Article

Approximate Solutions for Time-Fractional Fornberg–Whitham Equation with Variable Coefficients

Fahad Alsidrani [1,2] , **Adem Kılıçman** [2,3,*] and **Norazak Senu** [2,3]

1 Department of Mathematics, College of Science and Arts, Qassim University, Al Methnab 51931, Qassim, Saudi Arabia
2 Institute for Mathematical Research, Universiti Putra Malaysia, Serdang 43400, Selangor, Malaysia
3 Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, Serdang 43400, Selangor, Malaysia
* Correspondence: akilic@upm.edu.my

Abstract: In this research, three numerical methods, namely the variational iteration method, the Adomian decomposition method, and the homotopy analysis method are considered to achieve an approximate solution for a third-order time-fractional partial differential Equation (TFPDE). The equation is obtained from the classical (FW) equation by replacing the integer-order time derivative with the Caputo fractional derivative of order $\eta = (0, 1]$ with variable coefficients. We consider homogeneous boundary conditions to find the approximate solutions for the bounded space variable $l < \chi < L$ and $l, L \in \mathbb{R}$. To confirm the effectiveness of the proposed methods of non-integer order η, the computation of two test problems was presented. A comparison is made between the obtained results of the (VIM), (ADM), and (HAM) through tables and graphs. The numerical results demonstrate the effectiveness of the three numerical methods.

Keywords: fractional Fornberg–Whitham equation; approximate solution; partial differential equation; Riemann–Liouville derivatives; Caputo's derivatives; variational iteration method; Adomian decomposition method; homotopy analysis method

1. Introduction

The concept of fractional partial differential equations (FPDEs) has been the focus of many studies and an essential topic in computational mathematics due to their various applications in scientific fields. The fractional derivative allows for a more accurate description of the diffusion process, considering the effects of long-range interactions and memory effects in most biological systems and phenomena in physics [1]. In recent years, researchers have demonstrated that many phenomena are successfully described by mathematical models of non-integer order using mathematical tools, for example, the Keller–Segel model for chemotaxis [1], fractional Riccati differential equations [2], and diffusion wave equations [3,4]. The cost of solving large nonlinear systems and related large linear systems after linearization can vary depending on various factors, including the complexity of the system and the method used for solving the system. It is worth noticing that different discretization methods for fractional diffusion equations (FDEs) have been proposed to solve a large linear system. In [5], Donatelli et al. have studied the diffusion equation, which arises in many applications that involves fractional derivatives in the case of variable coefficients (FDE). The proposed method was based on spectral analysis and structure preserving preconditioners for solving the (FDE). The method involves discretization in the space of the fractional diffusion equation, which leads to a linear system with coefficient matrices having a Toeplitz-like structure. In addition, they have shown that the variable coefficient matrix sequence belongs to the generalized locally Toeplitz (GLT) sequences. In [6], Donatelli et al. have implemented a finite volume (FV) method to discretize the space-fractional diffusion Equation (SFDE) with variable coefficients and obtain a large linear system resulting in a

sequence of coefficient matrices. The fractional derivative is considered in the Riesz-space fractional derivative. They showed that the resulting sequence of coefficient matrices belongs to the generalized locally Toeplitz (GLT) sequences. They also developed a good preconditioner and multigrid method to efficiently solve the obtained linear system of equations. In [7], Lin et al. developed fast algorithms for solving the linear systems that arise from the discretized time-dependent space-fractional diffusion Equation (SFDE) with non-constant coefficients. They also proved the convergence of two iteration schemes, one pre-smoother and the other post-smoother. In [8], Bu et al. presented a numerical method for solving a large linear system of the multi-term time-fractional advection–diffusion Equation (MTADE) using the finite element multigrid method. The method is based on the fractional derivative in the Riesz–Caputo sense, and the finite element approximation is considered in the space direction and time direction, respectively. They also discussed the stability and convergence of fully discrete schemes of (MTADE) in two situations.

The fractional derivative (FD) is even more significant in modeling real-life situations; for example, the fractional partial nonlinear Fornberg–Whitham (FPNFW) equation is a mathematical model that describes the evolution of nonlinear dispersive waves in fluid dynamics and the behavior of waves in plasmas.

Consider the nonlinear time-fractional Fornberg–Whitham equation

$$\psi_\zeta^\eta(\chi,\zeta) - \psi_{\chi\chi\zeta}(\chi,\zeta) + \psi_\chi(\chi,\zeta) + \psi(\chi,\zeta)\psi_\chi(\chi,\zeta) - \psi(\chi,\zeta)\psi_{\chi\chi\chi}(\chi,\zeta) - 3\psi_\chi(\chi,\zeta)\psi_{\chi\chi}(\chi,\zeta) = 0 \qquad (1)$$

where $\psi(\chi,\zeta)$ is the fluid velocity, $0 < \eta \leq 1$ is the order of fractional equation, $\zeta > 0$ is the time, and χ is the spatial coordinate. In addition, when $\eta = 1$, Equation (1) is reduced to the original Fornberg–Whitham equation, which was first proposed by Whitham in 1967 for studying the qualitative behavior of wave breaking [9]. In 1978, Fornberg and Whitham [10] obtained a peaked solution of the form $\psi(\chi,\zeta) = \mathcal{K}exp\left(-1/2|\chi - 4\zeta/3|\right)$, where $\mathcal{K}$ is an arbitrary constant. In the literature, several mathematical methods have been implemented to obtain the approximate solutions of fractional differential equations, such as the Adomian decomposition method (ADM), variational iteration method (VIM), homotopy analysis method (HAM), homotopy perturbation method (HPM), Hermite wavelet method (HWM), optimal homotopy asymptotic method (OHAM), Shehu decomposition method (SDM), variational iteration transform method (VITM), Laplace decomposition method (LDM), direct power series method (DPSM), and others.

In [11], Kumar et al. solved the time-fractional Fornberg–Whitham equation involving the Atangana–Baleanu (AB) fractional derivative of non-integer order of the function $\psi(\chi,\zeta)$ by using the Laplace decomposition method (LDM). This method is a mix of Adomian's decomposition method and the Laplace transform approach. The existence of the solution and the uniqueness of the solution of the nonlinear Fornberg–Whitham equation of fractional order were examined. In [12], Gupta and Singh used the homotopy perturbation method (HPM) to find the approximate numerical solution of the time-fractional Fornberg–Whitham equation where the derivatives are taken in the Caputo sense. In [13], Merdan et al. implemented a differential transformation method (DTM) to obtain an approximate analytical solution of the fractional Fornberg–Whitham equation. In [14], Alderremy et al. used the natural transform decomposition method (NTDM) to obtain the approximate numerical solution of the fractional Fornberg–Whitham equation in view of the Caputo operator. In [15], Fayçal and Omrani used two powerful techniques, namely the homotopy analysis method (HAM) and the Adomian's decomposition method (ADM) to obtain an approximate analytical solution of the nonlinear Fornberg–Whitham equation, where $\eta = 1$, and concluded that these methods have perfect accuracy and reductions in the size of calculations. In [16], Wang et al. combined He's (HPM) and the fractional complex transform to find an approximate solution to the nonlinear time-fractional Fornberg–Whitham equation. Recently, in [17], Sartanpara et al. used the p-Homotopy analysis Shehu transform method for the time-fractional Fornberg–Whitham equation with the derivative of the fractional-order in the Caputo sense to obtain the approximate analytical solution.

In [18], Hijaz et al. numerically solved the Fornberg–Whitham classical type and modified type equations via the variational iteration algorithm-I. They used an auxiliary parameter to speed up the convergence rate to the exact solution. In [19], Shah et al. implemented modified techniques, namely the Shehu decomposition method (SDM) and the variational iteration transform method (VITM), to achieve an approximate analytical solution for the time-fractional Fornberg–Whitham equation. The fractional derivative is considered in the Caputo sense. In [20], Iqbal et al. successfully applied two modified methods to investigate the approximate solutions of the fractional Fornberg–Whitham equation. There was agreement between the numerical results obtained by the modified decomposition method (MDM) and modified variational iteration method (MVIM) involving fractional-order derivatives with Mittag–Leffler kernel.

In this paper, we consider the third-order time-fractional partial differential Equation (TFPDE) with variable coefficients

$$
\begin{aligned}
&\frac{\partial^\eta \psi(\chi,\zeta)}{\partial\zeta^\eta} - \alpha(\chi)\frac{\partial^3 \psi(\chi,\zeta)}{\partial\chi^2\partial\zeta} + 2\kappa(\chi)\frac{\partial\psi(\chi,\zeta)}{\partial\chi} + \beta(\chi)\psi(\chi,\zeta)\frac{\partial\psi(\chi,\zeta)}{\partial\chi} \\
&- \gamma(\chi)\psi(\chi,\zeta)\frac{\partial^3\psi(\chi,\zeta)}{\partial\chi^3} - \omega(\chi)\frac{\partial\psi(\chi,\zeta)}{\partial\chi}\frac{\partial^2\psi(\chi,\zeta)}{\partial\chi^2} = 0, \quad l < \chi < L, \quad l, L \in \mathbb{R}
\end{aligned}
\tag{2}
$$

with the initial and homogeneous boundary conditions

$$
\begin{aligned}
\psi(\chi,0) &= \phi(\chi) \\
\psi(l,\zeta) &= 0, \quad \psi(L,\zeta) = 0, \quad \zeta > 0
\end{aligned}
\tag{3}
$$

where $\alpha(\chi), \kappa(x), \beta(\chi), \gamma(\chi), \omega(\chi)$ are the variable coefficients and $p-1 < \eta \le p, (p \in \mathbb{N})$ is a parameter describing the order of the time-fractional equation. The reason behind including variable coefficients in the time-fractional Fornberg–Whitham equation is that it becomes a more accurate model for the propagation of waves. This makes it a useful tool for studying a variety of phenomena, including fluid dynamics, plasma physics, and others. It is important to note that Equation (2) contains different interesting nonlinear equations. In particular, for $\eta = 1$, when $\beta = 6 \, \gamma = -1$ and $\alpha = \kappa = \omega = 0$ in Equation (2), we obtain the well-known Korteweg–de Vries Equation (KdV), as given in [21]

$$
\psi_\zeta + 6\psi\psi_\chi + \psi\psi_{\chi\chi\chi} = 0, \quad \chi \in \mathbb{R}, \quad \zeta > 0.
\tag{4}
$$

When $\alpha = \gamma = 1$, $\kappa = 0$, $\beta = 4$, and $\omega = 3$ in Equation (2), we obtain the Degasperis–Procesi equation (DPE), as given in [22]

$$
\psi_\zeta - \psi_{\chi\chi\zeta} + 4\psi\psi_\chi - \psi\psi_{\chi\chi\chi} - 3\psi_\chi\psi_{\chi\chi} = 0, \quad \chi \in \mathbb{R}, \quad \zeta > 0.
\tag{5}
$$

When $\alpha = \omega = 1$, $\beta = 3$, $\gamma = 2$ in Equation (2), and $\kappa \in \mathbb{R}$ is a parameter related to the critical shallow water speed, we obtain the Camassa–Holm Equation (CHE), as given in [21]

$$
\psi_\zeta - \psi_{\chi\chi\zeta} + 2\kappa\psi_\chi + 3\psi\psi_\chi - 2\psi\psi_{\chi\chi\chi} - \psi_\chi\psi_{\chi\chi} = 0, \quad \chi \in \mathbb{R}, \quad \zeta > 0.
\tag{6}
$$

When $\alpha = \beta = \gamma = 1$, $\kappa = \frac{1}{2}$ and $\omega = 3$ in Equation (2), we obtain the Fornberg–Whitham Equation (FWE), as given in [9]

$$
\psi_\zeta - \psi_{\chi\chi\zeta} + \psi_\chi + \psi\psi_\chi - \psi\psi_{\chi\chi\chi} - 3\psi_\chi\psi_{\chi\chi} = 0, \quad \chi \in \mathbb{R}, \quad \zeta > 0
\tag{7}
$$

We organize the paper as follows. Section 2 defines preliminary definitions and some properties of the Riemann–Liouville integral and Caputo fractional derivative. In Section 3, the analysis of (VIM) for the nonlinear fractional equation is established. In Section 4, the analysis of (ADM) for the nonlinear fractional equation is established. In Section 5, the analysis of (HAM) for the nonlinear fractional equation is established. Section 6

illustrates the methods for solving time-fractional partial differential equations (TFPDEs) with suitable initial conditions.

2. Problem Formulation and Preliminaries

In this section, we present the definitions of partial Riemann–Liouville integrals, partial Riemann–Liouville derivatives, and Caputo time-fractional derivatives with some properties of the Caputo fractional derivatives, which will be used later.

Definition 1 ([23]). *Let* $\eta \in (0,1)$ *and* $\psi \in L^{-1}(D)$. *The partial Riemann–Liouville fractional integrals of order* η *of a function* $\psi(\chi, \zeta)$ *with respect to* ζ *are defined as*

$$_a^{RL}\mathcal{I}_\zeta^\eta(\psi(\chi, \zeta)) = \frac{1}{\Gamma(\eta)} \int_a^\zeta (\zeta - \xi)^{\eta-1} \psi(\chi, \xi) d\xi, \tag{8}$$

for almost all $(\chi, \zeta) \in D$ *and* $\Gamma(\eta)$ *is the well-known Gamma function.*

Definition 2 ([23]). *Let* $p - 1 < \eta \le p, p \in \mathbb{N}$ *and* $\psi \in L^{-1}(D)$. *The partial Riemann–Liouville fractional derivatives of order* η *of a function* $\psi(\chi, \zeta)$ *with respect to* ζ *are defined as*

$$_a\mathcal{D}_\zeta^\eta(\psi(\chi, \zeta)) = \frac{\partial^p}{\partial \zeta^p} {}_a\mathcal{I}_\zeta^{p-\eta}(\psi(\chi, \zeta)) = \frac{\partial^p}{\partial \zeta^p}\left(\frac{1}{\Gamma(p-\eta)} \int_a^\zeta (\zeta - \xi)^{p-\eta-1} \psi(\chi, \xi) d\xi\right) \tag{9}$$

for almost all $(\chi, \zeta) \in D$.

For $\psi \in C_\mu$, $\mu \ge -1$, $\gamma > -1$ *and* $\eta, \beta \ge 0$, *the operator* $_a\mathcal{I}_\zeta^\eta$ *satisfies the following properties* [24].

(1) $_a\mathcal{I}_\zeta^\eta {}_a\mathcal{I}_\zeta^\beta(\psi(\zeta)) = {}_a\mathcal{I}_\zeta^\beta {}_a\mathcal{I}_\zeta^\eta(\psi(\zeta)) = {}_a\mathcal{I}_\zeta^{\eta+\beta}(\psi(\zeta))$
(2) $_a\mathcal{I}_\zeta^\eta(\zeta - a)^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\eta+\gamma+1)}(\zeta - a)^{\eta+\gamma}$

Definition 3 ([25]). *Let* p *be the smallest integer that exceeds* η, *and the Caputo time-fractional derivative operator of order* $\eta > 0$ *of a function* $\psi(\chi, \zeta)$ *is defined as*

$$^C\mathcal{D}_\zeta^\eta(\psi(\chi, \zeta)) = \begin{cases} \mathcal{I}_\zeta^{p-\eta}\left(\frac{\partial^p \psi(\chi, \zeta)}{\partial \zeta^p}\right) = \frac{1}{\Gamma(p-\eta)} \int_0^\zeta (\zeta - \xi)^{p-\eta-1} \frac{\partial^p \psi(\chi, \xi)}{\partial \xi^p} d\xi, & p - 1 < \tau < p \\ \frac{\partial^p \psi(\chi, \zeta)}{\partial \zeta^p} & \eta = p \in \mathbb{N} \end{cases} \tag{10}$$

The operator $\mathcal{D}_\zeta^\eta$ *satisfies the following properties* [24]. *Let* $\zeta > 0$, $p - 1 < \eta \le p$, $(p \in \mathbb{N})$, *then*

(1) $\mathcal{I}_\zeta^\eta \mathcal{D}_\zeta^\eta(\psi(\chi, \zeta)) = \psi(\chi, \zeta) - \sum_{k=0}^{p-1} \frac{\zeta^k}{k!} \frac{\partial^k}{\partial \zeta^k} \psi(\chi, \zeta)\big|_{\zeta=0}$
(2) $\mathcal{D}_\zeta^\eta \mathcal{I}_\zeta^\eta(\psi(\chi, \zeta)) = \psi(\chi, \zeta)$

3. Analysis of Variational Iteration Method

In this section, we discuss the (VIM) solution for the time-fractional partial differential Equation (TFPDE) with variable coefficients. This method can reduce the size of calculations and directly handle both linear and nonlinear equations, homogeneous or non-homogeneous [26,27].

We consider the following time-fractional partial differential equation

$$\mathcal{D}_\zeta^\eta \psi(\chi, \zeta) + \mathcal{L}\psi(\chi, \zeta) + \mathcal{N}\psi(\chi, \zeta) = \mathcal{G}(\chi, \zeta) \tag{11}$$

where $\mathcal{D}_\zeta^\eta$ is the Caputo fractional derivative of order $p - 1 < \eta \le p$, $(p \in \mathbb{N})$, $\mathcal{L}$ is the linear operator, $\mathcal{N}$ is the nonlinear operator, and $\mathcal{G}(\chi, \zeta)$ is a known analytical function. According to the variational iteration method [28,29], to solve the third-order time-fractional partial differential Equation (TFPDE) with variable coefficients in Equation (2), the correction functional can be constructed as follows

$$\psi_{k+1}(\chi,\zeta) = \psi_k(\chi,\zeta) + \mathcal{I}_\zeta^\gamma \left[\lambda \left(\mathcal{D}_\zeta^\eta \psi_k(\chi,\zeta) + \mathcal{L}\tilde{\psi}_k(\chi,\zeta) + \mathcal{N}\tilde{\psi}_k(\chi,\zeta)\right)\right]$$

$$= \psi_k(\chi,\zeta) + \frac{1}{\Gamma(\gamma)} \int_0^\zeta (\zeta-\xi)^{\gamma-1}\lambda(\xi) \left(\frac{\partial^\eta \psi_k(\chi,\xi)}{\partial \xi^\eta} - \alpha(\chi)\frac{\partial^3 \tilde{\psi}_k(\chi,\xi)}{\partial \chi^2 \partial \xi} + 2\kappa(\chi)\frac{\partial \tilde{\psi}_k(\chi,\xi)}{\partial \chi}\right.$$

$$\left. + \beta(\chi)\tilde{\psi}_k(\chi,\xi)\frac{\partial \tilde{\psi}_k(\chi,\xi)}{\partial \chi} - \gamma(\chi)\tilde{\psi}_k(\chi,\xi)\frac{\partial^3 \tilde{\psi}_k(\chi,\xi)}{\partial \chi^3} - \omega(\chi)\frac{\partial \tilde{\psi}_k(\chi,\xi)}{\partial \chi}\frac{\partial^2 \tilde{\psi}_k(\chi,\xi)}{\partial \chi^2}\right)\mathrm{d}\xi, \tag{12}$$

where $\mathcal{I}_\zeta^\gamma$ denotes the Riemann–Liouville integral operator of order $\gamma = \eta + 1 - p$, subject to the initial and boundary conditions in Equation (3), where λ is a general Lagrange multiplier which can be identified by variational theory, ψ_k is the kth approximate solution and $\tilde{\psi}_k$ is considered as a restricted variation, i.e., $\delta\tilde{\psi}_k = 0$. This method first requires determining the Lagrange multiplier λ, and it can be easily identified as

$$\lambda(\xi) = \frac{(-1)^p(\xi-\zeta)^{p-1}}{(p-1)!} \tag{13}$$

where p is the highest order of the differential equation. By determining the value of Lagrange multiplier λ, the successive approximations ψ_{k+1} will be calculated using the given initial function ψ_0.

Making the above correction functional stationary and noticing that $\delta\tilde{\psi}_k = 0$, we obtain

$$\delta\psi_{k+1}(\chi,\zeta) = \delta\psi_k(\chi,\zeta) + \frac{1}{\Gamma(\gamma)}\delta \int_0^\zeta (\zeta-\xi)^{\gamma-1}\lambda(\xi)\left(\frac{\partial^\eta \psi_k(\chi,\xi)}{\partial \xi^\eta} - \alpha(\chi)\frac{\partial^3 \tilde{\psi}_k(\chi,\xi)}{\partial \chi^2 \partial \xi} + 2\kappa(\chi)\frac{\partial \tilde{\psi}_k(\chi,\xi)}{\partial \chi}\right.$$

$$\left. + \beta(\chi)\tilde{\psi}_k(\chi,\xi)\frac{\partial \tilde{\psi}_k(\chi,\xi)}{\partial \chi} - \gamma(\chi)\tilde{\psi}_k(\chi,\xi)\frac{\partial^3 \tilde{\psi}_k(\chi,\xi)}{\partial \chi^3} - \omega(\chi)\frac{\partial \tilde{\psi}_k(\chi,\xi)}{\partial \chi}\frac{\partial^2 \tilde{\psi}_k(\chi,\xi)}{\partial \chi^2}\right)\mathrm{d}\xi, \tag{14}$$

$$= \delta\psi_k(\chi,\zeta) + \frac{1}{\Gamma(\gamma)}\delta \int_0^\zeta (\zeta-\xi)^{\gamma-1}\lambda(\xi)\left(\frac{\partial^p \psi_k(\chi,\xi)}{\partial \xi^p}\right)\mathrm{d}\xi$$

This yields the Lagrange multipliers $\lambda(\xi) = -1$ for $p = 1$, and substituting this value of the Lagrange multiplier into the corrections functional Equation (12) gives the iteration formula for $0 < \eta \leq 1$

$$\psi_{k+1}(\chi,\zeta) = \psi_k(\chi,\zeta) - \mathcal{I}_\zeta^\eta\left(\frac{\partial^\eta \psi_k(\chi,\zeta)}{\partial \zeta^\eta} - \alpha(\chi)\frac{\partial^3 \psi_k(\chi,\zeta)}{\partial \chi^2 \partial \zeta} + 2\kappa(\chi)\frac{\partial \psi_k(\chi,\zeta)}{\partial \chi}\right.$$

$$\left. + \beta(\chi)\psi_k(\chi,\zeta)\frac{\partial \psi_k(\chi,\zeta)}{\partial \chi} - \gamma(\chi)\psi_k(\chi,\zeta)\frac{\partial^3 \psi_k(\chi,\zeta)}{\partial \chi^3} - \omega(\chi)\frac{\partial \psi_k(\chi,\zeta)}{\partial \chi}\frac{\partial^2 \psi_k(\chi,\zeta)}{\partial \chi^2}\right) \tag{15}$$

Considering the given initial condition values, $\psi_0(\chi,\zeta) = \phi(\chi)$, and using this selection in Equation (15), we obtain the following successive approximations for $k = 0, 1, \cdots$

$$\psi_1(\chi,\zeta) = \psi_0(\chi,\zeta) - \mathcal{I}_\zeta^\eta\left(\frac{\partial^\eta \psi_0(\chi,\zeta)}{\partial \zeta^\eta} - \alpha(\chi)\frac{\partial^3 \psi_0(\chi,\zeta)}{\partial \chi^2 \partial \zeta} + 2\kappa(\chi)\frac{\partial \psi_0(\chi,\zeta)}{\partial \chi}\right.$$

$$\left. + \beta(\chi)\psi_0(\chi,\zeta)\frac{\partial \psi_0(\chi,\zeta)}{\partial \chi} - \gamma(\chi)\psi_0(\chi,\zeta)\frac{\partial^3 \psi_0(\chi,\zeta)}{\partial \chi^3} - \omega(\chi)\frac{\partial \psi_0(\chi,\zeta)}{\partial \chi}\frac{\partial^2 \psi_0(\chi,\zeta)}{\partial \chi^2}\right) \tag{16}$$

$$\psi_2(\chi,\zeta) = \psi_1(\chi,\zeta) - \mathcal{I}_\zeta^\eta\left(\frac{\partial^\eta \psi_1(\chi,\zeta)}{\partial \zeta^\eta} - \alpha(\chi)\frac{\partial^3 \psi_1(\chi,\zeta)}{\partial \chi^2 \partial \zeta} + 2\kappa(\chi)\frac{\partial \psi_1(\chi,\zeta)}{\partial \chi}\right.$$

$$\left. + \beta(\chi)\psi_1(\chi,\zeta)\frac{\partial \psi_1(\chi,\zeta)}{\partial \chi} - \gamma(\chi)\psi_1(\chi,\zeta)\frac{\partial^3 \psi_1(\chi,\zeta)}{\partial \chi^3} - \omega(\chi)\frac{\partial \psi_1(\chi,\zeta)}{\partial \chi}\frac{\partial^2 \psi_1(\chi,\zeta)}{\partial \chi^2}\right) \tag{17}$$

$$\vdots$$

Thus, the corrections' functional Equation (15) will give a sequence of approximations $\psi_k(\chi, \zeta) = \sum_{m=0}^{k-1} \psi_m(\chi, \zeta)$.

Therefore, the solution of Equation (2) is given by

$$\psi(\chi, \zeta) = \lim_{k \to \infty} \psi_k(\chi, \zeta) \tag{18}$$

4. Analysis of Adomian Decomposition Method

In this section, we discuss the (ADM) solution for the time-fractional partial differential Equation (TFPDE) with variable coefficients. This method provides an analytical approximation to a rather wide class of nonlinear and stochastic equations without linearization, perturbation, closure approximations, or discretization methods resulting in massive numerical computation [30,31].

We consider the following time-fractional partial differential equation

$$\mathcal{D}_\zeta^\eta \psi(\chi, \zeta) + \mathcal{L}\psi(\chi, \zeta) + \mathcal{N}\psi(\chi, \zeta) = \mathcal{G}(\chi, \zeta) \tag{19}$$

where $\mathcal{D}_\zeta^\eta$ is the Caputo fractional derivative of order $p - 1 < \eta \le p$, $(p \in \mathbb{N})$, $\mathcal{L}$ is the linear operator, $\mathcal{N}$ is the nonlinear operator, and $\mathcal{G}(\chi, \zeta)$ is a known analytical function. To solve the third-order time-fractional partial differential Equation (TFPDE) with the variable coefficients shown in Equation (2), by Adomin decomposition method, we express this equation in the operator form as

$$\mathcal{D}_\zeta^\eta \psi(\chi, \zeta) - \alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi(\chi, \zeta) + 2\kappa(\chi)\mathcal{L}_\chi\psi(\chi, \zeta) + \beta(\chi)\psi(\chi, \zeta)\mathcal{L}_\chi\psi(\chi, \zeta)$$
$$- \gamma(\chi)\psi(\chi, \zeta)\mathcal{L}_{\chi\chi\chi}\psi(\chi, \zeta) - \omega(\chi)\mathcal{L}_\chi\psi(\chi, \zeta)\mathcal{L}_{\chi\chi}\psi(\chi, \zeta) = 0 \tag{20}$$

with the initial and boundary conditions shown in Equation (3), which are equivalent to

$$\mathcal{D}_\zeta^\eta \psi(\chi, \zeta) - \alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi(\chi, \zeta) + 2\kappa(\chi)\mathcal{L}_\chi\psi(\chi, \zeta) + \mathcal{N}\psi(\chi, \zeta) = 0 \tag{21}$$

Solving Equation (21) for $\mathcal{D}_\zeta^\eta \psi(\chi, \zeta)$, we obtain

$$\mathcal{D}_\zeta^\eta \psi(\chi, \zeta) = \alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi(\chi, \zeta) - 2\kappa(\chi)\mathcal{L}_\chi\psi(\chi, \zeta) - \mathcal{N}\psi(\chi, \zeta) \tag{22}$$

where $\alpha(\chi)$, $\kappa(\chi)$, $\beta(\chi)$, $\gamma(\chi)$, and $\omega(\chi)$ are continuous functions, η is the parameter describing the order of the time-fractional derivative, the notations $\mathcal{D}_\zeta^\eta = \frac{\partial^\eta}{\partial\zeta^\eta}$, $\mathcal{L}_{\chi\chi\zeta} = \frac{\partial^3}{\partial\chi^2\partial\zeta}$, $\mathcal{L}_\chi = \frac{\partial}{\partial\chi}$, $\mathcal{L}_{\chi\chi} = \frac{\partial^2}{\partial\chi^2}$, and $\mathcal{L}_{\chi\chi\chi} = \frac{\partial^3}{\partial\chi^3}$ are the symbolize of the linear operators, and $\mathcal{N}\psi(\chi, \zeta) = \beta(\chi)\psi(\chi, \zeta)\mathcal{L}_\chi\psi(\chi, \zeta) - \gamma(\chi)\psi(\chi, \zeta)\mathcal{L}_{\chi\chi\chi}\psi(\chi, \zeta) - \omega(\chi)\mathcal{L}_\chi\psi(\chi, \zeta)\mathcal{L}_{\chi\chi}\psi(\chi, \zeta)$ symbolizes the nonlinear operators. Applying the operator $\mathcal{I}_\zeta^\eta$ on both sides of Equation (22), with the basic properties of the operator $\mathcal{D}_\zeta^\eta$, we obtain

$$\mathcal{I}_\zeta^\eta \mathcal{D}_\zeta^\eta \psi(\chi, \zeta) = \mathcal{I}_\zeta^\eta \left[\alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi(\chi, \zeta) - 2\kappa(\chi)\mathcal{L}_\chi\psi(\chi, \zeta) - \mathcal{N}\psi(\chi, \zeta) \right] \tag{23}$$

$$\psi(\chi, \zeta) - \sum_{k=0}^{p-1} \frac{\zeta^k}{k!} \frac{\partial^k}{\partial\zeta^k}\psi(\chi, \zeta)\bigg|_{\zeta=0} = \mathcal{I}_\zeta^\eta \left[\alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi(\chi, \zeta) - 2\kappa(\chi)\mathcal{L}_\chi\psi(\chi, \zeta) - \mathcal{N}\psi(\chi, \zeta) \right] \tag{24}$$

$$\psi(\chi, \zeta) = \sum_{k=0}^{p-1} \frac{\zeta^k}{k!} \frac{\partial^k}{\partial\zeta^k}\psi(\chi, \zeta)\bigg|_{\zeta=0} + \mathcal{I}_\zeta^\eta \left[\alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi(\chi, \zeta) - 2\kappa(\chi)\mathcal{L}_\chi\psi(\chi, \zeta) - \mathcal{N}\psi(\chi, \zeta) \right] \tag{25}$$

Since the Adomain decomposition method is in the form of an infinite series

$$\psi(\chi, \zeta) = \sum_{m=0}^{\infty} \psi_m(\chi, \zeta) \tag{26}$$

and the nonlinear term $\mathcal{N}\psi(\chi,\zeta)$ can be decomposed into an infinite series of polynomials given by

$$\mathcal{N}\psi(\chi,\zeta) = \sum_{m=0}^{\infty} \mathcal{A}_m(\chi,\zeta) \tag{27}$$

where $\mathcal{A}_m(\chi,\zeta)$ are the Adomain polynomials of $\psi_0, \psi_1, \cdots, \psi_m$ defined by

$$\mathcal{A}_m(\psi_0, \psi_1, \cdots, \psi_m) = \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[\mathcal{N}\left(\sum_{j=0}^{\infty}\lambda^j\psi_j\right)\right]_{\lambda=0}, \quad m = 0, 1, \ldots \tag{28}$$

Therefore, the first Adomain polynomials for $\mathcal{N}\psi(\chi,\zeta)$ are defined by

$$\mathcal{A}_0 = \mathcal{N}(\psi_0)$$

$$\mathcal{A}_1 = \psi_1\left(\frac{d\mathcal{N}(\psi_0)}{d\psi_0}\right)$$

$$\mathcal{A}_2 = \psi_2\left(\frac{d\mathcal{N}(\psi_0)}{d\psi_0}\right) + \frac{\psi_1^2}{2!}\left(\frac{d^2\mathcal{N}(\psi_0)}{d\psi_0^2}\right)$$

$$\mathcal{A}_3 = \psi_3\left(\frac{d\mathcal{N}(\psi_0)}{d\psi_0}\right) + \psi_1\psi_2\left(\frac{d^2\mathcal{N}(\psi_0)}{d\psi_0^2}\right) + \frac{\psi_1^3}{3!}\left(\frac{d^3\mathcal{N}(\psi_0)}{d\psi_0^3}\right) \tag{29}$$

$$\mathcal{A}_4 = \psi_4\left(\frac{d\mathcal{N}(\psi_0)}{d\psi_0}\right) + \psi_1\psi_3\left(\frac{d^2\mathcal{N}(\psi_0)}{d\psi_0^2}\right) + \frac{\psi_2^2}{2!}\left(\frac{d^2\mathcal{N}(\psi_0)}{d\psi_0^2}\right) + \frac{\psi_1^2\psi_2}{2!}\left(\frac{d^3\mathcal{N}(\psi_0)}{d\psi_0^3}\right) + \frac{\psi_1^3}{4!}\mathcal{N}^{(4)}(\psi_0)$$

$$\vdots$$

Substituting Equations (26) and (27) into Equation (25), we obtain

$$\sum_{m=0}^{\infty}\psi_m(\chi,\zeta) = \sum_{k=0}^{p-1}\frac{\zeta^k}{k!}\frac{\partial^k}{\partial\zeta^k}\psi(\chi,\zeta)\bigg|_{\zeta=0} + \mathcal{I}_\zeta^\eta\left[\alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\sum_{m=0}^{\infty}\psi_m(\chi,\zeta) - 2\kappa(\chi)\mathcal{L}_\chi\sum_{m=0}^{\infty}\psi_m(\chi,\zeta) - \sum_{m=0}^{\infty}\mathcal{A}_m(\chi,\zeta)\right] \tag{30}$$

The Adomain decomposition method transforms Equation (30) into a set of recursive relations given by

$$\psi_0(\chi,\zeta) = \sum_{k=0}^{p-1}\frac{\zeta^k}{k!}\frac{\partial^k}{\partial\zeta^k}\psi(\chi,\zeta)\bigg|_{\zeta=0}$$

$$\psi_{m+1}(\chi,\zeta) = \mathcal{I}_\zeta^\eta\left[\alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi_m(\chi,\zeta) - 2\kappa(\chi)\mathcal{L}_\chi\psi_m(\chi,\zeta) - \mathcal{A}_m(\chi,\zeta)\right], \quad m \geq 0 \tag{31}$$

Let the expression

$$\psi_m(\chi,\zeta) = \sum_{k=0}^{m-1}\psi_k(\chi,\zeta) \tag{32}$$

be the *m*-term approximation of ψ. Using the above recursive relation Equation (31), we can obtain the first terms of (ADM) series solution for $m = 0, 1, \cdots$

$$\psi_1(\chi,\zeta) = \mathcal{I}_\zeta^\eta\left[\alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi_0(\chi,\zeta) - 2\kappa(\chi)\mathcal{L}_\chi\psi_0(\chi,\zeta) - \mathcal{A}_0(\chi,\zeta)\right] \tag{33}$$

$$\psi_2(\chi,\zeta) = \mathcal{I}_\zeta^\eta\left[\alpha(\chi)\mathcal{L}_{\chi\chi\zeta}\psi_1(\chi,\zeta) - 2\kappa(\chi)\mathcal{L}_\chi\psi_1(\chi,\zeta) - \mathcal{A}_1(\chi,\zeta)\right]$$

$$\vdots \tag{34}$$

Therefore, the approximate solution is

$$\psi(\chi,\zeta) = \lim_{m\to\infty}\psi_m(\chi,\zeta) \tag{35}$$

5. Analysis of Homotopy Analysis Method

In this section, we discuss the (HAM) solution for the time fractional partial differential Equation (TFPDE) with variable coefficients. Liao proposed a powerful and efficient method for nonlinear problems [32–34].

We consider the following time-fractional partial differential equation

$$\mathcal{D}_\zeta^\eta \psi(\chi, \zeta) + \mathcal{L}\psi(\chi, \zeta) + \mathcal{N}\psi(\chi, \zeta) = \mathcal{G}(\chi, \zeta) \tag{36}$$

where $\mathcal{D}_\zeta^\eta$ is the Caputo fractional derivative of order $p - 1 < \eta \le p$, $(p \in \mathbb{N})$, $\mathcal{L}$ is the linear operator, $\mathcal{N}$ is the nonlinear operator, and $\mathcal{G}(\chi, \zeta)$ is a known analytical function. To solve the third-order time-fractional partial differential Equation (TFPDE) with variable coefficients Equation (2) by homotopy analysis method, we consider the nonlinear operator

$$
\begin{aligned}
\mathcal{N}\big[\varphi(\chi, \zeta; \rho)\big] = {} & \frac{\partial^\eta \varphi(\chi, \zeta; \rho)}{\partial \zeta^\eta} - \alpha(\chi) \frac{\partial^3 \varphi(\chi, \zeta; \rho)}{\partial \chi^2 \partial \zeta} + 2\kappa(\chi) \frac{\partial \varphi(\chi, \zeta; \rho)}{\partial \chi} + \beta(\chi) \varphi(\chi, \zeta; \rho) \frac{\partial \varphi(\chi, \zeta; \rho)}{\partial \chi} \\
& - \gamma(\chi) \varphi(\chi, \zeta; \rho) \frac{\partial^3 \varphi(\chi, \zeta; \rho)}{\partial \chi^3} - \omega(\chi) \frac{\partial \varphi(\chi, \zeta; \rho)}{\partial \chi} \frac{\partial^2 \varphi(\chi, \zeta; \rho)}{\partial \chi^2} = 0, \quad \zeta > 0
\end{aligned}
\tag{37}
$$

and the linear operator

$$\mathcal{L}[\varphi(\chi, \zeta; \rho)] = \mathcal{D}_\zeta^\eta[\varphi(\chi, \zeta; \rho)] = \frac{\partial^\eta \varphi(\chi, \zeta; \rho)}{\partial \zeta^\eta} \tag{38}$$

subject to the initial and boundary conditions Equation (3), with the property $\mathcal{D}_\zeta^\eta(k) = 0$, where k is the integration constant. According to Liao [33], we can construct the zero-order deformation equation

$$(1 - \rho)\mathcal{D}_\zeta^\eta\big[\varphi(\chi, \zeta; \rho) - \psi_0(\chi, \zeta)\big] = \rho \hbar \mathcal{H}(\chi, \zeta)\mathcal{N}\big[\varphi(\chi, \zeta; \rho)\big] \tag{39}$$

where $\rho \in [0, 1]$ is an embedding parameter, $\mathcal{D}_\zeta^\eta$ is an auxiliary linear operator, $\varphi(\chi, \zeta; \rho)$ is a mapping function for $\psi(\chi, \zeta)$, $\psi_0(\chi, \zeta)$ is an initial guess of $\psi(\chi, \zeta)$, $\hbar$ is a nonzero auxiliary parameter, and $\mathcal{H}(\chi, \zeta)$ is a nonzero auxiliary function. Obviously, for $\rho = 0$ and $\rho = 1$, we have

$$
\begin{aligned}
\varphi(\chi, \zeta; 0) &= \psi_0(\chi, \zeta) \\
\varphi(\chi, \zeta; 1) &= \psi(\chi, \zeta)
\end{aligned}
\tag{40}
$$

Thus, as ρ moves from 0 to 1, the solution $\varphi(\chi, \zeta; \rho)$ varies from the initial guess $\psi_0(\chi, \zeta)$ to the solution $\psi(\chi, \zeta)$. Expanding $\varphi(\chi, \zeta; \rho)$ into the Taylor series with respect to the embedding parameter ρ, we obtain

$$\varphi(\chi, \zeta; \rho) = \psi_0(\chi, \zeta) + \sum_{m=1}^\infty \psi_m(\chi, \zeta)\rho^m \tag{41}$$

where

$$\psi_m(\chi, \zeta) = \frac{1}{m!} \frac{\partial^m \varphi(\chi, \zeta; \rho)}{\partial \rho^m}\bigg|_{\rho=0} \tag{42}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series Equation (41) converges at $\rho = 1$, then we have

$$\varphi(\chi, \zeta; 1) = \psi(\chi, \zeta) = \psi_0(\chi, \zeta) + \sum_{m=1}^\infty \psi_m(\chi, \zeta) \tag{43}$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao [35]. For $\hbar = -1$ and $\mathcal{H}(\chi, \zeta) = 1$, Equation (39) becomes

$$(1 - \rho)\mathcal{D}_\zeta^\eta\big[\varphi(\chi, \zeta; \rho) - \psi_0(\chi, \zeta)\big] + \rho \mathcal{N}\big[\varphi(\chi, \zeta; \rho)\big] = 0 \tag{44}$$

According to Equation (42), the governing equation can be deduced from the zero-order deformation Equation (39).

Define the vector

$$\vec{\psi}_m = \{\psi_0(\chi, \zeta), \psi_1(\chi, \zeta), \ldots, \psi_m(\chi, \zeta)\}$$

By differentiating Equation (39) m number of times with respect to the embedding parameter ρ then setting $\rho = 0$, and finally dividing them by $m!$, we obtain the so-called mth-order deformation equation

$$\mathcal{D}_\zeta^\eta[\psi_m(\chi, \zeta) - \chi_m \psi_{m-1}(\chi, \zeta)] = \hbar \mathcal{H}(\chi, \zeta) \mathcal{R}_m[\vec{\psi}_{m-1}(\chi, \zeta)] \tag{45}$$

where

$$\mathcal{R}_m[\vec{\psi}_{m-1}(\chi, \zeta)] = \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\varphi(\chi, \zeta; \rho)]}{\partial \rho^{m-1}}\bigg|_{\rho=0} \tag{46}$$

and

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases} \tag{47}$$

Applying the operator $\mathcal{I}_\zeta^\eta$ on both sides of Equation (45), with the basic properties of the operator $\mathcal{D}_\zeta^\eta$, we obtain the solution of the above mth-order deformation equation, with the assumption $\mathcal{H}(\chi, \zeta) = 1$

$$\psi_m(\chi, \zeta) = \chi_m \psi_{m-1}(\chi, \zeta) + \hbar \mathcal{I}_\zeta^\eta \mathcal{R}_m[\vec{\psi}_{m-1}(\chi, \zeta)] \tag{48}$$

where

$$\begin{aligned}
\mathcal{R}_m[\vec{\psi}_{m-1}(\chi, \zeta)] &= \frac{\partial^\eta \psi_{m-1}(\chi, \zeta)}{\partial \zeta^\eta} - \alpha(\chi)\frac{\partial^3 \psi_{m-1}(\chi, \zeta)}{\partial \chi^2 \partial \zeta} + 2\kappa(\chi)\frac{\partial \psi_{m-1}(\chi, \zeta)}{\partial \chi} \\
&+ \sum_{k=0}^{m-1}\left(\beta(\chi)\psi_k(\chi, \zeta)\frac{\partial \psi_{m-1-k}(\chi, \zeta)}{\partial \chi} - \gamma(\chi)\psi_k(\chi, \zeta)\frac{\partial^3 \psi_{m-1-k}(\chi, \zeta)}{\partial \chi^3} - \omega(\chi)\frac{\partial \psi_k(\chi, \zeta)}{\partial \chi}\frac{\partial^2 \psi_{m-1-k}(\chi, \zeta)}{\partial \chi^2}\right)
\end{aligned} \tag{49}$$

By using the above relation Equation (48) with the initial and boundary conditions Equation (3), we can obtain the first terms of the (HAM) series solution for $m = 1, 2, \cdots$

$$\begin{aligned}
\psi_1(\chi, \zeta) &= \chi_1 \psi_0(\chi, \zeta) + \hbar \mathcal{I}_\zeta^\eta \mathcal{R}_1[\vec{\psi}_0(\chi, \zeta)] \\
&= \hbar \mathcal{I}_\zeta^\eta\left[\frac{\partial^\eta \psi_0(\chi, \zeta)}{\partial \zeta^\eta} - \alpha(\chi)\frac{\partial^3 \psi_0(\chi, \zeta)}{\partial \chi^2 \partial \zeta} + 2\kappa(\chi)\frac{\partial \psi_0(\chi, \zeta)}{\partial \chi}\right. \\
&\left.+ \beta(\chi)\psi_0(\chi, \zeta)\frac{\partial \psi_0(\chi, \zeta)}{\partial \chi} - \gamma(\chi)\psi_0(\chi, \zeta)\frac{\partial^3 \psi_0(\chi, \zeta)}{\partial \chi^3} - \omega(\chi)\frac{\partial \psi_0(\chi, \zeta)}{\partial \chi}\frac{\partial^2 \psi_0(\chi, \zeta)}{\partial \chi^2}\right]
\end{aligned} \tag{50}$$

$$\begin{aligned}
\psi_2(\chi, \zeta) &= \chi_2 \psi_1(\chi, \zeta) + \hbar \mathcal{I}_\zeta^\eta \mathcal{R}_2[\vec{\psi}_1(\chi, \zeta)] \\
&= \psi_1(\chi, \zeta) + \hbar \mathcal{I}_\zeta^\eta\left[\frac{\partial^\eta \psi_1(\chi, \zeta)}{\partial \zeta^\eta} - \alpha(\chi)\frac{\partial^3 \psi_1(\chi, \zeta)}{\partial \chi^2 \partial \zeta} + 2\kappa(\chi)\frac{\partial \psi_1(\chi, \zeta)}{\partial \chi}\right. \\
&+ \beta(\chi)\psi_0(\chi, \zeta)\frac{\partial \psi_1(\chi, \zeta)}{\partial \chi} - \gamma(\chi)\psi_0(\chi, \zeta)\frac{\partial^3 \psi_1(\chi, \zeta)}{\partial \chi^3} - \omega(\chi)\frac{\partial \psi_0(\chi, \zeta)}{\partial \chi}\frac{\partial^2 \psi_1(\chi, \zeta)}{\partial \chi^2} \\
&\left.+ \beta(\chi)\psi_1(\chi, \zeta)\frac{\partial \psi_0(\chi, \zeta)}{\partial \chi} - \gamma(\chi)\psi_1(\chi, \zeta)\frac{\partial^3 \psi_0(\chi, \zeta)}{\partial \chi^3} - \omega(\chi)\frac{\partial \psi_1(\chi, \zeta)}{\partial \chi}\frac{\partial^2 \psi_0(\chi, \zeta)}{\partial \chi^2}\right]
\end{aligned} \tag{51}$$

$$\vdots$$

Therefore, we obtain an accurate approximation of Equation (2)

$$\psi(\chi, \zeta) = \sum_{k=0}^{m} \psi_k(\chi, \zeta) \tag{52}$$

6. Applications and Results

In this section, we apply (VIM), (ADM), and (HAM) to obtain the approximate solutions to the third-order time-fractional partial differential Equation (TFPDE) with variable coefficients and suitable initial conditions.

Example 1. *Consider the third-order time-fractional partial differential Equation (TFPDE) with variable coefficients*

$$\frac{\partial^\eta \psi(\chi, \zeta)}{\partial \zeta^\eta} - \chi \frac{\partial^3 \psi(\chi, \zeta)}{\partial \chi^2 \partial \zeta} + 2\chi \frac{\partial \psi(\chi, \zeta)}{\partial \chi} + \chi \psi(\chi, \zeta) \frac{\partial \psi(\chi, \zeta)}{\partial \chi}$$
$$- \chi \psi(\chi, \zeta) \frac{\partial^3 \psi(\chi, \zeta)}{\partial \chi^3} - \chi \frac{\partial \psi(\chi, \zeta)}{\partial \chi} \frac{\partial^2 \psi(\chi, \zeta)}{\partial \chi^2} = 0, \quad \zeta > 0, \quad 0 \le \chi \le 1, \quad 0 < \eta \le 1 \tag{53}$$

and the initial and boundary conditions

$$\psi(\chi, 0) = e^\chi$$
$$\psi(0, \zeta) = 0, \quad \psi(1, \zeta) = 0, \quad \zeta > 0 \tag{54}$$

Applying VIM: the iteration formula for Equation (53) can be constructed as

$$\psi_{k+1}(\chi, \zeta) = \psi_k(\chi, \zeta) - \mathcal{I}_\zeta^\eta \left(\frac{\partial^\eta \psi_k(\chi, \zeta)}{\partial \zeta^\eta} - \chi \frac{\partial^3 \psi_k(\chi, \zeta)}{\partial \chi^2 \partial \zeta} + 2\chi \frac{\partial \psi_k(\chi, \zeta)}{\partial \chi} \right.$$
$$\left. + \chi \psi_k(\chi, \zeta) \frac{\partial \psi_k(\chi, \zeta)}{\partial \chi} - \chi \psi_k(\chi, \zeta) \frac{\partial^3 \psi_k(\chi, \zeta)}{\partial \chi^3} - \chi \frac{\partial \psi_k(\chi, \zeta)}{\partial \chi} \frac{\partial^2 \psi_k(\chi, \zeta)}{\partial \chi^2} \right) \tag{55}$$

Considerin the given initial condition values, and using this selection in Equation (55), we obtain the following successive approximations

$$\psi_0(\chi, \zeta) = e^\chi$$
$$\psi_1(\chi, \zeta) = e^\chi \left(1 + \frac{\chi(e^\chi - 2)\zeta^\eta}{\Gamma(\eta + 1)} \right)$$
$$\psi_2(\chi, \zeta) = e^\chi \left(1 + \frac{\chi(e^\chi - 2)\zeta^\eta}{\Gamma(\eta + 1)} + \frac{2\chi(2\chi e^\chi + 2e^\chi - \chi - 2)\zeta^{2\eta-1}}{\Gamma(2\eta)} - \frac{4\chi(-3\chi e^{2\chi} - 4e^{2\chi} + 2\chi e^\chi + 3e^\chi - \chi - 1)\zeta^{2\eta}}{\Gamma(2\eta + 1)} \right. \tag{56}$$
$$\left. - \frac{\Gamma(2\eta + 1)\chi e^\chi (24 e^\chi \chi^2 - 14\chi^2 e^{2\chi} + 52\chi e^\chi - 23\chi e^{2\chi} - 4\chi^2 + 12 e^\chi - 4 e^{2\chi} - 20\chi - 8)\zeta^{3\eta}}{\Gamma(3\eta + 1)\Gamma(\eta + 1)^2} \right)$$

Hence, the approximate solution for Equation (53) is

$$\psi_2(\chi, \zeta) = e^\chi \left(3 + \frac{2\chi(e^\chi - 2)\zeta^\eta}{\Gamma(\eta + 1)} + \frac{2\chi(2\chi e^\chi + 2e^\chi - \chi - 2)\zeta^{2\eta-1}}{\Gamma(2\eta)} - \frac{4\chi(-3\chi e^{2\chi} - 4e^{2\chi} + 2\chi e^\chi + 3e^\chi - \chi - 1)\zeta^{2\eta}}{\Gamma(2\eta + 1)} \right.$$
$$\left. - \frac{\Gamma(2\eta + 1)\chi e^\chi (24 e^\chi \chi^2 - 14\chi^2 e^{2\chi} + 52\chi e^\chi - 23\chi e^{2\chi} - 4\chi^2 + 12 e^\chi - 4 e^{2\chi} - 20\chi - 8)\zeta^{3\eta}}{\Gamma(3\eta + 1)\Gamma(\eta + 1)^2} \right) \tag{57}$$

Applying ADM: The recursive relations for Equation (53) can be constructed as

$$\psi_0(\chi, \zeta) = \sum_{k=0}^{p-1} \frac{\zeta^k}{k!} \frac{\partial^k}{\partial \zeta^k} \psi(\chi, \zeta) \Big|_{\zeta=0}$$
$$\psi_{m+1}(\chi, \zeta) = \mathcal{I}_\zeta^\eta \left[\chi \mathcal{L}_{\chi\chi\zeta} \psi_m(\chi, \zeta) - 2\chi \mathcal{L}_\chi \psi_m(\chi, \zeta) - \mathcal{A}_m(\chi, \zeta) \right], \quad m \ge 0 \tag{58}$$

Using the above recursive relations, we can obtain the first terms of the (ADM) series solution

$$
\begin{aligned}
\psi_0(\chi,\zeta) &= e^\chi \\
\psi_1(\chi,\zeta) &= \frac{\chi e^\chi (e^\chi - 2)\zeta^\eta}{\Gamma(\eta+1)} \\
\psi_2(\chi,\zeta) &= \chi e^\chi \left(\frac{(4\chi e^\chi + 4e^\chi - 2\chi - 4)\zeta^{2\eta-1}}{\Gamma(2\eta)} + \frac{(2\chi e^{3\chi} + e^{3\chi} - 4\chi e^{2\chi} - 2e^{2\chi} - 4\chi e^\chi - 2e^\chi + 4\chi + 4)\zeta^{2\eta}}{\Gamma(2\eta+1)} \right)
\end{aligned}
\tag{59}
$$

Hence, the (ADM) series solution for Equation (53) is

$$
\psi(\chi,\zeta) = e^\chi \left(1 + \frac{\chi(e^\chi - 2)\zeta^\eta}{\Gamma(\eta+1)} + \frac{\chi(4\chi e^\chi + 4e^\chi - 2\chi - 4)\zeta^{2\eta-1}}{\Gamma(2\eta)} + \frac{\chi(2\chi e^{3\chi} + e^{3\chi} - 4\chi e^{2\chi} - 2e^{2\chi} - 4\chi e^\chi - 2e^\chi + 4\chi + 4)\zeta^{2\eta}}{\Gamma(2\eta+1)} \right)
\tag{60}
$$

Applying HAM: the mth-order deformation equation for Equation (53) is given by

$$
\begin{aligned}
\psi_m(\chi,\zeta) &= \chi_m \psi_{m-1}(\chi,\zeta) + \hbar \mathcal{I}_\zeta^\eta \left(\frac{\partial^\eta \psi_{m-1}(\chi,\zeta)}{\partial \zeta^\eta} - \chi \frac{\partial^3 \psi_{m-1}(\chi,\zeta)}{\partial \chi^2 \partial \zeta} + 2\chi \frac{\partial \psi_{m-1}(\chi,\zeta)}{\partial \chi} \right. \\
&\left. + \sum_{k=0}^{m-1} \left(\chi \psi_k(\chi,\zeta) \frac{\partial \psi_{m-1-k}(\chi,\zeta)}{\partial \chi} - \chi \psi_k(\chi,\zeta) \frac{\partial^3 \psi_{m-1-k}(\chi,\zeta)}{\partial \chi^3} - \chi \frac{\partial \psi_k(\chi,\zeta)}{\partial \chi} \frac{\partial^2 \psi_{m-1-k}(\chi,\zeta)}{\partial \chi^2} \right) \right)
\end{aligned}
\tag{61}
$$

where

$$
\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}
\tag{62}
$$

Using the above relation Equation (61), we can obtain the first terms of (HAM) series solution

$$
\begin{aligned}
\psi_0(\chi,\zeta) &= e^\chi \\
\psi_1(\chi,\zeta) &= -\frac{\chi e^\chi (e^\chi - 2)\hbar\zeta^\eta}{\Gamma(\eta+1)} \\
\psi_2(\chi,\zeta) &= \chi e^\chi \left(-\frac{(e^\chi - 2)(\hbar+1)\hbar\zeta^\eta}{\Gamma(\eta+1)} + \frac{4(3\chi e^{2\chi} + 4e^{2\chi} - 2\chi e^\chi - 3e^\chi + \chi + 1)\hbar^2\zeta^{2\eta}}{\Gamma(2\eta+1)} + \frac{4((\chi+1)e^\chi - \frac{\chi}{2} - 1)\hbar^2\zeta^{2\tau-1}}{\Gamma(2\eta)} \right)
\end{aligned}
\tag{63}
$$

Hence, the (HAM) series solution for Equation (53) is

$$
\psi(\chi,\zeta) = e^\chi \left(1 - \frac{\chi(e^\chi - 2)\hbar^2\zeta^\eta}{\Gamma(\eta+1)} + \frac{4\chi(3\chi e^{2\chi} + 4e^{2\chi} - 2\chi e^\chi - 3e^\chi + \chi + 1)\hbar^2\zeta^{2\eta}}{\Gamma(2\eta+1)} + \frac{4\chi(\chi e^\chi + e^\chi - \frac{\chi}{2} - 1)\hbar^2\zeta^{2\tau-1}}{\Gamma(2\eta)} \right)
\tag{64}
$$

Example 2. *Consider the third-order time-fractional partial differential Equation (TFPDE) with variable coefficients*

$$
\begin{aligned}
&\frac{\partial^\eta \psi(\chi,\zeta)}{\partial \zeta^\eta} - \chi^2 \frac{\partial^3 \psi(\chi,\zeta)}{\partial \chi^2 \partial \zeta} + 2\frac{\partial \psi(\chi,\zeta)}{\partial \chi} + \psi(\chi,\zeta)\frac{\partial \psi(\chi,\zeta)}{\partial \chi} \\
&- \psi(\chi,\zeta)\frac{\partial^3 \psi(\chi,\zeta)}{\partial \chi^3} - \frac{\partial \psi(\chi,\zeta)}{\partial \chi}\frac{\partial^2 \psi(\chi,\zeta)}{\partial \chi^2} = 0, \quad \zeta > 0, \quad 0 \le \chi \le 1, \quad 0 < \eta \le 1
\end{aligned}
\tag{65}
$$

and the initial and boundary conditions

$$
\begin{aligned}
\psi(\chi,0) &= \chi^2 \\
\psi(0,\zeta) &= 0, \quad \psi(1,\zeta) = 0, \quad \zeta > 0
\end{aligned}
\tag{66}
$$

Applying VIM: the iteration formula for Equation (65) can be constructed as

$$\psi_{k+1}(\chi,\zeta) = \psi_k(\chi,\zeta) - \mathcal{I}_\zeta^\eta \left(\frac{\partial^\eta \psi_k(\chi,\zeta)}{\partial \zeta^\eta} - \chi^2 \frac{\partial^3 \psi_k(\chi,\zeta)}{\partial \chi^2 \partial \zeta} + 2 \frac{\partial \psi_k(\chi,\zeta)}{\partial \chi} \right.$$
$$\left. + \psi_k(\chi,\zeta) \frac{\partial \psi_k(\chi,\zeta)}{\partial \chi} - \psi_k(\chi,\zeta) \frac{\partial^3 \psi_k(\chi,\zeta)}{\partial \chi^3} - \frac{\partial \psi_k(\chi,\zeta)}{\partial \chi} \frac{\partial^2 \psi_k(\chi,\zeta)}{\partial \chi^2} \right) \tag{67}$$

Considering the given initial condition values, and using this selection in Equation (66), we obtain the following successive approximations

$$\psi_0(\chi,\zeta) = \chi^2$$
$$\psi_1(\chi,\zeta) = \chi^2 - \frac{(2\chi^3 - 8\chi)\zeta^\eta}{\Gamma(\eta+1)}$$
$$\psi_2(\chi,\zeta) = \chi^2 - \frac{(2\chi^3 - 8\chi)\zeta^\eta}{\Gamma(\eta+1)} + \frac{12\chi^3 \zeta^{2\eta-1}}{\Gamma(2\eta)} + \frac{(-10\chi^4 + 132\chi^2 - 32)\zeta^{2\eta}}{\Gamma(2\eta+1)} + \frac{\Gamma(2\eta+1)(12\chi^5 - 304\chi^3 + 448\chi)\zeta^{3\eta}}{\Gamma(3\eta+1)\Gamma(\eta+1)^2} \tag{68}$$

Hence, the approximate solution for Equation (65) is

$$\psi(\chi,\zeta) = 3\chi^2 - \frac{2(2\chi^3 - 8\chi)\zeta^\eta}{\Gamma(\eta+1)} + \frac{12\chi^3 \zeta^{2\eta-1}}{\Gamma(2\eta)} + \frac{(-10\chi^4 + 132\chi^2 - 32)\zeta^{2\eta}}{\Gamma(2\eta+1)} + \frac{\Gamma(2\eta+1)(12\chi^5 - 304\chi^3 + 448\chi)\zeta^{3\eta}}{\Gamma(3\eta+1)\Gamma(\eta+1)^2} \tag{69}$$

Applying ADM: the recursive relations for Equation (65) can be constructed as

$$\psi_0(\chi,\zeta) = \sum_{k=0}^{p-1} \frac{\zeta^k}{k!} \frac{\partial^k}{\partial \zeta^k} \psi(\chi,\zeta) \Big|_{\zeta=0} \tag{70}$$
$$\psi_{m+1}(\chi,\zeta) = \mathcal{I}_\zeta^\eta \left[\chi^2 \mathcal{L}_{\chi\chi\zeta} \psi_m(\chi,\zeta) - 2\mathcal{L}_\chi \psi_m(\chi,\zeta) - \mathcal{A}_m(\chi,\zeta) \right], \quad m \geq 0$$

Using the above recursive relations, we can obtain the first terms of the (ADM) series solution

$$\psi_0(\chi,\zeta) = \chi^2$$
$$\psi_1(\chi,\zeta) = -\frac{(2\chi^3 - 8\chi)\zeta^\eta}{\Gamma(\eta+1)} \tag{71}$$
$$\psi_2(\chi,\zeta) = \frac{4(3\chi^5 - 18\chi^3 + 3\chi^2 + 24\chi - 4)\zeta^{2\eta}}{\Gamma(2\eta+1)} - \frac{12\chi^3 \zeta^{2\eta-1}}{\Gamma(2\eta)}$$

Hence, the (ADM) series solution for Equation (65) is

$$\psi(\chi,\zeta) = \chi^2 - \frac{(2\chi^3 - 8\chi)\zeta^\eta}{\Gamma(\eta+1)} + \frac{4(3\chi^5 - 18\chi^3 + 3\chi^2 + 24\chi - 4)\zeta^{2\eta}}{\Gamma(2\eta+1)} - \frac{12\chi^3 \zeta^{2\eta-1}}{\Gamma(2\eta)} \tag{72}$$

Applying HAM: the *m*th-order deformation equation for Equation (65) is given by

$$\psi_m(\chi,\zeta) = \chi_m \psi_{m-1}(\chi,\zeta) + \hbar \mathcal{I}_\zeta^\eta \left(\frac{\partial^\eta \psi_{m-1}(\chi,\zeta)}{\partial \zeta^\eta} - \chi^2 \frac{\partial^3 \psi_{m-1}(\chi,\zeta)}{\partial \chi^2 \partial \zeta} + 2 \frac{\partial \psi_{m-1}(\chi,\zeta)}{\partial \chi} \right.$$
$$\left. + \sum_{k=0}^{m-1} \left(\psi_k(\chi,\zeta) \frac{\partial \psi_{m-1-k}(\chi,\zeta)}{\partial \chi} - \psi_k(\chi,\zeta) \frac{\partial^3 \psi_{m-1-k}(\chi,\zeta)}{\partial \chi^3} - \frac{\partial \psi_k(\chi,\zeta)}{\partial \chi} \frac{\partial^2 \psi_{m-1-k}(\chi,\zeta)}{\partial \chi^2} \right) \right) \tag{73}$$

where

$$\chi_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases} \tag{74}$$

Using the above relation Equation (73), we can obtain the first terms of (HAM) series solution

$$\psi_0(\chi, \zeta) = \chi^2$$

$$\psi_1(\chi, \zeta) = \frac{(2\chi^3 - 8\chi)\hbar\zeta^\eta}{\Gamma(\eta + 1)}$$

$$\psi_2(\chi, \zeta) = \frac{2\chi(\chi - 2)(\chi + 2)(\hbar + 1)\hbar\zeta^\eta}{\Gamma(\eta + 1)} + \frac{2(5\chi^4 - 66\chi^2 + 16)\hbar^2\zeta^{2\eta}}{\Gamma(2\eta + 1)} - \frac{12\chi^3\hbar^2\zeta^{2\eta-1}}{\Gamma(2\eta)}$$

(75)

Hence, the (HAM) series solution for Equation (65) is

$$\psi(\chi, \zeta) = \chi^2 + \frac{2\chi(\chi - 2)(\chi + 2)(\hbar + 2)\hbar\zeta^\eta}{\Gamma(\eta + 1)} + \frac{2(5\chi^4 - 66\chi^2 + 16)\hbar^2\zeta^{2\eta}}{\Gamma(2\eta + 1)} - \frac{12\chi^3\hbar^2\zeta^{2\eta-1}}{\Gamma(2\eta)}$$

(76)

7. Conclusions

This paper presents three numerical methods considered to achieve an approximate solution for a third-order time-fractional partial differential equation (TFPDE). The equation is obtained from the classical (FW) equation by replacing the integer-order time derivative with the Caputo fractional derivative of order $\eta = (0, 1]$ with variable coefficients. The numerical results and graphs have been implemented using Maple 2022. In Figures 1–4 the graphical simulations for the approximate series solutions and the comparison in the form of absolute errors were presented to show the rate of change of the solutions when $\eta = 0.75$ and $\hbar = -1$. In Figures 5–8 we have shown the behavior of the solution with respect to the different values of η. As we can see in Tables 1 and 2, as ζ increases to 1, and χ increases to 1, the absolute errors slowly decrease. The agreement between the numerical results obtained by variational iteration method (VIM), Adomian decomposition method (ADM), and homotopy analysis method (HAM) involves fractional-order derivatives.

Table 1. This table shows the absolute errors for $\mathcal{W} = $ Abso.Error(VIM,ADM), $\mathcal{U} = $ Abso.Error(VIM,HAM), and $\mathcal{V} = $ Abso.Error(ADM,HAM) for Example 1.

	ζ	χ	$\mathcal{W}$	$\mathcal{U}$	$\mathcal{V}$
	0	0	2	2	0
	0.1	0.1	2.541025554	1.991394551	0.549631003
	0.2	0.2	3.977083804	1.738334153	2.238749651
	0.3	0.3	7.163139295	1.104283175	6.058856120
	0.4	0.4	14.00596958	0.26301164	13.74295794
$\eta = 0.25$	0.5	0.5	28.80747376	0.69271694	28.11475682
	0.6	0.6	61.19402329	7.56126741	53.63275588
	0.7	0.7	132.1993350	35.0418419	97.15749309
	0.8	0.8	286.4869258	117.4884319	168.9984939
	0.9	0.9	616.2555191	331.9641125	284.2914066
	1	1	1306.952037	842.2302549	464.7217817
	0	0	2	2	0
	0.1	0.1	2.289636497	2.099416352	0.190220145
	0.2	0.2	3.053697547	2.085683315	0.968014232
	0.3	0.3	4.954885502	1.897138477	3.057747025
	0.4	0.4	9.382828821	1.54535058	7.837478241
$\eta = 0.50$	0.5	0.5	19.47518759	1.73147878	17.74370881
	0.6	0.6	42.52650372	5.62861132	36.89789240
	0.7	0.7	95.44637321	23.41453671	72.03183650
	0.8	0.8	216.6411190	82.8281667	133.8129523
	0.9	0.9	490.9981703	252.3203983	238.6777720
	1	1	1100.897757	689.6370408	411.2607162

Table 1. *Cont.*

	ζ	χ	$\mathcal{W}$	$\mathcal{U}$	$\mathcal{V}$
	0	0	2	2	0
	0.1	0.1	2.219088227	2.152357150	0.066731077
	0.2	0.2	2.628656823	2.243794656	0.384862167
	0.3	0.3	3.627226957	2.269200185	1.358026772
	0.4	0.4	6.084771319	2.229972343	3.854798976
$\eta = 0.75$	0.5	0.5	11.96807065	2.40088841	9.567182238
	0.6	0.6	25.89804802	4.30750395	21.59054406
	0.7	0.7	58.93797052	13.61183285	45.32613767
	0.8	0.8	137.4703988	47.6301328	89.84026597
	0.9	0.9	323.2302447	153.3892413	169.8410034
	1	1	756.9909123	448.5531203	308.4377920
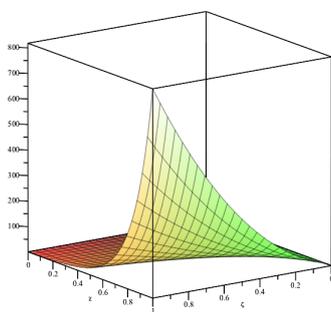
(**a**) $\psi(2, VIM)$
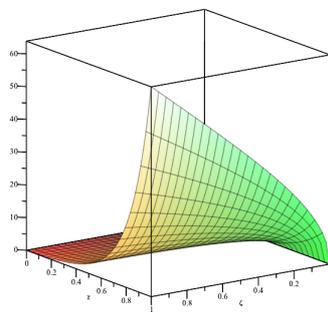
(**b**) $\psi(2, ADM)$
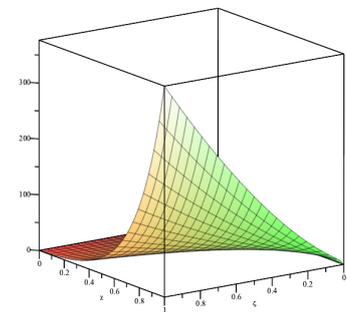
(**c**) $\psi(2, HAM)$

Figure 1. Graphical simulation of the second-level approximate solution $\psi_2(\chi, \zeta)$ when $\eta = 0.75$ and $\hbar = -1$ for Example 1.
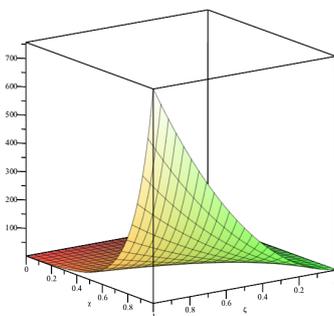

(**a**) $AbsoErorr(VIM, ADM)$
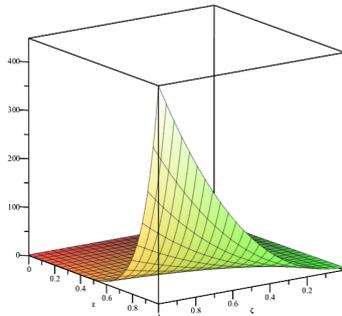
(**b**) $AbsoError(VIM, HAM)$
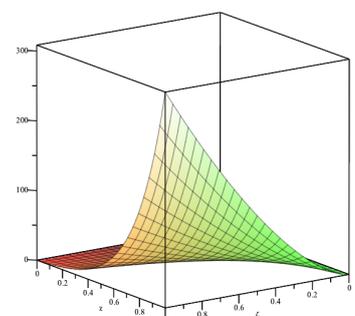
(**c**) $AbsoError(ADM, HAM)$

Figure 2. Graphical simulation of the absolute error when $\eta = 0.75$ and $\hbar = -1$ for Example 1.

**(a)** $\psi(1, VIM)$ **(b)** $\psi(1, ADM)$ **(c)** $\psi(1, HAM)$
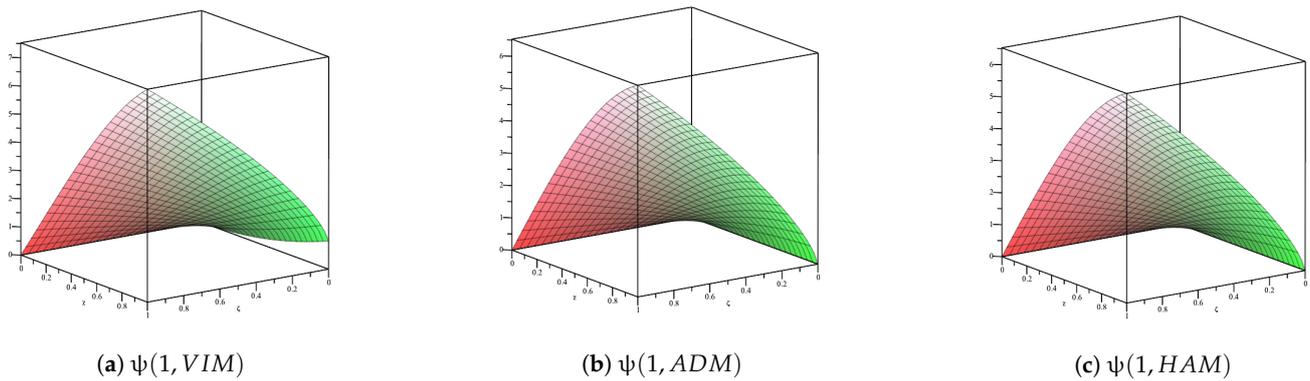
Figure 3. Graphical simulation of the first-level approximate solution $\psi_1(\chi, \zeta)$ when $\eta = 0.75$ and $\hbar = -1$ for Example 2.


(a) $AbsoErorr(VIM, ADM)$ **(b)** $AbsoError(VIM, HAM)$ **(c)** $AbsoError(ADM, HAM)$
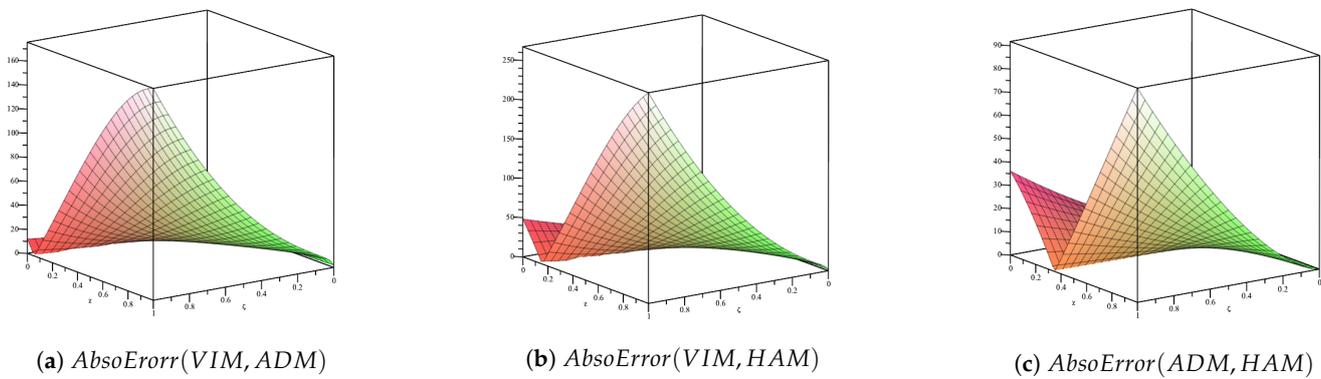
Figure 4. Graphical simulation of the absolute error when $\eta = 0.75$ and $\hbar = -1$ for Example 2.
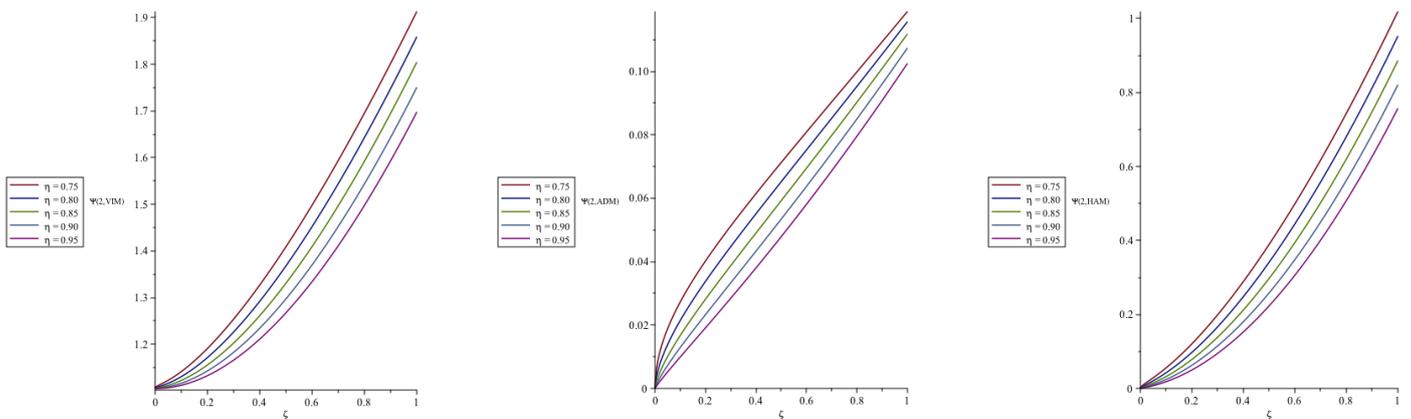
Figure 5. Plot of the second-level approximate solution $\psi_2(\chi, \zeta)$ at $\chi = 0.1$ for different values of η when $\hbar = -1$ for Example 1.
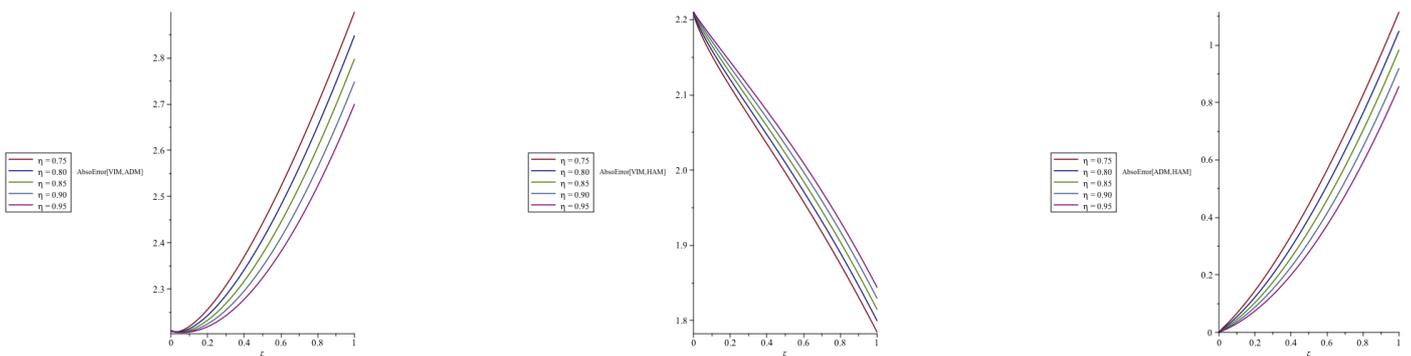

Figure 6. Plot of the absolute errors at $\chi = 0.1$ for different values of η when $\hbar = -1$ for Example 1.
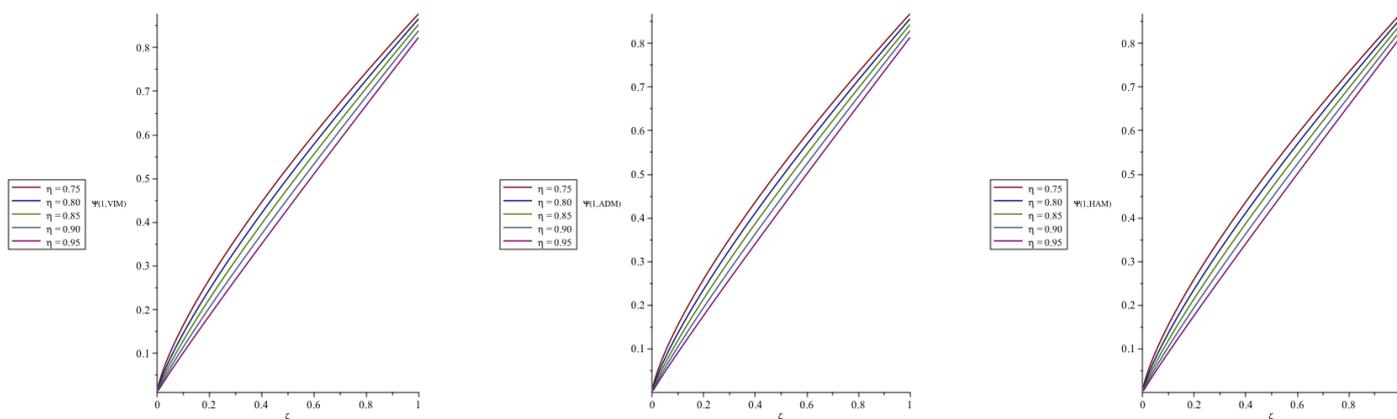
Figure 7. Plot of the first-level approximate solution $\psi_1(\chi, \zeta)$ at $\chi = 0.1$ for different values of η when $\hbar = -1$ for Example 2.

Table 2. This table shows the absolute errors for $\mathcal{W} = $ Abso.Error(VIM,ADM), $\mathcal{U} = $ Abso.Error(VIM,HAM), and $\mathcal{V} = $ Abso.Error(ADM,HAM) for Example 2.

	ζ	χ	$\mathcal{W}$	$\mathcal{U}$	$\mathcal{V}$
	0	0	0	0	0
	0.1	0.1	1.163766699	12.05048415	13.21425085
	0.2	0.2	17.02984022	5.10644444	11.92339578
	0.3	0.3	42.93553452	37.84546649	5.090068030
	0.4	0.4	76.65912143	82.86707406	6.20795263
$\eta = 0.25$	0.5	0.5	115.6963634	137.0397754	21.34341200
	0.6	0.6	157.0968959	196.8876146	39.79071872
	0.7	0.7	197.4322793	258.4945593	61.06227998
	0.8	0.8	232.7973816	317.5061666	84.70878500
	0.9	0.9	258.8244368	369.1656052	110.3411684
	1	1	270.7035747	408.3658330	137.6622584
	0	0	0	0	0
	0.1	0.1	0.7564569989	4.459744999	3.703288000
	0.2	0.2	2.570402463	2.155229537	4.725632000
	0.3	0.3	12.49304336	10.02229136	2.470752000
	0.4	0.4	30.39666054	33.87621254	3.479551997
$\eta = 0.50$	0.5	0.5	56.60474123	69.97974123	13.37500000
	0.6	0.6	90.31573016	117.6308022	27.31507200
	0.7	0.7	129.5188924	174.7947804	45.27588800
	0.8	0.8	170.9122578	238.0579858	67.14572800
	0.9	0.9	209.8305525	302.5997445	92.76919200
	1	1	240.1867606	362.1867606	122
	0	0	0	0	0
	0.1	0.1	0.2413076664	1.122257724	0.8809500575
	0.2	0.2	0.1238826704	1.465903435	1.589786105
	0.3	0.3	2.829465294	1.811451685	1.018013609
	0.4	0.4	9.761568790	11.41702283	1.655454036
$\eta = 0.75$	0.5	0.5	22.59293105	29.70740172	7.114470668
	0.6	0.6	42.43691686	58.35320436	15.91628749
	0.7	0.7	69.50572834	98.00145900	28.49573066
	0.8	0.8	102.7699335	147.9479522	45.17801869
	0.9	0.9	139.6218079	205.8265090	66.20470108
	1	1	175.5469776	267.3218165	91.77483894
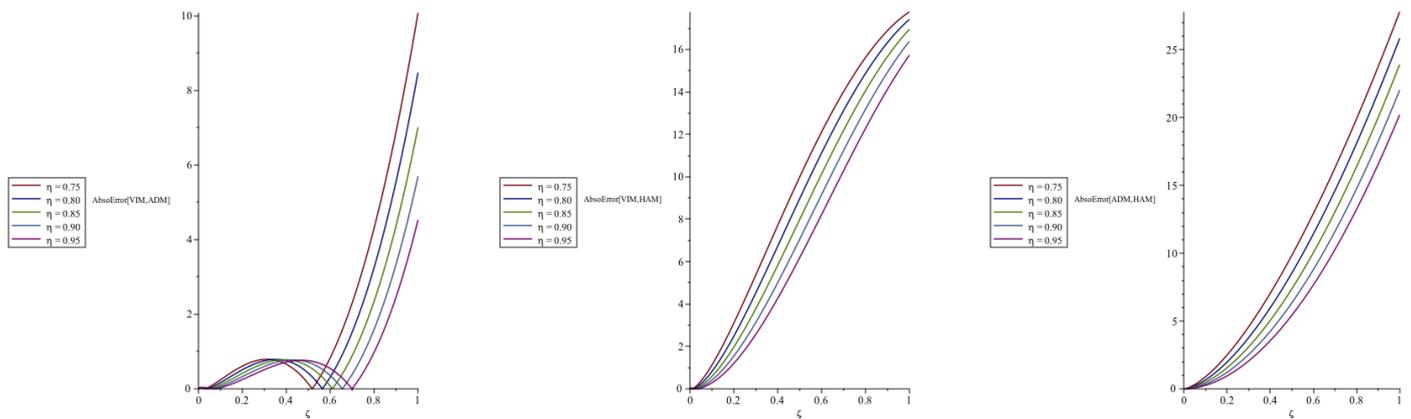
Figure 8. Plot of the absolute errors at $\chi = 0.1$ for different values of η when $\hbar = -1$ for Example 2.

References

1. Azevedo, J.; Cuevas, C.; Henriquez, E. Existence and asymptotic behaviour for the time-fractional Keller–Segel model for chemotaxis. *Math. Nachr.* **2019**, *292*, 462–480. [CrossRef]
2. Zabidi, N.A.; Abdul Majid, Z.; Kilicman, A.; Rabiei, F. Numerical Solutions of Fractional Differential Equations by Using Fractional Explicit Adams Method. *Mathematics* **2020**, *8*, 1675. [CrossRef]
3. Jafari, H.; Seifi, S. Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation. *Commun. Nonlinear Sci. Numer. Simul.* **2009**, *14*, 2006–2012. [CrossRef]
4. Jafari, H.; Golbabai, A.; Seifi, S.; Sayevand, K. Homotopy analysis method for solving multi-term linear and nonlinear diffusion–wave equations of fractional order. *Comput. Math. Appl.* **2010**, *59*, 1337–1344. [CrossRef]
5. Donatelli, M.; Mazza, M.; Serra-Capizzano, S. Spectral analysis and structure preserving preconditioners for fractional diffusion equations. *J. Comput. Phys.* **2016**, *307*, 262–279. [CrossRef]
6. Donatelli, M.; Mazza, M.; Serra-Capizzano, S. Spectral analysis and multigrid methods for finite volume approximations of space-fractional diffusion equations. *SIAM J. Sci. Comput.* **2018**, *40*, A4007–A4039. [CrossRef]
7. Lin, X.l.; Ng, M.K.; Sun, H.W. A multigrid method for linear systems arising from time-dependent two-dimensional space-fractional diffusion equations. *J. Comput. Phys.* **2017**, *336*, 69–86. [CrossRef]
8. Bu, W.; Liu, X.; Tang, Y.; Yang, J. Finite element multigrid method for multi-term time fractional advection diffusion equations. *Int. J. Model. Simul. Sci. Comput.* **2015**, *6*, 1540001. [CrossRef]
9. Whitham, G.B. Variational methods and applications to water waves. *Proc. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1967**, *299*, 6–25.
10. Fornberg, B.; Whitham, G.B. A numerical and theoretical study of certain nonlinear wave phenomena. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Sci.* **1978**, *289*, 373–404.
11. Kumar, D.; Singh, J.; Baleanu, D. A new analysis of the Fornberg-Whitham equation pertaining to a fractional derivative with Mittag–Leffler-type kernel. *Eur. Phys. J. Plus* **2018**, *133*, 1–10. [CrossRef]
12. Gupta, P.K.; Singh, M. Homotopy perturbation method for fractional Fornberg–Whitham equation. *Comput. Math. Appl.* **2011**, *61*, 250–254. [CrossRef]
13. Merdan, M.; Gökdoğan, A.; Yıldırım, A.; Mohyud-Din, S.T. Numerical simulation of fractional Fornberg-Whitham equation by differential transformation method. In *Abstract and Applied Analysis*; Hindawi: London, UK, 2012; Volume 2012.
14. Alderremy, A.; Khan, H.; Shah, R.; Aly, S.; Baleanu, D. The analytical analysis of time-fractional Fornberg–Whitham equations. *Mathematics* **2020**, *8*, 987. [CrossRef]
15. Abidi, F.; Omrani, K. The homotopy analysis method for solving the Fornberg–Whitham equation and comparison with Adomian's decomposition method. *Comput. Math. Appl.* **2010**, *59*, 2743–2750. [CrossRef]
16. Wang, K.L.; Liu, S.Y. He's fractional derivative and its application for fractional Fornberg-Whitham equation. *Therm. Sci.* **2017**, *21*, 2049–2055. [CrossRef]
17. Sartanpara, P.P.; Meher, R.; Meher, S. The generalized time-fractional Fornberg–Whitham equation: An analytic approach. *Partial. Differ. Equ. Appl. Math.* **2022**, *5*, 100350. [CrossRef]

18. Ahmad, H.; Seadawy, A.R.; Ganie, A.H.; Rashid, S.; Khan, T.A.; Abu-Zinadah, H. Approximate Numerical solutions for the nonlinear dispersive shallow water waves as the Fornberg–Whitham model equations. *Results Phys.* **2021**, *22*, 103907. [CrossRef]
19. Shah, N.A.; Dassios, I.; El-Zahar, E.R.; Chung, J.D.; Taherifar, S. The Variational Iteration Transform Method for Solving the Time-Fractional Fornberg–Whitham Equation and Comparison with Decomposition Transform Method. *Mathematics* **2021**, *9*, 141. [CrossRef]
20. Iqbal, N.; Yasmin, H.; Ali, A.; Bariq, A.; Al-Sawalha, M.M.; Mohammed, W.W. Numerical methods for fractional-order Fornberg-Whitham equations in the sense of Atangana-Baleanu derivative. *J. Funct. Spaces* **2021**, *2021*, 1–10. [CrossRef]
21. Lenells, J. Traveling wave solutions of the Camassa-Holm and Korteweg-de Vries equations. *J. Nonlinear Math. Phys.* **2004**, *11*, 508–520. [CrossRef]
22. Chen, J. The stability of solutions for the generalized degasperis-procesi equation with variable coefficients. *Math. Probl. Eng.* **2015**, *2015*, 207427. [CrossRef]
23. Jiang, J.; Feng, Y.; Li, S. Variational Problems with Partial Fractional Derivative: Optimal Conditions and Noether's Theorem. *J. Funct. Spaces* **2018**, *2018*, 1–14. [CrossRef]
24. Singh, B.K.; Kumar, P. Fractional variational iteration method for solving fractional partial differential equations with proportional delay. *Int. J. Differ. Equ.* **2017**, *2017*, 5206380. [CrossRef]
25. Dehghan, M.; Manafian, J.; Saadatmandi, A. Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numer. Methods Partial Differ. Equ. Int. J.* **2010**, *26*, 448–479. [CrossRef]
26. Wazwaz, A.M. The variational iteration method: A reliable analytic tool for solving linear and nonlinear wave equations. *Comput. Math. Appl.* **2007**, *54*, 926–932. [CrossRef]
27. Wazwaz, A.M. The variational iteration method for analytic treatment for linear and nonlinear ODEs. *Appl. Math. Comput.* **2009**, *212*, 120–134. [CrossRef]
28. He, J.H. Variational iteration method–a kind of non-linear analytical technique: Some examples. *Int. J. Non-Linear Mech.* **1999**, *34*, 699–708. [CrossRef]
29. He, J. A new approach to nonlinear partial differential equations. *Commun. Nonlinear Sci. Numer. Simul.* **1997**, *2*, 230–235. [CrossRef]
30. Adomian, G. A review of the decomposition method in applied mathematics. *J. Math. Anal. Appl.* **1988**, *135*, 501–544. [CrossRef]
31. Adomian, G. *Solving Frontier Problems of Physics: The Decomposition Method*; Springer Science & Business Media: New York, NY, USA, 2013; Volume 60.
32. Liao, S.J. The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems. Ph.D. Thesis, Shanghai Jiao Tong University Shanghai, Shanghai, China, 1992.
33. Liao, S. On the homotopy analysis method for nonlinear problems. *Appl. Math. Comput.* **2004**, *147*, 499–513. [CrossRef]
34. Liao, S. Comparison between the homotopy analysis method and homotopy perturbation method. *Appl. Math. Comput.* **2005**, *169*, 1186–1194. [CrossRef]
35. Liao, S. *Beyond Perturbation: Introduction to the Homotopy Analysis Method*; Chapman and Hall/CRC: Boca Raton, FL, USA, 2003.