quantum lemma was utilized in the demonstrations. Some consequences and an example were presented to illustrate the generated results. Using Matlab, Figure 1 confirms the results obtained in Section 3. Some interesting applications to special means have been presented. Many consequences arise in certain special cases of the parameter and an interesting problem to study may be to use these methods to prove q-fractional inequalities and similar inequalities for different kinds of convexities. The present study could be used to better guide the exploration of mineral resources. In our opinion this research could be very useful in structural geology, stratigraphic optimization, economic exploitation of mineral deposits, and building materials such as stones or gypsum.

We are confident that the ideas and techniques investigated here will inspire further studies on functional analysis and statistics.

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