



Article Approximate Controllability of Fractional Stochastic Evolution Inclusions with Non-Local Conditions

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Abstract: This article investigates the approximate controllability of non-linear fractional stochastic differential inclusions with non-local conditions. We establish a set of sufficient conditions for their approximate controllability and provide results in terms of controllability for the fractional stochastic control system. Our approach relies on using fractional calculus and the fixed-point theorem for multiple-valued operators. Finally, we present an illustrative example to support our findings.

Keywords: approximate controllability; fractional calculus; fractional evolution inclusions; stochastic differential inclusions; multi-valued maps

MSC: 34K37; 34B15

1. Introduction

Fractional calculus is a branch of mathematics that deals with the generalization of differentiation and integration to non-integer orders. Instead of dealing with whole numbers, it involves fractions for which the order of differentiation or integration can be a non-integer value. This theory has applications in various fields, such as physics, engineering, and finance, where it can be used to describe complex systems that exhibit fractional behavior including fractals and long-range dependence. Fractional calculus is a relatively new and developing field that is still being explored and has the potential to offer new insights into the behavior of complex systems [1] (Kilbas et al. [2], Zhou [3,4]).

Fractional differential inclusions play an important role in several fields, such as physics [5], mechanics, and engineering. The reader should refer to the monograph [6] and its references for information on the fundamentals of fractional differential equations and exceptions. Many recent articles have investigated mild solutions and controllability challenges for various types of differential inclusions; see [7] and the citations therein. We direct the reader to [8–10] for one method of solving fractional differential equations in impulsive stochastic functional differential systems with state-dependent delay in Hilbert spaces. In pharmacotherapy, some of the kinetics of evolution processes are not adequately captured by the effect of instantaneous signals. Consider a person's circulatory equilibrium; when drugs reach the bloodstream, they are absorbed gradually and continuously by the system. Hernández and O'Regan [11,12] and Hernández, Pierri, and O'Regan [13] introduced a new class of differential equations with non-instantaneous impulses. An updated model was introduced by Wang and Feckan [14].

The extensive development of the controllability concept for abstract regular and non-linear controlled systems in limited and unlimited dimensional areas was described in [15–17]. Deterministic models also change due to environmental noise, which is random and appears to be random. Therefore, it is crucial to change from a deterministic to a stochastic system to enhance model performance. Stochastic differentiated equations are



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Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https:// creativecommons.org/licenses/by/ 4.0/). used in the design and analysis of problems in mechanical, electronic, and control engineering and other physical sciences. An impulse neutral stochastic functional differentiation system with state-dependent delay has been shown to be approximately controllable in Hilbert fields [18]. However, within the past 20 years, mathematicians, scientists, and engineers have become fascinated with fractional differential equations because of their applications in numerous fields of research and industry. Differential models of an arbitrary order may be used to describe the storage and heredity properties of a number of crucial materials and processes. A mild solution to a class of impulsive fractional partial semi-linear differential equations was proved by Shu et al. [19] via solution operators and classical fixed-point theorems. For the category of fractional neutrality stochastic integrodifferential problems with indefinite delay, Cui and Yan [20] looked into the possibility of mild solutions.

Furthermore, the issue of the existence of fractional differential inclusion methods has been investigated by a number of authors for various types of dynamical processes, since fractional differential inclusions have been used in the quantitative modeling of several problems in both the economics and optimal control fields [21]. Yan and Zhang [22] derived a collection of sufficient conditions for the presence of solutions to spontaneous fractional partial neutral stochastic integro-differential inclusions in order to use the non-linear substitute of the Leray–Schauder type for multi-valued maps based on O'Regan and the characteristics of the solution operator.

Recently, Duan et al. [23] used the fixed-point theory of condensed multi-valued mapping to study the exact stability of non-linear stochastic impulsive evolution differential inclusions with infinite delay. The approximate controllability of fractional stochastic differential inclusions with non-local conditions has been studied in recent years. Balasubramaniam et al. [24] investigated this issue in a Hilbert space with non-local circumstances for fractional impulse integro-differential processes. Yan [25] demonstrated the approximate controllability of control systems driven by a class of partial fractional neutrality integro-differential inclusions with state-dependent delay. Sakthivel et al. [26,27] used fixed-point techniques and fractional calculus to investigate the approximate controllability of fractional deterministic and stochastic differential systems. Abuasbeh et al. [28] explored several existence and controllability theories for the Caputo order $q \in (1,2)$ of delay and fractional functional integro-evolution equations (FFIEEs). Moumen et al. [29] discussed the approximate controllability of a class of fractional stochastic evolution equations (FSEEs) in the Hilbert space using the Hilfer derivative. However, to the best of our knowledge, the issue of approximate controllability for fractional stochastic differential inclusions with non-local conditions has not yet been examined. In this article, we address this gap by studying the approximate controllability of stochastic differential inclusions with non-local conditions.

$$\begin{cases} {}^{c}D^{\beta}x(\zeta) \in Ax(\zeta) + Bu(\zeta) + F(\zeta, x(\zeta)) + G(\zeta, x(\zeta)) \frac{d\omega(\zeta)}{d\zeta}, \ \zeta \in J = [0, b], 1 < \beta < 2, \\ x(0) + h(x) = x_0, \ x'(0) = x_1. \end{cases}$$
(1)

1 (7)

where ${}^{c}D^{\beta}x(\zeta)$ represents the Caputo fractional derivative of order $\beta \in (1, 2)$, and *A* is the infinitesimal generator defined on *H*, a separable Hilbert space with the inner product (\cdot, \cdot) and norm $|| \cdot ||$. Let *k* be another separable Hilbert space with the inner product $(\cdot, \cdot)_{k}$. In 1960, Kalman initially brought forth the idea of controllability. The existence of a controlling function that directs a system's reaction from its initial state to ultimate state was demonstrated by this concept, which was significant in the research on control systems. Condensing mappings and compact semigroup concepts are typically applied to investigate the management of evolution inclusions. If the compactness of the semigroup is not taken for granted, one should use the fixed-point theorem while the appropriate operator is neither compact nor condensing. Zhou et al. [30,31] explored the controllability of fractional evolution inclusions/equations without presuming that the semigroup was compacted. The stability of fractional evolution equations under weak topology conditions was only briefly discussed in their research.

2. Preliminaries

In this section, we provide some definitions and preliminary results that will be useful for understanding this paper. Let a filtered complete probability space $(w(\zeta), f, \{f\}_{\zeta \ge 0}, P)$ be a fulfilled condition, which means that the filtration is a continuous non-decreasing family and f_0 contains all P-null sets. Let a Q-Weiner process $\omega^* = \{\omega^*(\zeta)\}$ be defined on $(w(\zeta), f, \{f\}_{\zeta \ge 0}, P)$. We assume that $e_k, k \ge 1$, and there is a complete ortho-normal system in K. Let the space of all Hilbert–Schmidt operators $L_2^0 = L_2(Q^{\frac{1}{2}}K, C)$ from $Q^{\frac{1}{2}}K$ to C. Let the Banach space be $L_2(\omega, f, C)$ for all f_b measurable square integrable random variables in C. Let \tilde{C} be indicated by $C([0, b], L_2(\omega, f, C))$ and the Banach space of all the continuous functions from [0, b] to $L_2(\omega, f, C)$ fulfil the conditions $\sup_{\zeta \in G} F ||x(\zeta)||^2 \le \infty$.

Now, we present a few basic definitions and results for multiple-valued maps. For further explanations regarding multiple-valued maps, see the books of Deimling [32] and Hu and Papageorgious [33]. We use the notation P(C) for the family of all subsets of C and denote

$$P_{bd}(C) = \{ \mathcal{Y} \in P(C) : \mathcal{Y} \text{ is bounded} \}, P_{cl}(C) = \{ \mathcal{Y} \in P(C) : \mathcal{Y} \text{ is closed} \},$$

$$P_{cv}(C) = \{ \mathcal{Y} \in P(C) : \mathcal{Y} \text{ is convex} \}, P_{cp}(C) = \{ \mathcal{Y} \in P(C) : \mathcal{Y} \text{ is compact} \}.$$

A multi-valued map $f : C \to P(C)$ has a closed valued if f(x) is closed for all $x \in C$. f is bounded on bounded sets if $f(\tilde{C}) = \bigcup_{x \in \tilde{C}} f(x)$ is bounded in C, i.e., $\sup_{x \in \tilde{C}} \{\sup\{\|\mathcal{Y}\| : \mathcal{Y} \in f(x)\}\} \le \infty$.

Definition 1 ([34]). The multi-valued map $f : G \times C \rightarrow P_{bd,cl,cv}(C)$ is said to be l^2 – Caratheodory if

(*i*) For every $v \in C, \zeta \mapsto E(\zeta, v)$ is measurable; (*ii*) For almost all $\zeta \in G, \zeta \mapsto F(\zeta, v)$ is u.s.c.; (*iii*) For each $n > 0, \zeta$ there exists $h_n \in l^1(G, L^+)$ such that

$$||E(\zeta, v)||^2 = \sup_{\zeta \in F(\zeta, v)} F||\check{f}||^2 \le h_n(\zeta), \text{ for all } ||v||_c^2 \le n \text{ and for a.e. } \zeta \in G.$$

Lemma 1 ([32]). Let C be a Hilbert space on a compact real interval G. Consider that J is an l^2 – Caratheodory multiple-valued map. For each $u \in \tilde{C}$, let Ψ be a linear continuous map from $l^2(G, C)$ to $\tilde{C}(G, C)$, *i.e.*, a closed operator

$$\Psi_{\circ}R_{J}: \tilde{C}(G,C) \to P_{cp,cv}(\tilde{C}(G,C)), x \mapsto (\Psi_{\circ}R_{J})(x) = \Psi(R_{J,x}),$$

in $\tilde{C}(G, C) \times C(G, C)$. The set $R_{J,x} = \{g \in l^2(l(K, C)) : g(\zeta) \in J(\zeta, u(\zeta)), \text{ for a.e. } \zeta \in G\}$ is non-empty.

Definition 2 ([35]). *For a function* $f : [0, \infty) \to S$ *, the Caputo derivative of order* β *can be stated as*

$${}^{c}D^{\beta}f(\zeta) = \frac{1}{\Gamma(m-\beta)} \int_{0}^{\zeta} (\zeta-\varsigma)^{m-\beta-1} f^{(m)}(\varsigma) d\varsigma = I^{m-\beta} f^{m}(\zeta),$$

for $m - 1 < \beta < m, m \in M$. If $0 < \beta \leq 1$, then

$$^{C}D^{\beta}f(\zeta) = rac{1}{\Gamma(1-\beta)}\int_{0}^{\zeta}(\zeta-\zeta)^{-\beta}f'(\zeta)d\zeta$$

For order $\beta > 0$, the Laplace transform of the Caputo derivative is given below

$$L\{{}^{c}D^{\beta}f(\zeta):\lambda\} = \lambda^{\beta}f(\lambda) - \Sigma_{k=0}^{m-1}\lambda^{\beta-k-1}f^{k}(0); m-1 < \beta < m.$$

3. Main Results

In this section, we initially establish the mild solution for the system in (1). With the hypothesis that the controllability operator has an induced inverse on a fraction space, we specifically transformed the controllability problem into a fixed-point theorem. Furthermore, we demonstrated that the approximate controllability of (1) was indicated by the approximate controllability of the associated linear system, subject to certain assumptions. The following hypotheses were required in order to verify the results. **Set of Assumptions**

Hypothesis 1. The operators $T_q(\zeta)$ and $R_q(\zeta)$ are compact when $q \in (0, 1)$.

Hypothesis 2. The multiple-valued map $E : J \times C \rightarrow P_{bd,cl,cv}(C)$ is an l^2 – Caratheodory function satisfying the following conditions:

(*i*) For every $\zeta \in G$, the function $E(\zeta, .) : C \to P_{bd,cl,cv}(C)$ is u.s.c., and for every $x \in C$, the function F(., x) is measurable. Additionally, for every fixed $x \in \tilde{C}$, the set

$$R_{E,x} = \{\check{f} \in l^2(\omega, C) : f(\zeta) \in F(\zeta, x)\}$$

is non-empty.

(ii) For every non-negative k, there exists a non-negative function $N_f(k)$ that is unbounded on K such that

 $\sup_{E||x||^2 \le K} ||F(\zeta, x)||^2 \le N_f(k).$

where $||E(\zeta, x)||^2 = \sup_{\check{f} \in E(\zeta, x)} F||f||^2$.

Hypothesis 3. The multiple-valued map $J : G \times C \rightarrow P_{bd,cl,cv}(l(K,C))$ is a l^2 – Caratheodory function satisfying the following conditions:

(*i*) For each $\zeta \in G$, the function $J(\zeta, \cdot) : C \to P_{bd,cl,cv}(l(K,C))$ is u.c.s.; for every $x \in C$, the function $G(\cdot, x)$ is measurable; and for each fixed $x \in C$, there exists the set

$$R_{J,x} = \{g \in l^2(l(K,H)) : g(\zeta) \in J(\zeta,x), \}$$

(ii) For every non-negative k, there exists a non-negative function $N_g(k)$ that is unbounded on K in such a way that

$$\sup_{F||x||^2 \le k} ||J(\zeta, x)||^2 \le N_g(k)$$

Hypothesis 4. The function $h : \tilde{C} \to C$ is completely continuous, and there exist the positive constants μ_1 and μ_2 such that

$$||h(x)||^2 \le \mu_1 ||x||_{\tilde{C}}^2 + \mu_2.$$

Hypothesis 5. *There exists a real number* r > 0 *such that*

$$\frac{L_1 + 4M_S^2 \frac{b^{2q-1}}{2q-1} [bN_f(r) + Tr(Q)N_g(r)](1 + \frac{4}{\alpha^2} N_B^4 M_S^4 \frac{b^{4q-2}}{4q-3})}{1 - L_2} < r$$

where

$$L_{1} = 8M_{T}^{2}(F||x_{0}||^{2} + \mu_{2}) + \frac{4}{\alpha^{2}}N_{B}^{4}M_{S}^{4}\frac{b^{4q-2}}{4q-3}\{8||Fx_{b}||^{2} + 8\int_{0}^{b}F||\phi(\varsigma)||_{l_{2}^{0}}^{2}d\varsigma + 8M_{T}^{2}(F||x_{0}||^{2} + \mu_{2})\}$$

$$L_{2} = 8M_{T}^{2}\mu_{1}\left(1 + \frac{4}{\alpha^{2}}N_{B}^{4}M_{S}^{4}\frac{b^{4q-2}}{4q-3}\right).$$

In this section, we investigate a mild solution for a control problem monitored by non-linear fractional stochastic evolution inclusions with non-local conditions. The system (1) is equivalent to the following integral equation:

$$x(\zeta) = x(0) + x'(0)\zeta + \frac{1}{\Gamma(\beta)} \int_0^{\zeta} (\zeta - \varsigma)^{\beta - 1} \left[Ax(\varsigma) + Bu(\varsigma) + F(\varsigma, x(\varsigma)) \right] d\varsigma$$

+
$$\frac{1}{\Gamma(\beta)} \int_0^{\zeta} (\zeta - \varsigma)^{\beta - 1} G(\varsigma, x(\varsigma) d\omega(\varsigma).$$
(2)

We introduce Mainardi's Wright-type function $M_q(\omega)$, $M_r(\omega) = \frac{1}{2} o(\omega^{-\frac{1}{2}}) \ge 0$ $q \in (0, 1)$

$$M_{q}(\omega) = \frac{1}{q\omega^{(1+\frac{1}{q})}} \omega(\omega^{z}) \ge 0, q \in (0, 1),$$

$$\omega_{q}(\omega) = \frac{1}{\pi} \sum_{m=1}^{\infty} (-1)^{m-1} (\omega)^{-qm-1} \frac{\sigma(mq+1)}{n!} sin(m\pi q).$$

Lemma 2. *If integral Equation (2) holds, then for* $\zeta \in J$ *,*

$$\begin{aligned} x(\zeta) &= T_q(\zeta)(x_0 - h(x)) + S_q(\zeta)x_1 + \frac{1}{\Gamma(q)} \int_0^{\zeta} R_q(\zeta)(\zeta - \varsigma)^{q-1} Bu(\varsigma) d\varsigma + \frac{1}{\Gamma(q)} \int_0^{\zeta} R_q(\zeta)(\zeta - \varsigma)^{q-1} f(\varsigma, x(\varsigma)) d\varsigma \\ &+ \frac{1}{\Gamma(q)} \int_0^{\zeta} R_q(\zeta)(\zeta - \varsigma)^{q-1} g(\varsigma, x(\varsigma)) d\omega(\varsigma). \end{aligned}$$

where

$$T_q(\zeta) = \int_0^\infty M_q(\omega) c(\zeta^q \omega) d\omega$$
$$S_q(\zeta) = \int_0^\zeta T_q(\zeta) d\zeta$$
$$R_q(\zeta) = \int_0^\infty q \omega M_q \omega \zeta(\zeta^q \omega) d\omega$$

Proof. Let $\lambda > 0$; the Laplace transform is given as follows:

$$\mu(\lambda) = \frac{1}{\lambda}(x_0 - h(x)) + \frac{1}{\lambda^2}x_1 + \frac{1}{\lambda^\beta}A\mu(\lambda) + \frac{1}{\lambda^\beta}\nu(\lambda),$$

where $\mu(\lambda) = \int_0^\infty e^{-\lambda \zeta} x(\zeta) d\zeta$. $\nu(\lambda) = \int_0^\infty e^{-\lambda \zeta} (Bu(\zeta) + f(\zeta, x(\zeta)) + g(\zeta, x(\zeta))(\zeta) d\omega(\zeta)) d\zeta$

This implies that, if $\lambda > 0$,

$$\begin{split} \mu(\lambda) &= \lambda^{\frac{\beta}{2}-1} \int_0^\infty e^{-\lambda^{\frac{\beta}{2}\zeta}} c(\zeta) \big(x(0) - h(x) \big) d\zeta + \lambda^{-1} \lambda^{\frac{\beta}{2}-1} \int_0^\infty e^{-\lambda^{\frac{\beta}{2}\zeta}} c(\zeta) x_1 d\zeta \\ &+ \int_0^\infty e^{-\lambda^{\frac{\beta}{2}\zeta}} s(\zeta) \mu(\lambda) d\zeta + \int_0^\infty e^{-\lambda^{\frac{\beta}{2}\zeta}} s(\zeta) \nu(\lambda) d\zeta. \end{split}$$

Consider the one-sided probability density function whose Laplace transform is

$$\int_0^\infty e^{-\lambda^\omega} \phi_q(\omega) d\omega = e^{-\lambda^q}.$$
(3)

Let $q = \frac{\beta}{2}$ for $q \in (\frac{1}{2}, 1)$; using Equation (3), we have

$$\begin{split} \lambda^{q-1} \int_{0}^{\infty} e^{-\lambda^{q\zeta}} c(\zeta)(x(0) - h(x)) d\zeta &= \int_{0}^{\infty} q(\lambda\zeta)^{q-1} e^{-(\lambda\zeta)} c(\zeta^{q})(x(0) - h(x)) d\zeta \\ &= \int_{0}^{\infty} (-\frac{1}{\lambda}) \frac{d}{d\zeta} \bigg(\int_{0}^{\infty} e^{-\lambda\zeta} \phi(\omega) d\omega \bigg) c(\zeta^{q})(x(0) - h(x)) d\zeta \\ &= \int_{0}^{\infty} \int_{0}^{\infty} \omega \phi_{q}(\omega) e^{-\lambda\zeta\omega} c(\zeta^{q})(x(0) - h(x)) d\omega d\zeta \\ &= \int_{0}^{\infty} e^{-\lambda\zeta} \big[\int_{0}^{\infty} \phi_{q}(\omega) c(\frac{\zeta^{q}}{\omega^{q}})(x(0) - h(x)) d\omega \big] d\zeta \\ &= L \big[\int_{0}^{\infty} M_{q}(\omega) c(\zeta^{q}\omega)(x(0) - h(x)) d\omega \big] d\zeta \\ &= L \big[T_{q}(\zeta)(x(0) - h(x)) \big] (\lambda). \end{split}$$

since $L[H_1(t)](\lambda) = \lambda^{-1}$.

Using the Laplace convolution theorem, we obtain

$$\lambda^{-1}\lambda^{q-1} \int_0^\infty e^{-\lambda^{q\zeta}} c(\zeta) x_1 d\zeta = L[H_1(\zeta)](\lambda) L[T_q(\zeta) x_1](\lambda)$$
$$= L[(H_1 * T_q)(\zeta) x_1](\lambda).$$
(5)

Similarly,

$$\int_{0}^{\infty} e^{-\lambda q\zeta} s(\zeta) \mu(\lambda) d\zeta = \int_{0}^{\infty} q\zeta^{q-1} e^{(-\lambda q\zeta)} s(\zeta^{q}) \mu(\lambda) d\zeta$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} q\zeta^{q-\zeta} \phi_{q}(\omega) e^{-\lambda q\zeta} s(\zeta^{q}) \mu(\lambda) d\omega d\zeta$$

$$= L \left[\int_{0}^{\infty} q\zeta^{q-1} M_{q}(\omega) s(\zeta^{q}\omega) d\omega \right] (\lambda) L \left[F(\zeta, x(\zeta)) + Gx(\zeta) + Bu(\zeta) \right] (\lambda)$$

$$= L \left[\int_{0}^{\zeta} (\zeta - \zeta)^{q-1} S_{q}(\zeta - \zeta) (F(\zeta, x(\zeta)) + Gx(\zeta)) + Bu(\zeta) \right].$$
(6)

Using Equations (4)–(6) in (2), we get

$$\begin{aligned} x(\zeta) &= T_q(\zeta)(x_0 - h(x)) + S_q(\zeta)x_1 + \frac{1}{\Gamma(q)} \int_0^{\zeta} R_q(\zeta)(\zeta - \zeta)^{q-1} Bu(\zeta)d\zeta + \frac{1}{\Gamma(q)} \int_0^{\zeta} R_q(\zeta)(\zeta - \zeta)^{q-1} f(\zeta, x(\zeta))d\zeta \\ &+ \frac{1}{\Gamma(q)} \int_0^{\zeta} R_q(\zeta)(\zeta - \zeta)^{q-1} g(\zeta, x(\zeta))d\omega(\zeta). \end{aligned}$$

4. Approximate Controllability Theorems

Theorem 1. Assume that (H1) - (H5) are satisfied; then, the fractional control system (1) has a mild solution on *G*.

Proof. For $\alpha > 0$, we define the multi-valued operator $\Phi : \tilde{C} \to P(\tilde{C})$ by

$$\begin{split} \Phi(x) &= \{ z \in \tilde{C} : z(\zeta) = T_q(\zeta - \varsigma)g_i(\zeta_i - \varsigma) + S_q(\zeta - \varsigma)x_1 + \int_0^{\zeta} R_q(\zeta - \varsigma)Bx_u^{\alpha}(\varsigma)d\varsigma \\ &+ \int_0^{\zeta} R_q(\zeta - \varsigma)f(\varsigma)d\varsigma + \int_0^{\zeta} R_q(\zeta - \varsigma)g(\varsigma)d\omega^*(\varsigma) \}. \end{split}$$

The operator Φ will be shown as having a fixed point. There are numerous steps to the verification.

Step 1 : For $\alpha > 0$, $\Phi(u)$ is convex for each $u \in \tilde{C}$. In fact, if $z_1, z_2 \in \Phi(u)$, then there exists $f_1, f_2 \in R_{F,u}$ and $g_1, g_2 \in R_{J,u}$ such that for each $\zeta \in G$, we have

$$\begin{aligned} z_{i}(\zeta) &= T_{q}(\zeta-\varsigma)g_{i}(\zeta_{i}-\varsigma) + S_{q}(\zeta-\varsigma)x_{1} + \int_{0}^{\zeta} R_{q}(\zeta-\varsigma)f_{i}(\varsigma)d\varsigma + \int_{0}^{\zeta} R_{q}(\zeta-\varsigma)g_{i}(\varsigma)d\omega^{*}(\varsigma) \\ &+ \int_{0}^{\zeta} R_{q}(\zeta-\xi)B\left\{B^{*}R_{q}^{*}(b-\xi)\left[(\alpha I + \Psi_{0}^{b})^{-1}[Eu_{b} - T_{q}(b)(u_{0} - h(u))] + \int_{0}^{\zeta} (\alpha I + \Psi_{\varsigma}^{b})^{-1}\Phi(\varsigma)d\omega(\varsigma)\right] \\ &- B^{*}R_{q}^{*}(b-\xi)\int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q}(b-\varsigma)f_{i}(\varsigma)d\varsigma \\ &- B^{*}R_{q}^{*}(b-\xi) - \int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}\zeta R_{q}(\zeta-\varsigma)g_{i}(\varsigma)d\omega^{*}(\varsigma)\right\}d\xi. \end{aligned}$$

Let $\lambda \in [0, 1]$; then, for each $\zeta \in G$, we have

$$\begin{split} \lambda z_{1}(\zeta) + (1-\lambda)z_{2}(\zeta) &= T_{q}(\zeta)(x_{0}-h(x)) + S_{q}(\zeta-\zeta)x_{1} \\ &+ \int_{0}^{\zeta} T_{q}(\zeta-\zeta)[\lambda f_{1}(\varsigma) + (1-\lambda)f_{2}]d\varsigma + \int_{0}^{\zeta} R_{q}(\zeta-\zeta)[\lambda g_{1}(\varsigma) \\ &+ (1-\lambda)g_{2}(\varsigma)]d\omega^{*}(\varsigma) + \int_{0}^{\zeta} R_{q}(\zeta-\zeta)B\{B^{*}R_{q}^{*}(b-\zeta)[(\alpha I + \Psi_{0}^{b})^{-1}[Eu_{b} - T_{q}(b) \\ &\times (u_{0} - h(u))] + \int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}\Phi(\varsigma)d\omega(\varsigma)] - B^{*}R_{q}^{*}(b-\zeta)\int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q} \\ &\times (b-\varsigma)[\lambda f_{1}(\varsigma) + (1-\lambda)f_{2}(\varsigma)]d\varsigma. \end{split}$$

Since $R_{E,x}$ and $R_{J,x}$ are convex, $\lambda f_1 + (1 - \lambda)f_2 \in R_{E,x}$, $\lambda g_1 + (1 - \lambda)g_2 \in R_{G,x}$. Then, $\lambda z_1 + (1 - \lambda) z_2 \in \phi(x).$

Step 2: ϕ_2 maps bounded sets into closed sets in \tilde{C} .

Consider a set $B_k = \{x \in \tilde{C} : ||x||_{\tilde{C}}^2 \le k, 0 \le \zeta \le b\}$, where *k* is a non-negative constant. B_k is definitely bounded and convex. There exists a positive constant *L* such that for each $z \in \phi, x \in B_k$, one has $F ||z(\zeta)||^2 \leq L$.

Let $z \in \phi_2$, $x \in B_k$. Then, there exist $\check{f} \in R_{E,x}$ and $g \in R_{J,x}$ such that for each $\zeta \in G$,

$$z(\zeta) = T_q(\zeta)g_i(\varsigma, x(\zeta_i - h(x)) + S_q(\zeta))x_1 + \int_0^{\zeta} R_q(\zeta - \varsigma)Bu_x^{\alpha}(\varsigma)d\varsigma + \int_0^{\zeta} R_q(\zeta - \varsigma)f(\varsigma)d\varsigma + \int_0^{\zeta} R_q(\zeta - \varsigma)g(\varsigma)d\omega^*(\varsigma)\}.$$

Next, we have

$$\begin{split} F||U_{x}^{\alpha}(\zeta)||^{2} &\leq \frac{1}{\alpha^{2}}N_{B}^{2}M_{S}^{2}(b-\zeta)^{2q-2}\left\{4||Fx_{b}+\int_{0}^{b}\Phi(\zeta)d\omega^{*}(\zeta)||^{2}+4F||T_{q}(b)(x_{0}-h(x))||^{2} \\ &+ 4F||\int_{0}^{b}R_{q}(b-\zeta)f(\zeta)d\zeta||^{2}+4F||\int_{0}^{b}R_{q}(b-\zeta)g(\zeta)d\omega^{*}(\zeta)||^{2}\right\} \\ F||U_{x}^{\alpha}(\zeta)||^{2} &\leq \frac{4}{\alpha^{2}}N_{B}^{2}M_{S}^{2}(b-\zeta)^{2q-2}\left\{2||F_{b}||^{2}||+2\int_{0}^{b}F||\Phi(\zeta)||_{l_{2}^{0}}^{2}d\zeta+M_{T}^{2}(2F||x_{0}||^{2} \\ &+ 2F||h(x)||^{2})+bF\int_{0}^{b}||R_{q}(b-\zeta)f(\zeta)||^{2}+Tr(Q)F\int_{0}^{b}||R_{q}(b-\zeta)g(\zeta)||^{2}d\zeta\right\} \\ F||U_{x}^{\alpha}(\zeta)||^{2} &\leq \frac{4}{\alpha^{2}}N_{B}^{2}M_{S}^{2}\left\{2||Ex_{b}||^{2}+2\int_{0}^{b}F||\Phi(\zeta)||_{l_{2}^{0}}^{2}d\zeta+2M_{T}^{2}(F||x_{0}||^{2}+\mu_{1}k+\mu_{2}) \\ &+ bM_{S}^{2}\frac{b^{2q-1}}{2q-1}M_{f}(k)+Tr(Q)M_{S}^{2}M_{g}(k)\frac{b^{2q-1}}{2q-1}\right\} \\ F||U_{x}^{\alpha}(\zeta)||^{2} &\leq (b-\zeta)^{2q-2}N_{U}, \end{split}$$

$$F||U_x^{\alpha}(\zeta)||^2 \leq (b-\zeta)^{2d}$$

where

$$\begin{split} N_{U} &= \frac{4}{\alpha^{2}} N_{B}^{2} M_{S}^{2} \bigg\{ 2||Fx_{b}||^{2} + 2 \int_{0}^{b} F||\Phi(\varsigma)||_{l_{2}^{0}}^{2} d\varsigma + 2M_{T}^{2}(F||x_{0}||^{2} + \mu_{1}k + \mu_{2}) + bM_{S}^{2} \frac{b^{2q-1}}{2q-1} M_{f}(k) \\ &+ Tr(Q) M_{S}^{2} \frac{b^{2q-1}}{2q-1} M_{g}(k) \bigg\}. \end{split}$$

Now, we have

$$\begin{split} F||z(\zeta)||^{2} &\leq 4F||T_{q}(\zeta)(x_{0}-h(x))||^{2}+4F||S_{q}(\zeta-\varsigma)x_{1}||^{2}+4F||\int_{0}^{\zeta}R_{q}(\zeta-\varsigma)Bx_{x}^{\alpha}(\varsigma)d\varsigma||^{2} \\ &+ 4F||\int_{0}^{\zeta}R_{q}(\zeta-\varsigma)f(\varsigma)d\varsigma||^{2}+4F||\int_{0}^{\zeta}R_{q}(\zeta-\varsigma)g(\varsigma)d\omega^{*}(\varsigma)||^{2} \\ F||z(\zeta)||^{2} &\leq 8M_{T}^{2}(F||x_{0}||^{2}+\mu_{1}k+\mu_{2})+4bN_{B}^{2}M_{S}^{2}\frac{b^{4q-3}}{4q-3}M_{U}+4bM_{S}^{2}\frac{b^{2q-1}}{2q-1}M_{f}(k) \\ &+ 4Tr(Q)M_{S}^{2}\frac{b^{2q-1}}{2q-1}M_{g}(k) \end{split}$$

 $F||z(\zeta)||^2 = l.$

Hence, Φ maps bounded sets into a closed set in \tilde{C} .

Step 3: Φ maps bounded sets into equi-continuous sets of \tilde{C} . Let $0 < \tau_1 < \tau_2 \leq b$, $\epsilon > 0$. For each $z \in \Phi(x)$ and $x \in B_k$, there exists $f \in R_{E,x}$ and $g \in R_{G,x}$ such that

$$\begin{split} z(\zeta) &= T_q(\zeta)(x_0 - h(x)) + S_q(\zeta))x_1 + \int_0^{\zeta} R_q(\zeta - \varsigma)Bu_x^{\kappa}(\varsigma)d\varsigma + \int_0^{\zeta} R_q(\zeta - \varsigma)f(\varsigma)d\varsigma \\ &+ \int_0^{\zeta} (\zeta - \varsigma)R_q(\zeta - \varsigma)g(\varsigma)d\omega^{*}(\varsigma). \\ F||z(\tau_2) - z(\tau_1)||^2 &\leq 10F||[T_q(\tau_2) - T_q(\tau_1)](x_0 - h(x))||^2 + 10F||S_q(\tau_2) - S_q(\tau_1)x_1||^2 + 10F||\int_0^{\tau_1 - \epsilon} \\ &\times [R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)]Bx_u^{\kappa}(\varsigma)d\varsigma||^2 + 10F||\int_{\tau_1 - \epsilon}^{\tau_1} [R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)]Bx_u^{\kappa}(\varsigma)d\varsigma||^2 \\ &+ 10F||\int_{\tau_1 - \epsilon}^{\tau_2} [R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)]f(\varsigma)d\varsigma||^2 + 10F||\int_{\tau_1}^{\tau_1 - \epsilon} [R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)]f(\varsigma)d\varsigma||^2 \\ &+ 10F||\int_{\tau_1 - \epsilon}^{\tau_1 - \epsilon} [R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)]g(\varsigma)d\omega^{*}(\varsigma)||^2 + 10F||\int_{\tau_1}^{\tau_2} R_q(\tau_2 - \varsigma)f(\varsigma)d\varsigma||^2 \\ &+ 10F||\int_0^{\tau_1 - \epsilon} [R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)]g(\varsigma)d\omega^{*}(\varsigma)||^2 + 10F||\int_{\tau_1 - \epsilon}^{\tau_2} [R_q(\tau_2 - \varsigma) \\ &- R_q(\tau_1 - \varsigma)]g(\varsigma)d\omega^{*}(\varsigma)||^2 + 10F||\int_{\tau_1}^{\tau_2} R_q(\tau_2 - \varsigma)g(\varsigma)d\omega^{*}(\varsigma)||^2 d\varsigma \\ F||z(\tau_2) - z(\tau_1)||^2 &\leq 10F||[T_q(\tau_2) - T_q(\tau_1)](x_0 - h(x))||^2 + ||S_q(\tau_2) - S_q(\tau_1)x_1||^2 + 10bN_b^2M_U \\ &\times \int_0^{\tau_1 - \epsilon} (b - \varsigma)^{2q-2}||R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)||^2d\varsigma + 10\epsilon N_b^2M_U \int_{\tau_1 - \epsilon}^{\tau_1} (b - \varsigma)^{2q-2} \\ &\times ||R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)||^2d\varsigma + 10(\tau_2 - \tau_1)N_b^2M_U \int_{\tau_1}^{\tau_2} (b - \varsigma)^{2q-2} - ||R_q(\tau_2 - \varsigma)||^2d\varsigma \\ + 10bM_f(k) \int_0^{\tau_1 - \epsilon} ||R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)||^2d\varsigma + 10\epsilon M_f(k) \int_{\tau_1 - \epsilon}^{\tau_1} ||R_q(\tau_2 - \varsigma)||^2d\varsigma \\ + 10Tr(Q)M_g(k) \int_0^{\tau_1 - \epsilon} ||R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)||^2d\varsigma \\ + 10Tr(Q)M_g(k) \int_{\tau_1}^{\tau_1} ||R_q(\tau_2 - \varsigma) - R_q(\tau_1 - \varsigma)||^2d\varsigma + 10Tr(Q)M_g(k) \int_{\tau_1}^{\tau_1} ||R_q(\tau_2 - \varsigma)|^2d\varsigma. \end{split}$$

The right hand side of the inequality presented above converges as $\tau_1 \rightarrow \tau_2$ with ϵ being sufficient small. Thus, the compactness of $T_q(\zeta)$, $S_q(\zeta)$, and $R_q(\zeta)$, achieve continuity in the uniform operator topology accordingly. Therefore, Φ is equi-continuous.

Step 4: We have to show that $W(\zeta) = \{z(\zeta) : z \in \Phi(B_k)\}$ is relatively compact for $\zeta \in G$. In this case, $\zeta = 0$ is trivial.

Let $\zeta \in (0, b]$ be fixed for each $\in (0, \zeta)$. For each $z \in \Phi(x)$ and $x \in B_k$, there exist $f \in S_{F,x}$ and $g \in S_{J,x}$ such that

$$\begin{split} z(\zeta) &= T_q(\zeta)(x_0 - hx) + S_q(\zeta)x_1 + \int_0^{\zeta} R_q(\zeta - \varsigma)Bx_x^{\alpha}(\varsigma)d\varsigma + \int_0^{\zeta} R_q(\zeta - \varsigma)f(\varsigma)d\varsigma \\ &+ \int_0^{\zeta} R_q(\zeta - \varsigma)g(\varsigma)d\omega^*(\varsigma).z^F(\zeta) = T_q(\zeta)(x_0 - h(x)) + R_q(\varepsilon) \int_0^{\zeta - \varepsilon} R_q(\zeta - \varepsilon - s) \\ &\times Bx_x^{\alpha}(\varsigma)d\varsigma + R_q(\varepsilon) \int_0^{\zeta - \varepsilon} R_q(\zeta - \varepsilon - \varsigma)f(\varsigma)d\varsigma + R_q(\varepsilon) \int_0^{\zeta - \varepsilon} R_q(\zeta - \varepsilon - \varsigma)g(\varsigma)d\omega^*(\varsigma). \\ F||z(\zeta) - z^F(\zeta)||^2 &\leq 3F||\int_{\zeta - x_i}^{\zeta} R_q(\zeta - \varsigma)Bx_x^{\alpha}(\varsigma)d\varsigma||^2 + 3F||\int_{\zeta - \varepsilon}^{\zeta} R_q(\zeta - \varsigma)f(\varsigma)d\varsigma||^2 \\ &+ 3F||\int_{\zeta - \varepsilon}^{\zeta} R_q(\zeta - \varsigma)g(\varsigma)d\omega^*(\varsigma)||^2 \\ F||z(\zeta) - z^F(\zeta)||^2 &\leq 3bN_B^2M_S^2\frac{\epsilon^{4q-3}}{4q-3}M_U + 3bM_S^2\frac{\epsilon^{2q-1}}{2q-1}M_f(K) + 3Tr(Q)M_S^2\frac{\epsilon^{2q-1}}{2q-1}M_g(K). \end{split}$$

Since, $\epsilon \to 0$, we can see that there are compact sets arbitrarily close to $W(\zeta)$ for every $\zeta \in (0, b]$. Thus, $W(\zeta)$ is relatively compact in *X*.

Step 5: Φ has a closed graph.

Let $x^m \to x^*$ and $z^m \to z^*$ as $n \to \infty$. When $z^* \in \Phi(x^*)$ is satisfied, there exist $f^m \in R_{F,x^m}$ and $g^m \in R_{J,x^m}$ such that for every $\zeta \in G$,

$$\begin{aligned} z^{m}(\zeta) &= T_{q}(\zeta)(x_{0} - h(x^{m})) + S_{q}(\zeta)x_{1} + \int_{0}^{\zeta} R_{q}(\zeta - \varsigma)f^{m}d\varsigma + \int_{0}^{\zeta} R_{q}(\zeta - \varsigma)g^{m}(\varsigma)d\omega^{*}(\varsigma) \\ &+ \int_{0}^{\zeta} R_{q}(\zeta - \xi)B\{B^{*}R_{q}^{*}(b - \xi)[(\alpha I + \Psi_{0}^{b})^{-1}[Fx_{b} - T(b)(x_{0} - h(x^{m}))] \\ &+ \int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}\Phi(\varsigma)d\omega^{*}(\varsigma)] - B^{*}R_{q}^{*}(b - \xi)\int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q}(b - \varsigma)f^{m}(\varsigma)d\varsigma \\ &- B^{*}R_{q}^{*}(b - \xi)\int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q}(b - \varsigma)g^{m}(\varsigma)d\omega^{*}(\varsigma)\}d\xi. \end{aligned}$$

We must prove that there exist $f^* \in R_{F,x_*}$ and $g^* \in R_{J,x_*}$ such that for each $\zeta \in G$,

$$\begin{aligned} z^{*}(\zeta) &= T_{q}(\zeta)(x_{0} - h(x^{*})) + S_{q}(\zeta)x_{1} + \int_{0}^{\zeta} R_{q}(\zeta - \varsigma)f^{*}(\varsigma)d\varsigma + \int_{0}^{t} R_{q}(\zeta - \varsigma)g^{*}(\varsigma)d\omega^{*}(\varsigma) \\ &+ \int_{0}^{\zeta} R_{q}(\zeta - \xi)B\{B^{*}S_{q}^{*}(b - \xi)[(\alpha I + \Psi_{0}^{b})^{-1}[Ex_{b} - T(b)(x_{0} - h(x^{*}))] \\ &+ \int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}\Phi(\varsigma)d\omega^{*}(\varsigma)] - B^{*}R_{q}^{*}(b - \xi)\int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q}(b - \varsigma)f^{*}(\varsigma)d\varsigma \\ &- B^{*}R_{q}^{*}(b - \xi)\int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q}(b - \varsigma)g^{*}(\varsigma)d\omega^{*}(\varsigma)\}d\xi. \end{aligned}$$

Since h is continuous, we obtain

$$\begin{split} \|\|z^{m}(\zeta) - z^{*}(\zeta)\|^{2} &= \left\| \left(z^{m}(\zeta) - T_{q}(\zeta)(x_{0} - h(x^{m})) + S_{q}(\zeta)x_{1} + \int_{0}^{\zeta} R_{q}(\zeta - \zeta)BB^{*}R_{q}^{*}(b - \zeta)[(\alpha I + \Psi_{0}^{b})^{-1} + \Psi_{0}^{b})^{-1} \right) \right\| \\ &\times [Fx_{b} - T(b)(x_{0} - h(x^{m}))] + \int_{0}^{b} (\alpha I + \Psi_{s}^{b})^{-1}\Phi(\varsigma)d\omega^{*}(\varsigma)]d\xi) - (z^{*}(\zeta) - T_{q}(\zeta) \\ &\times (x_{0} - x(u)) + \int_{0}^{\zeta} R_{q}(\zeta - \zeta)BB^{*}R_{q}^{*}(b - \zeta)[(\alpha I + \Psi_{0}^{b})^{-1}[Fx_{b} - T(b)(x_{0} - h(x^{*}))] \\ &+ \int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1}\Phi(\varsigma)d\omega^{*}(\varsigma)]d\xi \Big\|^{2} \to 0, \end{split}$$

Consider the continuous operator Ψ : $L^2(\Omega^*, C) \times L^2(L(K, G)) \to \tilde{C}(G, C)$.

$$\begin{split} (f,g) &\to \Psi(f,g)(\zeta) &= \int_{0}^{\zeta} R_{q}(\zeta-\varsigma)[f(\varsigma) + BB^{*}R_{q}^{*}(b-\varsigma)(\int_{0}^{b}(\alpha I + \Psi_{r}^{b})^{-1}R_{q}(b-\tau)f(\tau)d\tau)]d\varsigma \\ &+ \int_{0}^{t} R_{q}(\zeta-\varsigma)[g(\varsigma) + BB^{*}R_{q}^{*}(b-\varsigma)(\int_{0}^{b}(\alpha I + \Psi_{\tau}^{b})^{-1}R_{q}(b-\tau)g(\tau)d\omega^{*}(\varsigma))]d\omega^{*}(\varsigma) \\ &\times z^{m}(\zeta) - T_{q}(\zeta)(x_{0} - h(x^{m})) + \int_{0}^{\zeta} R_{q}(\zeta-\zeta)BB^{*}R_{q}^{*}(b-\zeta)[(\alpha I + \Psi_{0}^{b})^{-1}[Fx_{b} - T(b) \\ &\times (x_{0} - h(x^{m}))] + \int_{0}^{b}(\alpha I + \Psi_{\varsigma}^{b})^{-1}\Phi(\varsigma)d\omega^{*}(\varsigma)]d\zeta \in \Psi(R_{E}, J, x^{m}). \end{split}$$

Since $x^m \to x^*$, it follows from (1) that

$$\begin{aligned} z^{*}(\zeta) - T_{q}(\zeta)(x_{0} - h(x^{*})) &+ \int_{0}^{\zeta} R_{q}(\zeta - \zeta) BB^{*}R_{q}^{*}(b - \zeta)[(\alpha I + \Psi_{0}^{b})^{-1}[Fx_{b} - T(b)(x_{0} - h(x^{*}))] \\ &+ \int_{0}^{b} (\alpha I + \Psi_{\zeta}^{b})^{-1} \Phi(\zeta) d\omega^{*}(\zeta)] d\zeta \in \Psi(R_{F}, J, x^{*}). \end{aligned}$$

This demonstrates that $z^* \in \Phi(x^*)$. As a result, the graph for Φ is closed. Steps 1 to 5 together with the Arzela–Ascoli theorem result in Φ being a compact multiple-valued map with convex closed values.

Step 6: The solution for the operator Φ is found.

Create an open ball with the coordinates $B(0,r) \in \tilde{C}$, where r meets the inequality. As a result of the above procedures, we are aware that Φ satisfies every requirement of the lemma. Therefore, if we can demonstrate that the second assumption of the lemma is incorrect, we can show that the system in (1) has at least one mild solution. Let $x \in \tilde{C}$ be a possible solution for $\lambda x \in \Phi x$ for some $\lambda > 1$ with $F||x||_r^2 = r$. Then, we have

$$\begin{aligned} x(\zeta) &= \lambda^{-1} T_q(\zeta) (x_0 - h(x)) + \lambda^{-1} S_q(\zeta) x_1 + \lambda^{-1} \int_0^{\zeta} R_q(\zeta - \zeta) B x_x^{\alpha}(\zeta) d\zeta + \lambda^{-1} \int_0^{\zeta} R_q(\zeta - \zeta) f(\zeta) d\zeta \\ &+ \lambda^{-1} \int_0^{\zeta} R_q(\zeta - \zeta) g(\zeta) d\omega^*(\zeta). \end{aligned}$$

Next, using the assumption, we get

$$\begin{split} F||x(\zeta)||^2 &\leq 8M_T^2(F||x_0||^2 + \mu_1 F||x||^2 + \mu_2) + \frac{4}{\alpha^2} bN_B^4 M_S^4 \int_0^{\zeta} (b-\zeta)^{4q-4} \{8||Fx_b||\}^2 + 8\int_0^b F||\phi(\zeta)||_{L_2^0}^2 d\zeta \\ &+ 8N_T^2(F||x_0||^2 + \mu_1 F||x||^2 + \mu_2) + 4bN_S^2 \int_0^{\zeta} (b-\zeta)^{2q-2} M_f(F||x||^2) d\zeta \\ &+ 4Tr(Q)M_S^2 \int_0^{\zeta} (b-\zeta)^{2q-2} M_g(F||x||^2) d\zeta \} d\zeta + 4bM_b^2 \int_0^{\zeta} (\zeta-\zeta)^{2q-2} M_f(F||x||^2) d\zeta \\ &+ 4Tr(Q)M_S^2 \int_0^{\zeta} (\zeta-\zeta)^{2q-2} M_g(F||x||^2) d\zeta. \end{split}$$

Taking the supremum over ζ , we obtain

$$\begin{split} F||x||^{2} &\leq 8M_{T}^{2}(F||x_{0}||^{2} + \mu_{1}F||x||^{2} + \mu_{2}) + \frac{4}{\alpha^{2}}bN_{B}^{4}M_{S}^{4}\int_{0}^{\zeta}(b-\xi)^{4q-4}\{8||Fx_{b}||^{2} + 8\int_{0}^{\zeta}F||\phi||_{L_{2}^{0}}^{2}d\zeta \\ &+ 8M_{T}^{2}(F||x_{0}||^{2} + \mu_{1}F||x||^{2} + \mu_{2}) + 4bM_{S}^{2}\int_{0}^{\zeta})\zeta - \zeta)^{2q-2}M_{f}(F||x||^{2})d\zeta \\ &+ 4Tr(Q)M_{S}^{2}\int_{0}^{\zeta}(\zeta-\zeta)^{2q-2}M_{g}(F||x||^{2})d\zeta\}d\xi + 4bM_{S}^{2}\int_{0}^{\zeta}(\zeta-\zeta)^{2q-2}M_{f}(F||x||^{2})d\zeta \\ &+ 4Tr(Q)M_{S}^{2}\int_{0}^{\zeta}(\zeta-\zeta)^{2q-2}M_{g}(F||x||^{2})d\zeta. \end{split}$$

Substituting $F||x||_{\gamma}^2 = r$

$$r \leq 8M_T^2(F||x_0||^2 + \mu_1 r + \mu_2) + \frac{4}{\alpha^2}bN_B^4M_S^4\frac{b^{4q-3}}{4q-3}\{8||Fx_b||^2 + 8\int_0^{\zeta}F||\phi||_{L_2^0}^2d\zeta + 8M_T^2(F||x_0||^2 + \mu_1 r + \mu_2)$$

$$+ 4bM_{S}^{2}\frac{b^{2q-1}}{2q-1}M_{f}(r)d\varsigma + 4Tr(Q)M_{S}^{2}\frac{b^{2q-1}}{2q-1}M_{g}(r)\} + 4bM_{S}^{2}\frac{b^{2q-1}}{2q-1}M_{f}(r) + 4Tr(Q)M_{S}^{2}\frac{b^{2q-1}}{2q-1}M_{g}(r)$$

$$r \leq \frac{l_{1} + 4M_{S}^{2}\frac{b^{2q-1}}{2q-1}[bM_{f}(r) + Tr(Q)M_{g}(r)](1 + \frac{4}{\alpha^{2}}N_{B}^{4}M_{S}^{4}\frac{b^{4q-2}}{4q-3})}{1 - l_{2}},$$

which is contradiction. Thus, the operator inclusion $x \in \Phi x$ has a solution in B[0, r]. Therefore, the fractional stochastic inclusion (1) has a mild solution on *G*. \Box

Theorem 2. Assume that the functions E and J are uniformly bounded on their respective domains and that the assumptions (H1)–(H5) are true. Additionally, the fractional stochastic system in (1) is approximately controllable on J if the fractional linear differential inclusion is approximately controllable.

Proof. Let x^{α} be a fixed point on Φ . By the stochastic Fubini theorem, it is easy to see that

-h

$$\begin{aligned} x_{\alpha}(b) &= x_{b} - \alpha(\alpha I + \Psi_{0}^{b})^{-1}[Fx_{b} - T_{q}(b)(x_{0} - h(x^{\alpha})] - \alpha \int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1} \Phi(\varsigma) d\omega^{*}(\varsigma) \\ &+ \alpha \int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1} R_{q}(b - \varsigma) f^{\alpha}(\varsigma) d\varsigma + \alpha \int_{0}^{b} (\alpha I + \Psi_{\varsigma}^{b})^{-1} R_{q}(b - \varsigma) g^{\alpha}(\varsigma) d\omega^{*}(\varsigma). \end{aligned}$$

where

$$\begin{array}{ll} f^{\alpha} & \in & R_{F,x^{\alpha}} = \{f^{\alpha} \in l^{2}(\Omega^{*},H) : f^{\alpha}(\zeta) \in E(\zeta,x^{\alpha}(\zeta))\}, \\ g^{\alpha} & \in & R_{J,x^{\alpha}} = \{g^{\alpha} \in l^{2}(\Omega^{*},H) : g^{\alpha}(\zeta) \in J(\zeta,x)^{\alpha}(\zeta)\}. \end{array}$$

Assuming that *E* and *G* are true, it follows that *D* exists in such a way that

$$||f^{\alpha}(s)||^{2} + ||g^{\alpha}||^{2} \le D.$$

Then, there exists a subsequence indicated as $\{f^{\alpha}(\varsigma), g^{\alpha}(\varsigma)\}$ converging to $\{f(\varsigma), g(\varsigma)\}$. The compactness of $R_q(\zeta)$ implies that

$$R_q(b-\varsigma)f^{\alpha}(\varsigma) \to R_q(b-\varsigma)f(\varsigma), R_q(b-\varsigma)g^{\alpha}(\varsigma) \to R_q(b-\varsigma)g(\varsigma)$$

From the above equation, we have

$$\begin{split} F||x^{\alpha}(b) - x_{b}||^{2} &\leq 6||\alpha(\alpha I + \Psi_{0}^{b})^{-1}[Fx_{b} - T_{q}(b)(x_{0} - h(x^{\alpha}))]||^{2} \\ &+ 6F(\int_{0}^{b} ||\alpha(\alpha I + \Psi_{\varsigma}^{b})^{-1}\Psi(\varsigma)||_{l_{2}^{0}}^{2}d\varsigma) \\ &+ 6F(\int_{0}^{b} ||\alpha(\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q}(b - \varsigma)[f^{\alpha}(\varsigma) - f(\varsigma)]||d\varsigma)^{2} \\ &+ 6F(\int_{0}^{b} ||\alpha(\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q}(b - \varsigma)f(\varsigma)||d\varsigma)^{2} \\ &+ 6F(\int_{0}^{b} ||\alpha(\alpha I + \Psi_{\varsigma}^{b})^{-1}R_{q}(b - \varsigma)g(\varsigma)||_{l_{2}^{0}}^{2}d\varsigma). \end{split}$$

For all $0 \leq \varsigma \leq b$, the operator $\alpha (\alpha I + \Psi_c^b)^{-1} \to 0$ strongly as $\alpha \to 0^+$ and $||\alpha (\alpha I +$ $|\Psi_{c}^{b}|^{-1}\Psi(\zeta)|| \leq 1$. Hence, according to the Lebesque dominated convergence theorem, we obtain $F \|x^{\alpha}(b) - \bar{x}_b\|^2 \to 0$ as $\alpha \to 0^+$. This represents the approximate controllability of the system in (1). \Box

5. Example

Assume the following fractional stochastic partial differential equations with non-local conditions of the form

$$\begin{cases} \frac{\partial^{\frac{3}{2}}}{\partial \zeta_{2}^{\frac{3}{2}}} x(\zeta, y) = x_{yy}(\zeta, y) + \mu(\zeta, y) + L_{1}(\zeta, x(\zeta, y)) + L_{2}(\zeta, x(\zeta, y)) \frac{d\omega^{*}(\zeta)}{d\zeta}, \zeta \in J = [0, 1] \\ x(\zeta, 0) = x(\zeta, 1) = 0 \\ x(0, y) + \sum_{i=1}^{n} c_{i} x(\zeta_{i}, y) = x_{0}(y), 0 \le y \le 1, \end{cases}$$
(7)

where $\omega(\zeta)$ indicates a standard cylindrical Wiener process on $(\Omega^*, \nu, \{\nu_{\zeta}\}, P); x_0 \in l^2(0, 1)$ μ : $[0,1] \times (0,1) \rightarrow (0,1)$ is continuous in ζ ; $l_1, l_2 : R \rightarrow P(R)$ is continuous; and $c_i > 0$. Let $H = U = l^2(0, 1)$ and define the operator $A :\to H$ by Az = z'' with domain $D(A) = \{z \in U : z \in U\}$ *H*, *z*, *z'* are absolutely continuous, $z'' \in H$, z(0) = z(1) = 0}. Then, *A* generates an analytic semi-group $T(\zeta)$ given by

$$T(\zeta)z = \sum_{n=1}^{\infty} e^{-n^2 \zeta}(z, e_n) e_n, z \in H,$$

where $e_n(z) = \sqrt{2sin(nz)}$, n = 1, 2, ... is a complete ortho-normal set of eigenvectors of A. From these expressions, it follows that $\{T(\zeta), \zeta > 0\}$ is a uniformly bounded compact semi-group, so that $R(\lambda, A) = (\lambda I - A)^{-1}$ is a compact operator for $\lambda \in \rho(A)$. Then,

$$Az = \sum_{n=1}^{\infty} n^2(z, e_n) e_n, z \in H.$$

Let $X(\zeta)(z) = x(\zeta, z)$ and define the bounded linear operator $B : X \to H$ by Bx(t)(y) = $\mu(\zeta,z), 0 \leq z \leq 1$. Further, define $(\zeta, x(\zeta))(z) = l_1(\zeta, x(\zeta,z)) = \frac{e^{-1}}{1+e^{-1}} sin(x(\zeta,z)),$ $J(\zeta, x(\zeta))(z) = l_2(\zeta, x(\zeta, z)) = \frac{e^{-1}}{1+e^{-1}} sin(x(\zeta, z))$, and $h(x)(z) = \sum_{i=1}^n c_i x(\zeta_i, z)$. Then, the conditions (H2)–(H4) are verified. On the other hand, the linear system is approximately controllable. Therefore, with A, E, J, and B, the above system can be written in an abstract form. Thus, all the conditions of the above theorem are satisfied. Hence, according to the above theorem, the stochastic control system is approximately controllable on G.

6. Conclusions

In this paper, a mild solution for the approximate controllability system of fractional stochastic differential inclusions with non-local conditions was identified with modifications and generalizations from the existing relevant literature. The following contributions were made. A mild solution for a control problem governed by fractional stochastic evolution inclusions using the Caputo derivative with non-local conditions was obtained with

the help of the fixed-point theorem of convex multiple-valued maps. We established a set of sufficient conditions for their approximate controllability and provided results in terms of controllability for the fractional stochastic control system. This solution was implemented to show that the introduced stochastic control problem and cylindrical Wiener problem had a convenient invariant set under linear perturbation.

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