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Starlike Functions Based on Ruscheweyh $q-$Differential Operator defined in Janowski Domain

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Abstract: In this paper, we make use of the concept of $q-$calculus in the theory of univalent functions, to obtain the bounds for certain coefficient functional problems of Janowski type starlike functions and to find the Fekete–Szegö functional. A similar results have been done for the function $\wp^{-1}$. Further, for functions in newly defined class we determine coefficient estimates, distortion bounds, radius problems, results related to partial sums.

Keywords: starlike functions; convex functions; subordination; Fekete–Szegö inequality; Hadamard product; analytic functions

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1. Introduction, Definitions and Preliminaries

Denote by $\mathcal{A}$ the class of analytic functions $\wp$ inside open unit disc $\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ of the form

$$\wp(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n, \xi \in \mathbb{U}. \tag{1}$$

Let $\mathfrak{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. For two analytic functions $\wp(\xi)$ given by (1) and $\ell(\xi) = \xi + \sum_{n=2}^{\infty} b_n \xi^n, \xi \in \mathbb{U}$, the convolution (Hadamard product) of $\wp(\xi)$ and $\ell(\xi)$ is defined as:

$$\wp(\xi) * \ell(\xi) = \sum_{n=0}^{\infty} a_n b_n \xi^n.$$

Let $\wp, \ell \in \mathcal{A}$. We say that $\wp$ is subordinate to ℓ if there exists a Schwarz function $w(\xi)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(\xi)| < 1$ $(\xi \in \mathbb{U})$, such that $\wp(\xi) = \ell(\omega(\xi))$ $(\xi \in \mathbb{U})$. This subordination is denoted by

$$\wp \prec \ell \quad \text{or} \quad \wp(\xi) \prec \ell(\xi) \quad (\xi \in \mathbb{U}).$$

In particular, if the function ℓ is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$\wp(0) = \ell(0) \quad \textit{and} \quad \wp(\mathbb{U}) \subset \ell(\mathbb{U}).$$

The well-known subclasses of $\mathfrak{S}$ that are are *starlike* and *convex* in $\mathbb{U}$ is defined as below:

$$\mathfrak{S}^* := \left\{ \wp \in \mathcal{A} : \Re\left(\frac{\xi\wp'(\xi)}{\wp(\xi)}\right) > 0, \ \xi \in \mathbb{U} \right\} \tag{2}$$

$$\mathfrak{C} := \left\{ \wp \in \mathcal{A} : \Re\left(\frac{(\xi\wp'(\xi))'}{\wp'(\xi)}\right) > 0, \ \xi \in \mathbb{U} \right\}. \tag{3}$$

, respectively. Equivalently, we have

$$\mathfrak{S}^*(\varphi) = \left\{ \wp \in \mathcal{A} : \frac{\xi \wp'(\xi)}{\wp(\xi)} \prec \varphi(\xi) \right\},$$

$$\mathfrak{C}(\varphi) = \left\{ \wp \in \mathcal{A} : \frac{(\xi \wp'(\xi))'}{\wp'(\xi)} \prec \varphi(\xi) \right\},$$

where

$$\varphi(\xi) = \frac{1+\xi}{1-\xi}. \tag{4}$$

Janowski [1] defined the generalized function class $\mathfrak{S}^*[D,E]$ of starlike functions named as Janwoski function class as follows. A function $\wp$ is called in the class $\mathfrak{S}^*[D,E]$ if

$$\frac{\xi \wp'(\xi)}{\wp(\xi)} \prec \frac{1+D\xi}{1+E\xi} \qquad (-1 \leq E < D \leq 1)$$

or

$$\frac{\xi \wp'(\xi)}{\wp(\xi)} = \frac{(D+1)p(\xi) - (D-1)}{(E+1)p(\xi) - (E-1)} \qquad (-1 \leq E < D \leq 1). \tag{5}$$

The mentioned classes with the restriction $-1 \leq E < D \leq 1$ reduce to the popular *Janowski starlike* and *Janowski convex functions*, respectively. By replacing $D = 1 - 2\vartheta$ and $E = -1$, where $0 \leq \vartheta < 1$, we obtain the classes, namely the class of starlike and convex functions of order ϑ $(0 \leq \vartheta < 1)$ introduced by Robertson in [2], given, respectively, by

$$\mathfrak{S}^*(\vartheta) := \left\{ \wp \in \mathcal{A} : \Re\left(\frac{\xi \wp'(\xi)}{\wp(\xi)}\right) > \vartheta,\ \xi \in \mathbb{U} \right\}$$

and

$$\mathfrak{C}(\vartheta) := \left\{ \wp \in \mathcal{A} : \Re\left(\frac{(\xi \wp'(\xi))'}{\wp'(\xi)}\right) > \vartheta,\ \xi \in \mathbb{U} \right\}.$$

It is well known that $\mathfrak{S}^*(\vartheta) \subset \mathfrak{S}$ and $\mathfrak{C}(\vartheta) \subset \mathfrak{S}$. By virtue of the well known Alexander's relation, we see that $\wp \in \mathfrak{C}(\vartheta)$ in $\mathbb{U}$ if and only if $\xi \wp'(\xi) \in \mathfrak{S}^*(\vartheta)$ for each $0 \leq \vartheta < 1$. The classes defined above plays an imperative role in the progress of Geometric Function Theory (GFT). Various stimulating properties of $\mathfrak{S}$ have been studied from different viewpoints and perspective.The new class of $\mathcal{A}$ are defined by integral and differential operators in terms of convolution and we observe that this formalism (convolution product) brings an ease in further mathematical investigation and also helps to understand the geometric and symmetric properties of $f \in \mathfrak{S}$ better. The reputation of convolution in the theory of operators may easily be understood from the papers in [3–11]. We briefly recall here the notion of $q-$operators , i.e., $q-$difference operators that play vital role in the theory of hypergeometric series, special functions and quantum physics. The application of $q-$calculus was originated by Jackson [12] (see [13–17]). Kanas and Răducanu [14] have used the fractional q-calculus operators to examine certain function classes of $\mathcal{A}$.

Consider $0 < q < 1$ and a non-negative integer n. The $q-$integer number or basic number n is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1},\ [0]_q = 0.$$

We denote by $[t]_q = \dfrac{1-q^t}{1-q}$, where is t non-integer number .

The $q-$shifted factorial is defined as follows

$$[0]_q! = 1,\ \ [n]_q! = [1]_q[2]_q \ldots [n]_q.$$

Note that $\lim_{q\to 1^-}[n]_q = n$ and $\lim_{q\to 1^-}[n]_q! = n!$.

The Jackson's $q-$*derivative operator* (or $q-$*difference operator*) for a function $\wp \in \mathcal{A}$ is defined by

$$\mathcal{D}_q\wp(\xi) = \begin{cases} \dfrac{\wp(q\xi)-\wp(\zeta)}{\zeta(q-1)} & ,\xi \neq 0 \\ \wp'(0) & ,\xi = 0. \end{cases} \tag{6}$$

Note that

$$\mathcal{D}_q\xi^n = [n]_q\xi^{n-1}, \quad n \in \mathbb{N} = \{1,2,\dots\}, \xi \in \mathbb{U}.$$

Further, we define the operator $\mathcal{D}_q^m\wp(\xi), m \in \mathbb{N}$ as follows

$$\mathcal{D}_q^0\wp(\xi) = \wp(\xi) \text{ and } \mathcal{D}_q^m\wp(\xi) = \mathcal{D}_q(\mathcal{D}_q^{m-1}\wp(\xi)).$$

For $t \in \mathbb{R}$ and $n \in \mathbb{N}$, the $q-$*generalized Pochhammer* symbol is defined by

$$[t]_n = [t]_q[t+1]_q[t+2]_q\dots[t+n-1]_q.$$

Moreover, for $t > 0$ the $q-$Gamma function is given by

$$\Gamma_q(t+1) = [t]_q\Gamma_q(t) \text{ and } \Gamma_q(1) = 1.$$

By Ruscheweyh differential operator [18], lately Kanas and Răducanu [14] introduced the Ruscheweyh $q-$*differential operator* defined by

$$\mathcal{R}_q^m\wp(\xi) = F_{q,m+1}(\xi) * \wp(\xi) \quad \xi \in \mathbb{U},\ m > -1 \tag{7}$$

where $\wp \in \mathcal{A}$ and

$$F_{q,m+1}(\xi) = \xi + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+m)}{[n-1]_q!\Gamma_q(1+m)}\xi^n. \tag{8}$$

From (7) we have

$$\mathcal{R}_q^0\wp(\xi) = \wp(\xi), \qquad \mathcal{R}_q^1\wp(\xi) = \xi\mathcal{D}_q\wp(\xi)$$

and

$$\mathcal{R}_q^n\wp(\xi) = \frac{\xi\mathcal{D}_q^n(\xi^{n-1}\wp(\xi))}{[n]_q!} \quad n \in \mathbb{N}.$$

For $\wp \in \mathcal{A}$ given by (1), in view of (7) and (8), we obtain

$$\mathcal{R}_q^m\wp(\xi) = \xi + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+m)}{[n-1]_q!\Gamma_q(1+m)}a_n\xi^n \quad \xi \in \mathbb{U}. \tag{9}$$

Note that

$$\lim_{q\to 1^-} F_{q,m+1}(\xi) = \frac{\zeta}{(1-\xi)^{m+1}}$$

and

$$\lim_{q\to 1^-} \mathcal{R}_q^m\wp(\xi) = \wp(\xi) * \frac{\xi}{(1-\xi)^{m+1}}.$$

Moreover,

$$\mathcal{D}_q(\mathcal{R}_q^m\wp(\xi)) = 1 + \sum_{n=2}^{\infty}[n]_q\Upsilon_q(n,m)a_n\xi^{n-1} \tag{10}$$

where

$$\Upsilon_n = \Upsilon_q(n,m) = \frac{\Gamma_q(n+m)}{[n-1]_q!\Gamma_q(1+m)}. \tag{11}$$

In this article motivated by the works in [3–11,17], using the operator defined in (9) we introduce a new class of $\mathcal{A}$ as below:

$$\mathfrak{S}_q^{m,n}(D,E) = \left\{ \wp \in \mathcal{A} : \frac{\xi \mathcal{D}_q(\mathcal{R}_q^m \wp(\xi))}{\mathcal{R}_q^m \wp(\xi)} \prec \frac{1+D\xi}{1+E\xi} \right\} \quad (\xi \in \mathbb{U}), \tag{12}$$

where $-1 \leq E < D \leq 1$ and obtain Fekete–Szegö functional. Further, coefficient estimates, characteristic properties and partial sums results are derived.
By fixing the values of D and E one can state new classes $\mathfrak{S}_q^{m,n}(1-2\alpha,-1) = \mathfrak{S}_q^{m,n}(\alpha)$ analogues to the classes studied in [2] and $\mathfrak{S}_q^{m,n}(1-1) = \mathfrak{S}_q^{m,n}(\varphi)$ where φ is given by (4).

2. The Fekete–Szegö Inequality for $f \in \mathfrak{S}_q^{m,n}(D,E)$

To prove the Fekete–Szegö inequality for $\wp \in \mathfrak{S}_q^{m,n}(D,E)$ we use the following:

Lemma 1 ([19,20]). *If $P(\xi) = 1 + p_1\xi + p_2\xi^2 + \ldots$ and is in $P \in \mathcal{P}$ the class of functions of positive real part in $\mathbb{U}$, then*

$$|p_n| \leq 1,\ n \geq 1, \tag{13}$$

and for $\hbar \in \mathbb{C}$ complex number

$$\left| p_2 - \hbar p_1^2 \right| \leq 2\max\{1, |1-2\hbar|\}. \tag{14}$$

If $\hbar$ is a real parameter, then

$$\left| p_2 - \hbar p_1^2 \right| \leq \begin{cases} -4\hbar + 2 & (\hbar \leq 0) \\ 2 & (0 \leq \hbar \leq 1) \\ 4\hbar - 2 & (\hbar \geq 1). \end{cases} \tag{15}$$

When $\hbar > 1$ or $\hbar < 0$, equality (15) holds true if and only if

$$P_1(\xi) = \frac{1+\xi}{1-\xi}$$

or one of its rotations. When $0 < \hbar < 1$, then (15) holds if and only if

$$P_2(\xi) = \frac{1+\xi^2}{1-\xi^2}$$

or one of its rotations. When $\hbar = 0$, equality (15) holds if and only if

$$P_3(\xi) = \left(\frac{1+c}{2}\right)\frac{1+\xi}{-\xi+1} + \left(\frac{1-c}{2}\right)\frac{-\xi+1}{1+\xi} \qquad (0 \leq c \leq 1)$$

or one of its rotations. When $\hbar = 1$, then (15) holds true if $P(\xi)$ is a reciprocal of one of the functions such that the equality holds true in the case when $\hbar = 0$.

Theorem 1. *If $\wp \in \mathcal{A}$ and be given by (1), belongs to $\mathfrak{S}_q^{m,n}(D,E)$, then*

$$|a_2| \leq \frac{|D-E|}{q\Upsilon_2}, \tag{16}$$

$$|a_3| \leq \frac{|D-E|}{q(1+q)\Upsilon_3}\max\left\{1, \left|\frac{1+2E-D}{-E+D} - \left(\frac{-E+D}{q}\right)\right|\right\}. \tag{17}$$

and for a complex number $\aleph$,

$$\left|a_3 - \aleph a_2^2\right| \leq \frac{D-E}{\Upsilon_3} \max\{1, |\Xi(\aleph, D, E)|\}, \tag{18}$$

where

$$\Xi(\aleph, D, E) = \frac{1+2E-D}{D-E} - (\frac{D-E}{q}) + \frac{(1+q)\aleph(D-E)\Upsilon_3}{q\Upsilon_2^2},$$

and

$$\Upsilon_n = \frac{\Gamma_q(n+m)}{[n-1]_q!\Gamma_q(1+m)}.$$

Proof. We show that the relations (16), (17), (18) and (29) hold true for $\wp \in \mathfrak{S}_q^{m,n}(D,E)$. If $f \in \mathfrak{S}_q^{m,n}(D,E)$,

$$\frac{\xi \mathcal{D}_q(\mathcal{R}_q^m \wp(\xi))}{\mathcal{R}_q^m \wp(\xi)} \prec \frac{1+D\xi}{1+E\xi} \tag{19}$$

which yields,

$$\frac{\xi \mathcal{D}_q(\mathcal{R}_q^m \wp(\xi))}{\mathcal{R}_q^m \wp(\xi)} = \frac{1+Dw(\xi)}{1+Ew(\xi)} = G(w(\xi)), \quad (-1 \leq E < D \leq 1).$$

We can write $w(\xi)$ as follow

$$w(\xi) = \frac{1-h(\xi)}{1+h(\xi)} = \frac{p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots}{2 + p_1\xi + p_2\xi^2 + p_3\xi^3 + \cdots}.$$

but

$$G(w(\xi)) = 1 + \frac{1}{2}(D-E)p_1\xi + \frac{1}{4}\left(2(D-E)p_2 - (1+E)p_1^2\right)\xi^2 + \cdots, \tag{20}$$

and

$$\frac{\xi \mathcal{D}_q(\mathcal{R}_q^m \wp(\xi))}{\mathcal{R}_q^m \wp(\xi)} = 1 + (-1+[2]_q)\Upsilon_2 a_2\xi + \left((-1+[3]_q)\Upsilon_3 a_3 - (-1+[2]_q)\Upsilon_2^2 a_2^2\right)\xi^2 + \ldots \tag{21}$$

Equivalently,

$$\frac{\xi \mathcal{D}_q(\mathcal{R}_q^m \wp(\xi))}{\mathcal{R}_q^m \wp(\xi)} = 1 + q\Upsilon_2 a_2\xi + q\left((q+1)\Upsilon_3 a_3 - \Upsilon_2^2 a_2^2\right)\xi^2 + \ldots \tag{22}$$

If we compare (20) and (22) we get

$$a_2 = \frac{D-E}{2q\Upsilon_2} p_1, \tag{23}$$

$$a_3 = \frac{D-E}{2q(q+1)\Upsilon_3}\left(p_2 - \frac{p_1^2}{2}\left[\frac{1+E}{D-E} - (\frac{D-E}{q})\right]\right). \tag{24}$$

and applying (13) to (23) and (14) to (24), we get

$$|a_2| \leq \frac{|D-E|}{q\Upsilon_2}, \tag{25}$$

$$|a_3| \leq \frac{|D-E|}{q(q+1)\Upsilon_3} \max\left\{1, \left|\frac{1+2E-D}{D-E} - (\frac{D-E}{q})\right|\right\}. \tag{26}$$

In addition, from (23) and (24), we get

$$\left|a_3 - \aleph a_2^2\right| = \frac{|D-E|}{2q(q+1)\Upsilon_3}\left|p_2 - \frac{p_1^2}{2}\Xi(\aleph, D, E)\right|, \tag{27}$$

where

$$\Xi(\aleph, D, E) = \frac{1+E}{D-E} - (\frac{D-E}{q}) + \frac{(1+q)\aleph(D-E)\Upsilon_3}{q\Upsilon_2^2}. \tag{28}$$

If we apply (14) to (27) we attain the required results. In addition, for real $\aleph$, using (15) to above (27). □

Theorem 2. *If $\wp \in \mathcal{A}$ and be given by (1), belongs to $\mathfrak{S}_q^{m,n}(D, E)$ then for a real parameter $\aleph$, we have*

$$\left|a_3 - \aleph a_2^2\right| \leq \frac{|D-E|}{q(q+1)\Upsilon_3}\begin{cases} 1 - \Psi(\aleph, D, E) & (\aleph < \sigma_1) \\ 1 & (\sigma_1 \leq \aleph \leq \sigma_2) \\ \Psi(\aleph, D, E) - 1 & (\aleph > \sigma_2), \end{cases} \tag{29}$$

where

$$\Psi(\aleph, D, E) = \frac{1}{2}\left(\frac{1+E}{D-E} - (\frac{D-E}{q}) + \frac{(1+q)\aleph(D-E)\Upsilon_3}{q\Upsilon_2^2}\right), \tag{30}$$

$$\sigma_1 = \frac{q\Upsilon_2^2}{(1+q)(D-E)\Upsilon_3}\left(\frac{D-E}{q} - \frac{1+E}{D-E}\right)$$

and

$$\sigma_2 = \frac{q\Upsilon_2^2}{(1+q)(D-E)\Upsilon_3}\left(\frac{D-E}{q} + \frac{2D-3E-1}{D-E}\right).$$

Proof. For real $\aleph$ using (15) to above (27), we get the required results. □

3. The Coefficient Inequalities for $\wp^{-1} \in \mathfrak{S}_q^{m,n}(D, E)$

The Koebe one quarter theorem [21] ensures that the image of $\mathbb{U}$ under every univalent function $\wp \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function $\wp$ has an inverse $\wp^{-1}$ satisfying

$$\wp^{-1}(\wp(\xi)) = \xi,\ (\xi \in \mathbb{U})\text{and}\wp(\wp^{-1}(w)) = w, (|w| < r_0(\wp),\ r_0(\wp) \geq \frac{1}{4}).$$

A function $\wp \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $\wp$ and $\wp^{-1}$ are univalent in $\mathbb{U}$. We notice that the class of bi-univalent functions defined in the unit disk $\mathbb{U}$ is not empty. For example, the functions ξ, $\frac{\xi}{1-\xi}$, $-\log(1-\xi)$ and $\frac{1}{2}\log\frac{1+\xi}{1-\xi}$ are members of bi-univalent function class; however, the Koebe function is not a member.

Theorem 3. *If $\wp \in \mathfrak{S}_q^{m,n}(D, E)$ and the inverse function of $\wp$, $\wp^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$, the Koebe domain of the class $\wp \in \mathfrak{S}_q^{m,n}(D, E)$, then*

$$|d_2| = \frac{|D-E|}{q\Upsilon_2} \tag{31}$$

$$|d_3| = \frac{|D-E|}{q(q+1)\Upsilon_3}\max\{1, |\Xi(2, D, E) - 1|\} \tag{32}$$

and for any $\hbar \in \mathbb{C}$, we have

$$| d_3 - \hbar d_2^2 | \leq \frac{|D-E|}{q(q+1)\Upsilon_3} \max\{1, |\Xi(2,D,E) + \hbar \frac{(q+1)(D-E)\Upsilon_3}{q\Upsilon_2^2} - 1| \tag{33}$$

where $\Xi(2,D,E) = \frac{1+E}{D-E} - (\frac{D-E}{q}) + \frac{2(1+q)(D-E)\Upsilon_3}{q\Upsilon_2^2}$.

Proof. As

$$\wp^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n \tag{34}$$

is the inverse function of $\wp$, it can be seen that

$$\xi = \wp^{-1}(\wp(\xi)) = \wp\{\wp^{-1}(\xi)\}. \tag{35}$$

From (1) and (35), we obtain that

$$\xi = \wp^{-1}(\xi + \sum_{n=2}^{\infty} a_n \xi^n). \tag{36}$$

We can obtain from (35) and (36),

$$\xi + (a_2 + d_2)\xi^2 + (a_3 + 2a_2 d_2 + d_3)\xi^3 + \ldots = \xi. \tag{37}$$

By equating corresponding coefficients, of the relation (37), we obtain

$$d_2 = -a_2 \tag{38}$$
$$d_3 = 2a_2^2 - a_3. \tag{39}$$

From relations (23) and (38)

$$d_2 = -\frac{D-E}{2q\Upsilon_2} p_1; \tag{40}$$

$$\tag{41}$$

To find $|d_3|$, from (39) we have

$$|d_3| = |a_3 - 2a_2^2|$$

thus, by using (27) for real $(\aleph = 2)$we have

$$\begin{aligned} |d_3| = \left|a_3 - 2a_2^2\right| &= \frac{|D-E|}{2q(q+1)\Upsilon_3} v \left| p_2 - \frac{p_1^2}{2} \Xi(2,D,E) \right|; \\ &= \frac{|D-E|}{q(q+1)\Upsilon_3} \max\{1, |\Xi(2,D,E) - 1|\} \end{aligned} \tag{42}$$

where

$$\Xi(2,D,E) = \frac{1+E}{-E+D} - (\frac{D-E}{q}) + \frac{2(1+q)(D-E)\Upsilon_3}{q\Upsilon_2^2}. \tag{43}$$

For any complex number $\hbar$, we consider

$$\begin{aligned} d_3 - \hbar d_2^2 &= \frac{D-E}{2q(q+1)\Upsilon_3}\left(p_2 - \frac{p_1^2}{2}\Xi(2,D,E)\right) - \hbar\frac{(D-E)^2}{4q^2\Upsilon_2^2}p_1^2. \\ &= \frac{D-E}{2q(q+1)\Upsilon_3}\left(p_2 - \frac{p_1^2}{2}\left[\Xi(2,D,E) + \hbar\frac{(q+1)(D-E)\Upsilon_3}{q\Upsilon_2^2}\right]\right). \end{aligned} \tag{44}$$

Taking modulus on both sides of (44) and by using Lemma 1 and (13), we get:

$$|d_3 - \hbar d_2^2| \le \frac{|D-E|}{q(q+1)\Upsilon_3}\max\{1, |\Xi(2,D,E) + \hbar\frac{(q+1)(D-E)\Upsilon_3}{q\Upsilon_2^2} - 1|, \tag{45}$$

and this completes our proof. □

4. Characterization Properties

Employing techniques given by Silverman [22] we discuss certain characteristic properties of $f \in \mathfrak{S}_q^{m,n}(D,E)$ such as partial sums results, necessary and sufficient conditions, radii of close-to-convexity, distortion bounds, radii of starlikeness and convexity.

Theorem 4. *If $\wp \in \mathcal{A}$ and be given by (1), belongs to $\mathfrak{S}_q^{m,n}(D,E)$ then*

$$\sum_{n=2}^{\infty}\left(|D - [n]_q E| + (-1 + [n]_q)\right)\Upsilon_n |a_n| \le |-E + D|, \tag{46}$$

where Υ_n given by (11).

Proof. Let $\wp \in \mathfrak{S}_q^{m,n}(D,E)$ and by (12) we have

$$\frac{\xi\mathcal{D}_q(\mathcal{R}_q^m\wp(\xi))}{\mathcal{R}_q^m\wp(\xi)} = \frac{1 + Dw(\xi)}{Ew(\xi) + 1}\ (\xi \in \mathbb{U}) \tag{47}$$

where $w(\xi)$ is a Schwarz function. Equivalently

$$\left|\frac{\xi\mathcal{D}_q(\mathcal{R}_q^m\wp(\xi)) - \mathcal{R}_q^m\wp(\xi)}{D\mathcal{R}_q^m\wp(\xi) - E\xi\mathcal{D}_q(\mathcal{R}_q^m\wp(\xi))}\right| < 1.$$

Thus,

$$\begin{aligned} &\left|\frac{\xi\mathcal{D}_q(\mathcal{R}_q^m\wp(\xi)) - \mathcal{R}_q^m\wp(\xi)}{D\mathcal{R}_q^m\wp(\xi) - E\xi\mathcal{D}_q(\mathcal{R}_q^m\wp(\xi))}\right| \\ = &\left|\frac{\sum_{n=2}^{\infty}([n]_q - 1)\Upsilon_n a_n\xi^n}{(D-E)\xi + \sum_{n=2}^{\infty}(D - E[n]_q)\Upsilon_n a_n\xi^n}\right| \\ \le &\frac{\sum_{n=2}^{\infty}([n]_q - 1)\Upsilon_n|a_n|t^{n-1}}{|D-E| - \sum_{n=2}^{\infty}|D - E[n]_q|\Upsilon_n|a_n|t^{n-1}} < 1. \end{aligned}$$

Allowing $t \to 1$, simple computation yields (46). □

Example 1. *For*

$$\wp(\xi) = \xi + \sum_{n=2}^{\infty}\frac{|D-E|}{[(-1+[n]_q) + |D - [n]_q E|]\Upsilon_n}k_n\xi^n,\ \xi \in \mathbb{U},$$

such that $\sum_{n=2}^{\infty} k_n = 1$, *we get*

$$\begin{aligned} & \sum_{n=2}^{\infty} \left[\left| D - [n]_q E \right| + \left(-1 + [n]_q \right) \right] \Upsilon_n |a_n| \\ = & \sum_{n=2}^{\infty} \left[\left| D - [n]_q E \right| + \left(-1 + [n]_q \right) \right] \Upsilon_n \left(\frac{|D-E|}{\left[\left| D - [n]_q E \right| + \left(-1 + [n]_q \right) \right] \Upsilon_n} k_n \right) \\ = & |D-E| \sum_{n=2}^{\infty} k_n = |D-E|. \end{aligned}$$

Then $\wp(\xi) \in \mathfrak{S}_q^{m,n}(D,E)$, *and we observe that* (46) *is sharp.*

Corollary 1. *Let* $\wp \in \mathfrak{S}_q^{m,n}(D,E)$ *given by* (1)*. Then*

$$|a_n| \leq \frac{|D-E|}{\left[\left| D - [n]_q E \right| + \left(-1 + [n]_q \right) \right] \Upsilon_n}, \text{ for } n \geq 2, \tag{48}$$

where Υ_n *is defined by* (11)*. The approximation is sharp for*

$$\wp(\xi) = \xi - \frac{|D-E|}{\left[\left| D - [n]_q E \right| + \left(-1 + [n]_q \right) \right] \Upsilon_n} \xi^n, \; n \geq 2. \tag{49}$$

Theorem 5. *If* $\wp \in A$ *is in the class* $\mathfrak{S}_q^{m,n}(D,E)$, *then*

$$t - \frac{|D-E|}{[|D-(1+q)E| + q]\Upsilon_2} t^2 \leq |\wp(\xi)| \leq t + \frac{|D-E|}{[|D-(1+q)E| + q]\Upsilon_2} t^2. \tag{50}$$

For the function define by

$$\wp(\xi) = \xi - \frac{|D-E|}{[|D-(1+q)E| + q]\Upsilon_2} \xi^2 \quad |\xi| = t < 1, \tag{51}$$

the approximation is sharp.

Proof. We consider

$$\begin{aligned} |\wp(\xi)| &= \left| \xi + \sum_{n=2}^{\infty} a_n \xi^n \right| \\ &\leq |\xi| + \sum_{n=2}^{\infty} a_n |\xi|^n \\ &= t + \sum_{n=2}^{\infty} a_n |t|^n, \end{aligned}$$

since for $|\xi| = t < 1$ we get $t^n < t^2$ for $n \geq 2$ and

$$|\wp(\xi)| \leq t + t^2 \sum_{n=2}^{\infty} |a_n|. \tag{52}$$

Comparably

$$|\wp(\xi)| \geq t - t^2 \sum_{n=2}^{\infty} |a_n|. \tag{53}$$

From the relation (46) we have

$$\sum_{n=2}^{\infty}\left[\left|D-[n]_q E\right|+\left(-1+[n]_q\right)\right]\Upsilon_n|a_n| \leq |D-E|,$$

but

$$\begin{aligned}[|D-(1+q)E|+q]\Upsilon_2\sum_{n=2}^{\infty}|a_n| &\leq \sum_{n=2}^{\infty}\left[\left|D-E[n]_q\right|+\left(-1+[n]_q\right)\right]\Upsilon_n|a_n| \\ &\leq |-E+D|.\end{aligned}$$

Thus

$$\sum_{n=2}^{\infty}a_n \leq \frac{|D-E|}{[|D-(1+q)E|+q]\Upsilon_2}. \tag{54}$$

Using (54) in(52) and (53), we obtain the desired result. □

Theorem 6. *If $\wp \in \mathfrak{S}_q^{m,n}(D,E)$, then*

$$1-\frac{2|D-E|}{[q+|D-(1+q)E|]\Upsilon_2}t \leq \left|\wp'(\xi)\right| \leq 1+\frac{2|D-E|}{[|D-(1+q)E|+q]\Upsilon_2}t. \tag{55}$$

The equality holds for $\wp$ given in (51).

Proof. The proof is quite analogous by way of Theorem 5, so omitted. □

Theorem 7. *Let $\wp_i \in \mathfrak{S}_q^{m,n}(D,E)$ given by*

$$\wp_i(\xi)=\xi+\sum_{n=2}^{\infty}a_{i,n}\xi^n, \text{ where } i=1,2,3,...,k. \tag{56}$$

Then $H \in \mathfrak{S}_q^{m,n}(D,E)$, where

$$H(\xi)=\sum_{i=1}^{k}c_i\wp_i(\xi) \text{ where } \sum_{i=1}^{k}c_i=1. \tag{57}$$

Proof. By Theorem 4, we have

$$\sum_{n=2}^{\infty}\left[\left|D-[n]_q E\right|+\left(-1+[n]_q\right)\right]\Upsilon_n|a_n| \leq |-E+D|.$$

In addition,

$$\begin{aligned}H(\xi) &= \sum_{i=1}^{k}c_i\left(\xi+\sum_{n=2}^{\infty}a_{i,n}\xi^n\right) \\ &= \xi+\sum_{n=2}^{\infty}\left(\sum_{i=1}^{k}c_i a_{i,n}\right)\xi^n.\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[\left|D-E[n]_q\right|+\left(-1+[n]_q\right)\right]\Upsilon_n\left|\sum_{i=1}^{k}c_i a_{i,n}\right| \\
\leq\ & \sum_{i=1}^{k}\left[\sum_{n=2}^{\infty}\left[\left|D-E[n]_q\right|+\left(-1+[n]_q\right)\right]\Upsilon_n|a_{i,n}|\right]c_i \\
=\ & \sum_{i=1}^{k}|-E+D|c_i=|-E+D|\sum_{i=1}^{k}c_i=|-E+D|,
\end{aligned}$$

thus, $H(\xi)\in\mathfrak{S}_q^{m,n}(D,E)$. □

Theorem 8. *If $\wp_i\in\mathfrak{S}_q^{m,n}(D,E)$, be given by (56) then*

$$\mathcal{G}(\xi)=\xi+\frac{1}{k}\sum_{n=2}^{\infty}\left(\sum_{i=1}^{k}a_{n,i}\xi^n\right), \tag{58}$$

where $\mathcal{G}$ is the arithmetic mean of $\wp_i$, and $\mathcal{G}\in\mathfrak{S}_q^{m,n}(D,E)$.

Proof. To show $\mathcal{G}(\xi)\in\mathfrak{S}_q^{m,n}(D,E)$, by Theorem 4 it is adequate to show that

$$\sum_{n=2}^{\infty}\left[\left|-E[n]_q+D\right|+\left(-1+[n]_q\right)\right]\Upsilon_n\left(\frac{1}{k}\sum_{i=1}^{k}|a_{i,n}|\right)\leq|D-E|.$$

We consider that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[\left|D-[n]_qE\right|+\left(-1+[n]_q\right)\right]\Upsilon_n\left(\frac{1}{k}\sum_{i=1}^{k}|a_{i,n}|\right) \\
=\ & \frac{1}{k}\sum_{i=1}^{k}\left(\sum_{n=2}^{\infty}\left[\left|-E[n]_q+D\right|+\left(-1+[n]_q\right)\right]\Upsilon_n|a_{i,n}|\right) \\
\leq\ & \frac{1}{k}\sum_{i=1}^{k}|D-E|=|D-E|.
\end{aligned}$$

We observe that $\mathcal{G}\in\mathfrak{S}_q^{m,n}(D,E)$. □

Theorem 9. *If $\wp\in\mathfrak{S}_q^{m,n}(D,E)$, then $\wp$ is a starlike functions of order ϑ $(0\leq\vartheta<1)$, $|\xi|<t^*$,*

$$t^*=\inf_{n\geq 2}\left(\frac{(1-\vartheta)\left[\left|D-[n]_qE\right|+\left(-1+[n]_q\right)\right]}{|-E+D|(n-\vartheta)}\Upsilon_n\right)^{\frac{1}{n-1}}.$$

The equality holds for $\wp$ given in (49) .

Proof. Let $\wp\in\mathfrak{S}_q^{m,n}(D,E)$. We know that $\wp$ is in a starlike functions of order ϑ, if

$$\left|\frac{\xi\wp'(\xi)}{\wp(\xi)}-1\right|<-\vartheta+1.$$

By simple computation we get

$$\sum_{n=2}^{\infty}\left(\frac{-\vartheta+n}{1-\vartheta}\right)|a_n||\xi|^{n-1}<1. \tag{59}$$

Since $\wp \in \mathfrak{S}_q^{m,n}(D,E)$, from (46) we get

$$\sum_{n=2}^{\infty} \frac{\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{|-E+D|} \Upsilon_n |a_n| < 1. \tag{60}$$

The relation (59) will holds true if

$$\begin{aligned} &\sum_{n=2}^{\infty} \left(\frac{-\vartheta+n}{1-\vartheta}\right) |a_n| |\xi|^{n-1} \\ < &\sum_{n=2}^{\infty} \frac{\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{|D-E|} \Upsilon_n |a_n|, \end{aligned}$$

which implies that

$$\begin{aligned} |\xi|^{-1+n} &< \left(\frac{(1-\vartheta)\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{|D-E|(n-\vartheta)} \Upsilon_n\right), \\ |\xi| &< \left(\frac{(1-\vartheta)\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{|D-E|(-\vartheta+n)} \Upsilon_n\right)^{\frac{1}{-1+n}}, \end{aligned}$$

which yields the starlikeness of the family. □

Theorem 10. *If $\wp \in \mathfrak{S}_q^{m,n}(D,E)$, then $\wp$ is a close-to-convex functions of order ϑ $(0 \le \vartheta < 1)$, $|\xi| < t_1^*$,*

$$t_1^* = \inf_{n \ge 2} \left(\frac{(-\vartheta+1)\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{n|-E+D|} \Upsilon_n\right)^{\frac{1}{-1+n}}.$$

Proof. Let $\wp \in \mathfrak{S}_q^{m,n}(D,E)$. If $\wp$ is close-to-convex function of order ϑ, then we have

$$\left|\wp'(\xi) - 1\right| < 1 - \vartheta,$$

that is

$$\sum_{n=2}^{\infty} \frac{n}{-\vartheta+1} |a_n| |\xi|^{-1+n} < 1. \tag{61}$$

Since $\wp \in \mathfrak{S}_q^{m,n}(D,E)$, by (46) we have

$$\sum_{n=2}^{\infty} \frac{\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{|D-E|} \Upsilon_n |a_n| < 1. \tag{62}$$

The relation (59) will holds true if

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{n}{1-\vartheta} |a_n| |\xi|^{-1+n} \\ < &\sum_{n=2}^{\infty} \frac{\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{|D-E|} \Upsilon_n |a_n|. \end{aligned}$$

Or, equivalently

$$\begin{aligned} |\xi|^{n-1} &< \left(\frac{(1-\vartheta)\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{n|-E+D|} \Upsilon_n\right), \\ |\xi| &< \left(\frac{(1-\vartheta)\left[|D-[n]_q E| + \left(-1+[n]_q\right)\right]}{n|D-E|} \Upsilon_n\right)^{\frac{1}{-1+n}}, \end{aligned}$$

which yields the desired result. □

5. Partial Sums

The partial sums results were examined in [23] by Silverman, for $\wp \in \mathfrak{S}^*(\vartheta)$ and $\wp \in \mathfrak{C}(\vartheta)$ and $\wp$ is as assumed in (1) and established through

$$\begin{aligned} \wp_1(\xi) &= \xi, \\ \wp_j(\xi) &= \xi + \sum_{n=2}^{j} a_n \xi^n. \end{aligned}$$

Partial sums for different subclasses was investigated by several author's, we can see [24,25] and references cited therein. In this section we investigate sharp lower bounds for

$$\Re\left(\frac{\wp(\xi)}{\wp_j(\xi)}\right), \ \Re\left(\frac{\wp_j(\xi)}{\wp(\xi)}\right), \ \Re\left(\frac{\wp'(\xi)}{\wp_j'(\xi)}\right) \text{ and } \Re\left(\frac{\wp_j'(\xi)}{\wp'(\xi)}\right).$$

Theorem 11. *If $\wp \in \mathcal{A}$ and be given by (1), belongs to $\mathfrak{S}_q^{m,n}(D,E)$ and holds (46), then*

$$\Re\left(\frac{\wp(\xi)}{\wp_j(\xi)}\right) \geq 1 - \frac{1}{\Theta_{j+1}} \quad (\forall \xi \in \mathbb{U}) \tag{63}$$

and

$$\Re\left(\frac{\wp_j(\xi)}{\wp(\xi)}\right) \geq \frac{\Theta_{j+1}}{1+\Theta_{j+1}} \quad (\forall \xi \in \mathbb{U}), \tag{64}$$

where

$$\Theta_j = \frac{\left[|D - [n]_q E| + ([n]_q - 1)\right]}{|D-E|} \Upsilon_n. \tag{65}$$

Proof. To prove (63), we set:

$$\begin{aligned} \Theta_{j+1}\left[\frac{\wp(\xi)}{\wp_j(\xi)} - \left(1 - \frac{1}{\Theta_{j+1}}\right)\right] &= \frac{1 + \sum\limits_{n=2}^{j} a_n \xi^{n-1} + \Theta_{j+1} \sum\limits_{n=j+1}^{\infty} a_n \xi^{n-1}}{1 + \sum\limits_{n=2}^{j} a_n \xi^{n-1}} \\ &= \frac{1+\psi_1(\xi)}{1+\psi_2(\xi)}. \end{aligned}$$

Taking

$$\frac{1+\psi_1(\xi)}{1+\psi_2(\xi)} = \frac{1+w(\xi)}{1-w(\xi)},$$

by simple computation,

$$w(\xi) = \frac{\psi_1(\xi) - \psi_2(\xi)}{2 + \psi_1(\xi) + \psi_2(\xi)}.$$

Thus

$$w(\xi) = \frac{\Theta_{j+1} \sum\limits_{n=j+1}^{\infty} a_n \xi^{n-1}}{2 + 2\sum\limits_{n=2}^{j} a_n \xi^{n-1} + \Theta_{j+1} \sum\limits_{n=j+1}^{\infty} a_n \xi^{n-1}},$$

which leads the inequality:

$$|w(\xi)| \leq \frac{\Theta_{j+1} \sum\limits_{n=j+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{j} |a_n| - \Theta_{j+1} \sum\limits_{n=j+1}^{\infty} |a_n|}, \quad |\xi| < 1.$$

We get $|w(\xi)| \leq 1$ if and only if

$$2\Theta_{j+1} \sum_{n=j+1}^{\infty} |a_n| \leq -2\sum_{n=2}^{j} |a_n| + 2,$$

which yields that:

$$\Theta_{j+1} \sum_{n=j+1}^{\infty} |a_n| + \sum_{n=2}^{j} |a_n| \leq 1. \tag{66}$$

To prove (63), it suffices to show that the left hand side of (66) is bounded above by the following sum:

$$\sum_{n=2}^{\infty} \Theta_n |a_n|,$$

which is equivalent to

$$\sum_{n=2}^{j} (\Theta_n - 1)|a_n| + \sum_{n=j+1}^{\infty} (\Theta_n - \Theta_{j+1})|a_n| \geq 0. \tag{67}$$

From (67), it evidence that the proof of approximation in (63) is now completed.

To prove (64), we consider:

$$\begin{aligned}(1+\Theta_{j+1})\left(\frac{\wp_j(\xi)}{\wp(\xi)} - \frac{\Theta_{j+1}}{1+\Theta_{j+1}}\right) &= \frac{1 + \sum\limits_{n=2}^{j} a_n \xi^{-1+n} - \Theta_{j+1} \sum\limits_{n=j+1}^{\infty} a_n \xi^{-1+n}}{1 + \sum\limits_{n=2}^{\infty} a_n \xi^{-1+n}} \\ &= \frac{1+w(\xi)}{1-w(\xi)},\end{aligned}$$

where

$$|w(\xi)| \leq \frac{(1+\Theta_{j+1}) \sum\limits_{n=j+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=2}^{j} |a_n| - (\Theta_{j+1} - 1) \sum\limits_{n=j+1}^{\infty} |a_n|} \leq 1. \tag{68}$$

The relation (68) is equivalent to

$$\sum_{n=2}^{j} |a_n| + \Theta_{j+1} \sum_{n=j+1}^{\infty} |a_n| \leq 1. \tag{69}$$

The left hand side of (69) is bounded above by $\sum\limits_{n=2}^{\infty} \Theta_n |a_n|$, which completes the proof of the assertion(64), thus the proof of Theorem 11 is completed. □

Theorem 12. *If $\wp \in \mathcal{A}$ be given by (1), belongs to $\mathfrak{S}_q^{m,n}(D,E)$ and satisfies (46), then*

$$\Re\left(\frac{\wp'(\xi)}{\wp_j'(\xi)}\right) \geq 1 - \frac{1+j}{\Theta_{j+1}} \quad (\forall \zeta \in \mathbb{U}) \tag{70}$$

and

$$\Re\left(\frac{\wp_j'(\xi)}{\wp'(\xi)}\right) \geq \frac{\Theta_{j+1}}{\Theta_{j+1}+1+j} \quad (\forall \xi \in \mathbb{U}), \tag{71}$$

where Θ_j is as in (65).

Proof. The proof of this theorem is much akin to that of Theorem 11, and we will omit the details. □

6. Conclusions

We have considered the results like the necessary and sufficient conditions, partial sums type results, the Fekete–Szegö inequalities, close-to-convexity, the radii of starlikeness and distortions bounds. In addition, inspiring further researchers working in the field of Geometric Function Theory and draw the attention of the interested readers towards recent articles (see, [26–29]). In conclusion, we suggest, the recently-published review-cum-expository review article by Srivastava ([26], p. 340), who piercing out the fact that the results for the above-mentioned or new $q-$ analogues can easily (and possibly trivially) be translated into the corresponding results for the so-called $(p;q)-$analogues (with $0 < |q| < p \leq 1$) by applying some obvious parametric and argument variations with the additional parameter p being redundant. In addition, we trust that, this paper will stimulate a number of researchers to extend this idea for meromorphic functions, also new classes can be defined by convoluting with certain probability distribution series and also further subordinating with generalized telephone numbers [30].

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