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Investigating Asymptotic Stability for Hybrid Cubic Integral Inclusion with Fractal Feedback Control on the Real Half-Axis

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Abstract: In this paper, we discuss the existence of solutions for a hybrid cubic delayed integral inclusion with fractal feedback control. We are seeking solutions for these hybrid cubic delayed integral inclusions that are defined, continuous, and bounded on the semi-infinite interval. Our proof is based on the technique associated with measures of noncompactness by a given modulus of continuity in the space in $BC(R_+)$. In addition, some sufficient conditions are investigated to demonstrate the asymptotic stability of the solutions of that integral inclusion. Finally, some cases analyzed are in the presence and absence of the control variable, and two examples are provided in order to indicate the validity of the assumptions.

Keywords: hybrid integral inclusion; existence results; measure of noncompactness; darbo fixed-point theorem; fractal feedback control

MSC: 45G10; 47H08; 45M10

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1. Introduction and Background

In the last few years, many researchers have focused their work on some cubic integral equations. They have extended their results for quadratic integral equations to a specific set of cubic integral equations on a bounded interval, for example, Refs. [1–5]. The cubic integral equations can be considered as a generalization of quadratic integral equations, which are applicable to many real-world problems. Furthermore, in [6], the authors investigated some findings for the existence of solutions to a cubic functional integral equation related to a control variable; this has broader implications than those discussed in [2–5].

The investigations in [2–5] are on bounded intervals. However, in this article, we establish our results on unbounded intervals.

Caballero et al. [1] provided the first contribution to the solvability of the cubic integral equations; they proved the existence of nondecreasing solutions to Urysohn cubic integral equations.

The importance of dealing with problems involving control variables is due to the unforeseen factors that continually upset ecosystems in the actual world, which may result in modifications to biological traits such as rates of survival. Ecology has a practical interest in the question of whether an ecosystem can withstand those unpredictable disruptive events that continue for a short period of time. In the context of control variables, the disturbance functions are what we refer to as control variables.

Cichoń (1996) initiated abstract control problems for differential conclusions [7]. His results were also applied to a semilinear optimal control problem.

In [8], Chen established some averaging conditions for a nonautonomous Lotka–Volterra system that is regulated via feedback by producing a suitable Lyapunov function (Lyapunov functional).

A class of feedback-controlled nonlinear functional-integral equations exist, are asymptotically stable, and are globally attractive, as found by Nasertayoob, using the measure of noncompactness in conjunction with Darbo's fixed point theorem [9]. Moreover, under appropriate circumstances, the authors of [10] investigated whether a nonlinear neutral delay population system with feedback control has a positive periodic solution. The proof depends on the strict-set-contraction operators' fixed-point theorem [10].

A functional integral equation incorporating a control parameter function that fulfils a constraint functional equation is the subject of El-Sayed et al.'s research [11]. Additional findings of existence may be found in [12], where researchers looked at a nonlinear functional integral equation restricted by a functional equation with a parameter.

In this article, we demonstrate that solutions exist and are asymptotically stable for a class of nonlinear functional-integral inclusions

$$\frac{x(\mathfrak{r}) - f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r})))}{f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r})))} \in G\left(\mathfrak{r}, v(\mathfrak{r}), x(\phi_3(\mathfrak{r})) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))\, d\varsigma\right), \quad \mathfrak{r} \in R_+ = [0, \infty); \tag{1}$$

with the fractal feedback control

$$\frac{dv(\mathfrak{r})}{d\mathfrak{r}^\beta} = -\alpha v(\mathfrak{r}) + f_3(\mathfrak{r}, x(\mathfrak{r})),\ v_0 = v(0),\ \alpha \geq 0,\ \beta \in (0,1),$$

where $\frac{d}{d\mathfrak{r}^\beta}$ is the fractal derivative of order β (see [13,14]) and f_i $(i = 1,2,3)$ and h satisfy some conditions and G is a Lipschitzian set-valued map.

This is the first attempt to discuss a class of nonlinear functional-integral inclusions with fractal feedback control. the nonlinear problem affected by an external source is studied in R_+. The existence of a control variable v that satisfies the fractal derivation equation is established. We study the existence and the stability of solutions for the hybrid cubic functional integral inclusion (1) in $BC(R_+)$.

The basic tools used in our research are the fixed point theorem of the Darbo type [15] and the strategy of measure of noncompactness.

The measure of noncompactness and Darbo's fixed point theorem are beneficial methods and techniques to study the nonlinear functional–integral equations that arise in some real-world problems [16–20].

Banaś in [16–22] successfully used the method connected to a measure of noncompactness in the Banach space $BC(R_+)$ (which consists of all bounded and continuous functions on R_+) to determine the existence of asymptotically stable solutions to some integral and quadratic integral equations (see [18,19]). Furthermore, for the solvability of some problems in the half-line axis, see [23–27].

To be able to recall the definitions of the keywords' global attractivity, local attractivity, and asymptotic stability of the solution, see [18,27,28].

What follows in the article is arranged as follows: In Section 1, we outline some previous results to explain our motivation and the innovation of the work. Section 2 states and demonstrates the existence of a result for a single-valued problem by a direct application of Darbo's fixed point theorem [15]. Additionally, the asymptotic stability of the solution to our problem will be studied. Next, we extend our results to the multi-valued problem in Section 4. Finally, in Section 4, we will provide an illustration of our main result with two examples and with some special cases of the studied problem.

2. Single Valued Problem

To achieve our aims, we initially study the single-valued problem corresponding to a class of nonlinear functional-integral inclusions (1) using a fractal feedback control

$$\begin{aligned} x(\mathfrak{r}) &= f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r}))) + f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r}))) \\ &\quad . \quad g\left(\mathfrak{r}, v_x, x(\phi_3(\mathfrak{r})) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))) \, d\varsigma\right), \ \mathfrak{r} \geq 0 \\ v_x(\mathfrak{r}) &= v(0)e^{-\alpha\mathfrak{r}^\beta} + \beta \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)} \, \varsigma^{\beta-1} \, f_3(\varsigma, x(\varsigma)) d\varsigma, \end{aligned} \tag{2}$$

depending on the following conditions:

(i) Given continuous functions $\phi_i : R_+ \to R_+$ (for $i = 1,2,3,4$) , with $\phi_i \to \infty$ as $\mathfrak{r} \to \infty$.

(ii) The functions $\psi : R_+ \to R_+$ are continuous, non-decreasing, and $\psi(\mathfrak{r}) \leq \mathfrak{r}$.

(iii) $f_1 : R_+ \times R \to R$ and $f_2 : R_+ \times R \to R\backslash\{0\}$ are continuous functions, and there exists a continuous function $\eta_i(\mathfrak{r})$ $(i = 1,2)$ such that

$$|f_i(\mathfrak{r}, \mu) - f_i(\mathfrak{r}, \nu)| \leq \eta_i(\mathfrak{r}) \, |\mu - \nu|.$$

for each $\mathfrak{r} \in R_+$ and for all $\mu, \nu \in R$. Moreover, the function $\mathfrak{r} \to f_i(\mathfrak{r}, 0)$ belongs to the space $BC(R_+)$, and we have

$$|f_i(\mathfrak{r}, \mu)| \leq \eta_i|\mu| + N_i,$$

$$\text{where } \eta_i = \sup_{\mathfrak{r} \in R_+} |\eta_i(\mathfrak{r})| < 1, \ \ N_i = \sup_{\mathfrak{r} \in I} |\{f_i(\mathfrak{r}, 0)| : \mathfrak{r} \in R_+\} < \infty.$$

and $\lim_{\mathfrak{r}\to\infty} \eta_i(\mathfrak{r}) = 0, \quad \lim_{\mathfrak{r}\to\infty} f_i(\mathfrak{r}, 0) = 0.$

(iv) $f_3 : R_+ \times R \to R$ is a Carathéodory function that is measurable in $\mathfrak{r} \in R_+$, $\forall \mu \in \mathbb{R}$ and continuous in $\mu \in \mathbb{R}$; $\forall \mathfrak{r} \in R_+$, there are two integrable functions $a, b : R_+ \to$ such that

$$|f_3(\mathfrak{r}, \mu)| \leq a(\mathfrak{r}) + b(\mathfrak{r})|\mu|, \ \ \mathfrak{r} \in R_+.$$

(v) Let $g(\mathfrak{r}, \mu, \nu) : R_+ \times R_+ \times R_+ \to R_+$ be a Lipschitz function with a Lipschitz constant $l > 0$ such that

$$|g(\mathfrak{r}, \mu_1, \mu_2) - g(\mathfrak{r}, \nu_1, \nu_2)| \leq l \left(|\mu_1 - \nu_1| + |\mu_2 - \nu_2|\right),$$

$\forall \ \mathfrak{r} \in R_+$ and $\forall \ \mu_i, \nu_i \in R, i = 1,2$. In addition, $\mathfrak{r} \to g(\mathfrak{r}, 0, 0)$ belongs to the space $BC(R_+)$, and we obtain

$$|g(\mathfrak{r}, \mu_1, \mu_2)| \leq l \left(|\mu_1| + |\mu_2|\right) + M, \text{ where } M = \sup_{\mathfrak{r} \in I} |\{g(\mathfrak{r}, 0, 0)| : \ \mathfrak{r} \in R_+\} < \infty.$$

(vi) The function $h : R_+ \times R \to R_+$, is a Carathéodory function that is measurable in $\mathfrak{r} \in J, \forall \ \mu, \in R$ and continuous in $\mu \in R$, $\forall \ \mathfrak{r} \in J$, and there exist measurable and bounded functions $k_1, k_2 : R_+ \times R_+ \to R_+$, such that

$$|h(\mathfrak{r}, \varsigma, \mu)| \leq k_1(\mathfrak{r}, \varsigma) + k_2(\mathfrak{r}, \varsigma)|\mu|, \ \ \mathfrak{r}, \varsigma \in [0, \infty),$$

and

$$\sup_{\mathfrak{r} \in [0,T]} \int_0^{\mathfrak{r}} k_1(\mathfrak{r}, \varsigma) d\varsigma = k_1, \quad \lim_{\mathfrak{r}\to\infty} \int_0^{\mathfrak{r}} k_1(\mathfrak{r}, \varsigma) d\varsigma = 0,$$

$$\sup_{\mathfrak{r} \in [0,T]} \int_0^{\mathfrak{r}} k_2(\mathfrak{r}, \varsigma) d\varsigma = k_2, \quad \lim_{\mathfrak{r}\to\infty} \int_0^{\mathfrak{r}} k_2(\mathfrak{r}, \varsigma) d\varsigma = 0.$$

(vii) The following equation has a positive solution r. A positive solution to what follows the equation is r

$$\eta_2\, l\, k_2 r^3 + \left[(k_1\eta_2 + k_2 N_2 + \eta_2 V_2) l\right] r^2 \\ + \left[\eta_1 + \eta_2 M + l N_2 (k_1 + V_2) + \eta_2 l (V + V_1)\ - 1\right] r + N_1 + N_2 M + N_2 l (V + V_1) = 0,$$

such that

$$\eta_1 + \eta_2\, \left[M + l(V + V_1 + V_2\, r) + l r [k_1 + k_2 r]\right] + l[N_2 + \eta_2\, r][k_1 + k_2 r] \le 1.$$

Remark 1. *For any function x belonging to $BC(R_+)$, the solution to a fractal feedback control equation in (1)*

$$\frac{dv(\mathfrak{r})}{d\mathfrak{r}^\beta} = -\alpha v(\mathfrak{r}) + f_3(\mathfrak{r}, x(\mathfrak{r})),\ v_0 = v(0),\ \alpha \ge 0,\ \beta \in (0,1),$$

indicated by $v_x(\mathfrak{r})$, and it can be stated as follows:

$$v_x(\mathfrak{r}) = v(0) e^{-\alpha \mathfrak{r}^\beta} + \beta \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, f_3(\varsigma, x(\varsigma)) d\varsigma.$$

Chen [8] prove that with the positive initial condition $v_x(0) > 0$, the solution $v_x(\mathfrak{r})$ is globally attractive and bounded above by positive constants. Then

$$\begin{aligned} |v_x(\mathfrak{r}) &\le v(0) e^{-\alpha \mathfrak{r}^\beta} + \beta \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, [a(\varsigma)\ +\ b(\varsigma)] |x(\varsigma)| d\varsigma \\ &\le V + \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, a(\varsigma)\, d\varsigma + \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, b(\varsigma) \|x\| d\varsigma \\ &\le V + \sup_{\mathfrak{r} \in R_+} \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, a(\varsigma)\, d\varsigma + \|x\| \sup_{\mathfrak{r} \in R_+} \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, b(\varsigma) d\varsigma \\ &\le V + V_1 + V_2\, r, \end{aligned}$$

where

$$\sup_{\mathfrak{r} \in R_+} v(0)\, e^{-\alpha \mathfrak{r}^\beta} = V, \quad \sup_{\mathfrak{r} \in R_+} \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, a(\varsigma) d\varsigma = V_1, \text{ and}$$
$$\sup_{\mathfrak{r} \in R_+} \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, b(\varsigma) \|x\| d\varsigma = V_2.$$

Theorem 1. *Assume that assumptions (i)–(vii) hold. Then, the Equation (2) has at least one solution $x = x(t)$ that belongs to the space in the space $BC(R_+)$. Additionally, the solutions to Equation (2) are locally attractive.*

Proof. Let B_r be the ball described by

$$B_r = \{x \in BC(R_+) :\ \|x\| \le r\}.$$

By defining the operator $\mathbb{F}$ on the space $BC(R_+)$ by

$$\begin{aligned} &\mathbb{F}x(\mathfrak{r}) \\ &= f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r}))) + f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r}))) g\left(\mathfrak{r}, v_x(\mathfrak{r}), x(\phi_3(\mathfrak{r}))\ \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma))) d\varsigma\right),\ \mathfrak{r} \in [0, \infty) \end{aligned}$$

corresponding to our assumptions, note that for any function $x \in BC(R_+)$, the function $\mathbb{F}x$ is continuous on the interval $[0, \infty)$.

We will demonstrate that for some $r > 0$, $\mathbb{F}B_r \subset B_r$, we obtain

$$
\begin{aligned}
& |\ \mathbb{F}x(\mathfrak{r})\ | \\
= \ & \left| f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r}))) + f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r})))g\left(\mathfrak{r}, v_x(\mathfrak{r}), x(\phi_3(\mathfrak{r}))\int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))d\varsigma\right)\right| \\
\leq \ & |f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r}))| + |f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r}))|\left|g\left(\mathfrak{r}, v_x(\mathfrak{r}), x(\phi_3(\mathfrak{r})))\int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))d\varsigma\right)\right| \\
\leq \ & [f_1(\mathfrak{r}, 0)| + \eta_1|x(\phi_1(\mathfrak{r}))|] \\
+ \ & [f_2(\mathfrak{r}, 0)| + \eta_2|x(\phi_2(\mathfrak{r}))|]\big[|g(\mathfrak{r}, 0, 0)| + l(|v_x(\mathfrak{r})| + |x(\phi_3(\mathfrak{r}))|\int_0^{\psi(\mathfrak{r})} |h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))|d\varsigma)\big] \\
\leq \ & [f_1(\mathfrak{r}, 0)| + \eta_1|x(\phi_1(\mathfrak{r}))|] \\
+ \ & [f_2(\mathfrak{r}, 0)| + \eta_2|x(\phi_2(v))|]\big[|g(\mathfrak{r}, 0, 0)| \\
+ \ & l(V + V_1 + V_2\, r + |x(\phi_3(\mathfrak{r}))|\int_0^{\psi(\mathfrak{r})} [k_1(\mathfrak{r}, \varsigma) + k_2(\mathfrak{r}, \varsigma)|x(\phi_4(\varsigma))|]d\varsigma\big] \\
\leq \ & [f_1(\mathfrak{r}, 0)| + \eta_1\|x\|] + [f_2(\mathfrak{r}, 0)| + \eta_2\|x\|]\big[|g(\mathfrak{r}, 0, 0)| \\
+ \ & l(V + V_1 + V_2\, r + \|x\|\int_0^{t} [k_1(\mathfrak{r}, \varsigma) + k_2(\mathfrak{r}, \varsigma)\|x\|]d\varsigma\big] \\
\leq \ & [N_1 + \eta_1\, r] + [N_2 + \eta_2\, r][M + l\ (V + V_1 + V_2\, r) + l\ r\ (k_1 + k_2\, r)] = r.
\end{aligned}
$$

Taking assumption (iv) into consideration, from the above estimate, we conclude that the operator $\mathbb{F}$ transforms the ball B_r into itself. There exists a positive solution $r = r_0$ to the equation

$$
\begin{aligned}
& \eta_2\, l\, k_2 r^3 + \big[(k_1\eta_2 + k_2 N_2 + \eta_2 V_2)l\big]\ r^2 + \big[\eta_1 + \eta_2 M + lN_2(k_1 + V_2) + \eta_2 l(V + V_1)\ - 1\big]r \\
& \ + N_1 + N_2 M + N_2 l(V + V_1) = 0.
\end{aligned}
$$

Now, we show that $\mathbb{F}$ is continuous on the ball B_r. In order to do this, let us fix $\epsilon > 0$ and select $x, y \in B_r$ such that $\|x - y\| \leq \epsilon$. Then, for $\mathfrak{r} \in I$, we obtain

$$
\begin{aligned}
& |\ \mathbb{F}x(\mathfrak{r}) - \mathbb{F}y(\mathfrak{r})\ | \\
& = \left| f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r}))\ +\ f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r}))\ g\left(\mathfrak{r}, v_x, x(\phi_3(\mathfrak{r}))\int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))d\varsigma\right)\right. \\
& \left. - f_1(\mathfrak{r}, y(\phi_1(\mathfrak{r}))\ +\ f_2(\mathfrak{r}, y(\phi_2(\mathfrak{r}))\ g\left(\mathfrak{r}, v_y, y(\phi_3(\mathfrak{r}))\int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, y(\phi_4(\varsigma)))d\varsigma\right)\right| \\
& \leq |f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r}))) - f_1(\mathfrak{r}, y(\phi_1(\mathfrak{r})))| \\
& + |f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r}))) - f_2(\mathfrak{r}, y(\phi_2(\mathfrak{r})))|\ \left|g\left(\mathfrak{r}, v_x(\mathfrak{r}), x(\phi_3(\mathfrak{r}))\int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))ds\right)\right| \\
& + |f_2(\mathfrak{r}, y(\phi_2(\mathfrak{r})))| \\
& \times\ \left|g\left(\mathfrak{r}, v_x(\mathfrak{r}), x(\phi_3(\mathfrak{r}))\int_0^{\phi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))d\varsigma\right) - g\left(\mathfrak{r}, v_y(\mathfrak{r}), y(\phi_3(\mathfrak{r}))\int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, y(\phi_4(\varsigma)))d\varsigma\right)\right| \\
& \leq\ \eta_1\ |x(\phi_1(\mathfrak{r})) - y(\phi_1(\mathfrak{r}))| \\
& + \eta_2\ |x(\phi_2(\mathfrak{r})) - y(\phi_2(\mathfrak{r}))|\ \big[(|g(\mathfrak{r}, 0, 0)| + l\big(|v_x(\mathfrak{r})| + |x(\phi_3(\mathfrak{r}))|\big)\int_0^{\psi(\mathfrak{r})} |h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))|d\varsigma)\big] \\
& + l[|f_2(\mathfrak{r}, 0)| + \eta_2|y(\mathfrak{r})|]\big[|v_x(\mathfrak{r}) - v_y(\mathfrak{r})| \\
& + \big|x(\phi_3(\mathfrak{r}))\int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))d\varsigma - y(\phi_3(\mathfrak{r}))\int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, y(\phi_4(\varsigma)))d\varsigma\big|\big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \eta_1 \,|x(\phi_1(\mathfrak{r})) - y(\phi_1(\mathfrak{r}))| + \eta_2 \,|x(\phi_2(\mathfrak{r})) - y(\phi_2(\mathfrak{r}))|\,]\big[M + l(v(0)|e^{-\alpha\mathfrak{r}}| \\
&+ \int_0^{\mathfrak{r}} e^{-\alpha\varsigma}[a(\varsigma) \;+\; b(\varsigma)]|x(\varsigma)|d\varsigma + \;\|x\| \int_0^{\psi(\mathfrak{r})} [k_1(\mathfrak{r},\varsigma) + k_2(\mathfrak{r},\varsigma)|x(\phi_4(\varsigma))|]d\varsigma)\big] \\
&+ l\,[N_2 + \eta_2\|y\|]\bigg[\int_0^{\mathfrak{r}} e^{-\alpha\varsigma}[|f_3(\varsigma,x(\varsigma)) - f_3(\varsigma,y(\varsigma))|]d\varsigma \\
&+ \big|x(\phi_3(\mathfrak{r})) - y(\phi_3(\mathfrak{r}))\big| \int_0^{\psi(\mathfrak{r})} |h(\mathfrak{r},\varsigma,x(\phi_4(\varsigma)))|d\varsigma \\
&+ |y(\phi_3(\mathfrak{r}))| \int_0^{\psi(\mathfrak{r})} |\, h(\mathfrak{r},\varsigma,x(\phi_4(\varsigma))) - h(\mathfrak{r},\varsigma,y(\phi_4(\varsigma)))\, |d\varsigma\bigg] \\
&\leq \eta_1 \,|x(\phi_1(\mathfrak{r})) - y(\phi_1(\mathfrak{r}))| \\
&+ \eta_2 \,|x(\phi_2(\mathfrak{r})) - y(\phi_2(\mathfrak{r}))|\,]\big[M + l(V + V_1 + V_2\, r) + l\,\|x\| \int_0^{\psi(\mathfrak{r})} [k_1(\mathfrak{r},\varsigma) + k_2(\mathfrak{r},\varsigma)\|x\|]d\varsigma)\big] \\
&+ l\,[N_2 + \eta_2\|y\|]\bigg[\int_0^{\mathfrak{r}} e^{-\alpha\varsigma}[|f_3(\varsigma,x(\varsigma))| + |f_3(\varsigma,y(\varsigma))|]d\varsigma \\
&+ \big|x(\phi_3(\mathfrak{r})) - y(\phi_3(\mathfrak{r}))\big| \int_0^{\psi(\mathfrak{r})} [k_1(\mathfrak{r},\varsigma) + k_2(\mathfrak{r},\varsigma)\|x\|]d\varsigma \\
&+ \|y\| \int_0^{\psi(\mathfrak{r})} |\, h(\mathfrak{r},\varsigma,x(\phi_4(\varsigma)))\big| + \big|h(\mathfrak{r},s,y(\phi_4(\varsigma)))\, |d\varsigma\bigg].
\end{aligned}
$$

Take into account the next two cases

(i) Select $T > 0$ so that for $\mathfrak{r} \geq T$, the given inequalities are true:

$$\int_0^{\mathfrak{r}} e^{-\alpha\varsigma}(a(\varsigma) + b(\varsigma)r)\, d\varsigma \leq \epsilon_1,$$

and

$$r \int_0^{\mathfrak{r}} [k_1(\mathfrak{r},\varsigma) + k_2(\mathfrak{r},\varsigma)r]\, d\varsigma \leq \epsilon_2.$$

Then, we have

$$
\begin{aligned}
&|\; \mathbb{F}x(\mathfrak{r}) - \mathbb{F}y(t)\; | \\
&\leq \eta_1 \,\|x - y\| + \eta_2\|x - y\|\big[(M + l(V + V_1 + V_2\, r) \\
&+ l\|x\| \int_0^{\mathfrak{r}} [k_1(\mathfrak{r},\varsigma) + k_2(\mathfrak{r},\varsigma)\|x\|]d\varsigma)\big] \\
&+ l[N_2 + \eta_2\|y\|]\bigg[\int_0^{\mathfrak{r}} e^{-\alpha\varsigma}(a(\varsigma) + b(\varsigma)\|x\|)ds + \int_0^{\mathfrak{r}} e^{-\alpha\varsigma}(a(\varsigma) + b(\varsigma)\|y\|)d\varsigma \\
&+ \|x - y\| \int_0^{\mathfrak{r}} [k_1(\mathfrak{r},\varsigma) + k_2(\mathfrak{r},\varsigma)\|x\|]d\varsigma \\
&+ \|y\|\big(\int_0^{\mathfrak{r}} [k_1(\mathfrak{r},\varsigma) + k_2(\mathfrak{r},\varsigma)\|x\|]d\varsigma + \int_0^{\mathfrak{r}} [k_1(\mathfrak{r},\varsigma) + k_2(\mathfrak{r},\varsigma)\|y\|]d\varsigma\big)\bigg] \\
&\leq \big(\eta_1 + \eta_2\,\big[M + l(V + V_1 + V_2\, r) + lr[k_1 + k_2 r]\big] + l[N_2 + \eta_2\, r][k_1 + k_2 r]\big)\,\epsilon \\
&+ 2l[N_2 + \eta_2\, r](\epsilon_1 + \epsilon_2).
\end{aligned}
$$

(ii) For $\mathfrak{r} \leq T$, define the function $\omega_h^T(x, \omega^T(\phi_4, \epsilon))$ and $\omega^T(f_3, \epsilon)$, where, for $\epsilon > 0$, we denote

$$\omega_h^T(x, \omega^T(\phi_4, \epsilon)) = \sup\{|h(\mathfrak{r},\varsigma,x(\varsigma)) - h(\mathfrak{r},\varsigma,y(\varsigma))| : \; \mathfrak{r},\varsigma \in [0,T], \quad \|x - y\| \leq \epsilon\},$$

$$\omega^T(f_3, \epsilon) = \sup\{|f_3(\mathfrak{r},x(\varsigma)) - f_3(\mathfrak{r},y(\varsigma))| : \; t \in [0,T], \quad \|x - y\| \leq \epsilon\}.$$

Considering that the function h has uniform continuity, we conclude that

$$\omega_h^T(x,\omega^T(\phi_4,\epsilon)),\ \ \omega^T(f_3,\epsilon) \to 0 \quad \text{as} \ \ \epsilon \to 0.$$

Therefore, using the above estimate in this case, we have

$$\begin{aligned}
&| \ \mathbb{F}x(\mathfrak{r}) - \mathbb{F}y(\mathfrak{r}) \ | \\
&\leq \eta_1 \ \|x-y\| + \eta_2 \ \|x-y\| \left[\ M + l(V+V_1+V_2\ r) + l\ \|x\| [k_1 + k_2\|x\|]\right)\right] \\
&+ l \ [N_2 + \eta_2 \|y\|] \left[\omega^T(f_3,\epsilon) + \|x-y\| [k_1 + k_2\|x\|] \ + \|y\| \ \omega_h^T(x,\omega^T(\phi_4,\epsilon)) \right] \\
&\leq \left[\eta_1 \ + \eta_2 \ \left[\ M + l(V+V_1+V_2\ r) + lr[k_1+k_2\ r]\right] + l\ [N_2 + \eta_2\ r][k_1+k_2 r]\right]\epsilon.
\end{aligned}$$

Finally, from the two cases (i) and (ii), and considering the previous information, we come to the conclusion that the operator $\mathbb{F}$ continuously maps the ball B_r into itself.

Now, let us take a nonempty subset X of B_r. Let $T > 0$ and $\epsilon > 0$ be given, and choose a function $x \in X$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in [0,T]$ such that $|\mathfrak{r}_2 - \mathfrak{r}_1| \leq \epsilon, \ \ \mathfrak{r}_1 \leq \mathfrak{r}_2$, then

$$\begin{aligned}
& \quad | \ \mathbb{F}x(\mathfrak{r}_2) - \mathbb{F}x(\mathfrak{r}_1) \ | \\
= & \quad \Big| f_1(\mathfrak{r}_2, x(\phi_1(\mathfrak{r}_2(\mathfrak{r})))) \\
+ & \quad f_2(\mathfrak{r}_2, x(\phi_2(\mathfrak{r}_2(t))) g\left(t_2, v_x(\mathfrak{r}_2), x(\phi_3(t_2(\mathfrak{r})) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2,\varsigma,x(\phi_4(\varsigma)))d\varsigma\right) \\
- & \quad f_1(\phi_1(\mathfrak{r}_1), x(\phi_1(\mathfrak{r}_1))) \\
+ & \quad f_2(\phi_2(\mathfrak{r}_1), x(\phi_2(\mathfrak{r}_1))) g\left(\mathfrak{r}_1, v_x(\mathfrak{r}_1), x(\phi_3(\mathfrak{r}_1)) \int_0^{\psi(\mathfrak{r}_1)} h(\mathfrak{r}_1,\varsigma,x(\phi_4(\varsigma)))d\varsigma\right)\Big| \\
\leq & \quad \big|f_1(\mathfrak{r}_2, x(\phi_1(\mathfrak{r}_2))) - f_1(\mathfrak{r}_1, x(\phi_1(\mathfrak{r}_1)))\big| \\
+ & \quad \Big|f_2(\mathfrak{r}_2, x(\phi_2(\mathfrak{r}_1))) g\left(\mathfrak{r}_2, v_x(\mathfrak{r}_2), x(\phi_3(\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2,\varsigma,x(\phi_4(\varsigma)))d\varsigma\right) \\
- & \quad f_2(\mathfrak{r}_1, x(\phi_2(\mathfrak{r}_1))) g\left(\mathfrak{r}_2, v_x(\mathfrak{r}_2), x(\phi_3(\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2,\varsigma,x(\phi_4(\varsigma)))d\varsigma\right)\Big| \\
+ & \quad \Big|f_2(\mathfrak{r}_1, x(\phi_2(\mathfrak{r}_1))) g\left(\mathfrak{r}_2, v_x(\mathfrak{r}_2), x(\phi_3(\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2,\varsigma,x(\phi_4(\varsigma)))d\varsigma\right) \\
- & \quad f_2(\mathfrak{r}_1, x(\phi_2(\mathfrak{r}_1))) g\left(\mathfrak{r}_1, v_x(\mathfrak{r}_1), x(\mathfrak{r}_1) \int_0^{\psi(\mathfrak{r}_1)} h(\mathfrak{r}_1,\varsigma,x(\phi_4(\varsigma)))d\varsigma\right)\Big| \\
\leq & \quad \big|f_1(\mathfrak{r}_2, x(\phi_1(\mathfrak{r}_2))) - f_1(\mathfrak{r}_1, x(\phi_1(\mathfrak{r}_1)))\big| \\
+ & \quad \big|f_2(\mathfrak{r}_2, x(\phi_2(\mathfrak{r}_2))) - f_2(\mathfrak{r}_1, x(\phi_2(\mathfrak{r}_1)))\big| \left|g\left(t_2, v_x(t_2), x(\phi_3(\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2,\varsigma,x(\phi_4(\varsigma)))d\varsigma\right)\right| \\
+ & \quad |f_2(\mathfrak{r}_1, x(\phi_2(\mathfrak{r}_1)))| \Bigg[\Bigg|g\left(\mathfrak{r}_2, v_x(\mathfrak{r}_2), x(\phi_3(\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2,\varsigma,x(\phi_4(\varsigma))\right) \\
- & \quad g\left(\mathfrak{r}_1, v_x(\mathfrak{r}_1), x(\phi_3(\mathfrak{r}_1)) \int_0^{\psi(\mathfrak{r}_1)} h(\mathfrak{r}_1,s,x(\phi_4(\varsigma)))d\varsigma\right)\Bigg|\Bigg] \\
\leq & \quad [|f_1(\mathfrak{r}_2, x(\phi_1(\mathfrak{r}_2))) - f_1(\mathfrak{r}_1, x(\phi_1(\mathfrak{r}_2)))| + |f_1(\mathfrak{r}_1, x(\phi_1(\mathfrak{r}_2))) - f_1(\mathfrak{r}_1, x(\phi_1(\mathfrak{r}_1)))|] \\
+ & \quad [|f_2(\mathfrak{r}_2, x(\phi_2(\mathfrak{r}_2))) - f_2(\mathfrak{r}_1, x(\phi_2(\mathfrak{r}_2)))| + |f_2(\mathfrak{r}_1, x(\phi_2(\mathfrak{r}_2))) - f_2(\mathfrak{r}_1, x(\phi_2(\mathfrak{r}_1)))|] \\
\times & \quad \left[|g(\mathfrak{r}_2,0,0)| + l(V+V_1+V_2\ r + |x(\phi_3(\mathfrak{r}_2))| \int_0^{\psi(\mathfrak{r}_2)} |h(t_2,\varsigma,x(\varsigma))|d\varsigma)\right] \\
+ & \quad [|f_2(\mathfrak{r}_1,0)| + \eta_2 |x(\phi_2(\mathfrak{r}_1))|] \Bigg[\Bigg|g\left(\mathfrak{r}_2, v_x(\mathfrak{r}_2), x(\phi((\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2,\varsigma,x(\varsigma))d\varsigma\right) \\
- & \quad g\left(\mathfrak{r}_1, v_x(\mathfrak{r}_2), x(\phi_3(\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2,\varsigma,x(\varsigma))d\varsigma\right)\Bigg|
\end{aligned}$$

$$
\begin{aligned}
+ \quad & \left| g\left(\mathfrak{r}_1, v_x(\mathfrak{r}_2), x(\phi_3(\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2, \varsigma, x(\varsigma)) d\varsigma \right) \right. \\
- \quad & \left. g\left(t_1, v_x(\mathfrak{r}_1), x(\phi_3(\mathfrak{r}_1)) \int_0^{\psi(\mathfrak{r}_1)} h(\mathfrak{r}_1, \varsigma, x(\varsigma)) d\varsigma \right) \right| \Big] \\
\leq \quad & [\theta_{f_1}(\delta) + \eta_1 |x(\phi_1(\mathfrak{r}_2)) - x(\phi_1(\mathfrak{r}_1))|] + [\theta_{f_2}(\delta) + \eta_2 |x(\phi_2(t_2)) - x(\phi_2(\mathfrak{r}_1))|] \\
\times \quad & [\, M + l(V + V_1 + V_2\, r) + l|x(\phi_3(\mathfrak{r}_2))| \int_0^{\psi(t_2)} [k_1(\mathfrak{r}_2, \varsigma) + k_2(\mathfrak{r}_2, \varsigma)|x(\phi_4(\varsigma)))|] d\varsigma] \\
+ \quad & [N_2 + \eta_2 |x(\phi_2(\mathfrak{r}_1))|] \Big[\theta_g(\delta) + l\, |v_x(\mathfrak{r}_2) - v_x(\mathfrak{r}_1)| \\
+ \quad & l\, \Big[|x(\phi_3(\mathfrak{r}_2)) \int_0^{\psi(\mathfrak{r}_2)} h(\mathfrak{r}_2, \varsigma, x(\varsigma)) d\varsigma - x(\phi_3(\mathfrak{r}_1)) \int_0^{\psi(\mathfrak{r}_1)} h(\mathfrak{r}_1, \varsigma, x(\phi(\varsigma))) \Big| d\varsigma \Big] \\
\leq \quad & [\theta_{f_1}(\delta) + \eta_1 |x(\phi_1(\mathfrak{r}_2)) - x(\phi_1(\mathfrak{r}_1))|] + [\theta_{f_2}(\delta) + \eta_2 |x(\phi_2(\mathfrak{r}_2)) - x(\phi_2(\mathfrak{r}_1))|] \\
\times \quad & [\, M + l(V + V_1 + V_2\, r) + l|x(\phi_3(\mathfrak{r}_2)| \int_0^{\psi(\mathfrak{r}_2)} [k_1(\mathfrak{r}_2, \varsigma) + k_2(\mathfrak{r}_2, \varsigma)|x(\phi_4(\varsigma)))|] d\varsigma] \\
+ \quad & [N_2 + \eta_2 |x(\phi_2(\mathfrak{r}_1))|] \Big[\theta_g(\delta) \\
+ \quad & l\theta_{v_x}(\delta) + l|x(\phi_3(\mathfrak{r}_2))| \int_0^{\psi(\mathfrak{r}_2)} [|h(\mathfrak{r}_2, \varsigma, x(\phi_4(\varsigma))) - h(\mathfrak{r}_1, \varsigma, x(\phi_4(\varsigma)))|] d\varsigma \\
+ \quad & l|x(\phi_3(\mathfrak{r}_2)) - x(\phi_3(\mathfrak{r}_1))| \int_0^{\psi(\mathfrak{r}_1)} |h(\mathfrak{r}_1, \varsigma, x(\phi_4(\varsigma)))| d\varsigma \\
+ \quad & l\big|x(\phi_3(\mathfrak{r}_2))\big| \int_{\psi(\mathfrak{r}_1)}^{\psi(\mathfrak{r}_2)} |h(\mathfrak{r}_1, \varsigma, x(\phi_4(\varsigma)))| d\varsigma \Big] \\
\leq \quad & [\theta_{f_1}(\delta) + \eta_1 |x(\phi_1(\mathfrak{r}_2)) - x(\phi_1(\mathfrak{r}_1))|] + [\theta_{f_2}(\delta) + \eta_2 |x(\phi_2(\mathfrak{r}_2)) - x(\phi_2(\mathfrak{r}_1))|] \\
\times \quad & [\, M + l(V + V_1 + V_2 r) + l|x(\phi_3(\mathfrak{r}_2))| \int_0^{\psi(\mathfrak{r}_2)} [k_1(\mathfrak{r}_2, \varsigma) + k_2(\mathfrak{r}_2, \varsigma)|x(\phi_4(\varsigma)))|] d\varsigma] \\
+ \quad & [N_2 + \eta_2 |x(\phi_2(\mathfrak{r}_1))|] \Big[\theta_g(\delta) + l\theta_{v_x}(\delta) + l|x(\phi_3(\mathfrak{r}_2))| \int_0^{\mathfrak{r}_2} \omega_h^T(x, \omega^T(\phi_4, \epsilon))\, d\varsigma \\
+ \quad & l|x(\phi_3(\mathfrak{r}_2)) - x(\phi_3(\mathfrak{r}_1))| \int_0^{\mathfrak{r}_1} [k_1(\mathfrak{r}_2, \varsigma) + k_2(\mathfrak{r}_2, \varsigma)|x(\phi_4(\varsigma)))|] d\varsigma \\
+ \quad & l\, \big|x(\phi_3(t_1))\big| \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} [k_1(\mathfrak{r}_2, \varsigma) + k_2(\mathfrak{r}_2, \varsigma)|x(\phi_4(\varsigma)))|] d\varsigma \Big] \\
\leq \quad & [\theta_{f_1}(\delta) + \eta_1 |x(\phi_1(\mathfrak{r}_2)) - x(\phi_1(\mathfrak{r}_1))|] \\
+ \quad & [\theta_{f_2}(\delta) + \eta_2 |x(\phi_2(\mathfrak{r}_2)) - x(\phi_2(\mathfrak{r}_1))|] \big[\, M + l\, (V + V_1 + v_2\, r) \\
+ \quad & l\|x\| \int_0^{\mathfrak{r}_2} [k_1(\mathfrak{r}_2, \varsigma) + k_2(\mathfrak{r}_2, \varsigma)\|x\|] d\varsigma] \\
+ \quad & [N_2 + \eta_2 \|x\|] \big[\theta_g(\delta) + l\theta_{v_x}(\delta) + l\|x\|\, \omega_h^T(x, \omega^T(\phi_4, \epsilon)) \\
+ \quad & l\, |x(\phi_3(\mathfrak{r}_2)) - x(\phi_3(\mathfrak{r}_1))| \int_0^{t_1} [k_1(\mathfrak{r}_2, \varsigma) + k_2(\mathfrak{r}_2, \varsigma)\|x\|] d\varsigma \\
+ \quad & l\, \|x\| \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} [k_1(\mathfrak{r}_2, \varsigma) + k_2(\mathfrak{r}_2, \varsigma)\|x\|] d\varsigma \\
\leq \quad & [\theta_{f_1}(\delta) + \eta_1\, \omega^T(x, \omega^T(\phi_1, \epsilon))] \\
+ \quad & [\theta_{f_2}(\delta) + \eta_2\, \omega^T(x, \omega^T(\phi_2, \epsilon))] \big[\, M + l(V + V_1 + V_2\, r) + l\, r\, [k_1 + k_2\, r] \big] \\
+ \quad & [N_2 + \eta_2\, r] \big[\theta_g(\delta) + l\theta_{v_x}(\delta) + lr\, \omega_h^T(x, \omega^T(\phi_4, \epsilon)) + l\, [\omega^T(x, \omega^T(\phi_3, \epsilon))]\, [k_1 + k_2\, r] \big],
\end{aligned}
$$

where we denote

$$\theta_{v_x}(\delta) = \sup\{|v_x(\mathfrak{r}_2) - v_x(\mathfrak{r}_1)| : \mathfrak{r}_1, \mathfrak{r}_2 \in [0,T],\ \mathfrak{r}_1 < \mathfrak{r}_2, |\mathfrak{r}_2 - \mathfrak{r}_1| < \delta,\ |x| \leq r\},$$
$$\theta_{f_i}(\delta) = \sup\{|f_i(\mathfrak{r}_2, x) - f_i(\mathfrak{r}_1, x)| : \mathfrak{r}_1, \mathfrak{r}_2 \in [0,T],\ \mathfrak{r}_1 < \mathfrak{r}_2, |\mathfrak{r}_2 - \mathfrak{r}_1| < \delta,\ |x| \leq r\}(i = 1,2),$$
$$\theta_g(\delta) = \sup\{|g(\mathfrak{r}_2, x) - g(\mathfrak{r}_1, x)| : \mathfrak{r}_1, \mathfrak{r}_2 \in [0,T],\ \mathfrak{r}_1 < \mathfrak{r}_2, |\mathfrak{r}_2 - \mathfrak{r}_1| < \delta,\ |x| \leq r\}.$$

We therefore, arrive at the following estimate:

$$\begin{aligned}
& \omega^T(\mathbb{F}x, \epsilon) \\
\leq\ & [\theta_{f_1}(\delta) + \eta_1\, \omega^T(x, \omega^T(\phi_1, \epsilon))] \\
+\ & [\theta_{f_2}(\delta) + \eta_2\, \omega^T(x, \omega^T(\phi_2, \epsilon))]\big[\, M + l(V + V_1 + V_2\, r) + l\, r\, [k_1 + k_2\, r]\big] \\
+\ & [N_2 + \eta_2\, r]\big[\theta_g(\delta) + l\theta_{v_x}(\delta) + lr\, \omega_h^T(x, \omega^T(\phi_4, \epsilon)) + l\, [\omega^T(x, \omega^T(\phi_3, \epsilon))]\, [k_1 + k_2\, r]\big],
\end{aligned} \tag{3}$$

subsequently, depending on the function $g, f_i : [0,\infty) \times B_r \to \mathbb{R},\ (i = 1,2)$ are uniform continuity, conditions (iii) and (iv), we have to deduce $\theta_{v_x}(\delta),\ \theta_g(\delta)$ and $\theta_{f_i}(\delta) \to 0,$ as $\delta \to 0$. Additionally, it is clear that $\omega^T(\phi_i, \epsilon) \to 0$ $(i = 1,2,3)$. As a result, when we combine the facts with the estimate (3), we obtain

$$w_0^T(FX) \quad \leq \quad \big[\eta_1 + \eta_2(M + l(V + V_1 + V_2 r) + lr(k_1 + k_2 r)) + l[N_2 + \eta_2 r](k_1 + k_2 r)\big] w_0^T(X).$$

Consequently, we obtain

$$w_0(FX) \leq \big[\eta_1 + \eta_2(M + l(V + V_1 + V_2 r) + lr(k_1 + k_2 r)) + l[N_2 + \eta_1 r](k_1 + k_2 r)\big] w_0(X). \tag{4}$$

In the following, we take a nonempty set $X \subset B_r$. Then for any $x,\ y \in X$, and fixed $\mathfrak{r} \geq 0$, we obtain

$$\begin{aligned}
& |\ \mathbb{F}x(\mathfrak{r}) - \mathbb{F}y(\mathfrak{r})\ | \\
& \leq\ \eta_1\, \|x - y\| + \eta_2\, \|x - y\|\, \Big[(M + l(V + V_1 + V_2\, r) + l\, \|x\| \int_0^{\mathfrak{r}} [k_1 + k_2\|x\|] d\varsigma\Big] \\
& + l\, [N_2 + \eta_2\|y\|] \bigg[\int_0^{\mathfrak{r}} e^{-\alpha\varsigma} [|f_3(\varsigma, x(\varsigma)) - f_3(\varsigma, y(\varsigma))|] d\varsigma \\
& + |x(\phi_3(\mathfrak{r})) - y(\phi_3(\mathfrak{r}))| \int_0^{\mathfrak{r}} [k_1 + k_2\|x\|] d\varsigma + \|y\| \int_0^{\mathfrak{r}} |\ h(\mathfrak{r}, \varsigma, x(\varsigma)) - h(\mathfrak{r}, \varsigma, y(\varsigma))\ | d\varsigma \bigg].
\end{aligned}$$

Hence, it is simple to arrive at the following inequality:

$$\begin{aligned}
& \mathrm{diam}(\mathbb{F}X)(\mathfrak{r}) \\
& \leq\ \eta_1\, \mathrm{diam}X(\mathfrak{r}) + \eta_2\, \mathrm{diam}X(\mathfrak{r})\big[\, M + l(V + V_1 + V_2\, r) + l\, r|[k_1 + k_2\, r]\big] \\
& +\ l\ \ [N_2 + \eta_2\, r]\mathrm{diam}X(\mathfrak{r})[k_1 + k_2\, r]\ \ + l\ \ [N_2 + \eta_2\, r]\omega^T(f_3, \epsilon) + r\omega_h^T(x, \omega^T(\phi_4, \epsilon))\big].
\end{aligned}$$

Now, considering our conditions, we discover this estimate:

$$\begin{aligned}
& \lim_{\mathfrak{r}\to\infty} \sup\ \mathrm{diam}\mathbb{F}X(\mathfrak{r}) \\
& \leq\ (\eta_1 + \eta_2\ \big[\, M + l(V + V_1 + V_2\, r) + l\ \ r|[k_1 + k_2\, r]\big] \\
& + l\, [N_2 + \eta_2\, r][k_1 + k_2\, r])\ \lim_{\mathfrak{r}\to\infty} \sup\ \mathrm{diam}X(\mathfrak{r}).
\end{aligned}$$

Then

$$\lim_{\mathfrak{r}\to\infty} \sup\ \mathrm{diam}\mathbb{F}X(\mathfrak{r}) \leq c\ \lim_{t\to\infty} \sup\ \mathrm{diam}X(\mathfrak{r}), \tag{5}$$

where we denote

$$c = \eta_1 + \eta_2\ \big[\, M + l(V + V_1 + V_2\, r) + l\ \ r|[k_1 + k_2\, r]\big] + l\, [N_2 + \eta_2\, r][k_1 + k_2\, r].$$

Clearly, given assumption (vi), we know that $c < 1$.

Finally, linking (4) and (5), using the formula that describes the measure of noncompactness [29,30], we arrive at the following inequality.

$$\mu(\mathbb{F}X) \leq c\,\mu(X). \tag{6}$$

Now, taking into account the condition that $c = \left[\, M + lV + l\;\; r|[k_1 + k_2\, r]\right] + l\,k\,[N + \eta\, r][\eta_3 + k_1 + k_2\, r]) < 1,$ and Darbo's fixed point theorem [15], we deduce that the ball B_r has a fixed point x for the operator $\mathbb{F}$. Clearly, the functional integral Equation (2) has a solution x. Additionally, consider that the ball B_r contains the image of the space $BC(R_+)$ under the operator F. We conclude that B_r contains the set Fix$\mathbb{F}$ of all the fixed points of $\mathbb{F}$. It is evident that all solutions to Equation (2) are included in the set Fix $\mathbb{F}$. On the other hand, we determine that the family $ker\mu$ includes the set Fix $\mathbb{F}$ [30]. Now, taking into account the description of sets belonging to $ker\mu$ (the kernel $ker\mu$ of this measure includes the nonempty and bounded subsets X of $BC(R_+)$ such that the thickness of the bundle generated by functions from X decreases to zero at infinity, and functions from X are locally equicontinuous on R_+ [26]), we conclude that all solutions to Equation (2) are globally asymptotically stable. □

3. Multi-Valued Problem

The multi-valued problems have attracted significant attention during the last few decades. The literature on this topic is now much enriched and contains a variety of results ranging from existence theory to the methods of solution for such problems (see [31–33]).

Now, consider the nonlinear functional–integral inclusions (1) with feedback control under the following assumption:

(v^*) Let $G(\mathfrak{r}, x) : R_+ \times R_+ \times R_+ \to 2^{R_+}$ satisfy the following assumptions:

- (a) The set $G(\mathfrak{r}, x, y)$ is a nonempty, closed, and convex subset for all $(\mathfrak{r}, x) \in R_+ \times R_+ \times R_+$.
- (b) The set-valued map $G : [0, \infty) \times R_+ \times R_+ \to 2^{R_+}$ is continuous and Lipschitzian set-valued map with a nonempty compact convex subset of 2^{R^+}, with a Lipschitz constant $l > 0$, such that

$$H_d(G(\mathfrak{r}, x_1, y_1), G(\mathfrak{r}, x_2, y_2)) \leq l(|x_1 - y_1| + |x_2 - y_2|), \;\; x_i, y_i \in R_+,\; i = 1, 2.$$

Existence Theorem

Now, from the main results obtained in Section 2, we deduce the following results for feedback control functional-integral inclusions (1).

Theorem 2. *Let assumptions (i)–(iv), (v^*), and (vi)–(vii) hold. Then, the functional inclusion (1) has at least one solution $x = x(t)$, which belongs to the space in the space $BC(R_+)$. Moreover, solutions of inclusion (1) are locally attractive.*

Proof. By replacing assumption (v) by (v^*) and using Theorem ([33], Section 9, Chapter 1, Theorem 1), we can deduce that the set of Lipschitz selections of G is nonempty and that there exists a Lipschitz function $g \in G$ with

$$|g(\mathfrak{r}, x_1, x_2) - g(\mathfrak{r}, y_1, y_2)| \leq l\,\left(|x_1 - y_1| + |x_2 - y_2|\right),$$

for each $\mathfrak{r} \in R_+$ and for all $x_i, y_i \in R, i = 1, 2$. Moreover, the function $\mathfrak{r} \to g(\mathfrak{r}, 0, 0)$ belongs to the space $BC(R_+)$, and we have

$$|g(\mathfrak{r}, x_1, x_2)| \leq l\,|x_1 - x_2| + M, \text{ where } M = \sup_{\mathfrak{r} \in I} |\{g(\mathfrak{r}, 0, 0)| : \mathfrak{r} \in R_+\} < \infty.$$

i.e., assumption (v) of Theorem 2 is satisfied. Therefore, all assumptions of Theorem 2 are met. and function g satisfies the differential Equation (2)

$$\begin{aligned} x(\mathfrak{r}) &= f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r}))) + f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r}))) \\ &\quad . \quad g\left(\mathfrak{r}, v_x, x(\phi_3(\mathfrak{r})) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))) \, d\varsigma\right), \ \mathfrak{r} \geq 0 \\ v_x(\mathfrak{r}) &= v(0) e^{-\alpha \mathfrak{r}^\beta} + \beta \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)} \varsigma^{\beta - 1} f_3(\varsigma, x(\varsigma)) d\varsigma. \end{aligned}$$

Therefore, any solution to Equation (2) is a solution of inclusion (1). □

4. General Discussion and Examples

In this section, we present some cases in the absence and presence of feedback control. Moreover, two illustrative examples are presented.

Remark 2. *Replace assumption (i) by*
(i^*) $\phi_i, : [0, \infty) \to [0, \infty)$, *for* $i = 1, 2, 3, 4$, *such that*

$$|\varphi(\mathfrak{r}) - \varphi(\varsigma)| \leq |\mathfrak{r} - \varsigma|, \quad \text{and} \quad \varphi(0) = 0.$$

In this case, we have that $\phi_i(\mathfrak{r})$ *for* $i = 1, 2, 3, 4$ *is continuous and* $\phi_i(\mathfrak{r}) \leq \mathfrak{r}$, *and under assumptions (i),* (ii^*)*, and (iii)–(iv) the functional Equation (2) has at least one solution* $x = x(\mathfrak{r})$*, which belongs to the space in the space* $BC(R_+)$.

- **In the case of the presence of a control variable**
  (1*) **Phanograph functional integral inclusion with feedback control**
  Letting $\psi(\mathfrak{r}) = \mathfrak{r}$ and $\phi_i(\mathfrak{r}) = \eta_i \mathfrak{r}, \mathfrak{r} \in I$ and $\eta_i \in (0,1), i = 1, 2, 3, 4$, then we have the Phanograph functional-integral inclusion

$$\frac{x(\mathfrak{r}) - f_1(\mathfrak{r}, x(\eta_1 \mathfrak{r}))}{f_2(\mathfrak{r}, x(\eta_2 \mathfrak{r}))} \in G\left(\mathfrak{r}, v(\mathfrak{r}), x(\eta_3 \mathfrak{r})) \int_0^{\mathfrak{r}} h(\mathfrak{r}, \varsigma, x(\eta_4 \, \varsigma)) \, d\varsigma\right), \ \mathfrak{r} \in R_+;$$

$$\frac{dv(\mathfrak{r})}{d\mathfrak{r}^\beta} = -\alpha v(\mathfrak{r}) + f_3(\mathfrak{r}, x(\mathfrak{r})), \ v_0 = v(0), \ \alpha \geq 0, \ \beta \in (0,1).$$

  (2*) **Retarded functional integral inclusion with feedback control**
  Let $\phi_i(\mathfrak{r}) = \mathfrak{r} - r_i$, $\mathfrak{r} \geq r_i > 0$, $i = 1, 2, 3, 4$ and $\phi_i(\mathfrak{r}) = 0$, $\mathfrak{r} < r_i$, $i = 1, 2, 3, 4$. Then, we have the functional retarded integral inclusion with feedback control

$$\begin{aligned} &\frac{x(\mathfrak{r}) - f_1(\mathfrak{r}, x(\mathfrak{r} - r_1))}{f_2(\mathfrak{r}, x(\mathfrak{r} - r_2))} \\ &\in G\left(\mathfrak{r}, v(\mathfrak{r}), x(\mathfrak{r} - r_3)) \int_0^{\mathfrak{r}} h(\mathfrak{r}, \varsigma, x(\varsigma - r_4)) \, d\varsigma\right), \ \mathfrak{r} \geq r_i > 0, \ i = 1, 2, ; \end{aligned}$$

$$\frac{dv(\mathfrak{r})}{d\mathfrak{r}^\beta} = -\alpha v(\mathfrak{r}) + f_3(\mathfrak{r}, x(\mathfrak{r})), \ v_0 = v(0), \ \alpha \geq 0, \ \beta \in (0,1).$$

  (3*) For $\alpha = 0$, then we obtain the functional retarded integral inclusion with feedback control

$$\begin{aligned} &\frac{x(\mathfrak{r}) - f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r})))}{f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r})))} \\ &\in G\left(\mathfrak{r}, v(\mathfrak{r}), x(\phi_3(\mathfrak{r})) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma))) \, d\varsigma\right), \ \mathfrak{r} \in R_+; \end{aligned}$$

$$\frac{dv(\mathfrak{r})}{d\mathfrak{r}^\beta} = f_3(\mathfrak{r}, x(\mathfrak{r})) \ \beta \in (0,1),$$

which gives

$$\frac{x(\mathfrak{r}) - f_1(\mathfrak{r}, x(\phi_1(\mathfrak{r})))}{f_2(\mathfrak{r}, x(\phi_2(\mathfrak{r})))} \in G\left(\mathfrak{r}, \beta \int_0^{\mathfrak{r}} e^{-\alpha(\mathfrak{r}^\beta - \varsigma^\beta)}\, \varsigma^{\beta-1}\, f_3(\varsigma, x(\varsigma)) d\varsigma, x(\phi_3(\mathfrak{r})) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma)))\, d\varsigma\right), \ \ \mathfrak{r} \in R_+.$$

(4*) Let $f_1(\mathfrak{r}, x) = 1$, $f_2(\mathfrak{r}, x)x$, $g(\mathfrak{r}, v, x) = x + v$, and $\phi_i(\mathfrak{r}) = \mathfrak{r}$; then, the cubic integral inclusion (1) takes the form

$$x(\mathfrak{r}) \in 1 + x(\mathfrak{r})\left[x(\mathfrak{r}) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\varsigma)) d\varsigma + v(\mathfrak{r})\right], \tag{7}$$

with
$$\frac{dv(\mathfrak{r})}{d\mathfrak{r}^\beta} = -\alpha v(\mathfrak{r}) + f_3(\mathfrak{r}, x(\mathfrak{r})),\ v_0 = v(0),\ \alpha \geq 0,\ \beta \in (0, 1).$$

- **In the case of the absence of control variable** $v(\mathfrak{r}) = 0$, we obtain some particular cases which that useful for the theory of qualitative analysis of some functional integral equations and important for some models and real problems.

(1) Let $f_i(\mathfrak{r}, x) = 1,\ (i = 1, 2)$; then, the integral inclusion (1) takes the form

$$x(\mathfrak{r}) \in g\big(\mathfrak{r}, x(\phi_3(\mathfrak{r})) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\varsigma)) d\varsigma\big), \tag{8}$$

and then, under the assumptions of Theorem 2, the functional integral inclusion (8) has at least one asymptotically stable solution $x \in BC(R_+)$.

(2) Let $f_1(\mathfrak{r}, x) = 1$, $f_2(\mathfrak{r}, x) = g(\mathfrak{r}, x) = x$, and $\phi_i(\mathfrak{r}) = \mathfrak{r}$; then, the cubic integral inclusion (1) takes the form

$$x(\mathfrak{r}) \in 1 + x^2(\mathfrak{r}) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\varsigma)) d\varsigma, \tag{9}$$

and under the assumptions of Theorem 2, then the cubic integral inclusion (9) has at least one asymptotically stable solution $x \in BC(R_+)$.

(3) Let $f_1(\mathfrak{r}, x(\mathfrak{r})) = f_2(\mathfrak{r}, x(\mathfrak{r})) = f(\mathfrak{r}, x(\mathfrak{r}))$ and $g(\mathfrak{r}, x) = 1 + x$, in the integral Equation (1); we obtain the cubic integral inclusion

$$x(\mathfrak{r}) \in 2f(\mathfrak{r}, x(\varphi_1(\mathfrak{r}))) + f(\mathfrak{r}, x(\varphi_2(\mathfrak{r})))x(\phi_3(\mathfrak{r})) \int_0^{\psi(\mathfrak{r})} h(\mathfrak{r}, \varsigma, x(\phi_4(\varsigma))) d\varsigma, \tag{10}$$

and under the assumptions of Theorem 2, then the cubic integral inclusion (10) has at least one asymptotically stable solution $x \in BC(R_+)$.

4.1. Example 1

Consider the following hybrid cubic functional integral inclusion:

$$\frac{x(\mathfrak{r}) - \frac{\mathfrak{r}}{4(1+\mathfrak{r}^2)} \arctan(\mathfrak{r} + x(\mathfrak{r}))}{\frac{1}{3\sqrt{\mathfrak{r}^2+24}} \frac{\sqrt{|x(\mathfrak{r})|}}{(1+\sqrt{|x(\mathfrak{r})|}}} \in \left[0, \frac{\mathfrak{r}}{1+\mathfrak{r}^2} \sin(\mathfrak{r} + x(\mathfrak{r}) \int_0^{\mathfrak{r}} [\frac{2\mathfrak{r} - \varsigma}{1+\mathfrak{r}^4} + \frac{\mathfrak{r}|x(\varsigma)|}{2\pi(\mathfrak{r}^2+1)(\varsigma+1)}] d\varsigma)\right], \quad \mathfrak{r} \geq 0, \tag{11}$$

with a fractal feedback control

$$\frac{dv(\mathfrak{r})}{d\mathfrak{r}^{\frac{1}{2}}} = -0.1v(\mathfrak{r}) + \frac{1}{8}e^{-\mathfrak{r}} cos(\mathfrak{r}) + \mathfrak{r}^2\, e^{\frac{-7}{5}\mathfrak{r}}\, x(\mathfrak{r}).$$

Now, we investigate the solvability of the integral Equation (11) on the space $BC(R_+)$. Take into account that this hybrid integral inclusion is a particular case of inclusion (1) with

$$\begin{aligned} f_1(\mathfrak{r},x(\mathfrak{r})) &= \frac{\mathfrak{r}}{4(1+\mathfrak{r}^2)}\ \arctan(\mathfrak{r}+x(\mathfrak{r})), \\ f_2(\mathfrak{r},x(\mathfrak{r})) &= \frac{1}{3\sqrt{\mathfrak{r}^2+24}}\frac{\sqrt{|x(\mathfrak{r})|}}{(1+\sqrt{|x(\mathfrak{r})|})}, \\ f_3(\mathfrak{r},x(\mathfrak{r})) &= \frac{1}{8}e^{-\mathfrak{r}}cos(\mathfrak{r})+\mathfrak{r}^2\,e^{\frac{-7}{5}\mathfrak{r}}\,x(\mathfrak{r}), \\ g(\mathfrak{r},x(\mathfrak{r})) &= \frac{1}{1+\mathfrak{r}}\ \sin(\mathfrak{r}+x(\mathfrak{r})), \\ h(\mathfrak{r},\varsigma,x(\varsigma)) &= \frac{2\mathfrak{r}-\varsigma}{1+\mathfrak{r}^4}+\frac{t|x(\varsigma)|}{2\pi(\mathfrak{r}^2+1)(\varsigma+1)}. \end{aligned}$$

Obviously, the function f_i, $(i = 1,2)$ is mutually continuous. Currently, for any $x,y\in R_+$ and $\mathfrak{r}\geq 0$,

$$\begin{aligned} |f_1(\mathfrak{r},x(\mathfrak{r}))-f_1(\mathfrak{r},y(\mathfrak{r}))| &\leq \frac{1}{8}|x(\mathfrak{r})-y(\mathfrak{r})|, \\ |f_2(\mathfrak{r},x(\mathfrak{r}))-f_2(\mathfrak{r},y(\mathfrak{r}))| &\leq \frac{1}{15}|x(\mathfrak{r})-y(\mathfrak{r})|, \\ |f_3(\mathfrak{r},x(\mathfrak{r}))| &\leq \frac{1}{8}e^{-\mathfrak{r}}+\mathfrak{r}^2\,e^{\frac{-7\mathfrak{r}}{5}}\ |x(\mathfrak{r})|. \end{aligned}$$

This indicates that condition (iv) is satisfied, with $N_1 = \frac{\pi}{16}$ and $N_2 = \frac{1}{15}$ and $\eta_1 = \frac{1}{8}$, $\eta_2 = \frac{1}{15}$, where $f_1(\mathfrak{r},0) = \frac{\mathfrak{r}}{1+\mathfrak{r}^2}$ arctan$(\mathfrak{r})$. Thus $\lim_{\mathfrak{r}\to\infty} f_1(\mathfrak{r},0) = 0$.
Also, $V = 0,1$, $V_1 = \frac{6}{13}$, and $V_2 = \frac{16}{27}$. On the other hand, we have

$$|g(\mathfrak{r},x(\mathfrak{r}))-g(\mathfrak{r},y(\mathfrak{r}))| \leq \frac{1}{1+\mathfrak{r}}\ |x(\mathfrak{r})-y(\mathfrak{r})| \leq \frac{|x(\mathfrak{r})-y(\mathfrak{r})|}{2}\ ,$$

where $l = \frac{1}{2}$ and $g(\mathfrak{r},0),= \frac{1}{1+\mathfrak{r}}$ sin$(\mathfrak{r})$. with $M = \frac{1}{2}$. Further, notice that the function $h(\mathfrak{r},\varsigma,x)$ satisfies assumption (v), where

$$|h(\mathfrak{r},\varsigma,x(\varsigma))| \leq \frac{2\mathfrak{r}-\varsigma}{1+\mathfrak{r}^4}+\frac{\mathfrak{r}|x(\varsigma)|}{2\pi(\mathfrak{r}^2+1)(\varsigma+1)}.$$

This indicates that we can insert $k_1(\mathfrak{r},\varsigma) = \frac{\mathfrak{r}(2\mathfrak{r}-\varsigma)}{2(1+\mathfrak{r}^4)}$ and $k_2(\mathfrak{r},\varsigma) = \frac{\mathfrak{r}}{2\pi(\mathfrak{r}^2+1)(\varsigma+1)}$.
To verify assumption (v), notice that

$$\lim_{\mathfrak{r}\to\infty}\int_0^{\mathfrak{r}} k_1(\mathfrak{r},\varsigma) = \lim_{\mathfrak{r}\to\infty}\int_0^{\mathfrak{r}} \frac{\mathfrak{r}(2\mathfrak{r}-\varsigma)}{2(1+\mathfrak{r}^4)}d\varsigma = \lim_{\mathfrak{r}\to\infty}\frac{3\mathfrak{r}^3}{4\mathfrak{r}^4+4} = 0,$$

and

$$\lim_{\mathfrak{r}\to\infty}\int_0^{\mathfrak{r}} k_2(\mathfrak{r},\varsigma) = \lim_{\mathfrak{r}\to\infty}\int_0^{\mathfrak{r}} \frac{\mathfrak{r}}{2\pi(\mathfrak{r}^2+1)(\varsigma+1)}d\varsigma = \lim_{\mathfrak{r}\to\infty}\frac{\mathfrak{r}\ln(\mathfrak{r}+1)}{2\pi\cdot(\mathfrak{r}^2+1)} = 0.$$

Moreover, we have $k_1 = 0.14246919...$ and $k_2 = 0.0906987$.

Finally, let us pay attention to the cubic equation of Theorem 2, which has the form

$$\begin{aligned} &\eta_2\, l\, k_2 r^3+\left[(k_1\eta_2+k_2N_2+\eta_2V_2)l\right]\, r^2 \\ &+\left[\eta_1+\eta_2M+lN_2(k_1+V_2)+\eta_2 l(V+V_1)\ -1\right]r+N_1+N_2M+N_2 l(V+V_1) = 0, \end{aligned} \tag{12}$$

and has the following root

$$r_1 = -21.3032,\ \ r_2 = 0.3.59388,\ \ \text{and}\ \ r_3 = 11.8394,$$

and it is easily seen that the root $r_0 = 0.3.59388$ of the previous equation satisfies the inequality such that

$$\eta_1 + \eta_2 \left[M + l(V + V_1 + V_2\, r_0) + l r_0 [k_1 + k_2 r_0]\right] + l[N_2 + \eta_2\, r_0][k_1 + k_2 r_0] \simeq 0.249183 \leq 1.$$

As a result, all the prerequisites of Theorem 2 are met. Hence, we conclude that inclusion (11) has at least one solution in the space $BC(R_+)$. Moreover, the solutions are locally attractive.

4.2. Example 2

Consider the problem (1) with a variable control v, when $f_1(\mathfrak{r}, x) = \frac{1}{1+\mathfrak{r}}$, $f_2(t, x) = x$, $G(t, v, x, w) = v + w$, $\phi_3(\mathfrak{r}) = \psi(\mathfrak{r}) = \phi_4(\mathfrak{r}) = \mathfrak{r}$ and $h(\mathfrak{r}, \varsigma, x) = \frac{\mathfrak{r}}{\mathfrak{r}+\varsigma} x(\varsigma)$. Then we obtain the following cubic integral inclusion involving the Chandrasekhar kernel

$$x(\mathfrak{r}) - \frac{1}{1+\mathfrak{r}} \in x(t)\left[v(\mathfrak{r}) + x(\mathfrak{r}) \int_0^{\mathfrak{r}} \frac{\mathfrak{r}}{\mathfrak{r}+\varsigma} x(\varsigma)\, d\varsigma\right], \quad \mathfrak{r} \in R_+ = [0, \infty);$$

with a fractal feedback control

$$\frac{dv(\mathfrak{r})}{d\mathfrak{r}^{\beta}} = -\alpha v(\mathfrak{r}) + f_3(\mathfrak{r}, x(\mathfrak{r})),\ v_0 = v(0),\ \alpha \geq 0.$$

When $f_2(t, x) = 1$, we obtain Chandrasekhar integral equation. In the radiative transfer, some problems are reduced by S. Chandrasekhar [34] to the well-known integral equation of Chandrasekhar type (see [35,36]).

5. Conclusions

It is known that in the more reasonable position, unanticipated factors may continuously disrupt a physical system. The alteration of the system's parameters typically causes these disruptions (perturbations). These perturbation functions can be thought of as control variables in the terminology of control theory. In particular, control variables may be taken into account while using integral equations.

The feedback control mechanism may be used in conjunction with harvesting, culling, or other biological control techniques. We recommend the reader to [8–10] for literature on feedback control system stability.

Inspired by these applications and by the existence results of differential and integral equations involving control variables that are obtained in [8,11,12], we have established the existence and the asymptotic stability of the solutions for a nonlinear cubic functional integral inclusion with a feedback control on the real half-line, using the technique associated with a measure of noncompactness [37,38].

Furthermore, by selecting an appropriate noncompactness measure, we have proved that such solutions are asymptotically stable in the Banach space $BC(R_+)$.

Our discussion is located in the class of bounded continuous functions $BC(R_+)$. Our work can be considered a formal generalization of the results of the theory of differential equations and inclusions Refs. [7,39–43].

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References

1. Caballero, J.; O'Regan, D.; Sadarangani, K. On nondecreasing solutions of cubic integral equations of Urysohn type. *Comment. Math. (Prace Math.)* **2004**, *44*, 39–53.
2. Awad, H.K.; Darwish, M.A. On Monotonic Solutions of a Cubic Urysohn Integral Equation with Linear Modification of the Argument. *Adv. Dyn. Syst. Appl.* **2018**, *13*, 91–99.
3. Awad, H.K.; Darwish, M.A.; Metwali, M.M.A. On a Cubic Integral Equation of Urysohn Type with Linear Perturbation of Second Kind. *JMA* **2018**, *41*, 29–38. [CrossRef]
4. Awad, H.K.; Darwish, M.A. On Erdélyi-Kober cubic fractional integral equation of Urysohn-Volterra type. *Differ. Equ. Control Process.* **2019**, *1*, 70–83.
5. El-Sayed, A.M.A.; Omar, Y.M.Y. Chandrasekhar quadratic and cubic integral equations via Volterra-Stieltjes quadratic integral equation. *Demonstr. Math.* **2021**, *54*, 25–36. [CrossRef]
6. El-Sayed, A.M.A.; Hashem, H.H.G.; Al-Issa, S.M. Analytical Contribution to a Cubic Functional Integral Equation with Feedback Control on the Real Half Axis. *Mathematics* **2023**, *11*, 1133. [CrossRef]
7. Cichoń, M. Differential inclusions and abstract control problems. *Bull. Aust. Math. Soc.* **1996**, *53*, 109–122. [CrossRef]
8. Chen, F. The permanence and global attractivity of Lotka–Volterra competition system with feedback controls. *Nonlinear Anal. Real World Appl.* **2006**, *7*, 133–143. [CrossRef]
9. Nasertayoob, P. Solvability and asymptotic stability of a class of nonlinear functional-integral equation with feedback control. *Commun. Nonlinear Anal.* **2018**, *5*, 19–27.
10. Nasertayoob, P.; Mansour Vaezpour, S. Positive periodic solution for a nonlinear neutral delay population equation with feedback control. *J. Nonlinear Sci. Appl.* **2013**, *6*, 152–161. [CrossRef]
11. El-Sayed, A.M.A.; Alrashdi, M.A.H. On a functional integral equation with constraint existence of solution and continuous dependence. *Int. J. Differ. Equ. Appl.* **2019**, *18*, 37–48.
12. El-Sayed, A.M.A.; Hamdallah, E.A.; Ahmed, R.G. On a nonlinear constrained problem of a nonlinear functional integral equation. *Appl. Anal. Optim.* **2022**, *6*, 95–107.
13. He, J.H. A new fractal derivation thermal science. *Year* **2011**, *15* (Suppl. 1), S145–S147.
14. Liang, Y.; Ninghu, S.; Wen, C. A time-space Hausdorff derivative model for anomalous transport in porous media. *Fract. Calc. Appl. Anal.* **2019**, *22*, 1517–1536. [CrossRef]
15. Darbo, G. Punti in transformazationi a condominio non compatto. *Rend. Sem. Math. Univ. Padova* **1955**, *4*, 84–92.
16. Banaś, J.; Dronka, J. Integral operators of Volterra-Stieltjes type, their properties and applications. *Math. Comput. Model.* **2000**, *32*, 1321–1331. [CrossRef]
17. Banaś, J.; Sadarangani, K. Solvability of Volterra-Stieltjes operator-integral equations and their applications. *Comput Math. Appl.* **2001**, *41*, 1535–1544. [CrossRef]
18. Banaś, J.; Rzepka, B. An application of a Measure of Noncompactness in the study of asymptotic stability. *Appl. Math. Lett.* **2003**, *16*, 1–6. [CrossRef]
19. Banaś, J.; Rzepka, B. On existence and asymptotic stability of solutions of nonlinear integral equation. *Math. Anal. Appl.* **2003**, *284*, 165–173. [CrossRef]
20. Banaś, J.; O'Regan, D. Volterra-Stieltjes integral operators. *Math. Comput. Model.* **2005**, *41*, 335–344. [CrossRef]
21. Banaś, J.; Mena, J.C. Some Properties of Nonlinear Volterra-Stieltjes Integral Operators. *Comput. Math. Appl.* **2005**, *49*, 1565–1573. [CrossRef]
22. Banaś, J.; Dubiel, A. Solvability of a Volterra-Stieltjes integral equation in the class of functions having limits at infinity. *Electron. J. Qual. Theory Differ. Equ.* **2017**, *53*, 1–17. [CrossRef]
23. Banaś, J.; Jleli, M.; Mursaleen, M.; Samet, B.; Vetro, C. (Eds.) *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*; Springer: Singapore; New Delhi, India, 2017.
24. Banaś, J.; Chlebowicz, A.; Woxsx, W. On measures of noncompactness in the space of functions defined on the half-axis with values in a Banach space. *J. Math. Anal. Appl.* **2020**, *489*, 124187. [CrossRef]
25. Banaś, J.; Chlebowicz, A. On solutions of an infinite system of nonlinear integral equations on the real half-axis. *Banach J. Math. Anal.* **2019**, *13*, 944–968. [CrossRef]
26. Banaś, J.; Dhage, B.C. Global asymptotic stability of solutions of a functional integral equation. *Nonlinear Anal.* **2008**, *69*, 1945–1952. [CrossRef]
27. Banaś, J.; O'Regan, D. On existence and local attractivity of solutions of a quadratic Volterra integral equation of fractional order. *J. Math. Anal. Appl.* **2008**, *345*, 573–582. [CrossRef]
28. Hu, X.; Yan, J. The global attractivity and asymptotic stability of solution of a nonlinear integral equation. *J. Math. Anal. Appl.* **2006**, *321*, 147–156. [CrossRef]
29. Banaś, J. Measures of noncompactness in the space of continuous tempered functions. *Demonstr. Math.* **1981**, *14*, 127–133. [CrossRef]
30. Banaś, J.; Goebel, K. Measure of noncompactnees in Banach spaces. In *Lecture Notes in Pure and Applied Mathematics*; Marcel Dekker: New York, NY, USA, 1980; Volume 60.
31. Aghajani, A.; Jalilian, Y. Existence and global attractivity of solutions of a nonlinear functional integral equation. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 3306–3312. [CrossRef]

32. Aghajani, A.; Banaś, J.; Sabzali, N. Some generalizations of Darbo fixed point theorem and applications. *Bull. Belg. Math. Soc. Simon Stevin*. **2013**, *20*, 345–358. [CrossRef]
33. Aubin, J.P.; Cellina, A. *Differential Inclusion*; Springer: Berlin/Heidelberg, Germany, 1984.
34. Chandrasekhar, S. *Radiative Transfer*; Oxford University Press: New York, NY, USA, 1960.
35. Chandrasekhar, S. On the radiative equilibrium of a stellar atmosphere XIV. *Astrophys. J*. **1947**, *105*, 164–203. [CrossRef]
36. Waziri, M.Y.; Leong, W.J.; Hassan, M.A.; Monsi, M. A low memory solver for integral equations of Chandrasekhar type in the Radiative transfer problems. *Math. Prob. Engin*. **2011**, *2011*, 467017.
37. Akhmerov, R.R.; Kamenskii, M.I.; Potapov, A.S.; Rodkina, A.E.; Sadovskii, B.N. Measure of noncompactness and condensing operators. In *Operator Theory: Advanced and Applications*; Brikhäuser Verlag: Basel, Switzerland, 1992; Volume 55.
38. Toledano, J.M.A.; Benavides, T.D.; Acedo, G.L. *Measures of Noncompactness in Metric Fixed Point Theory*; Brikhäuser Verlag: Basel, Switzerland, 1997.
39. Cichoń, M.; Cichoń, K.; Satco, B. Measure Differential Inclusions through Selection Principles in the Space of Regulated Functions. *Mediterr. J. Math*. **2018**, *15*, 148. [CrossRef]
40. Cichoń, K.; Satco, B. Differential inclusions and multivalued integrals, Discussiones Mathematicae Differential Inclusions. *Control Optim.* **2013**, *33*, 171–191.
41. Piazza, L.D.; Marraffa, V.; Satco, B. Approximating the solutions of differential inclusions driven by measures. *Ann. Mat. Pure Appl. 1923* **2019**, *198*, 2123–2140. [CrossRef]
42. Satco, B. Continuous dependence results for set-valued measure differential problems. *Electron. Qual. Theory Differ. Equ.* **2015**, *79*, 1–15. [CrossRef]
43. Satco, B. Existence results for Urysohn integral inclusions involving the Henstock integral. *J. Math. Anal. Appl.* **2007**, *336*, 44–53. [CrossRef]