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Solitary Wave Solutions to a Fractional Model Using the Improved Modified Extended Tanh-Function Method

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Abstract: Nonlinear fractional partial differential equations (NLFPDEs) are widely used in simulating a variety of phenomena arisen in several disciplines such as applied mathematics, engineering, physics, and a wide range of other applications. Solitary wave solutions of NLFPDEs have become a significant tool in understanding the long-term dynamics of these events. This article primarily focuses on using the improved modified extended tanh-function algorithm to determine certain traveling wave solutions to the space-time fractional symmetric regularized long wave (SRLW) equation, which is used to discuss space-charge waves, shallow water waves, etc. The Jumarie's modified Riemann-Liouville derivative is successfully used to deal with the fractional derivatives, which appear in the SRLW problem. We find many traveling wave solutions on the form of trigonometric, hyperbolic, complex, and rational functions. Furthermore, the performance of the employed technique is investigated in comparison to other techniques such as the Oncoming $exp(-\Theta(q))$-expansion method and the extended Jacobi elliptic function expansion strategy. Some obtained results are graphically displayed to show their physical features. The findings of this article demonstrate that the used approach enables us to handle more NLFPDEs that emerge in mathematical physics.

Keywords: SRLW equation; solitary solutions; fractional derivative; improved modified extended tanh-function method; traveling waves

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1. Introduction

Nonlinear fractional partial differential equations (NLFPDEs) were discovered in 1695 to describe several scientific applications that arise in a variety of disciplines, including biology, fluid dynamics, optical physics, atomic physics, circuit analysis, geochemistry, and several others [1]. The traveling wave feature is seen in many models and is essential for comprehending how these models will behave in the future. In other words, such solutions widely contribute to a great comprehension for the physical properties of the relevant model. Consequently, NLFPDEs have been successfully used to analyze traveling wave solutions of several real-world phenomena. Numerous valuable strategies such as qualitative strategies, algebraic approaches, general analytic processes, geometric-qualitative processes, approximate analytic strategies, and numerical techniques have been efficiently and successfully developed to extract some traveling wave solutions for NLFPDEs. Some of these techniques are the expansion method [2,3], homotopy perturbation technique [4], lumped Galerkin approach [5], Hirota bilinear strategy [6], F'/F-expansion technique [7], modified F-expansion approach [8], and others [9–14].

When L'Hopital wrote to Leibniz in 1695 to ask about the 1/2-derivative, the theory of fractional calculus was first taken into consideration [15]. Later, researchers began to take into account derivatives of particular types, including complex, fractional, and irrational derivatives. In fact, in earlier decades, certain informative definitions for fractional derivatives were developed. Some of these definitions, however, might provide varying consequences. To put it another way, they do not generate the same result for the derivative of a specific function. For instance, in 1812, Lapalce developed a useful definition for a fractional derivative of functions by utilizing integrals. Furthermore, Lacroix presented the n-th derivative for a given power function in 1819 [15]. However, there was no sold clue for the validity of these approaches on an arbitrary order. In 1832, the first Liouville definition, which depends on a formula for differentiating an exponential function, was derived by Liouville [16]. Riemann developed his convenient definition for fractional derivatives in his valuable paper given in [17]. In the late 19th century, the Riemann–Liouville definition for a fractional derivative of a given function was perfectly developed. For more definitions about fractional calculus, one can see refs. [15,18]. It is significant to point out that fractional calculus has become a prosperous discipline for the majority of mathematicians, engineers, and physicians after 1900.

In 1984, Seyler and Fenstermacher [19] presented the space-time fractional symmetric regularized long wave (SRLW) equation. This equation is employed to describe weakly nonlinear ion acoustic, space-charge waves as well as additional physical phenomena such shallow water waves, solitary waves, and ion-acoustic waves in plasma. The importance of this equation can be easily seen in its applications in nonlinear science. Consequently, a variety of practical techniques have been utilized to derive the analytical and approximative solutions to this equation. For instance, Xu [20] successfully obtained some generalized soliton solutions and periodic solutions for the SRLW problem by using the Exp-function approach. The fractional sub-equation method was nicely applied in [21] to develop five exact solutions for the SRLW problem. Shakeel and Mohyud-Din [22] employed a complex transformation to gain the corresponding ordinary differential equation of the SRLW problem, then they obtained some hyperbolic, trigonometric, and rational solutions for this equation using the fractional novel G'/G-expansion process. Furthermore, the modified F-expansion strategy and the new auxiliary approach were perfectly applied in [23,24], respectively, to find some exact solutions for the SRLW problem. In addition, several traveling wave solutions for the proposed problem were obtained in [25] using the direct method, which employs the Jacobi elliptic functions. Finally, Zhu and Qi [26] used two essential strategies called the extended complex and the G'/G^2-expansion techniques to extract some exact solutions for the SRLW equation.

The results of the literature review, which show that this equation has not received a necessary level of investigation, serve as the motivation for this article. Even though these investigations were significant, they were executed using extremely rudimentary methodologies, which limit the knowledge they provide on the traveling wave solutions to the problem under consideration. Despite the fact that scientists have developed numerous solutions for the SRLW equation using various methods, there is no comparison between the performance of the used techniques. However, the leading purpose of this article is to find some exact solutions for the proposed equation by employing the improved modified extended tanh-function technique. The nonlinear SRLW equation [19] reads as

$$D_t^{2\beta} w + D_x^{2\beta} w + w D_t^{\beta}(D_x^{\beta} w) + D_x^{\beta} w D_t^{\beta} w + D_t^{2\beta}(D_x^{2\beta} w) = 0, \qquad 0 < \beta \leq 1, \quad t > 0, \tag{1}$$

where $w(x,t)$ is a function that denotes the wave profile and β is a fractional order. According to [27], the improved modified extended tanh-function technique gives twenty-two different solutions (such as bell-shaped solitary wave solutions, kink-shaped solitary wave solutions, triangular type solutions, rational solutions, periodic solutions, exponential type solutions, and hyperbolic type solutions) under some constraints. Interestingly, under a successful choice of parameters, the employed approach produces traveling wave solutions with uniform structures. These kinds of solutions are therefore applicable to some events that occur in the real world. The performance of the proposed strategy is successfully compared with the performance of other methods. The surfaces of some solutions are illustrated in 3D figures, and some contours are also shown to exhibit the long behavior of the solutions. The traveling wave solutions attained from the suggested strategy imply that the method is uncomplicated to to use and is computationally feasible.

This article is designed as follows. In Section 2, we explain a convenient definition for the Jumarie's modified Riemann–Liouville derivative. In Section 3, we outline the description of the approach adopted. Section 4 presents some exact solutions for the considered equation. In Section 5, we provide a comprehensive explanation of the most significant developments found in this research paper, while Section 6 is devoted to the conclusions of this paper.

2. The Jumarie's Modified Riemann–Liouville Derivative

Definition 1 ([28]). *Let h be a function. Then, the γ-th order conformable fractional derivative of h is defined by*

$$D^{\gamma}(h)(\tau) = \lim_{\epsilon\to 0} \frac{h(\tau+\epsilon\tau^{1-\gamma}) - h(\tau)}{\epsilon}, \ \ \forall\, \tau > 0, \ \ \gamma \in (0,1).$$

If h is γ-differentiable in some interval $(0,\gamma)$, $\gamma > 0$ and $\lim_{\tau\to 0^+} h^{(\gamma)}(\tau)$ exists, then define $h^{(\gamma)}(0) = \lim_{\tau\to 0^+} h^{(\gamma)}(\tau)$.

The Jumarie's modified Riemann–Liouville derivative of order γ [29] is defined by the following equations.

$$D^{\gamma}_{\tau} h(\tau) = \begin{cases} \frac{1}{\Gamma(1-\gamma)} \int_0^{\tau} (\tau-\zeta)^{-\gamma-1}(h(\zeta)-h(0))\mathrm{d}\zeta, & \gamma < 0, \\ \\ \frac{1}{\Gamma(1-\gamma)} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^{\tau} (\tau-\zeta)^{-\gamma}(h(\zeta)-h(0))\mathrm{d}\zeta, & 0<\gamma<1, \\ \\ \left[h^{(\gamma-m)}(\tau)\right]^{(m)}, & m \le \gamma < m+1, \ \ m \ge 1. \end{cases}$$

Let $\gamma \in (0,1]$, and assume that u and w are γ-differentiable for $\tau > 0$. Then, the following are satisfied:

$$\begin{aligned} D^{\gamma}_{\tau}\tau^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\gamma)}\tau^{\beta-\gamma}, \qquad \beta > 0, \\ D^{\gamma}_{\tau} c &= 0, \qquad \text{c is a constant}, \\ D^{\gamma}_{\tau}(c_1 u(\tau) + c_2 w(\tau)) &= c_1 D^{\gamma}_{\tau} u(\tau) + c_2 D^{\gamma}_{\tau} w(\tau), \quad c_1 \text{ and } c_2 \text{ are constants}, \\ D^{\gamma}_{\tau}(u(\tau)w(\tau)) &= w(\tau) D^{\gamma}_{\tau} u(\tau) + u(\tau) D^{\gamma}_{\tau} w(\tau), \\ D^{\gamma}_{\tau} u[w(\tau)] &= u'_w[w(\tau)] D^{\gamma}_{\tau} w(\tau) = D^{\gamma}_{w} u[w(\tau)] \left(w'(\tau)\right)^{\gamma}, \\ D^{\gamma}_{\tau}\left(\frac{u(\tau)}{w(\tau)}\right) &= \frac{w(\tau) D^{\gamma}_{\tau}(u(\tau)) - u(\tau) D^{\gamma}_{\tau}(w(\tau))}{w^2(\tau)}, \ \ w(\tau) \neq 0. \end{aligned}$$

3. Improved Modified Extended Tanh-Function Technique

We will summarize the steps of the improved modified extended tanh-function technique as shown in [27]. First, consider a NLFPDE on the form

$$F_1(w, D^{\beta}_t w, D^{\beta}_x w, D^{\beta}_t D^{\beta}_x w, D^{2\beta}_{tt} w, D^{2\beta}_{xx} w, \ldots) = 0, \tag{2}$$

where F_1 is a polynomial in $w(x,t)$ and β is a fractional order. We then use the fractional complex transformation

$$w = w(x,t) = W(\xi), \qquad \xi = \frac{x^{\beta}}{\Gamma(1+\beta)} + \frac{\mu t^{\beta}}{\Gamma(1+\beta)}, \tag{3}$$

to convert Equation (2) into the following equation

$$F_2(W, W', W'', W''', \ldots) = 0, \tag{4}$$

where $W' = dW/d\xi$. The improved modified extended tanh-function technique presents the traveling wave solution of Equation (4) on the form

$$W(\xi) = \Sigma_{j=0}^{M} \gamma_j G^j(\xi) + \Sigma_{j=1}^{M} \delta_j G^{-j}(\xi), \tag{5}$$

where γ_j, δ_j, and μ are evaluated later. The value of M can be easily obtained by taking the balance of the nonlinear term with the highest derivative. The function $G(\xi)$ is the solution of the following Riccati differential equation

$$G'(\xi) = \alpha\sqrt{a_0 + a_1 G + a_2 G^2 + a_3 G^3 + a_4 G^4}, \tag{6}$$

where the constants $a_k \ \forall k$ are given under some restrictions and $\alpha = \pm 1$. Yang and Hon [27] proposed seven cases, each of which involve various traveling wave solutions. In Appendix A, we present the first three cases of the solutions of Equation (6) with their classifications as shown in [27]. We then find the value of M to be substituted into Equation (5). Next, we insert Equation (5) along with Equation (6) into Equation (4) to obtain an algebraic equation. Taking the coefficients of $G(\xi)$ leads to a system of equations whose solutions determine the values of γ_j, δ_j, and μ.

4. Traveling Wave Solutions of SRLW Equation

Inserting Equation (3) into Equation (1) gives the following ODE:

$$(\mu^2 + 1)W'' + \mu(WW'' + (W')^2) + \mu^2 W'''' = 0. \tag{7}$$

Integrate both sides of Equation (7) twice to have

$$(\mu^2 + 1)W + \frac{\mu}{2}W^2 + \mu^2 W'' = 0, \tag{8}$$

where the integration constants are taken by zero. Taking the balance of the nonlinear term with the second derivative gives $M = 2$. Hence, the traveling wave solution is given by

$$W(\xi) = \gamma_0 + \gamma_1 G(\xi) + \gamma_2 G^2(\xi) + \frac{\delta_1}{G(\xi)} + \frac{\delta_2}{G^2(\xi)}. \tag{9}$$

First case: if $a_0 = a_1 = a_3 = 0$, then

$$\gamma_0 = \pm\frac{8a_2\alpha^2}{\sqrt{4a_2\alpha^2 - 1}}, \quad \gamma_1 = \delta_1 = \delta_2 = 0, \quad \gamma_2 = \pm\frac{12a_4\alpha^2}{\sqrt{4a_2\alpha^2 - 1}}, \quad \mu = \mp\frac{1}{\sqrt{4a_2\alpha^2 - 1}}.$$

The traveling wave solutions are obtained from Equation (9) as follows:

$$W_{1,2}(\xi) = \pm\frac{8a_2\alpha^2}{\sqrt{4a_2\alpha^2 - 1}} \mp \frac{12a_2\alpha^2}{\sqrt{4a_2\alpha^2 - 1}} sech^2(\sqrt{a_2}\xi), \quad a_2 > 0, \ a_4 < 0, \tag{10}$$

$$W_{3,4}(\xi) = \pm\frac{8a_2\alpha^2}{\sqrt{4a_2\alpha^2 - 1}} \mp \frac{12a_2\alpha^2}{\sqrt{4a_2\alpha^2 - 1}} sec^2(\sqrt{-a_2}\xi), \quad a_2 < 0, \ a_4 > 0, \tag{11}$$

$$W_{5,6}(\xi) = \pm\frac{12\alpha^4}{\sqrt{4a_2\alpha^2 - 1}}\xi, \quad a_2 = 0, \ a_4 > 0, \tag{12}$$

where

$$\xi = \frac{x^\beta}{\Gamma(1+\beta)} \mp \frac{t^\beta}{\Gamma(1+\beta)\sqrt{4a_2\alpha^2 - 1}}.$$

Second case: if $a_1 = a_3 = 0$, then

- First family of solutions

$$\gamma_0 = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}},$$
$$\gamma_1 = \delta_1 = \delta_2 = 0,\ \gamma_2 = \pm 12a_4\alpha^2\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}},$$
$$\mu = \mp\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}.$$

Therefore, the traveling wave solutions are obtained by substituting the above values into Equation (9) as follows:

$$W_{7,8}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}$$
$$\mp 6\alpha^4 a_2\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}tanh^2\left(\sqrt{-\frac{a_2}{2}}\xi\right),\ a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},$$

$$W_{9,10}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}$$
$$\pm 6a_2\alpha^4\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}tan^2\left(\sqrt{\frac{a_2}{2}}\xi\right),\ a_2 > 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4}, \quad (13)$$

$$W_{11,12}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}$$
$$\mp \frac{12a_4\alpha^2 a_2 m^2}{a_4(2m^2 - 1)}\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}cn^2\left(\sqrt{\frac{a_2}{2m^2 - 1}}\xi\right),$$
$$a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2 m^2(1 - m^2)}{a_4(2m^2 - 1)^2},$$

$$W_{13,14}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}$$
$$\mp \frac{12a_4\alpha^2 m^2}{a_4(2 - m^2)}\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}dn^2\left(\sqrt{\frac{a_2}{2 - m^2}}\xi\right),$$
$$a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2(1 - m^2)}{a_4(2 - m^2)^2},$$

$$W_{15,16}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp \frac{12a_4a_2\alpha^2m^2}{a_4(1+m^2)}\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}sn^2\left(\sqrt{\frac{-a_2}{1+m^2}}\xi\right),$$
$$a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2m^2}{a_4(1+m^2)^2},$$

where

$$\xi = \frac{x^\beta}{\Gamma(1+\beta)} \mp \sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}\frac{t^\beta}{\Gamma(1+\beta)}.$$

- Second family of solutions

$$\gamma_0 = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}},$$
$$\gamma_1 = \gamma_2 = \delta_1 = 0,\ \delta_2 = \pm 12a_0\alpha^2\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}},$$
$$\mu = \mp\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}.$$

Now, the traveling wave solutions are obtained by substituting the above values into Equation (9) as follows:

$$W_{17,18}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp\left(\frac{24a_0a_4}{a_2}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}tanh^{-2}\left(\sqrt{-\frac{a_2}{2}}\xi\right),\ a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},$$

$$W_{19,20}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\pm\left(\frac{24a_0a_4}{a_2}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}tan^{-2}\left(\sqrt{\frac{a_2}{2}}\xi\right),\ a_2 > 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},$$

$$W_{21,22}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp\frac{12a_0\alpha^2a_4(2m^2-1)}{a_2m^2}\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}cn^{-2}\left(\sqrt{\frac{a_2}{2m^2-1}}\xi\right),$$
$$a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2m^2(1-m^2)}{a_4(2m^2-1)^2},$$

$$W_{23,24}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp \frac{12a_0\alpha^2a_4(2-m^2)}{m^2}\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}dn^{-2}\left(\sqrt{\frac{a_2}{2-m^2}}\xi\right),$$
$$a_2>0,\ a_4<0,\ a_0 = \frac{a_2^2(1-m^2)}{a_4(2-m^2)^2},$$

$$W_{25,26}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp \frac{12a_0\alpha^2a_4(1+m^2)}{a_2m^2}\sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}sn^{-2}\left(\sqrt{\frac{-a_2}{1+m^2}}\xi\right),$$
$$a_2<0,\ a_4>0,\ a_0 = \frac{a_2^2m^2}{a_4(1+m^2)^2},$$

where

$$\xi = \frac{x^\beta}{\Gamma(1+\beta)} \mp \sqrt{\frac{1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}\frac{t^\beta}{\Gamma(1+\beta)}.$$

- Third family of solutions

$$\gamma_0 = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}},$$
$$\gamma_1 = \delta_1 = \delta_2 = 0,\ \gamma_2 = \pm 12a_4\alpha^2\sqrt{\frac{-1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{1-16a_2^2\alpha^4-48a_0a_4\alpha^4}},$$
$$\mu = \mp\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}.$$

Hence, the traveling wave solutions are obtained by substituting the above values into Equation (9) as follows:

$$W_{27,28}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp 6\alpha^4a_2\sqrt{\frac{-1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{1-16a_2^2\alpha^4-48a_0a_4\alpha^4}}tanh^2\left(\sqrt{-\frac{a_2}{2}}\xi\right),\ a_2<0,\ a_4>0,\ a_0=\frac{a_2^2}{4a_4},$$

$$W_{29,30}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\pm 6\alpha^4a_2\sqrt{\frac{-1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{1-16a_2^2\alpha^4-48a_0a_4\alpha^4}}tan^2\left(\sqrt{\frac{a_2}{2}}\xi\right),\ a_2>0,\ a_4>0,\ a_0=\frac{a_2^2}{4a_4},$$

$$
\begin{aligned}
W_{31,32}(\xi) &= \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}} \\
&\mp \frac{12\alpha^2 a_2 m^2}{(2m^2 - 1)}\sqrt{\frac{-1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{1 - 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}cn^2\left(\sqrt{\frac{a_2}{2m^2 - 1}}\xi\right), \\
& a_2 > 0,\, a_4 < 0,\, a_0 = \frac{a_2^2 m^2(1 - m^2)}{a_4(2m^2 - 1)^2},
\end{aligned}
$$

$$
\begin{aligned}
W_{33,34}(\xi) &= \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}} \\
&\mp \frac{12\alpha^2 m^2}{(2 - m^2)}\sqrt{\frac{-1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{1 - 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}dn^2\left(\sqrt{\frac{a_2}{2 - m^2}}\xi\right), \\
& a_2 > 0,\, a_4 < 0,\, a_0 = \frac{a_2^2(1 - m^2)}{a_4(2 - m^2)^2},
\end{aligned}
$$

$$
\begin{aligned}
W_{35,36}(\xi) &= \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}} \\
&\mp \frac{12\alpha^4 a_2 m^2}{(1 + m^2)}\sqrt{\frac{-1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{1 - 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}sn^2\left(\sqrt{\frac{-a_2}{1 + m^2}}\xi\right), \\
& a_2 < 0,\, a_4 > 0,\, a_0 = \frac{a_2^2 m^2}{a_4(1 + m^2)^2},
\end{aligned}
$$

where

$$
\xi = \frac{x^\beta}{\Gamma(1+\beta)} \mp \sqrt{\frac{1 - 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}\frac{t^\beta}{\Gamma(1+\beta)}.
$$

- Fourth family of solutions

$$
\begin{aligned}
\gamma_0 &= \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}, \\
\gamma_1 &= \gamma_2 = \delta_1 = 0,\, \delta_2 = \pm 12a_0\alpha^2\sqrt{\frac{-1 + 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{1 - 16a_2^2\alpha^4 + 48a_0a_4\alpha^4}}, \\
\mu &= \mp\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 - 48a_0a_4\alpha^4}}.
\end{aligned}
$$

Using Equation (9), we can find several traveling wave solutions for the considered equation as follows:

$$W_{37,38}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp \frac{24a_0a_4}{a_2}\sqrt{\frac{-1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{1-16a_2^2\alpha^4+48a_0a_4\alpha^4}}tanh^{-2}\left(\sqrt{-\frac{a_2}{2}}\xi\right),\ a_2<0,\ a_4>0,\ a_0=\frac{a_2^2}{4a_4},$$

$$W_{39,40}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\pm \frac{24a_0a_4}{a_2}\sqrt{\frac{-1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{1-16a_2^2\alpha^4+48a_0a_4\alpha^4}}tan^{-2}\left(\sqrt{\frac{a_2}{2}}\xi\right),\ a_2>0,\ a_4>0,\ a_0=\frac{a_2^2}{4a_4},$$

$$W_{41,42}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp \frac{12a_0a_4\alpha^2(2m^2-1)}{a_2m^2}\sqrt{\frac{-1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{1-16a_2^2\alpha^4+48a_0a_4\alpha^4}}cn^{-2}\left(\sqrt{\frac{a_2}{2m^2-1}}\xi\right),$$
$$a_2>0,\ a_4<0,\ a_0=\frac{a_2^2m^2(1-m^2)}{a_4(2m^2-1)^2},$$

$$W_{43,44}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp \frac{12a_0\alpha^2a_4(2-m^2)}{m^2}\sqrt{\frac{-1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{1-16a_2^2\alpha^4+48a_0a_4\alpha^4}}dn^{-2}\left(\sqrt{\frac{a_2}{2-m^2}}\xi\right),$$
$$a_2>0,\ a_4<0,\ a_0=\frac{a_2^2(1-m^2)}{a_4(2-m^2)^2},$$

$$W_{45,46}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 - 3a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}$$
$$\mp \frac{12a_0a_4(m^2+1)}{a_2m^2}\sqrt{\frac{-1+4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{1-16a_2^2\alpha^4+48a_0a_4\alpha^4}}sn^{-2}\left(\sqrt{-\frac{a_2}{m^2+1}}\xi\right),$$
$$a_2<0,\ a_4>0,\ a_0=\frac{a_2^2m^2}{a_4(1+m^2)^2},$$

where

$$\xi = \frac{x^\beta}{\Gamma(1+\beta)} \mp \sqrt{\frac{1-4\sqrt{\left(a_2^2-3a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4-48a_0a_4\alpha^4}}\frac{t^\beta}{\Gamma(1+\beta)}.$$

- Fifth family of solutions

$$
\begin{aligned}
\gamma_0 &= \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}},\\
\gamma_1 &= \delta_1 = 0,\ \gamma_2 = \pm 12a_4\alpha^2\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}},\\
\delta_2 &= \pm 12a_0\alpha^2\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}},\\
\mu &= \mp\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}.
\end{aligned}
$$

Hence, the traveling wave solutions are obtained by substituting the above values into Equation (9) as follows:

$$
\begin{aligned}
W_{47,48}(\xi) = &\mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}\\
&\mp 6a_2\alpha^4\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}tanh^2\left(\sqrt{-\frac{a_2}{2}}\xi\right)\\
&\mp \frac{24a_0a_4}{a_2}\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}tanh^{-2}\left(\sqrt{-\frac{a_2}{2}}\xi\right),\\
&a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},
\end{aligned}
$$

$$
\begin{aligned}
W_{49,50}(\xi) = &\mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}\\
&\pm 6\alpha^4a_2\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}tan^2\left(\sqrt{\frac{a_2}{2}}\xi\right)\\
&\pm \frac{24a_0a_4}{a_2}\sqrt{\frac{1 - 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}tan^{-2}\left(\sqrt{\frac{a_2}{2}}\xi\right),\\
&a_2 > 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},
\end{aligned}
$$

$$W_{51,52}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}$$
$$\mp \frac{12\alpha^2 a_2 m^2}{(2m^2-1)}\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}cn^2\left(\sqrt{\frac{a_2}{2m^2-1}}\xi\right)$$
$$\mp \frac{12a_0\alpha^2 a_4(2m^2-1)}{a_2m^2}\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}cn^{-2}\left(\sqrt{\frac{a_2}{2m^2-1}}\xi\right),$$
$$a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2m^2(1-m^2)}{a_4(2m^2-1)^2},$$

$$W_{53,54}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}$$
$$\mp \frac{12\alpha^2 m^2}{(2-m^2)}\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}dn^2\left(\sqrt{\frac{a_2}{2-m^2}}\xi\right)$$
$$\mp \frac{12a_0\alpha^2 a_4(2-m^2)}{m^2}\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}dn^{-2}\left(\sqrt{\frac{a_2}{2-m^2}}\xi\right),$$
$$a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2(1-m^2)}{a_4(2-m^2)^2},$$

$$W_{55,56}(\xi) = \mp 4\left(-a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}$$
$$\mp \frac{12\alpha^4 a_2 m^2}{(1+m^2)}\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}sn^2\left(\sqrt{-\frac{a_2}{1+m^2}}\xi\right)$$
$$\mp \frac{12a_0a_4(1+m^2)}{a_2m^2}\sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}sn^{-2}\left(\sqrt{-\frac{a_2}{1+m^2}}\xi\right),$$
$$a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2m^2}{a_4(1+m^2)^2},$$

where

$$\xi = \frac{x^\beta}{\Gamma(1+\beta)} \mp \sqrt{\frac{1-4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}\frac{t^\beta}{\Gamma(1+\beta)}.$$

- Sixth family of solutions

$$\gamma_0 = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}},$$

$$\gamma_1 = \delta_1 = 0,\ \gamma_2 = \pm 12a_4\alpha^2\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}},$$

$$\delta_2 = \pm 12a_0\alpha^2\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}},$$

$$\mu = \mp\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}.$$

Utilizing Equation (9), the traveling wave solutions can be expressed as follows:

$$\begin{aligned}
W_{57,58}(\xi) &= \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}} \\
&\mp 6a_2\alpha^4\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}\left(tanh^2\left(\sqrt{-\frac{a_2}{2}}\xi\right)\right) \\
&\mp \frac{24a_0a_4}{a_2}\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}\left(tanh^{-2}\left(\sqrt{-\frac{a_2}{2}}\xi\right)\right), \\
& a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},
\end{aligned}$$

$$\begin{aligned}
W_{59,60}(\xi) &= \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}} \\
&\pm 6a_2\alpha^4\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}\left(tan^2\left(\sqrt{\frac{a_2}{2}}\xi\right)\right) \\
&\pm \frac{24a_0a_4}{a_2}\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}\left(tan^{-2}\left(\sqrt{\frac{a_2}{2}}\xi\right)\right), \\
& a_2 > 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},
\end{aligned}$$

$$\begin{aligned}
W_{61,62}(\xi) &= \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}} \\
&\mp \frac{12\alpha^2a_2m^2}{2m^2-1}\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}\left(cn^2\left(\sqrt{\frac{a_2}{2m^2-1}}\xi\right)\right) \\
&\mp \frac{12a_0\alpha^2a_4(2m^2-1)}{a_2m^2}\sqrt{\frac{1 + 4\sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}}{-1 + 16a_2^2\alpha^4 + 192a_0a_4\alpha^4}}\left(cn^{-2}\left(\sqrt{\frac{a_2}{2m^2-1}}\xi\right)\right), \\
& a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2m^2(1-m^2)}{a_4(2m^2-1)^2},
\end{aligned}$$

$$W_{63,64}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}$$
$$\mp \frac{12\alpha^2m^2}{2-m^2}\sqrt{\frac{1+4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}\left(dn^2\left(\sqrt{\frac{a_2}{2-m^2}}\xi\right)\right)$$
$$\mp \frac{12a_0\alpha^2a_4(2-m^2)}{m^2}\sqrt{\frac{1+4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}\left(dn^{-2}\left(\sqrt{\frac{a_2}{2-m^2}}\xi\right)\right),$$
$$a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2(1-m^2)}{a_4(2-m^2)^2},$$

$$W_{63,64}(\xi) = \pm 4\left(a_2\alpha^2 + \sqrt{\left(a_2^2 + 12a_0a_4\right)\alpha^4}\right)\sqrt{\frac{1+4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}$$
$$\mp \frac{12\alpha^4a_2m^2}{1+m^2}\sqrt{\frac{1+4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}\left(sn^2\left(\sqrt{-\frac{a_2}{1+m^2}}\xi\right)\right)$$
$$\mp \frac{12a_0a_4(1+m^2)}{a_2m^2}\sqrt{\frac{1+4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}\left(sn^{-2}\left(\sqrt{-\frac{a_2}{1+m^2}}\xi\right)\right),$$
$$a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2m^2}{a_4(1+m^2)^2},$$

where

$$\xi = \frac{x^\beta}{\Gamma(1+\beta)} \mp \sqrt{\frac{1+4\sqrt{\left(a_2^2+12a_0a_4\right)\alpha^4}}{-1+16a_2^2\alpha^4+192a_0a_4\alpha^4}}\frac{t^\beta}{\Gamma(1+\beta)}.$$

Third case: if $a_0 = a_1 = a_4 = 0$, then

$$\gamma_0 = \pm\frac{2a_2\alpha^2}{\sqrt{a_2\alpha^2-1}},\quad \gamma_1 = \pm\frac{3a_3\alpha^2}{\sqrt{a_2\alpha^2-1}},\quad \gamma_2 = \delta_1 = \delta_2 = 0,\quad \mu = \mp\frac{1}{\sqrt{a_2\alpha^2-1}}.$$

From Equation (9), we can extract some traveling wave solutions as follows:

$$W_{65,66}(\xi) = \pm\frac{2a_2\alpha^2}{\sqrt{a_2\alpha^2-1}} \mp \frac{3a_2\alpha^2}{\sqrt{a_2\alpha^2-1}}sech^2\left(\frac{\sqrt{a_2}}{2}\xi\right),\ \ a_2 > 0, \tag{14}$$

$$W_{67,68}(\xi) = \pm\frac{2a_2\alpha^2}{\sqrt{a_2\alpha^2-1}} \mp \frac{3a_2\alpha^2}{\sqrt{a_2\alpha^2-1}}sec^2\left(\frac{\sqrt{-a_2}}{2}\xi\right),\ \ a_2 < 0,$$

$$W_{69,70}(\xi) = \pm\frac{12\alpha^2}{i}\xi^{-2},\ a_2 = 0,$$

where

$$\xi = \frac{x^\beta}{\Gamma(1+\beta)} \mp \frac{1}{\sqrt{a_2\alpha^2-1}}\frac{t^\beta}{\Gamma(1+\beta)}.$$

5. Result and Discussion

This section is fundamentally dedicated to emphasizing the significant findings of the present research. The space-time fractional symmetric regularized long wave (SRLW) equation has numerous traveling wave solutions, which are mainly derived using the improved modified extended tanh-function approach. The Jumarie's modified Riemann–Liouville derivative is employed to deal with the fractional derivatives present in the SRLW problem. Trigonometric, hyperbolic, complex, and rational functions are utilized to demonstrate the wave propagation of this problem.

The proposed approach is used in this study for a variety of reasons. To begin with, this method provides a wide range of solutions presented in various forms such as bell-shaped solitary wave solutions, kink-shaped solitary wave solutions, triangular type solutions, rational solutions, periodic solutions, exponential type solutions, and hyperbolic type solutions. In particular, the proposed technique produces twenty-two different solutions [27]. The number of the traveling wave solutions increases in accordance with the obtained solutions of the algebraic system, which arises from Equation (5) along with Equation (6) into Equation (4). Implementing this strategy is often simple. Furthermore, the proposed technique is more convenient than other techniques. For example, Li et al. [30] invoked the Oncoming $exp(-\Theta(q))$-expansion process to only derive nine traveling wave solutions for the SRLW problem. In [31], three solutions on the form of rational and exponential functions were successfully obtained using the improved Bernoulli sub-equation function methodology. Moreover, by employing the extended Jacobi elliptic function expansion strategy, eight traveling wave solutions that are hyperbolic, trigonometric, and Jacobi elliptic functions were shown in [32]. Consequently, one can efficiently and successfully extract more traveling wave solutions for NLFPDEs using the improved modified extended tanh-function approach.

In order to present the graphical behaviors of the above-determined traveling wave solutions, we consider the time-fractional order β as $0 < \beta \leq 1$. Notably, the time-fractional order is taken by $\beta = 0.99$ and 1. Actually, the structure of traveling waves is significantly influenced by the values of the fractional order. In Figure 1 (left), we show a single solitary traveling wave (called pulses) for solution (10) $(W_1(\xi))$ under the parameters $a_0 = a_1 = a_3 = 0$, $a_2 = 1$, $a_4 = -20$. Figure 1 (right) illustrates the contour of this solution under the same parameters. The improved modified extended tanh-function algorithm produces internal solitary waves as shown in Figure 2 (left), which is the profile of solution (10) $(W_2(\xi))$ under the parameters $a_0 = a_1 = a_3 = 0$, $a_2 = 1$, $a_4 = -20$. It is worth noting that internal solitary waves are hump-shaped, which are physically similar to surface waves except that they travel horizontally within the fluid. This phenomenon takes place in lakes, seas, and oceans. In Figure 2 (right), we present the contour of this internal traveling wave when $a_0 = a_1 = a_3 = 0$, $a_2 = 1$, $a_4 = -20$. Some periodic traveling wave solutions are plotted for the proposed problem. For instance, Figure 3 (left) demonstrates a periodic propagation for solution (13) $(W_9(\xi))$ under the parameters $a_0 = 0.25$, $a_2 = a_4 = 1$. Figure 3 (right) shows the contour of solution (13) $(W_9(\xi))$ when $a_0 = 0.25$, $a_2 = a_4 = 1$. Moreover, Figure 4 (left) presents a 3D plot for a periodic solution of $W_{59}(\xi)$ when $a_1 = a_3 = 0$, $a_2 = 1.1$, $a_4 = 1 = \alpha = \beta$ inside the domain $-2\pi \leq x, t \leq 2\pi$. Figure 4 (right) displays a three-dimensional plot for the periodic solution $W_{59}(\xi)$ when $a_1 = a_3 = 0$, $a_2 = 1.1$, $a_4 = 1 = \alpha = \beta$. Finally, the solution $W_{61}(\xi)$ describes an internal solitary wave when we consider $a_1 = a_3 = 0$, $a_2 = 2$, $a_4 = -1$, $\alpha = \beta = m = 1$ inside the domain $-\pi \leq x, t \leq \pi$ (see Figure 5 (left)). The contour of solution $W_{61}(\xi)$ is depicted in Figure 5 (right).

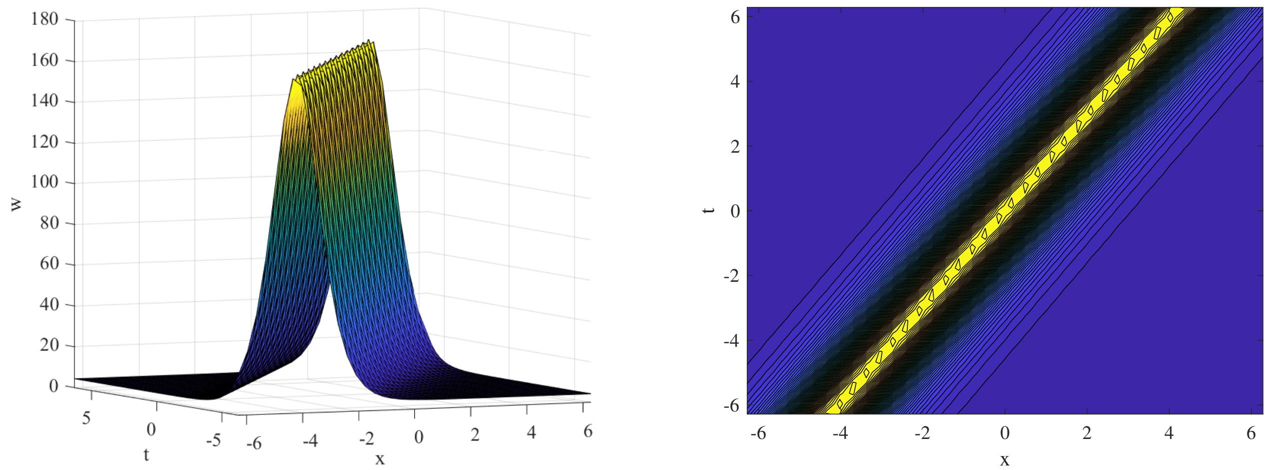

Figure 1. The behavior of solution (10) $(W_1(\xi))$ under the parameters $a_0 = a_1 = a_3 = 0$, $a_2 = 1$, $a_4 = -20$. The contour of this solution is shown in the right figure under the same parameters.

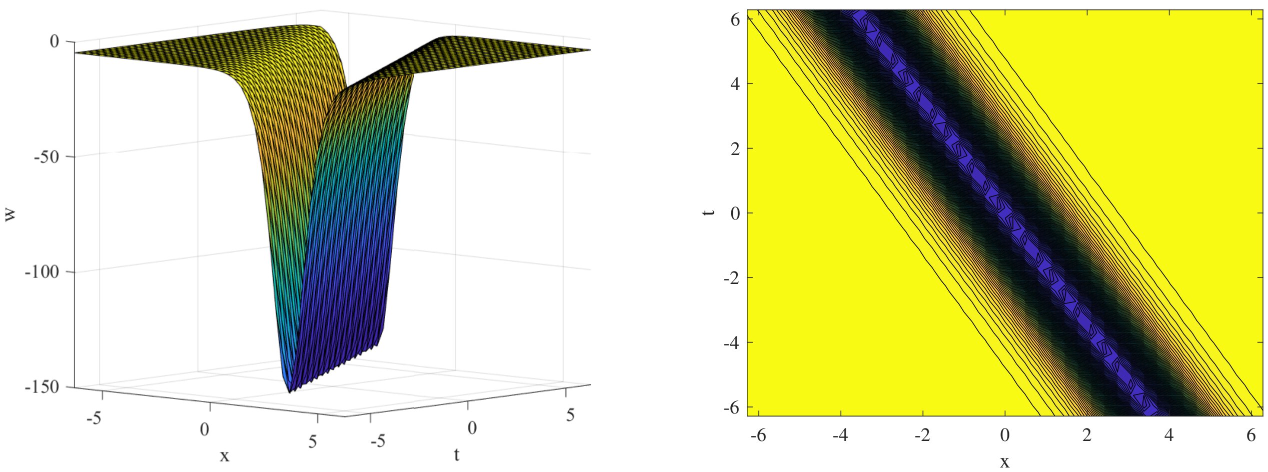

Figure 2. A traveling wave solution and a contour of solution (10) $(W_2(\xi))$ under the parameters $a_0 = a_1 = a_3 = 0$, $a_2 = 1$, $a_4 = -20$.

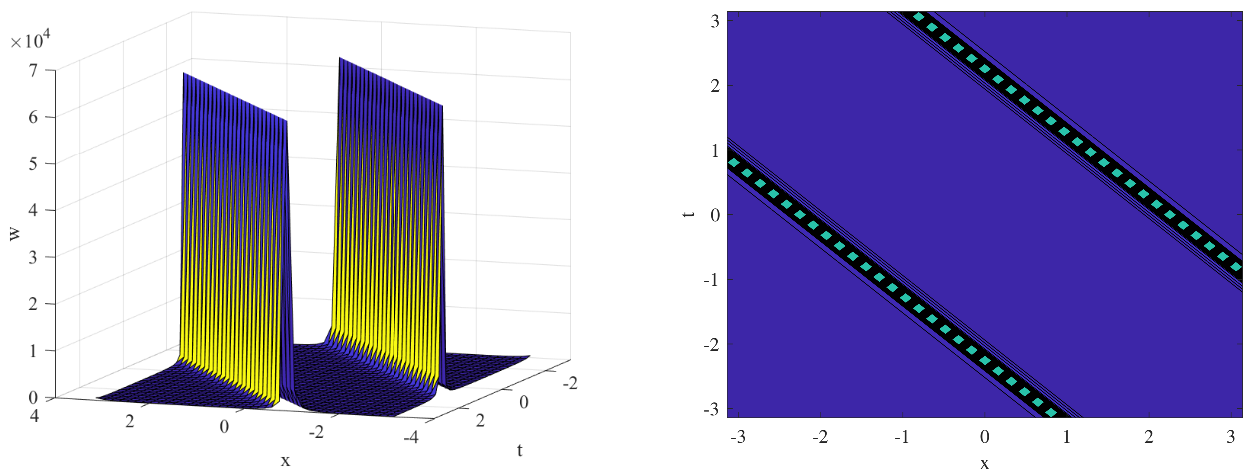

Figure 3. A traveling wave solution and a contour of solution (13) $(W_9(\xi))$ under the parameters $a_0 = 0.25$, $a_2 = a_4 = 1$.

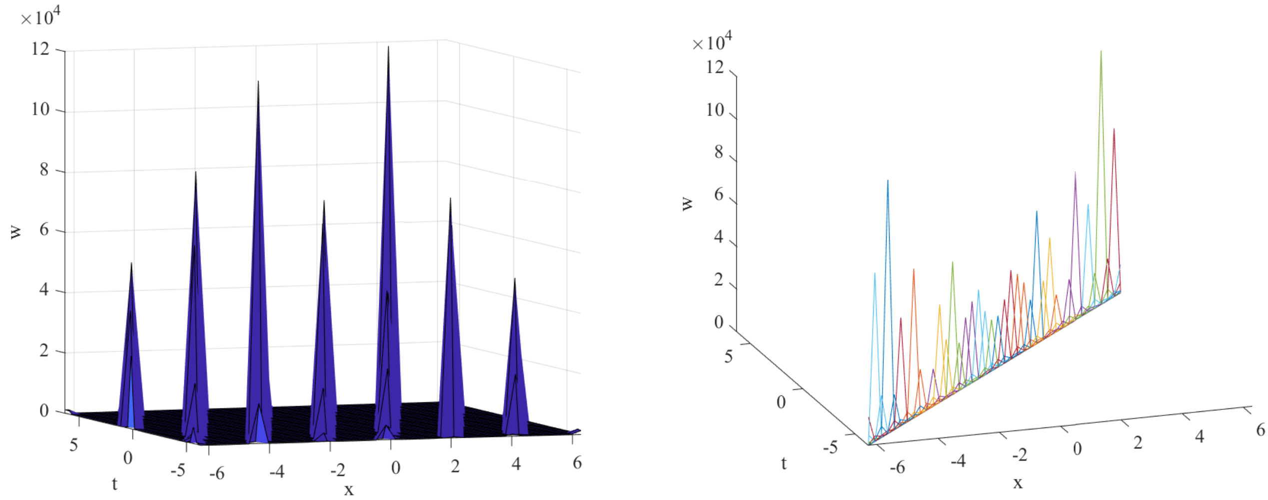

Figure 4. A 3D plot for a periodic solution of $W_{59}(\xi)$ under the values $a_1 = a_3 = 0$, $a_2 = 1.1$, $a_4 = 1 = \alpha = \beta$ inside the domain $-2\pi \leq x, t \leq 2\pi$.

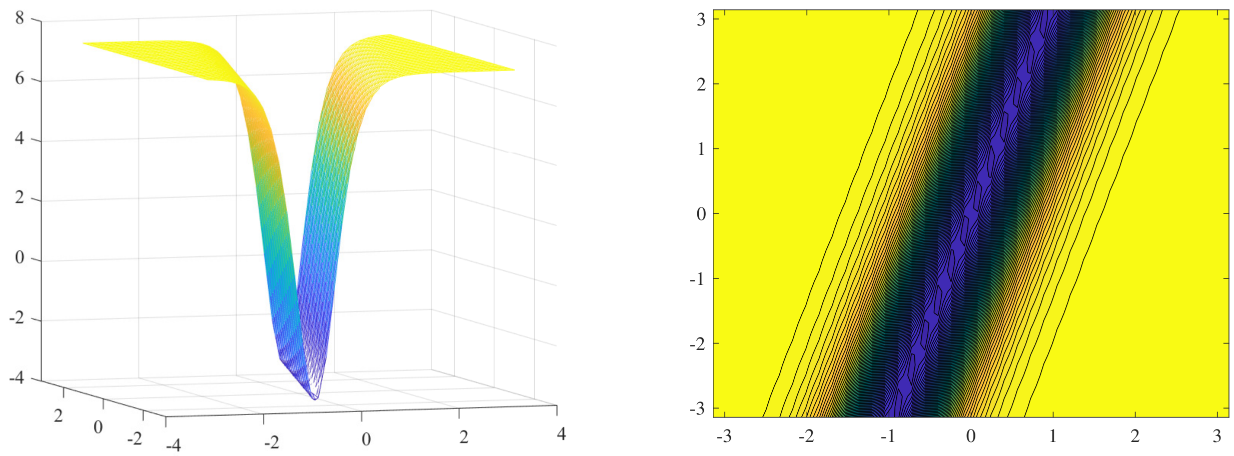

Figure 5. A traveling wave solution and its contour for the solution $W_{61}(\xi)$ under the values $a_1 = a_3 = 0$, $a_2 = 2$, $a_4 = -1$, $\alpha = \beta = m = 1$ inside the domain $-\pi \leq x, t \leq \pi$.

6. Conclusions

The algorithm of the improved modified extended tanh-function method has been successfully applied on the SRLW equation to extract its traveling wave solutions. The Jumarie's modified Riemann–Liouville derivative has also been invoked in this paper to deal with the fractional derivatives. We have derived 70 traveling wave solutions for the SRLW equation using the proposed approach. These solutions have been presented in the form of trigonometric, hyperbolic, and rational functions. The Jacobi elliptic functions have been used in determining these solutions. We found that the structure of the traveling waves become unknown when a very small value for the fractional order is used. For instance, the behavior of the periodic solution presented in Figure 4 becomes unknown when $\beta < 1$. From the comparison given in the previous section, we can state that the improved modified extended tanh-function technique produces more solutions than other techniques such as the Oncoming $exp(-\Theta(q))$-expansion method. The improved modified extended tanh-function method is effective, dependable, and versatile for developing new periodic, dark, bright, and bell-kink-type traveling wave solutions for a huge class of nonlinear PDEs.

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Appendix A

In this part, we present certain types of solutions for Equation (6) as shown in [27].

- First case: If $a_0 = a_1 = a_3 = 0$, then a bell-shaped solitary wave solution, a triangular type solution, and a rational solution for Equation (6) are shown as follows:

$$G(\xi) = \sqrt{-\frac{a_2}{a_4}} sech(\sqrt{a_2}\xi),\ a_2 > 0,\ a_4 < 0,$$
$$G(\xi) = \sqrt{-\frac{a_2}{a_4}} sec(\sqrt{-a_2}\xi),\ a_2 < 0,\ a_4 > 0,$$
$$G(\xi) = -\frac{\alpha}{\sqrt{a_4}\xi},\ a_2 = 0,\ a_4 > 0.$$

- Second case: If $a_1 = a_3 = 0$, then a kink-shaped solitary wave solution, a triangular type solution, and three Jacobi elliptic doubly periodic type solutions for Equation (6) are given as follows:

$$G(\xi) = \alpha\sqrt{-\frac{a_2}{2a_4}} tanh\left(\sqrt{-\frac{a_2}{2}}\xi\right),\ a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},$$
$$G(\xi) = \alpha\sqrt{\frac{a_2}{2a_4}} tan\left(\sqrt{\frac{a_2}{2}}\xi\right),\ a_2 > 0,\ a_4 > 0,\ a_0 = \frac{a_2^2}{4a_4},$$
$$G(\xi) = \sqrt{\frac{-a_2 m^2}{a_4(2m^2-1)}} cn\left(\sqrt{\frac{a_2}{2m^2-1}}\xi\right),\ a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2 m^2(1-m^2)}{a_4(2m^2-1)^2},$$
$$G(\xi) = \sqrt{\frac{-m^2}{a_4(2-m^2)}} dn\left(\sqrt{\frac{a_2}{2-m^2}}\xi\right),\ a_2 > 0,\ a_4 < 0,\ a_0 = \frac{a_2^2(1-m^2)}{a_4(2-m^2)^2},$$
$$G(\xi) = \alpha\sqrt{\frac{-a_2 m^2}{a_4(m^2+1)}} sn\left(\sqrt{-\frac{a_2}{m^2+1}}\xi\right),\ a_2 < 0,\ a_4 > 0,\ a_0 = \frac{a_2^2 m^2}{a_4(m^2+1)^2},$$

where m denotes a modulus.

- Third case: If $a_0 = a_1 = a_4 = 0$, then a bell-shaped solitary wave solution, a triangular type solution, and a rational solution for Equation (6) are given by

$$G(\xi) = -\frac{a_2}{a_3} sech^2\left(\frac{\sqrt{a_2}}{2}\xi\right),\ a_2 > 0,$$
$$G(\xi) = -\frac{a_2}{a_3} sec^2\left(\frac{\sqrt{-a_2}}{2}\xi\right),\ a_2 < 0,$$
$$G(\xi) = \frac{4}{a_3\xi^2},\ a_2 = 0.$$

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