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Certain Properties and Applications of Δ_h Hybrid Special Polynomials Associated with Appell Sequences

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Abstract: The development of certain aspects of special polynomials in line with the monomiality principle, operational rules, and other properties and their aspects is obvious and indisputable. The study presented in this paper follows this line of research. By using the monomiality principle, new outcomes are produced, and their differential equation and series representation is obtained, which are important in several branches of mathematics and physics. Thus, in line with prior facts, our aim is to introduce the Δ_h hybrid special polynomials associated with Hermite polynomials denoted by ${}_{\Delta_h H}Q_m(u,v,w;h)$. Further, we obtain some well-known main properties and explicit forms satisfied by these polynomials.

Keywords: Δ_h hybrid special polynomials; Appell polynomials; monomiality principle; explicit form

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1. Introduction and Preliminaries

Algebraic and enumerative combinatorics as well as applied mathematics all have an interest in the study of polynomial sequences. In engineering, biophysics, mathematical modeling, and approximation theory, numerous polynomials, namely, the tangent polynomials, Laguerre polynomials, Chebyshev polynomials, Legendre polynomials, and Jacobi polynomials, arise as the solutions of specific ordinary differential equations. Numerous problems in applied mathematics, theoretical physics, approximation theory, and other disciplines of mathematics include the Appell polynomial sequence, which is one of the significant classes of polynomial sequences [1]. Further, Appell polynomials obey all the axioms of an Abelian group under the composition operation.

Appell [1], in the eighteenth century, presented sequences of polynomials $Q_m(u)$ which satisfied the relation:

$$\frac{d}{du}Q_m(u) = m\, Q_{m-1}(u), \qquad m \in \mathbb{N}_0 \tag{1}$$

and possessed the generating relation listed below:

$$A(t)\exp(ut) = \sum_{k=0}^{\infty} Q_k(u)\frac{t^k}{k!}, \tag{2}$$

where $A(t)$, on the real line, is convergent with a Taylor expansion given by

$$A(t) = \sum_{k=0}^{\infty} Q_k \frac{t^k}{k!}, \qquad Q_0 \neq 0. \tag{3}$$

Particularly in recent years, a number of extensions of special functions in mathematical physics have seen a significant evolution. This new development provides the analytical basis for the vast majority of precisely solved problems in mathematical physics and engineering, which have several wide-ranging applications. The inducement of multitudinous-index and variable special functions is a significant advancement in the theory of generalized special functions. The significance of these functions has been acknowledged in both practical contexts and pure mathematics. These multitudinous-index and multitudinous-variable polynomials are needed to tackle the issues emerging in various disciplines of mathematics, from the theory of partial differential equations to abstract group theory. The idea of multiple-index, multiple-variable was initially created by Hermite [2]. The Hermite polynomials are found in physics, in numerical analysis as Gaussian quadrature, and in quantum harmonic oscillators and Schrödinger's equation.

Recently, Shahid Wani et al. established various doped polynomials of a special type and derived their numerous characteristics and properties, which are important from an engineering point of view, see, e.g., [3–6]. These properties include: summation formulae, determinant forms, approximation properties, explicit and implicit formulae, generating expressions, etc.

Let $g : I \subset \mathbb{R} \to \mathbb{R}$ and h $\in \mathbb{R}_+$, then the forward difference operator represented by Δ_h ([7] p. 2) is given by

$$\Delta_h[g](u) = g(u+h) - g(u).$$

Thus, for a finite difference of order $i \in \mathbb{N}$, it follows that

$$\Delta_h^i[g](u) = \Delta_h\big(\Delta_h^{i-1}[g](u)\big) = \sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l}g(u+lh), \tag{4}$$

where $\Delta_h^0 = I$ and $\Delta_h^1 = \Delta_h$, with I as the identity operator.

Recently, Costabile and Longo [8] made the first attempt in the direction of introducing Δ_h polynomial sequences namely Δ_h Appell polynomials and studied their several properties. The generating function for these polynomials $\mathcal{Q}_m(u;h)$ is defined by the following generating function:

$$\sum_{m=0}^{\infty}\mathcal{Q}_m(u;h)\frac{t^m}{m!} = \gamma(t)(1+ht)^{\frac{u}{h}}, \tag{5}$$

or by the relation

$$\Delta_h[\mathcal{Q}_m](u;h) = mh\mathcal{Q}_{m-1}(u;h), \tag{6}$$

respectively.

For $h \to 0$, the expression (5) reduces to Equations (2) and (6) reduces to (1), respectively.

Further, in [8], Δ_h Appell sequences $\mathcal{Q}_m(u)$, $\quad m \in \mathbb{N}$ were given by the product of two functions in power series $\gamma(t)(1+ht)^{\frac{u}{h}}$ by

$$\gamma(t)(1+ht)^{\frac{u}{h}} = \mathcal{Q}_0(u;h) + \mathcal{Q}_1(u;h)\frac{t}{1!} + \mathcal{Q}_2(u;h)\frac{t^2}{2!} + \cdots + \mathcal{Q}_m(u;h)\frac{t^m}{m!} + \cdots, \tag{7}$$

where

$$\gamma(t) = \gamma_{0,h} + \gamma_{1,h}\frac{t}{1!} + \gamma_{2,h}\frac{t^2}{2!} + \cdots + \gamma_{m,h}\frac{t^m}{m!} + \cdots. \tag{8}$$

Appell sequences of Δ_h form reduce to polynomials, for example, generalized falling factorials $(u)_m^h \equiv (u)_m$ [7], a Boole sequence $B_{lm}(u;\lambda)$ [7], a Bernoulli sequence of the second kind $b_m(u)$ [7], a Poisson–Charlier sequence $C_m(u;\gamma)$ ([7] p. 2).

The origins of monomiality can be traced to 1941 when Steffenson developed the poweroid notion [9], which was later refined by Dattoli [10]. The $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ operators exist and function as multiplicative and derivative operators for a polynomial set $\{b_m(u)\}_{m\in\mathbb{N}}$, which means that they hold the expressions

$$b_{m+1}(u) = \hat{\mathcal{M}}\{b_m(u)\} \tag{9}$$

and

$$m\, b_{m-1}(u) = \hat{\mathcal{D}}\{b_m(u)\}. \tag{10}$$

Then, the set $\{b_m(u)\}_{m\in\mathbb{N}}$ manipulated by multiplicative and derivative operators is referred to as a quasi-monomial and is required to obey the formula:

$$[\hat{\mathcal{D}}, \hat{\mathcal{M}}] = \hat{\mathcal{D}}\hat{\mathcal{M}} - \hat{\mathcal{M}}\hat{\mathcal{D}} = \hat{1}, \tag{11}$$

thus displays a Weyl group structure as a result.

The properties of the operators $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ can be used to determine the properties of the underlying set $\{b_m(u)\}_{m\in\mathbb{N}}$ when it is quasi-monomial. Thus, the following traits are accurate:

(i) $b_m(u)$ demonstrates the differential equation

$$\hat{\mathcal{M}}\hat{\mathcal{D}}\{b_m(u)\} = m\, b_m(u), \tag{12}$$

if $\hat{\mathcal{M}}$ and $\hat{\mathcal{D}}$ are notions of a differential operator.

(ii) The explicit form of $b_m(u)$ can be cast in the form as

$$b_m(u) = \hat{\mathcal{M}}^m\,\{1\}, \tag{13}$$

while taking $b_0(u) = 1$.

(iii) Moreover, the generating relation in exponential form for $b_m(u)$ can be cast in the form

$$e^{t\hat{\mathcal{M}}}\{1\} = \sum_{m=0}^{\infty} b_m(u)\frac{t^m}{m!}, \quad |t| < \infty \ , \tag{14}$$

by using identity (13).

These operational approaches are still used today in many areas of mathematical physics, quantum mechanics, and classical optics. Therefore, these techniques provide effective and potent tools of research, see for example [11–13].

By differentiating expression (5) with respect to t and u, respectively, we can construct the operators for the Δ_h Appell polynomials, which are provided by the expressions:

$$\mathcal{Q}_{m+1}(u;h) = \hat{\mathcal{M}}_{\mathcal{A}}\{\mathcal{Q}_m(u;h)\} = \left(\frac{u}{1+{}_u\Delta_h} + \frac{\gamma^{'}(\frac{{}_u\Delta_h}{h})}{\gamma(\frac{{}_u\Delta_h}{h})}\right)\{\mathcal{Q}_m(u;h)\} \tag{15}$$

and

$$\mathcal{Q}_{m-1}(u;h) = \hat{\mathcal{D}_{\mathcal{A}}}\{\mathcal{Q}_m(u;h)\} = \frac{log(1+(\frac{{}_u\Delta_h}{h}))}{mh}\{\mathcal{Q}_m(u;h)\}. \tag{16}$$

Moreover, using Formulas (15) and (16) as a reference to (12), we get the expression for a differential equation listed as:

$$\left(\frac{u}{1+{}_u\Delta_h} + \frac{\gamma^{'}(\frac{{}_u\Delta_h}{h})}{\gamma(\frac{{}_u\Delta_h}{h})} - \frac{m^2 h}{log(1+{}_u\Delta_h)}\right)\mathcal{Q}_m(u;h) = 0. \tag{17}$$

For, $h \to 0$, the expressions (15)–(17) reduce to the multiplicative and derivative operators and the differential equation satisfied by Appell polynomials $Q_m(u)$ given by expression (2) [1].

Recent years have witnessed a considerable evolution in the induction of multivariable and index functions in polynomial families of special functions. To handle the problems that emerge in a variety of mathematical fields, such as mathematical physics, engineering mathematics, approximation and automata theory, and abstract algebra, multivariate functions and indices of special functions are required. Currently, many mathematicians are doing research extensively on Δ_h and degenerate multivariate special polynomials of mathematical physics, see for example [8,14–19].

In light of the significance of these findings, revitalized and inspired by Costabile and Longo's work [8], here, we introduce Δ_h three-variable Hermite based Appell polynomials, which possess a generating expression of the form:

$$\gamma(t)\ (1+ht)^{\frac{u}{h}}\ (1+ht^2)^{\frac{v}{h}}\ (1+ht^3)^{\frac{w}{h}} = \sum_{m=0}^{\infty} {}_{\Delta_h H}Q_m(u,v,w;h)\frac{t^m}{m!}. \tag{18}$$

The manuscript is organized as follows: Three-variable Δ_h Hermite based Appell polynomials are introduced in Section 2 by proving the result given by expression (18). Moreover, additional results are proved to verify that these polynomials are of degree K, $k \in \mathbb{N}$, along with some of their specific features such as explicit series representations. In Section 3, the quasi-monomial characteristics of these polynomials are established, and their significant property as a differential equation is established. In Section 4, a few members of this polynomial family are established, and their related findings are found. In the last section, a conclusion is drawn.

2. Δ_h Three-Variable Hermite Based Appell Polynomials

Here, we offer a different, more generic approach for identifying three-variable Hermite based Appell sequences with Δ_h (Δ_h 3VHAP). Any Appell type polynomial family must satisfy (1) to (3). Therefore, in view of these facts, we have the following theorems.

Theorem 1. *Since Δ_h 3VHAP sequences are given by (18), we have*

$$\begin{aligned} {}_u\Delta_h[{}_{\Delta_h H}Q_m(u,v,w;h)] &= mh\ {}_{\Delta_h H}Q_{m-1}(u,v,w;h) \\ {}_v\Delta_h[{}_{\Delta_h H}Q_m(u,v,w;h)] &= m(m-1)h\ {}_{\Delta_h H}Q_{m-2}(u,v,w;h) \\ {}_w\Delta_h[{}_{\Delta_h H}Q_m(u,v,w;h)] &= m(m-1)(m-2)h\ {}_{\Delta_h H}Q_{m-2}(u,v,w;h) \end{aligned} \tag{19}$$

Theorem 2. *Further, for the power series*

$$\gamma(t) = \gamma_{0,h} + \gamma_{1,h}\frac{t}{1!} + \gamma_{2,h}\frac{t^2}{2!} + \cdots + \gamma_{m,h}\frac{t^m}{m!} + \cdots,\ \ \textit{where}\ \ \gamma_{0,h} \neq 0, \tag{20}$$

with γ_m, $m = 0,1,2,...$ as real coefficients, the Δ_h 3VHAP sequence ${}_{\Delta_h H}Q_m(u,v,w;h)$ $m \in \mathbb{N}$ is determined by the product of the series expansion $\gamma(t)(1+ht)^{\frac{u}{h}}\ (1+ht^2)^{\frac{v}{h}}\ (1+ht^3)^{\frac{w}{h}}$, that is

$$\begin{aligned} \gamma(t)\ (1+ht)^{\frac{u}{h}}\ (1+ht^2)^{\frac{v}{h}}\ (1+ht^3)^{\frac{w}{h}} &= {}_{\Delta_h H}Q_0(u,v,w;h) + {}_{\Delta_h H}Q_1(u,v,w;h)\tfrac{t}{1!} \\ &+ {}_{\Delta_h H}Q_2(u,v,w;h)\tfrac{t^2}{2!} + \cdots + {}_{\Delta_h H}Q_m(u,v,w;h)\tfrac{t^m}{m!} + \cdots. \end{aligned} \tag{21}$$

Proof. Expanding $(1+ht)^{\frac{u}{h}}\ (1+ht^2)^{\frac{v}{h}}\ (1+ht^3)^{\frac{w}{h}}$ by a Newton series for finite differences at $u = v = w = 0$ and ordering the product of the developments of functions $\gamma(t)$ and $(1+ht)^{\frac{u}{h}}\ (1+ht^2)^{\frac{v}{h}}\ (1+ht^3)^{\frac{w}{h}}$ with respect to the powers of t, then in view of expression (7), we observe the polynomials ${}_{\Delta_h H}Q_m(u,v,w;h)$ are expressed in Equation (21) as coefficients of $\frac{t^m}{m!}$ as the generating function of Δ_h three-variable Hermite based Appell polynomials. □

Next, the series representation in explicit form for the Δ_h 3VHAP sequence is derived. To derive it, we first derive the explicit form of the Δ_h 3VHAP sequence given by taking $\gamma(t) = 1$, in (18), i.e.,

$$(1+ht)^{\frac{u}{h}}\ (1+ht^2)^{\frac{v}{h}}\ (1+ht^3)^{\frac{w}{h}} = \sum_{m=0}^{\infty} {}_{\Delta_h}H_m(u,v,w;h)\frac{t^m}{m!}, \tag{22}$$

in the listed form as:

Theorem 3. *For, the Δ_h 3VHAP sequence, the succeeding explicit series formula holds true:*

$$_{\Delta_h}H_m(u,v,w;h) = \sum_{k=0}^{[\frac{m}{k}]}\sum_{l=0}^{[\frac{k}{3}]}\binom{m}{k}\binom{k}{3l}(u)^h_{m-k}\,(v)^h_{k-3l}\,(w)^h_l\,\frac{(2m)!}{m!}\,\frac{(3l)!}{l!}, \tag{23}$$

where $(u)^h_m \equiv (u)_m$ *and is given by*

$$(u)^h_m = u(u+h)(u+2h)\cdots(u+(m-1)h),\;\; m=1,2,\cdots,\;\; (u)^h_0 = 1. \tag{24}$$

Proof. Expanding $(1+ht)^{\frac{u}{h}}\,(1+ht^2)^{\frac{v}{h}}\,(1+ht^3)^{\frac{w}{h}}$ in terms of raising factorials given by (24), we have

$$(1+ht)^{\frac{u}{h}}(1+ht^2)^{\frac{v}{h}}(1+ht^3)^{\frac{w}{h}} = \sum_{m=0}^{\infty}\left(-\frac{u}{h}\right)_m(-h)^m\frac{t^m}{m!}\sum_{k=0}^{\infty}\left(-\frac{v}{h}\right)_k(-h)^k\frac{t^{2k}}{k!}\sum_{l=0}^{\infty}\left(-\frac{w}{h}\right)_l(-h)^l\frac{t^{3l}}{l!}. \tag{25}$$

In cognizance of the product rule of two series, namely, the Cauchy product in the last two series of the r.h.s. of the above expression, it follows that

$$(1+ht)^{\frac{u}{h}}\,(1+ht^2)^{\frac{v}{h}}\,(1+ht^3)^{\frac{w}{h}} = \sum_{m=0}^{\infty}(u)^h_m\frac{t^m}{m!}\,\sum_{k=0}^{\infty}\sum_{l=0}^{[\frac{k}{3}]}\binom{k}{3l}(v)^h_{k-3l}\,(w)^h_l\,\frac{(3l)!}{l!}\,\frac{t^k}{k!}. \tag{26}$$

Again, taking cognizance of the product rule of two series, namely, the Cauchy product in the first two series of the r.h.s. of the above expression, it follows that

$$(1+ht)^{\frac{u}{h}}(1+ht^2)^{\frac{v}{h}}(1+ht^3)^{\frac{w}{h}} = \sum_{m=0}^{\infty}\sum_{k=0}^{[\frac{m}{k}]}\sum_{l=0}^{[\frac{k}{3}]}\binom{m}{k}\binom{k}{3l}(u)^h_{m-k}\,(v)^h_{k-3l}\,(w)^h_l\,\frac{(2m)!}{m!}\,\frac{(3l)!}{l!}\,\frac{t^m}{m!}. \tag{27}$$

Inserting the series expansion of Δ_h three-variable Hermite polynomials given by (22) on the l.h.s. of the above equation and in the resultant equation, comparing the same powers of t, we are led to assertion (23). □

Next, we derive explicit forms of Δ_h 3VHAP sequences by proving the following results:

Theorem 4. *The Δ_h 3VHAP sequences hold the listed explicit form:*

$$_{\Delta_h H}Q_m(u,v,w;h) = \sum_{s=0}^{[\frac{m}{s}]}\binom{m}{s}\,\mathcal{Q}_{s,h}\;{}_{\Delta_h}H_{m-s}(u,v,w;h). \tag{28}$$

Proof. Inserting expressions (7) with $h=0$ and (22) in the l.h.s. of (18), we have

$$\sum_{s=0}^{\infty}\mathcal{Q}_{s,h}(u)\frac{t^s}{s!}\sum_{m=0}^{\infty}{}_{\Delta_h}H_m(u,v,w;h)\frac{t^m}{m!} = \sum_{m=0}^{\infty}{}_{\Delta_h H}Q_m(u,v,w;h)\frac{t^m}{m!}. \tag{29}$$

In cognizance of the C.P. rule in the l.h.s. of the (29) and then in the resultant equation, the coefficients of the same powers of t are compared, which leads us to (28). □

Theorem 5. *The Δ_h 3VHAP sequences hold the listed explicit form:*

$$_{\Delta_h H}Q_m(u,v,w;h) = \sum_{s=0}^{[\frac{m}{s}]}\binom{m}{s}\,\gamma_{s,h}\;{}_{\Delta_h}H_{m-s}(u,v,w;h). \tag{30}$$

Proof. Inserting expressions (8) and (22) in the l.h.s. of (18), we have

$$\sum_{s=0}^{\infty}\gamma_{s,h}\frac{t^s}{s!}\sum_{m=0}^{\infty}{}_{\Delta_h}H_m(u,v,w;h)\frac{t^m}{m!} = \sum_{m=0}^{\infty}{}_{\Delta_h H}Q_m(u,v,w;h)\frac{t^m}{m!}. \tag{31}$$

By comparing the coefficients of similar powers of t in the l.h.s. of the previous equation with the resulting equation using the C.P. rule, the claim in (30) is reached. □

Theorem 6. *For the Δ_h 3VHAP sequence, the succeeding explicit series formula holds true:*

$$_{\Delta_h H}Q_m(u,v,w;h) = \sum_{i=0}^{[\frac{m}{i}]} \sum_{k=0}^{[\frac{i}{k}]} \sum_{l=0}^{[\frac{k}{3}]} \binom{m}{i} \binom{i}{k} \binom{k}{3l} \gamma_{m-i,h} (u)_{i-k}^{h} (v)_{k-3l}^{h} (w)_{l}^{h} \frac{(2i)!}{i!} \frac{(3l)!}{l!}. \quad (32)$$

Proof. Inserting expressions (8) and (27) with m replaced by i in the l.h.s. of Equation (18), it follows that

$$\sum_{m=0}^{\infty} \gamma_{m,h} \frac{t^m}{m!} \sum_{k=0}^{[\frac{i}{k}]} \sum_{l=0}^{[\frac{k}{3}]} \binom{i}{k} \binom{k}{3l} (u)_{i-k}^{h} (v)_{k-3l}^{h} (w)_{l}^{h} \frac{(2i)!}{i!} \frac{(3l)!}{l!} \frac{t^i}{i!} = \sum_{m=0}^{\infty} {}_{\Delta_h H}Q_m(u,v,w;h) \frac{t^m}{m!}. \quad (33)$$

By comparing the coefficients of similar powers of t in the l.h.s. of the previous equation with the resulting equation using the C.P. rule, the claim in (32) is reached. □

3. Monomiality Principle

Here, we establish the quasi-monomial properties satisfied by Δ_h 3VHAP sequences, by proving the following results:

Theorem 7. *The Δ_h 3VHAP sequences satisfy the following multiplicative and derivative operators:*

$$\begin{gathered}\left(\frac{u}{1+{}_u\Delta_h} + \frac{2v_u\Delta_h}{h+{}_u\Delta_h{}^2} + \frac{3w_u\Delta_h{}^2}{h^2+{}_u\Delta_h{}^3} + \frac{\gamma'(\frac{{}_u\Delta_h}{h})}{\gamma(\frac{{}_u\Delta_h}{h})} \right) \{ {}_{\Delta_h H}Q_m(u,v,w;h) \} = \\ {}_{\Delta_h H}Q_{m+1}(u,v,w;h) = \hat{\mathcal{M}}_{\Delta_h} \{ {}_{\Delta_h H}Q_m(u,v,w;h) \}\end{gathered} \quad (34)$$

and

$$_{\Delta_h H}Q_{m-1}(u,v,w;h) = \hat{\mathcal{D}}_{\Delta_h} \{ {}_{\Delta_h H}Q_m(u,v,w;h) \} = \frac{log(1+{}_u\Delta_h)}{mh} \{ {}_{\Delta_h H}Q_m(u,v,w;h) \}, \quad (35)$$

respectively.

Proof. In view of finite difference operator Δ_h, we have

$$_u\Delta_h [{}_{\Delta_h H}Q_m(u,v,w;h)] = h\, t\, [{}_{\Delta_h H}Q_{m-1}(u,v,w;h)], \quad (36)$$

or

$$\frac{{}_u\Delta_h}{h} [{}_{\Delta_h H}Q_m(u,v,w;h)] = t\, [{}_{\Delta_h H}Q_{m-1}(u,v,w;h)]. \quad (37)$$

Differentiating (18) with respect to t and u, we have

$$\begin{gathered}{}_{\Delta_h H}Q_{m+1}(u,v,w;h) = \hat{\mathcal{M}}_{\Delta_h} \{ {}_{\Delta_h H}Q_m(u,v,w;h) \} = \\ \left(\frac{u}{1+ht} + \frac{2vt}{1+ht^2} + \frac{3wt^2}{1+ht^3} + \frac{\gamma'(t)}{\gamma(t)} \right) \{ {}_{\Delta_h H}Q_m(u,v,w;h) \}\end{gathered} \quad (38)$$

and

$$_{\Delta_h H}Q_{m-1}(u,v,w;h) = \hat{\mathcal{D}}_{\Delta_h} \{ {}_{\Delta_h H}Q_m(u,v,w;h) \} = \frac{log(1+ht)}{mh} \{ {}_{\Delta_h H}Q_m(u,v,w;h) \}, \quad (39)$$

respectively.

Using identity (37) in view of (9) and (10) in Equations (38) and (39), we are led to assertions (34) and (35). □

The Δ_h 3VHAP sequences satisfy the following differential equation:

$$\left(\frac{u}{1+{}_u\Delta_h}+\frac{2v_u\Delta_h}{h+{}_u\Delta_h{}^2}+\frac{3w_u\Delta_h{}^2}{h^2+{}_u\Delta_h{}^3}+\frac{\gamma'\left(\frac{u\Delta_h}{h}\right)}{\gamma\left(\frac{u\Delta_h}{h}\right)}-\frac{m^2h}{log(1+_u\Delta_h)}\right)\{_{\Delta_h H}Q_m(u,v,w;h)\}=0. \tag{40}$$

Proof. Making use of expressions (34) and (35) in (12), we are led to assertion (40). □

For $v, w \to 0$, expressions (34), (35) and (40) reduce to the multiplicative and derivative operators, and the differential expression satisfied by the Δ_h Appell polynomials $\mathcal{Q}_m(u;h)$ given by expressions (15)–(17).

For $h \to 0$, expressions (34), (35) and (40) reduces to the multiplicative and derivative operators, and the differential equation fulfilled by Appell polynomials $Q_m(u)$ given by expression (2).

4. Examples

A variety of Appell polynomial family members can be obtained depending on the proper choice for the function $\alpha(t)$. These members' names, generating expressions, and associated numbers are listed below:

The generating expression for $_{\Delta_h}\beta_m(u;h)$, i.e., Δ_h Bernoulli polynomials, is given by:

$$\frac{t}{(1+ht)^{\frac{1}{h}}-1}(1+ht)^{\frac{u}{h}}=\sum_{m=0}^{\infty}{}_{\Delta_h}\beta_m(u;h)\frac{t^m}{m!}, \qquad |t|<2\pi. \tag{41}$$

The generating expression for $_{\Delta_h}\mathcal{E}_m(u;h)$, i.e., Δ_h Euler polynomials, is given by

$$\frac{2}{(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{u}{h}}=\sum_{m=0}^{\infty}{}_{\Delta_h}\mathcal{E}_m(u;h)\frac{t^m}{m!}, \qquad |t|<\pi. \tag{42}$$

The generating expression for $_{\Delta_h}\mathcal{G}_m(w;h)$, i.e., Δ_h Genocchi polynomials, is given by

$$\frac{2t}{(1+ht)^{\frac{1}{h}}+1}(1+ht)^{\frac{u}{h}}=\sum_{m=0}^{\infty}{}_{\Delta_h}\mathcal{G}_m(u;h)\frac{t^m}{m!}, \qquad |t|<\pi. \tag{43}$$

For $h \to 0$, these polynomials reduce to the $\mathcal{B}_m(u)$, $\mathcal{E}_m(u)$, and $\mathcal{G}_m(u)$ polynomials [20].

The Δ_h polynomials and numbers of $\mathcal{B}_m(u)$, $\mathcal{E}_m(u)$, and $\mathcal{G}_m(u)$ are widely used in number theory, combinatorics, numerical analysis, and other fields of practical mathematics. The Bernoulli numbers may be found in many mathematical formulas, including the Taylor expansion, the sums of powers of natural numbers, and the trigonometric and hyperbolic tangent and cotangent functions. The Euler $\mathcal{E}_m$ numbers enter the Taylor expansion at the trigonometric and hyperbolic secant function origins. The Genocchi $\mathcal{G}_m$ numbers are helpful in graph theory, automata theory, and counting the number of up–down ascending sequences.

Hence, the following generating functions for Δ_h three-variable Hermite based Bernoulli, Euler, and Genocchi polynomials are valid given a reasonable choice of $\gamma(t)$ in (18):

$$\frac{t}{(1+ht)^{\frac{1}{h}}-1}\,(1+ht)^{\frac{u}{h}}\,(1+ht^2)^{\frac{v}{h}}\,(1+ht^3)^{\frac{w}{h}}=\sum_{m=0}^{\infty}{}_{\Delta_h H}\beta_m(u,v,w;h)\frac{t^m}{m!}, \tag{44}$$

$$\frac{2}{(1+ht)^{\frac{1}{h}}+1}\,(1+ht)^{\frac{u}{h}}\,(1+ht^2)^{\frac{v}{h}}\,(1+ht^3)^{\frac{w}{h}}=\sum_{m=0}^{\infty}{}_{\Delta_h H}\mathcal{E}_m(u,v,w;h)\frac{t^m}{m!}, \tag{45}$$

and

$$\frac{2t}{(1+ht)^{\frac{1}{h}}+1}\,(1+ht)^{\frac{u}{h}}\,(1+ht^2)^{\frac{v}{h}}\,(1+ht^3)^{\frac{w}{h}}=\sum_{m=0}^{\infty}{}_{\Delta_h H}\mathcal{G}_m(u,v,w;h)\frac{t^m}{m!}, \tag{46}$$

respectively. As a result, these polynomials can provide the relevant outcomes:

Theorem 8. *Since we observe Δ_h 3VH based Bernoulli, Euler, and Genocchi polynomials are given by (44)–(46), these polynomials satisfy the relations:*

$$\begin{aligned} {}_u\Delta_h[{}_{\Delta_h H}\beta_m(u,v,w;h)] &= mh\ {}_{\Delta_h H}\beta_{m-1}(u,v,w;h) \\ {}_v\Delta_h[{}_{\Delta_h H}\beta_m(u,v,w;h)] &= m(m-1)h\ {}_{\Delta_h H}\beta_{m-2}(u,v,w;h) \\ {}_w\Delta_h[{}_{\Delta_h H}\beta_m(u,v,w;h)] &= m(m-1)(m-2)h\ {}_{\Delta_h H}\beta_{m-2}(u,v,w;h), \end{aligned} \tag{47}$$

$$\begin{aligned} {}_u\Delta_h[{}_{\Delta_h H}\mathcal{E}_m(u,v,w;h)] &= mh\ {}_{\Delta_h H}\mathcal{E}_{m-1}(u,v,w;h) \\ {}_v\Delta_h[{}_{\Delta_h H}\mathcal{E}_m(u,v,w;h)] &= m(m-1)h\ {}_{\Delta_h H}\mathcal{E}_{m-2}(u,v,w;h) \\ {}_w\Delta_h[{}_{\Delta_h H}\mathcal{E}_m(u,v,w;h)] &= m(m-1)(m-2)h\ {}_{\Delta_h H}\mathcal{E}_{m-2}(u,v,w;h), \end{aligned} \tag{48}$$

and

$$\begin{aligned} {}_u\Delta_h[{}_{\Delta_h H}\mathcal{G}_m(u,v,w;h)] &= mh\ {}_{\Delta_h H}\mathcal{G}_{m-1}(u,v,w;h) \\ {}_v\Delta_h[{}_{\Delta_h H}\mathcal{G}_m(u,v,w;h)] &= m(m-1)h\ {}_{\Delta_h H}\mathcal{G}_{m-2}(u,v,w;h) \\ {}_w\Delta_h[{}_{\Delta_h H}\mathcal{G}_m(u,v,w;h)] &= m(m-1)(m-2)h\ {}_{\Delta_h H}\mathcal{G}_{m-2}(u,v,w;h), \end{aligned} \tag{49}$$

respectively.

Moreover, these polynomials meet the following explicit form in light of Equation (30):

Theorem 9. *The Δ_h 3VH based $\mathcal{B}_m(u)$, $\mathcal{E}_m(u)$, and $\mathcal{G}_m(u)$ polynomials hold the explicit form:*

$$ {}_{\Delta_h H}\beta_m(u,v,w;h) = \sum_{s=0}^{[\frac{m}{s}]} \binom{m}{s} \beta_{s,h}\ {}_{\Delta_h H_{m-s}(u,v,w;h)}, \tag{50}$$

$$ {}_{\Delta_h H}\mathcal{E}_m(u,v,w;h) = \sum_{s=0}^{[\frac{m}{s}]} \binom{m}{s} \mathcal{E}_{s,h}\ {}_{\Delta_h H_{m-s}(u,v,w;h)}, \tag{51}$$

and

$$ {}_{\Delta_h H}\mathcal{G}_m(u,v,w;h) = \sum_{s=0}^{[\frac{m}{s}]} \binom{m}{s} \mathcal{G}_{s,h}\ {}_{\Delta_h H_{m-s}(u,v,w;h)}, \tag{52}$$

respectively.

Further, in view of Equation (32), these polynomials satisfy the following explicit form:

Theorem 10. *For the Δ_h three-variable Hermite based Bernoulli, Euler, and Genocchi polynomials, the succeeding explicit series formulae*

$$ {}_{\Delta_h H}\beta_m(u,v,w;h) = \sum_{i=0}^{[\frac{m}{i}]} \sum_{k=0}^{[\frac{i}{k}]} \sum_{l=0}^{[\frac{k}{3}]} \binom{m}{i} \binom{i}{k} \binom{k}{3l} \gamma_{m-i,h} (u)^h_{i-k}\ (v)^h_{k-3l}\ (w)^h_l \frac{(2i)!}{i!} \frac{(3l)!}{l!}, \tag{53}$$

$$ {}_{\Delta_h H}\mathcal{E}_m(u,v,w;h) = \sum_{i=0}^{[\frac{m}{i}]} \sum_{k=0}^{[\frac{i}{k}]} \sum_{l=0}^{[\frac{k}{3}]} \binom{m}{i} \binom{i}{k} \binom{k}{3l} \gamma_{m-i,h} (u)^h_{i-k}\ (v)^h_{k-3l}\ (w)^h_l \frac{(2i)!}{i!} \frac{(3l)!}{l!}, \tag{54}$$

and

$$ {}_{\Delta_h H}\mathcal{G}_m(u,v,w;h) = \sum_{i=0}^{[\frac{m}{i}]} \sum_{k=0}^{[\frac{i}{k}]} \sum_{l=0}^{[\frac{k}{3}]} \binom{m}{i} \binom{i}{k} \binom{k}{3l} \gamma_{m-i,h} (u)^h_{i-k}\ (v)^h_{k-3l}\ (w)^h_l \frac{(2i)!}{i!} \frac{(3l)!}{l!}, \tag{55}$$

respectively, hold true.

Similarly, in the same fashion, other corresponding results for these polynomials can be established.

5. Conclusions

In this paper, we established Δ_h hybrid special polynomials and obtained their several properties. These hybrid special polynomials were established by convoluting Appell and Δ_h Hermite polynomials. Additionally, we give a determinant representation to them and also established their series representations. These presented results can be applied in any three-variable Δ_h Hermite based Appell type polynomials, such as Bernoulli, Euler, Genocchi, and tangent polynomials. Further, we established their explicit forms, generating relations and series expansions.

Further, future investigations and observations can be used to establish extended, generalized forms, integral representations, and other properties of the above-mentioned polynomials. Moreover, the determinant forms and summation formulae can also be a problem for new observations.

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