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Differential Subordination and Superordination Results for q-Analogue of Multiplier Transformation

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Abstract: The results obtained by the authors in the present paper refer to quantum calculus applications regarding the theories of differential subordination and superordination. These results are established by means of an operator defined as the q-analogue of the multiplier transformation. Interesting differential subordination and superordination results are derived by the authors involving the functions belonging to a new class of normalized analytic functions in the open unit disc U, which is defined and investigated here by using this q-operator.

Keywords: convex function; differential operator; q-analogue operator; differential subordination; best dominant; differential superordination; best subordinant

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1. Introduction

Quantum calculus is widely engaged in mathematical fields due to its numerous uses related to combinatorics [1], associated with orthogonal polynomials [2–4], regarding number theory [5], or involving basic hypergeometric functions [6]. Certain aspects concerning fundamentals of q-calculus and how it was embedded in mathematical theories can be seen in [7–9]. In 1911, Jackson introduced the notions of q-derivative [10] and q-integral [11].

The first applications of q-calculus in geometric function theory are seen in [12], where the authors define the class of q-starlike functions. Numerous applications of quantum calculus in geometric function theory have emerged in the recent years after the general context for such research was established by Srivastava in a book chapter published in 1989 [13]. Certain aspects regarding the use of quantum calculus in geometric function theory are highlighted in a recent paper [14], and other developments are emphasized in the review done by Srivastava in 2020 [15] alongside the multitude of q-operators derived by involving well-known differential and integral operators specific to geometric function theory.

New operators were defined using q-hypergeometric functions [16,17], subclasses of meromorphic functions were introduced and studied using q-hypergeometric functions [18,19] and q-hypergeometric polynomials were defined in [20]. Many investigations concerned the q-analogue of Ruscheweyh differential operators defined in [21] and the q-analogue of Sălăgean differential operators introduced in [22]. For example, differential subordinations are investigated involving a certain q-Ruscheweyh type derivative operator in [23], a q-Ruscheweyh derivative operator is used for the definition and coefficient estimates investigation of a new class of analytic functions in [24], and classes of analytic univalent functions are introduced and investigated in [25] using both Ruscheweyh and Sălăgean q-analogue operators. Subordination results involving the q-analogue of the Sălăgean differential operator are obtained in [26]. and a generalization of the Sălăgean q-differential operator is involved in the study of certain differential subordinations in [27]. New subclasses of univalent functions are introduced in [28] using Sălăgean q-differential operators, and a quasi-Hadamard product is associated with the study regarding certain starlike and convex functions with respect to symmetric points involving q-Sălăgean operators in [29]. A q-Bernardi integral operator is introduced in [30]. Recent results obtained by applying it to starlike and convex functions can be seen in [31,32] and for certain subclasses of p-valent functions in [33]. A q-Bernardi integral operator is introduced and studied regarding m-fold symmetric functions in [34].

The Srivastava–Attiya operator and the multiplier transformation are adapted to a quantum calculus approach in [35]. Further applications of the q-Srivastava–Attiya operator involving holomorphic and bi-univalent functions are proved in [36,37]. In recent investigations, the q-analogue of the multiplier transformation was used to define and study new subclasses of harmonic univalent functions [38] and to obtain fuzzy differential subordinations [39].

The motivation for introducing the new results contained in this paper resides from the nice results recently obtained by incorporating quantum calculus aspects into geometric function theory as listed above. Reading about the applications of the q-analogue of the multiplier transformation regarding the definition of new subclasses of univalent functions [38] and connected to the theory of fuzzy differential subordination [39], and considering the recent results involving another quantum calculus operator and the classical theories of differential subordination and superordination [40,41], we were inspired to further investigate the q-analogue of the multiplier transformation connected to the idea of introducing and studying new subclasses of univalent functions. For the investigation presented in this paper, the q-analogue of the multiplier transformation is applied for defining a new convex subclass of normalized analytic functions in the open unit disc U, which is further investigated using the methods of differential subordination and superordination theories.

The notions and preliminary known results used in the research are first introduced.

The investigation is set in the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and involves the class of holomorphic functions in the unit disc $\mathcal{H}(U)$.

Particular subclasses of $\mathcal{H}(U)$ involved are:

$$\mathcal{A}(p,n) = \{f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j \in \mathcal{H}(U)\},$$

with $\mathcal{A}_n = \mathcal{A}(1,n)$, $\mathcal{A} = \mathcal{A}_1 = \mathcal{A}(1,1)$ and

$$\mathcal{H}[a,n] = \{f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \in \mathcal{H}(U)\},$$

where $n, p \in \mathbb{N}$, $a \in \mathbb{C}$.

The differential subordination's theory established by Mocanu and Miller [42] concerns the following definitions:

The analytic function ϕ is subordinate to the analytic function γ in U, denoted as $\phi \prec \gamma$, when there exists an analytic function ω in U, such that $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in U$ and $\phi(z) = \gamma(\omega(z))$, $z \in U$. For univalent function γ, we have $\phi \prec \gamma$ if and only if $\phi(0) = \gamma(0)$ and $\phi(U) \subseteq \gamma(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and h be a univalent function in U. A solution of the differential subordination is an analytic function p in U that verifies the differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \qquad z \in U. \tag{1}$$

A dominant of the solutions of the differential subordination is the univalent function π such that $p \prec \pi$ for all p satisfying (1). The best dominant of (1) is a dominant $\widetilde{\pi}$ with the property $\widetilde{\pi} \prec \pi$ for all dominants π of (1).

The dual theory of differential superordination introduced by Mocanu and Miller in 2003 [43] is characterized by the following definitions:

As a dual notion, the analytic function ϕ is superordinate to the analytic function γ, denoted as $\gamma \prec \phi$, when there exists an analytic function ω in U, such that $\omega(0) = 0$,

$|\omega(z)| < 1, z \in U$ and $\gamma(z) = \phi(\omega(z)), z \in U$. For univalent function ϕ we have $\gamma \prec \phi$ if and only if $\phi(0) = \gamma(0)$ and $\gamma(U) \subseteq \phi(U)$.

Let $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$ and h be an analytic function in U. A solution of the differential superordination is the univalent fnction p such that $\psi(p(z), zp'(z); z)$ is univalent in U and verifies the differential superordination

$$h(z) \prec \psi(p(z), zp'(z); z), \quad z \in U. \tag{2}$$

A subordinant of the solutions of the differential superordination is the analytic function π such that $\pi \prec p$ for all p satisfying (2). The best subordinant of (2) is a univalent subordinant $\widetilde{\pi}$ with property $\pi \prec \widetilde{\pi}$ for all subordinants π of (2).

Let Q represent the set of analytic and injective functions on $\overline{U} \backslash E(f)$ with property $f'(t) \neq 0$ for $t \in \partial U \backslash E(f)$ and $E(f) = \{t \in \partial U : \lim_{z \to t} f(z) = \infty\}$; $Q(a)$ is the subclass of Q with the property $f(0) = a$.

The following lemmas are useful for the proofs of the new results contained in the next sections.

Lemma 1 (Mocanu and Miller [42])**.** *Let a convex function g in U and the function*

$$h(z) = n\alpha z g'(z) + g(z),$$

with $z \in U$, n a positive integer, and $\alpha > 0$.

When the function

$$g(0) + p_n z^n + p_{n+1} z^{n+1} + \cdots = p(z), \quad z \in U$$

is holomorphic in U and the differential subordination

$$\alpha z p'(z) + p(z) \prec h(z), \quad z \in U,$$

holds, then the differential subordination

$$p(z) \prec g(z)$$

holds as well, and the result is sharp.

Lemma 2 (Ruscheweyh and Hallenbeck ([44], Th. 3.1.6, p. 71))**.** *Let a convex function h such that $h(0) = a$, and $\alpha \in \mathbb{C}^*$ with Re $\alpha \geq 0$. When $p \in \mathcal{H}[a, n]$ and the differential subordination*

$$\frac{zp'(z)}{\gamma} + p(z) \prec h(z), \quad z \in U,$$

holds, then the differential subordinations

$$p(z) \prec g(z) \prec h(z), \quad z \in U,$$

hold for

$$g(z) = \frac{\alpha}{n z^{\frac{\alpha}{n}}} \int_0^z h(t) t^{\frac{\alpha}{n} - 1} dt, \quad z \in U.$$

Lemma 3 (Mocanu and Miller ([43], Th. 3.1.6, p. 71))**.** *Let a convex function h such that $h(0) = a$, and $\alpha \in \mathbb{C}^*$ with Re $\alpha \geq 0$. When $p \in Q \cap \mathcal{H}[a, n]$, $\frac{zp'(z)}{\alpha} + p(z)$ is a univalent function in U and the differential superordination*

$$h(z) \prec \frac{zp'(z)}{\alpha} + p(z), \; z \in U,$$

holds, then the differential superordination

$$g(z) \prec p(z), \ z \in U,$$

holds as well, for the convex function $g(z) = \frac{\alpha}{nz^{\frac{\alpha}{n}}} \int_0^z h(t) t^{\frac{\alpha}{n}-1} dt$, $z \in U$ *the best subordinant.*

Lemma 4 (Mocanu and Miller [43])**.** *Consider a convex function g in U and the function*

$$h(z) = \frac{zg'(z)}{\alpha} + g(z), \ z \in U,$$

with $\alpha \in \mathbb{C}^*$, $Re\ \alpha \geq 0$. *If* $p \in Q \cap \mathcal{H}[a,n]$, $\frac{zp'(z)}{\alpha} + p(z)$ *is a univalent function in U and the differential superordination*

$$\frac{zg'(z)}{\alpha} + g(z) \prec \frac{zp'(z)}{\alpha} + p(z), \ z \in U,$$

holds, then the differential superordination

$$g(z) \prec p(z), z \in U,$$

holds as well, for $g(z) = \frac{\alpha}{nz^{\frac{\alpha}{n}}} \int_0^z h(t) t^{\frac{\alpha}{n}-1} dt$, $z \in U$ *the best subordinant.*

We note the notions and notations of q-calculus.
For $0 < q < 1$, $n \in \mathbb{N}$, it is denoted that

$$[n]_q = \frac{1-q^n}{1-q},$$

and

$$[n]_q! = \begin{cases} \prod_{k=1}^{n} [k]_q, & n \in \mathbb{N}^*, \\ 1, & n = 0. \end{cases}$$

The q-derivative operator $\mathcal{D}_q$ is defined for a function $f \in \mathcal{A}$ by ([45]):

$$\mathcal{D}_q(f(z)) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

It can be observed that

$$\lim_{q \to 1} \mathcal{D}_q(f(z)) = \lim_{q \to 1} \frac{f(z) - f(qz)}{(1-q)z} = f'(z)$$

for f a differentiable function.
For $f(z) = z^k$, $\mathcal{D}_q(f(z)) = \mathcal{D}_q\left(z^k\right) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}$.
We now show the definition of the q-analogue of the multiplier transformation:

Definition 1 ([35])**.** *Denote by* $\mathcal{I}_q^{m,l}$ *the q-analogue of multiplier transformation*

$$\mathcal{I}_q^{m,l} f(z) := z + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q} \right)^m a_k z^k,$$

with $l > -1$, $q \in (0,1)$, *m a real number, and* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{A}$, $z \in U$.

Using the properties of q-calculus, we obtain

$$z\mathcal{D}_q\left(\mathcal{I}_q^{m,l}f(z)\right) = \left(1+\frac{[l]_q}{q^l}\right)\mathcal{I}_q^{m+1,l}f(z) - \frac{[l]_q}{q^l}\mathcal{I}_q^{m,l}f(z).$$

Using the q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$ given in Definition 1, a new subclass of normalized analytic functions in the open unit disc U is introduced in Section 2 of this paper. It is proved that this new class is convex and, using this property, subordination results are investigated in the theorems of Section 2 involving functions from the newly defined class, operator $\mathcal{I}_q^{m,l}$, and Lemmas 1 and 2. In Section 3, differential superordinations involving the operator $\mathcal{I}_q^{m,l}$ are considered for which the best subordinants are also found. Lemmas 3 and 4 are necessary for establishing the new results.

2. Differential Subordination Results

The new class of normalized analytic functions in the open unit disc U is defined using the q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$ given in Definition 1.

Definition 2. *Let $\alpha \in [0,1)$. The class $S_{m,l}^q(\alpha)$ consists of the functions $f \in \mathcal{A}$ with property*

$$Re\left(\mathcal{I}_q^{m,l}f(z)\right)' > \alpha, \quad z \in U. \tag{3}$$

The first result concerning the class $S_{m,l}^q(\alpha)$ establishes its convexity.

Theorem 1. *$S_{m,l}^q(\alpha)$ is a convex set.*

Proof. Consider the functions

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{jk}z^k, \ \ z \in U, \ \ j = 1,2,$$

belonging to the class $S_{m,l}^q(\alpha)$. It is enough to prove that the function

$$f(z) = \lambda_1 f_1(z) + \lambda_2 f_2(z)$$

belongs to the class $S_m(\delta,\alpha)$, with λ_1 and λ_2 positive real numbers such that $\lambda_1 + \lambda_2 = 1$.

The function f has the following form:

$$f(z) = z + \sum_{k=2}^{\infty}(\lambda_1 a_{1k} + \lambda_2 a_{2k})z^k, \quad z \in U,$$

and

$$\mathcal{I}_q^{m,l}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m (\lambda_1 a_{1k} + \lambda_2 a_{2k})z^k, \ \ z \in U. \tag{4}$$

Differentiating relation (4), we obtain

$$\left(\mathcal{I}_q^{m,l}f(z)\right)' = 1 + \sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m (\lambda_1 a_{1k} + \lambda_2 a_{2k})kz^{k-1}, \quad z \in U.$$

Hence

$$Re\left(\mathcal{I}_q^{m,l}f(z)\right)' = 1+Re\left(\lambda_1\sum_{k=2}^{\infty}k\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_{1k}z^{k-1}\right) + Re\left(\lambda_2\sum_{k=2}^{\infty}k\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_{2k}z^{k-1}\right). \quad (5)$$

Taking into account that $f_1, f_2 \in S_{m,l}^q(\alpha)$, we can write

$$Re\left(\lambda_j\sum_{k=2}^{\infty}k\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_{jk}z^{k-1}\right) > \lambda_k(\alpha-1), \quad j=1,2. \quad (6)$$

Using relation (6), we get from (5):

$$Re\left(\mathcal{I}_q^{m,l}f(z)\right)' > 1+\lambda_1(\alpha-1)+\lambda_2(\alpha-1)=\alpha, \quad z\in U,$$

which proved that the set $S_{m,l}^q(\alpha)$ is convex. $\square$

We next investigate a series of differential subordinations involving the convex functions of the class $S_{m,l}^q(\alpha)$ and the q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$.

Theorem 2. *Considering g a convex function, we define the function*

$$h(z)=\frac{zg'(z)}{a+2}+g(z), \ \ a>0,\ z\in U. \quad (7)$$

For $f\in S_{m,l}^q(\alpha)$, consider

$$F(z)=\frac{a+2}{z^{a+1}}\int_0^z t^a f(t)dt, \quad z\in U, \quad (8)$$

then the differential subordination

$$\left(\mathcal{I}_q^{m,l}f(z)\right)' \prec h(z), \ \ z\in U, \quad (9)$$

implies the sharp differential subordination

$$\left(\mathcal{I}_q^{m,l}F(z)\right)' \prec g(z), \ \ z\in U.$$

Proof. Relation (8) can be written as follows:

$$z^{a+1}F(z)=(a+2)\int_0^z t^a f(t)dt, \quad (10)$$

and differentiating it, we have

$$zF'(z)+(a+1)F(z)=(a+2)f(z) \quad (11)$$

and

$$z\left(\mathcal{I}_q^{m,l}F(z)\right)'+(a+1)\mathcal{I}_q^{m,l}F(z)=(a+2)\mathcal{I}_q^{m,l}f(z), \quad z\in U. \quad (12)$$

Differentiating the last relation, we get

$$\frac{z\left(\mathcal{I}_q^{m,l}F(z)\right)''}{a+2}+\left(\mathcal{I}_q^{m,l}F(z)\right)'=\left(\mathcal{I}_q^{m,l}f(z)\right)', \quad z\in U, \quad (13)$$

and subordination (9) can be written in the form

$$\frac{z\left(\mathcal{I}_q^{m,l}F(z)\right)''}{a+2}+\left(\mathcal{I}_q^{m,l}F(z)\right)'\prec\frac{zg'(z)}{a+2}+g(z). \tag{14}$$

Denoting

$$p(z)=\left(\mathcal{I}_q^{m,l}F(z)\right)'\in\mathcal{H}[1,1], \tag{15}$$

differential subordination (14) has the following form:

$$\frac{zp'(z)}{a+2}+p(z)\prec\frac{zg'(z)}{a+2}+g(z),\quad z\in U.$$

Applying Lemma 1, we get

$$p(z)\prec g(z),$$

that means

$$\left(\mathcal{I}_q^{m,l}F(z)\right)'\prec g(z),\quad z\in U$$

for g the best dominant. □

Theorem 3. *Denoting*

$$I_a(f)(z)=\frac{a+2}{z^{a+1}}\int_0^z t^a f(t)dt,\ \ a>0, \tag{16}$$

then

$$I_a\left[S_{m,l}^q(\alpha)\right]\subset S_{m,l}^q(\alpha^*), \tag{17}$$

where

$$\alpha^*=2\alpha-1+(a+2)(2-2\alpha)\int_0^1\frac{t^{a+1}}{t+1}dt. \tag{18}$$

Proof. We use the hypothesis of Theorem 3 for the convex function $h(z)=\frac{1+(2\alpha-1)z}{1+z}$ and, following the same steps as the proof of Theorem 2, the differential subordination

$$\frac{zp'(z)}{a+2}+p(z)\prec h(z),$$

with p defined by relation (15).

Applying Lemma 2, we get the differential subordinations

$$p(z)\prec g(z)\prec h(z),$$

equivalently with

$$\left(\mathcal{I}_q^{m,l}F(z)\right)'\prec g(z)\prec h(z),$$

where

$$g(z)=\frac{a+2}{z^{a+2}}\int_0^z t^{a+1}\frac{1+(2\alpha-1)t}{1+t}dt=$$
$$2\alpha-1+\frac{(a+2)(2-2\alpha)}{z^{a+2}}\int_0^z\frac{t^{a+1}}{t+1}dt.$$

Taking into account that g is a convex function with $g(U)$ symmetric to the real axis, we obtain

$$Re\left(\mathcal{I}_q^{m,l}F(z)\right)' \geq \min_{|z|=1} Reg(z) = Reg(1) = \alpha^* =$$

$$2\alpha - 1 + (a+2)(2-2\alpha)\int_0^1 \frac{t^{a+1}}{t+1}.$$

□

Theorem 4. *Considering the convex function g such that $g(0) = 1$, we define the function*

$$h(z) = zg'(z) + g(z), \quad z \in U.$$

If $f \in \mathcal{A}$ verifies the subordination

$$\left(\mathcal{I}_q^{m,l}f(z)\right)' \prec h(z), \quad z \in U, \tag{19}$$

then the sharp differential subordination

$$\frac{\mathcal{I}_q^{m,l}f(z)}{z} \prec g(z), \quad z \in U$$

holds.

Proof. Considering

$$p(z) = \frac{\mathcal{I}_q^{m,l}f(z)}{z} = \frac{z + \sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_k z^k}{z} = 1 + p_1 z + p_2 z^2 + ..., \quad z \in U,$$

evidently $p \in \mathcal{H}[1,1]$, so we can write

$$zp(z) = \mathcal{I}_q^{m,l}f(z), \quad z \in U,$$

and differentiating it, we get

$$\left(\mathcal{I}_q^{m,l}f(z)\right)' = zp'(z) + p(z), \quad z \in U.$$

Subordination (19) take the form

$$zp'(z) + p(z) \prec h(z) = zg'(z) + g(z), \quad z \in U,$$

and applying Lemma 1, we get

$$p(z) \prec g(z), \quad z \in U,$$

which means

$$\frac{\mathcal{I}_q^{m,l}f(z)}{z} \prec g(z), \quad z \in U.$$

□

Theorem 5. *Considering the convex function h with $h(0) = 1$, for $f \in \mathcal{A}$ that verifies the subordination*

$$\left(\mathcal{I}_q^{m,l}f(z)\right)' \prec h(z), \quad z \in U, \tag{20}$$

we get the differential subordination

$$\frac{\mathcal{I}_q^{m,l} f(z)}{z} \prec g(z), \quad z \in U,$$

for the convex function $g(z) = \frac{1}{z} \int_0^z h(t)dt$*, which is the best dominant.*

Proof. Denote

$$p(z) = \frac{\mathcal{I}_q^{m,l} f(z)}{z} = 1 + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q} \right)^m a_k z^{k-1} \in \mathcal{H}[1,1], \ z \in U.$$

Differentiating it the relation, we get

$$\left(\mathcal{I}_q^{m,l} f(z) \right)' = zp'(z) + p(z), \quad z \in U$$

and differential subordination (20) has the following form:

$$zp'(z) + p(z) \prec h(z), \quad z \in U.$$

After applying Lemma 2, we get

$$p(z) \prec g(z) = \frac{1}{z} \int_0^z h(t)dt, \quad z \in U,$$

written as

$$\frac{\mathcal{I}_q^{m,l} f(z)}{z} \prec g(z) = \frac{1}{z} \int_0^z h(t)dt, \quad z \in U$$

for g the best dominant. □

Theorem 6. *Considering a convex function g such that $g(0) = 1$, we define the function $h(z) = zg'(z) + g(z)$, $z \in U$. When $f \in \mathcal{A}$ verifies the subordination*

$$\left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} \right)' \prec h(z), \quad z \in U, \tag{21}$$

then we get the sharp differential subordination

$$\frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} \prec g(z), \ z \in U.$$

Proof. Denote

$$p(z) = \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} = \frac{z + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q} \right)^{m+1} a_k z^k}{z + \sum_{k=2}^{\infty} \left(\frac{[k+l]_q}{[1+l]_q} \right)^{m} a_k z^k}.$$

Differentiating it, we get $p'(z) = \frac{\left(\mathcal{I}_q^{m+1,l} f(z)\right)'}{\mathcal{I}_q^{m,l} f(z)} - p(z) \frac{\left(\mathcal{I}_q^{m,l} f(z)\right)'}{\mathcal{I}_q^{m,l} f(z)}$ written as $zp'(z) + p(z) = \left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} \right)'$.

Differential subordination (21) has the following form, for $z \in U$,

$$zp'(z) + p(z) \prec h(z) = zg'(z) + g(z),$$

and applying Lemma 1, we get the differential subordination, for $z \in U$,

$$p(z) \prec g(z),$$

written as

$$\frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} \prec g(z).$$

□

3. Differential Superordination Results

In this section, differential superordinations are investigated concerning the q-analogue of the multiplier transformation $\mathcal{I}_q^{m,l}$ and its derivatives of first and second order. The best subordinant is given for each of the investigated differential superordinations.

Theorem 7. *Considering $f \in \mathcal{A}$ and a convex function h in U such that $h(0) = 1$, denote $F(z) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t)dt$, $z \in U$, Re $a > -2$, and assume that $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ is a univalent function in U, $\left(\mathcal{I}_q^{m,l} F(z)\right)' \in Q \cap \mathcal{H}[1,1]$. If the differential superordination*

$$h(z) \prec \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{22}$$

holds, then we get the differential superordination

$$g(z) \prec \left(\mathcal{I}_q^{m,l} F(z)\right)', \quad z \in U,$$

with the convex function $g(z) = \frac{a+2}{z^{a+2}} \int_0^z h(t)t^{a+1}dt$ the best subordinant.

Proof. Using the relation $z^{a+1}F(z) = (a+2)\int_0^z t^a f(t)dt$ from Theorem 2 and differentiating it, we can write $zF'(z) + (a+1)F(z) = (a+2)f(z)$ in the following form: $z\left(\mathcal{I}_q^{m,l} F(z)\right)'$ $+(a+1)\mathcal{I}_q^{m,l} F(z) = (a+2)\mathcal{I}_q^{m,l} f(z)$, $z \in U$, which after differentiating it again, has the form

$$\frac{z\left(\mathcal{I}_q^{m,l} F(z)\right)''}{a+2} + \left(\mathcal{I}_q^{m,l} F(z)\right)' = \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U.$$

Using the last relation, superordination (22) can be written

$$h(z) \prec \frac{z\left(\mathcal{I}_q^{m,l} F(z)\right)''}{a+2} + \left(\mathcal{I}_q^{m,l} F(z)\right)'. \tag{23}$$

Define

$$p(z) = \left(\mathcal{I}_q^{m,l} F(z)\right)', \quad z \in U, \tag{24}$$

and replacing (24) in (23) we have $h(z) \prec \frac{zp'(z)}{a+2} + p(z)$, $z \in U$. Applying Lemma 3 considering $n = 1$ and $\alpha = a+2$, it yields $g(z) \prec p(z)$, equivalently with $g(z) \prec \left(\mathcal{I}_q^{m,l} F(z)\right)'$, $z \in U$, with the best subordinant $g(z) = \frac{a+2}{z^{a+2}} \int_0^z h(t)t^{a+1}dt$ convex function. □

Theorem 8. *Considering a convex function g, we define the function $h(z) = \frac{zg'(z)}{a+2} + g(z)$, with Re $a > -2$, $z \in U$. For $f \in \mathcal{A}$, denote $F(z) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t)dt$, $z \in U$ and assume that $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ is univalent in U and $\left(\mathcal{I}_q^{m,l} F(z)\right)' \in Q \cap \mathcal{H}[1,1]$. When the differential superordination*

$$h(z) \prec \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{25}$$

holds, then we get the differential superordination

$$g(z) \prec \left(\mathcal{I}_q^{m,l} F(z)\right)', \quad z \in U,$$

for $g(z) = \frac{a+2}{z^{a+2}} \int_0^z h(t)t^{a+1}dt$ the best subordinant.

Proof. Considering $p(z) = \left(\mathcal{I}_q^{m,l} F(z)\right)'$, $z \in U$, following the proof of Theorem 7 we can write the differential superordination (25) in the following form:

$$h(z) = \frac{zg'(z)}{a+2} + g(z) \prec \frac{zp'(z)}{a+2} + p(z), \quad z \in U.$$

Applying Lemma 4 for $\alpha = a + 2$ and $n = 1$, we obtain the differential superordination $g(z) \prec p(z) = \left(\mathcal{I}_q^{m,l} F(z)\right)'$, $z \in U$, having $g(z) = \frac{a+2}{z^{a+2}} \int_0^z h(t)t^{a+1}dt$ the best subordinant. $\square$

Theorem 9. *For $f \in \mathcal{A}$ denote $F(z) = \frac{a+2}{z^{a+1}} \int_0^z t^a f(t)dt$, $z \in U$, and $h(z) = \frac{1+(2\alpha-1)z}{1+z}$, where Re $a > -2$, $\alpha \in [0,1)$. Assume that $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ is univalent in U, $\left(\mathcal{I}_q^{m,l} F(z)\right)' \in Q \cap \mathcal{H}[1,1]$ and the differential superordination*

$$h(z) \prec \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{26}$$

is satisfied, then the differential superordination

$$g(z) \prec \left(\mathcal{I}_q^{m,l} F(z)\right)', \quad z \in U,$$

is satisfied for the convex function $g(z) = 2\alpha - 1 + \frac{(a+2)(2-2\alpha)}{z^{a+2}} \int_0^z \frac{t^{a+1}}{t+1}dt$, $z \in U$ as the best subordinant.

Proof. Denoting $p(z) = \left(\mathcal{I}_q^{m,l} F(z)\right)'$, following the proof of Theorem 7, superordination (26) can be written as $h(z) = \frac{1+(2\alpha-1)z}{1+z} \prec \frac{zp'(z)}{a+2} + p(z)$, $z \in U$.

Applying Lemma 3, we get the differential superordination $g(z) \prec p(z)$, with $g(z) = \frac{a+2}{z^{a+2}} \int_0^z \frac{1+(2\alpha-1)t}{1+t} t^{a+1}dt = 2\alpha - 1 + \frac{(a+2)(2-2\alpha)}{z^{a+2}} \int_0^z \frac{t^{a+1}}{t+1}dt \prec \left(\mathcal{I}_q^{m,l} F(z)\right)'$, $z \in U$, and g is the best subordinant and it is convex. $\square$

Theorem 10. *For $f \in \mathcal{A}$, consider a convex function h such that $h(0) = 1$ and assume that $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ is univalent and $\frac{\mathcal{I}_q^{m,l} f(z)}{z} \in Q \cap \mathcal{H}[1,1]$. When the superordination*

$$h(z) \prec \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{27}$$

holds, then the following differential superordination

$$g(z) \prec \frac{\mathcal{I}_q^{m,l} f(z)}{z}, \quad z \in U,$$

is satisfied, for the convex function $g(z) = \frac{1}{z}\int_0^z h(t)dt$ *the best subordinant.*

Proof. Denoting $p(z) = \frac{\mathcal{I}_q^{m,l} f(z)}{z} = \frac{z+\sum_{k=2}^{\infty}\left(\frac{[k+l]_q}{[1+l]_q}\right)^m a_k z^k}{z} \in \mathcal{H}[1,1]$, $z \in U$, we can write $\mathcal{I}_q^{m,l} f(z) = zp(z)$, and differentiating it, we have $\left(\mathcal{I}_q^{m,l} f(z)\right)' = zp'(z) + p(z)$, $z \in U$.

With this notation, differential superordination (27) becomes $h(z) \prec zp'(z) + p(z)$, $z \in U$, and applying Lemma 3, we get $g(z) \prec p(z) = \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, $z \in U$, for $g(z) = \frac{1}{z}\int_0^z h(t)dt$ the best subordinant and convex. □

Theorem 11. *Considering a convex function g in U we define the function h by $h(z) = zg'(z) + g(z)$. Assume $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ is univalent, $\frac{\mathcal{I}_q^{m,l} f(z)}{z} \in Q \cap \mathcal{H}[1,1]$ for $f \in \mathcal{A}$ and the superordination*

$$h(z) = zg'(z) + g(z) \prec \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{28}$$

is satisfied, then the differential superordination

$$g(z) \prec \frac{\mathcal{I}_q^{m,l} f(z)}{z}, \quad z \in U,$$

is satisfied for $g(z) = \frac{1}{z}\int_0^z h(t)dt$ *the best subordinant.*

Proof. Taking account the proof of Theorem 10 for $p(z) = \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, the superordination (28) can be written in the following form: $zg'(z) + g(z) \prec zp'(z) + p(z)$, $z \in U$.

Applying Lemma 4, we get the differential superordination $g(z) \prec p(z)$, equivalently with $g(z) = \frac{1}{z}\int_0^z h(t)dt \prec \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, $z \in U$, for g the best subordinant. □

Theorem 12. *Let $h(z) = \frac{1+(2\alpha-1)z}{1+z}$ with $0 \leq \alpha < 1$, $z \in U$. For $f \in \mathcal{A}$ suppose that $\left(\mathcal{I}_q^{m,l} f(z)\right)'$ is univalent and $\frac{\mathcal{I}_q^{m,l} f(z)}{z} \in Q \cap \mathcal{H}[1,1]$. If the differential superordination*

$$h(z) \prec \left(\mathcal{I}_q^{m,l} f(z)\right)', \quad z \in U, \tag{29}$$

holds, then we get the following differential superordination

$$g(z) \prec \frac{\mathcal{I}_q^{m,l} f(z)}{z}, \quad z \in U,$$

where the best subordinant is the convex function $g(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\ln(1+z)}{z}$, $z \in U$.

Proof. Following the proof of Theorem 10 for $p(z) = \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, superordination (29) takes the form $h(z) = \frac{1+(2\alpha-1)z}{1+z} \prec zp'(z) + p(z)$, $z \in U$.

Applying Lemma 3, we get the following differential superordination $g(z) \prec p(z)$, which can be written as $g(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = 2\alpha - 1 + \frac{2(1-\alpha)}{z}\ln(z+1) \prec \frac{\mathcal{I}_q^{m,l} f(z)}{z}$, $z \in U$. The function g is the best subordinant and it is convex. □

Theorem 13. *Considering a convex function h such that* $h(0) = 1$, *for* $f \in \mathcal{A}$, *suppose that* $\left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)'$ *is univalent and* $\frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} \in Q \cap \mathcal{H}[1,1]$. *If the differential superordination*

$$h(z) \prec \left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)', \quad z \in U, \tag{30}$$

holds, then we get the following differential superordination:

$$g(z) \prec \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}, \quad z \in U,$$

with the best subordinant is the convex function $g(z) = \frac{1}{z}\int_0^z h(t)dt$.

Proof. Let $p(z) = \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}$, after differentiating it, we can write $p'(z) = \frac{\left(\mathcal{I}_q^{m+1,l} f(z)\right)'}{\mathcal{I}_q^{m,l} f(z)} - p(z)\frac{\left(\mathcal{I}_q^{m,l} f(z)\right)'}{\mathcal{I}_q^{m,l} f(z)}$ in the form $zp'(z) + p(z) = \left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)'$.

Differential superordination (30) for $z \in U$ becomes $h(z) \prec zp'(z) + p(z)$.

Applying Lemma 3, we obtain the following differential superordination: $g(z) \prec p(z) = \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}$, $z \in U$, for the best subordinant $g(z) = \frac{1}{z}\int_0^z h(t)dt$ convex. □

Theorem 14. *Consider a convex function g and the function h defined by* $h(z) = zg'(z) + g(z)$. *For* $f \in \mathcal{A}$, *assume that* $\left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)'$ *is univalent and* $\frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} \in Q \cap \mathcal{H}[1,1]$. *If the differential superordination*

$$h(z) = zg'(z) + g(z) \prec \left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)', \quad z \in U, \tag{31}$$

holds, then we get the differential superordination

$$g(z) \prec \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}, \quad z \in U,$$

and the best subordinant is $g(z) = \frac{1}{z}\int_0^z h(t)dt$.

Proof. Following the proof of Theorem 13 for $p(z) = \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}$, superordination (31) has the form $h(z) = zg'(z) + g(z) \prec zp'(z) + p(z)$, $z \in U$.

Applying Lemma 4, it yields $g(z) \prec p(z)$, equivalently with $g(z) = \frac{1}{z}\int_0^z h(t)dt \prec \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}$, $z \in U$, and the best subordinant is g. □

Theorem 15. *Consider* $h(z) = \frac{1+(2\alpha-1)z}{1+z}$, *with* $0 \le \alpha < 1$. *For* $f \in \mathcal{A}$, *assume that* $\left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)'$ *is univalent and* $\frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)} \in Q \cap \mathcal{H}[1,1]$. *If the differential superordination*

$$h(z) \prec \left(\frac{z\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}\right)', \quad z \in U, \tag{32}$$

holds, then the differential superordination

$$g(z) \prec \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}, \quad z \in U,$$

holds, and the best subordinant is the convex function $g(z) = 2\alpha - 1 + 2(1-\alpha)\frac{\ln(1+z)}{z}$, $z \in U$.

Proof. Using the notation $p(z) = \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}$, differential superordination (32) can be written $h(z) = \frac{1+(2\alpha-1)z}{1+z} \prec zp'(z) + p(z)$, $z \in U$.

Applying Lemma 3, we get the differential superordination $g(z) \prec p(z)$, equivalently with $g(z) = \frac{1}{z}\int_0^z \frac{1+(2\alpha-1)t}{1+t}dt = 2\alpha - 1 + 2(1-\alpha)\frac{1}{z}\ln(z+1) \prec \frac{\mathcal{I}_q^{m+1,l} f(z)}{\mathcal{I}_q^{m,l} f(z)}$, $z \in U$.

The best subordinant is the convex function g. $\square$

4. Conclusions

The new results proved in this paper are related to a new class of analytic normalized functions in U, $S_{m,l}^q(\alpha)$, given in Definition 2, using the q-analogue of the multiplier transformation $I_q^{m,l}$ shown in Definition 1. In Section 2 of the paper, the class is introduced and its convexity property is proved. Using this attribute of the functions belonging to class $S_{m,l}^q(\alpha)$, sharp differential subordinations are next investigated in five theorems. In Theorem 2, the best dominant for the differential subordination is also provided; in Theorem 3, a certain inclusion relation is proved for the class $S_{m,l}^q(\alpha)$. In Section 3 of the paper, differential superordinations are established in the nine theorems involving the q-analogue of the multiplier transformation $I_q^{m,l}$, its first derivative $\left(I_q^{m,l} f(z)\right)'$, second derivative $\left(I_q^{m,l} f(z)\right)''$, and the expression $\frac{zI_q^{m+1,l} f(z)}{I_q^{m,l} f(z)}$ and its derivative.

For future studies, the subordination and superordination results obtained here can inspire investigations where other q-operators are used instead of q-analogue of the multiplier transformation $I_q^{m,l} f(z)$. In addition, since the best dominant of the differential subordination in Theorem 2 is given, and the best subordinants are provided for the differential superordinations studied in Section 3, conditions for univalence of the operator $I_q^{m,l} f(z)$ investigated here could be further obtained. Other classes of univalent functions could be defined using the q-analogue of the multiplier transformation $I_q^{m,l} f(z)$ and different subordination relations. Coefficient estimates could also be studied for the class $S_{m,l}^q(\alpha)$.

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