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Analysis of a Fractional Variational Problem Associated with Cantilever Beams Subjected to a Uniformly Distributed Load

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Abstract: In this paper, we investigate the existence and uniqueness of minimizers of a fractional variational problem generalized from the energy functional associated with a cantilever beam under a uniformly distributed load. We apply the fractional Euler–Lagrange condition to formulate the minimization problem as a boundary value problem and obtain existence and uniqueness results in both L^2 and L^∞ settings. Additionally, we characterize the continuous dependence of the minimizers on varying loads in the energy functional. Moreover, an approximate solution is derived via the homotopy perturbation method, which is numerically demonstrated in various examples. The results show that the deformations are larger for smaller orders of the fractional derivative.

Keywords: cantilever beam; existence and uniqueness of minimizers; fractional boundary value problem; Euler–Lagrange theorem; homotopy perturbation method



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1. Introduction

Fractional calculus is a branch of mathematics concerned with derivatives and integrals of non-integer order. It has been applied in various fields including chemistry, biology, engineering, epidemic modeling, and viscoelasticity [1–4]. Several researchers have investigated differential equations of arbitrary order, beginning with the existence and uniqueness of solutions and moving on to analytical and computational techniques to find solutions [5–8].

Although there are several engineering applications for the construction of bridges and buildings, attentive analyses of elastic beam equations are required to ensure the stability of the structure. A cantilever beam is a rigid structural element that is rigidly fixed at a single point on one side while the other side is free. In structural engineering, the behavior of cantilever beams is often analyzed using classical beam theory, which is based on the assumptions of small deformations and linear elastic material behavior. In the context of a cantilever beam, fractional calculus can be used to analyze the response of the beam to external loads and predict its dynamic behavior under different conditions [9,10]. For example, the motion of a cantilever beam subjected to a harmonic load can be described using fractional differential equations. In addition, fractional calculus can be used to study the beam's natural frequencies and mode shapes, which are important characteristics that influence its behavior [11,12].

Overall, employing fractional calculus in the analysis of cantilever beams can provide more accurate predictions of the behavior of these structures under various load conditions. By considering the effects of fractional derivatives on the stiffness and strength of a cantilever beam, it is possible to better understand and predict its response to external

loads. This can be useful for the design and optimization of cantilever beams in a variety of applications. For example, the existence and uniqueness of solutions to the boundary value problem of the cantilever beam were studied in the framework of quantum calculus in [13] and the ψ -Hilfer derivative in [14]. In addition, the deflection of the cantilever beam based on fractional calculus was also studied in [15,16]. Moreover, there are several methods available for solving fractional differential equations including integral transform techniques such as the Laplace and Fourier transforms, fixed-point techniques, and the Adomian decomposition method. These methods can be used to analyze the behavior of cantilever beams with fractional calculus and obtain the corresponding solutions.

One significant method for determining an elastica's equilibrium forms is to derive the condition for stationary of the total energy. Then, the corresponding boundary value problems associated with ordinary differential equations are used to determine the equilibrium shapes. Della et al. [17] analyzed the equilibrium configuration of an elastica with one end clamped under a uniformly distributed load, which is depicted in Figure 1.

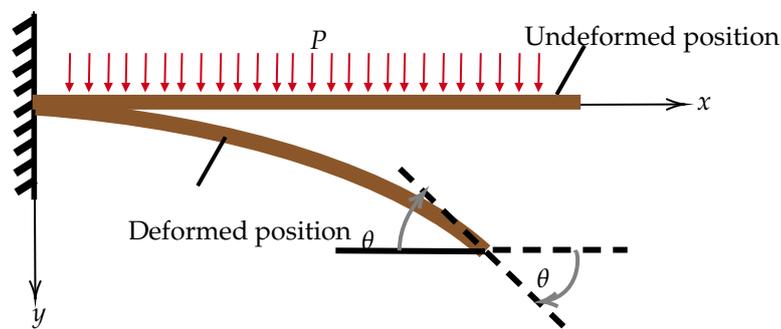


Figure 1. Undeformed and deformed positions of a cantilever beam subjected to a uniformly distributed load.

They investigated the sufficient conditions for the stability and instability of the equilibrium shape of the elastica obtained from the minimization problem of the total energy functional described by

$$\begin{cases} \min E(\theta) = \frac{1}{2} \int_0^L |\theta'(s)|^2 ds - P \int_0^L (L - s) \sin \theta(s) ds, \\ \theta(0) = \theta'(L) = 0, \end{cases} \quad (1)$$

where P, L, s , and θ denote the concentrated load, the length of the beam, the arc length, and the tangent angle, respectively.

It is possible to use fractional calculus to analyze the behavior of cantilever beams. In this approach, the beam is modeled as a dynamic system with memory and the governing equations of motion are described by fractional calculus operators. The solutions of these equations can provide insights into the response of the beam under various loading conditions and can be used to design and optimize the beam's structural performance.

Motivated by previous works, we consider the generalization of the potential energy associated with the cantilever beam under the framework of fractional calculus and determine the shape that minimizes the functional E and satisfies the boundary conditions:

$$\begin{cases} \min E(\theta) = \frac{1}{2} \int_0^L |{}^C_0\mathcal{D}_s^\alpha \theta(s)|^2 ds - \frac{P}{\Gamma(\alpha + 1)} \int_0^L (L - s)^\alpha \sin \theta(s) ds, \\ \theta(0) = {}^C_0\mathcal{D}_s^\alpha \theta(s)|_{s=L} = 0, \end{cases} \quad (2)$$

where $P > 0$ and $\frac{1}{2} < \alpha < 1$ is the order of the Caputo fractional derivative. The properties of the minimizers of the total energy can be characterized by the corresponding Euler–Lagrange conditions.

The aim of this paper is to use Euler–Lagrange conditions for Problem (2) to characterize the boundary value problem. Then, the well-known fixed-point theorems of Schaefer and Banach are used to establish the existence and uniqueness of the solutions for these boundary value problems. Different load values are analyzed for continuous dependence. Finally, we approximate the analytical solution for various loads and fractional orders to demonstrate the theoretical results. The main contribution of this paper is to provide an analysis of the deflection of the cantilever beam through the fractional energy functional derived from the physical and geometrical aspects in terms of the tangent angle or curvature, which is complementary to [8,16]. This technique can be applied to analyze the deformation of the cantilever beam under different load types.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and essential theoretical results on fractional calculus and calculus of variations. In Section 3, the total energy is reformulated as a boundary value problem using Euler–Lagrange conditions. The existence and uniqueness results are proved via fixed-point techniques in Section 4. Based on techniques from nonlinear functional analysis, we analyze the continuous dependence of minimizers on the different loads in Section 5. Finally, the analytical solution is approximated by the homotopy perturbation method in Section 6. We also present numerical examples to support the validity of the analytical results.

2. Preliminary Background of Fractional Calculus and Calculus of Variations

In this section, we first give some essential definitions and properties of fractional differential operators and fractional integral operators.

Further details on this subject and its applications can be found, in [3,18,19].

2.1. Fractional Calculus

Let u be a real valued function defined on the interval $[a, b]$ and $\text{Re}(\alpha) > 0$.

Definition 1 (Fractional integral in the sense of Riemann–Liouville, [3]). *The left and right Riemann–Liouville fractional integral operators of order α of function u are defined, respectively, by*

$$({}_a\mathcal{I}_x^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} u(s) ds$$

and

$$({}_x\mathcal{I}_b^\alpha u)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} u(s) ds,$$

for all $x \in [a, b]$.

Definition 2 (Fractional derivative in the sense of Riemann–Liouville [3]). *The left and right Riemann–Liouville fractional derivatives of order α of a function u are defined, respectively, by*

$$({}_a\mathcal{D}_x^\alpha u)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-s)^{-\alpha} u(s) ds$$

and

$$({}_x\mathcal{D}_b^\alpha u)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (s-x)^{-\alpha} u(s) ds,$$

for all $x \in [a, b]$.

We denote by $AC([a, b])$ the space of the functions defined on $[a, b]$, which are absolutely continuous.

Definition 3 (Fractional derivative in the sense of Caputo, [3]). Let $u \in AC([a, b])$. The left and right Caputo fractional derivatives are defined, respectively, by

$$({}^C D_x^\alpha u)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-s)^{-\alpha} u'(s) ds$$

and

$$({}^C D_b^\alpha u)(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b (s-x)^{-\alpha} u'(s) ds,$$

for all $x \in [a, b]$.

Remark 1. For $0 < \alpha < 1$, the Riemann–Liouville and Caputo fractional derivatives satisfy the following relations:

$$({}^C D_x^\alpha u)(x) = ({}_a D_x^\alpha u)(x) - \frac{u(a)}{\Gamma(1-\alpha)} (x-a)^{-\alpha}$$

and

$$({}^C D_b^\alpha u)(x) = ({}_x D_b^\alpha u)(x) - \frac{u(b)}{\Gamma(1-\alpha)} (b-x)^{-\alpha}.$$

Lemma 1 (Fundamental Theorem of Caputo Calculus, [3]). Let $0 < \alpha < 1$ and let f be a differentiable function on $[a, b]$. We have

$${}_a \mathcal{I}_x^\alpha ({}^C D_x^\alpha u)(x) = u(b) - u(a)$$

and

$${}_x \mathcal{I}_b^\alpha ({}^C D_b^\alpha u)(x) = u(a) - u(b).$$

2.2. Fractional Calculus of Variations

The fractional calculus of variations involves finding a function y that optimizes (minimizes or maximizes) a certain functional that depends on y and its fractional derivatives. Consider the optimization problem for the functional given by

$$\mathcal{J}[y] = \int_a^b \mathcal{L}(x, y, {}^C D_x^\alpha y) dx \quad (3)$$

with a Lagrangian $\mathcal{L} \in C^1([a, b] \times \mathbb{R}^2)$ depending on y , which is a function of the independent variable x and its left Caputo fractional derivative of order $0 < \alpha < 1$. For $\zeta_a, \zeta_b \in \mathbb{R}$ given, we impose the boundary conditions:

$$y(a) = \zeta_a, \quad y(b) = \zeta_b. \quad (4)$$

Notice that \mathcal{J} becomes a functional for the classical calculus of variations when $\alpha = 1$. We next state the Euler–Lagrange equation for the above problem.

Theorem 1 (The Euler–Lagrange Equation for a Functional with Caputo Derivatives, [20]). Consider the optimization problem of the functional (3), where the Lagrangian \mathcal{L} belongs to $C^2([a, b] \times \mathbb{R}^2)$ under the boundary conditions (4). If $y \in C^1[a, b]$ is an optimal solution, the fractional Euler–Lagrange condition

$$\frac{\partial \mathcal{L}(x, y, {}^C D_x^\alpha y)}{\partial y} + {}^C D_b^\alpha \frac{\partial \mathcal{L}(x, y, {}^C D_x^\alpha y)}{{}^C D_x^\alpha y} = 0.$$

holds.

3. Boundary Value Problem Associated with Minimizers

To reformulate the minimization problem (2), we apply the Euler–Lagrange condition in Theorem 1, where the Lagrangian has the form

$$\mathcal{L}(s, \theta, {}^C\mathcal{D}_s^\alpha \theta) := \frac{1}{2} \left| {}^C\mathcal{D}_s^\alpha \theta(s) \right|^2 - \frac{P}{\Gamma(\alpha + 1)} (L - s)^\alpha \sin \theta(s).$$

Then, the extremum of the energy functional in Equation (2) is as follows:

$${}^C\mathcal{D}_L^\alpha ({}^C\mathcal{D}_s^\alpha \theta(s)) = \frac{P}{\Gamma(\alpha + 1)} (L - s)^\alpha \cos \theta(s). \tag{5}$$

To study the nonlinear problem (5), we first reformulate it into an integral equation in the following section.

Lemma 2. *The solution of (5) satisfies*

$$\begin{aligned} \theta(s) = \theta(0) + P \left({}^C\mathcal{I}_s^\alpha {}^C\mathcal{I}_L^\alpha (L - s)^\alpha \cos \theta(s) \right) \\ + \left(\frac{s}{L} \right)^\alpha \left(\theta(L) - \theta(0) - \frac{P}{\Gamma(\alpha + 1)} \left({}^C\mathcal{I}_s^\alpha {}^C\mathcal{I}_L^\alpha (L - s)^\alpha \cos \theta(s) \Big|_{s=L} \right) \right). \end{aligned}$$

Moreover, if $\theta(0) = 0$ and $\theta(L) = c$, we have that

$$\begin{aligned} \theta(s) &= \frac{P}{\Gamma(\alpha + 1)} \left({}^C\mathcal{I}_s^\alpha {}^C\mathcal{I}_L^\alpha (L - s)^\alpha \cos \theta(s) - \left(\frac{s}{L} \right)^\alpha {}^C\mathcal{I}_s^\alpha {}^C\mathcal{I}_L^\alpha (L - s)^\alpha \cos \theta(s) \Big|_{s=L} \right) + \left(\frac{s}{L} \right)^\alpha \theta(L) \\ &= \frac{P}{\Gamma(\alpha + 1)} \left(\frac{1}{(\Gamma(\alpha))^2} \int_0^s (s - x)^{\alpha-1} \int_x^L (\tau - x)^{\alpha-1} (L - \tau)^\alpha \cos \theta(\tau) d\tau dx \right. \\ &\quad \left. - \left(\frac{s}{L} \right)^\alpha \frac{1}{(\Gamma(\alpha))^2} \int_0^L (L - t)^{\alpha-1} \int_x^L (\tau - x)^{\alpha-1} (L - \tau)^\alpha \cos \theta(\tau) d\tau dt \right) \\ &\quad + \left(\frac{s}{L} \right)^\alpha \theta(L). \end{aligned}$$

Proof. We integrate (5) twice by applying the right fractional integral operator followed by the left fractional integral operator to obtain

$${}^C\mathcal{I}_s^\alpha \left({}^C\mathcal{I}_L^\alpha \left({}^C\mathcal{D}_L^\alpha \left({}^C\mathcal{D}_s^\alpha \theta(s) \right) \right) \right) - \frac{P}{\Gamma(\alpha + 1)} {}^C\mathcal{I}_s^\alpha \left({}^C\mathcal{I}_L^\alpha (L - s)^\alpha \cos \theta(s) \right) = 0.$$

Next, we apply the composition rule and property from Lemma 1 on $[s, L]$ to obtain

$${}^C\mathcal{I}_s^\alpha \left({}^C\mathcal{D}_s^\alpha \theta(s) - {}^C\mathcal{D}_s^\alpha \theta(s) \Big|_{s=L} \right) = \frac{P}{\Gamma(\alpha + 1)} {}^C\mathcal{I}_s^\alpha \left({}^C\mathcal{I}_L^\alpha (L - s)^\alpha \cos \theta(s) \right).$$

Since the value ${}^C\mathcal{D}_s^\alpha \theta(s) \Big|_{s=L}$ is a constant, this yields that

$$\theta(s) - \theta(0) - {}^C\mathcal{D}_s^\alpha \theta(s) \Big|_{s=L} \frac{s^\alpha}{\Gamma(\alpha + 1)} = \frac{P}{\Gamma(\alpha + 1)} {}^C\mathcal{I}_s^\alpha \left({}^C\mathcal{I}_L^\alpha (L - s)^\alpha \cos \theta(s) \right). \tag{6}$$

In the above equation, we see that the unknown value ${}^C\mathcal{D}_s^\alpha \theta(s) \Big|_{s=L}$ can be determined due to the boundary condition. We substitute $s = L$ into (6) to give

$${}^C\mathcal{D}_s^\alpha \theta(s) \Big|_{s=L} = \frac{\Gamma(\alpha + 1)}{L^\alpha} \left(\theta(L) - \theta(0) - \frac{P}{\Gamma(\alpha + 1)} \left({}^C\mathcal{I}_s^\alpha {}^C\mathcal{I}_L^\alpha (L - s)^\alpha \cos \theta(s) \Big|_{s=L} \right) \right). \tag{7}$$

As a consequence of (7) and (6), we obtain the integral form of (5) in the following:

$$\theta(s) = \theta(0) + P \left({}_0^C \mathcal{I}_s^\alpha {}_s^C \mathcal{I}_L^\alpha (L-s)^\alpha \cos \theta(s) + \left(\frac{s}{L} \right)^\alpha \left(\theta(L) - \theta(0) - \frac{P}{\Gamma(\alpha+1)} \left({}_0^C \mathcal{I}_s^\alpha {}_s^C \mathcal{I}_L^\alpha (L-s)^\alpha \cos \theta(s) \Big|_{s=L} \right) \right) \right).$$

Hence, the proof is complete. \square

Corollary 1. The solution of (5) subject to $\theta(0) = {}_0^C \mathcal{D}_s^\alpha \theta(s) \Big|_{s=L} = 0$ satisfies the integral equation

$$\theta(s) = \frac{P}{\Gamma(\alpha+1)} \int_0^L \mathcal{H}(s, \tau) (L-\tau)^\alpha \cos \theta(\tau) d\tau,$$

where

$$\mathcal{H}(s, \tau) = \frac{1}{(\Gamma(\alpha))^2} \times \begin{cases} \int_0^\tau (s-x)^{\alpha-1} (\tau-x)^{\alpha-1} dt, & 0 \leq \tau \leq s \leq L, \\ \int_0^s (s-x)^{\alpha-1} (\tau-x)^{\alpha-1} dt, & 0 \leq s \leq \tau \leq L. \end{cases}$$

Proof. From Equation (6) in Lemma 2, we obtain

$$\begin{aligned} \theta(s) &= \frac{P}{\Gamma(\alpha+1)} \left({}_0^C \mathcal{I}_s^\alpha {}_s^C \mathcal{I}_L^\alpha (L-s)^\alpha \cos \theta(s) \right) \\ &= \frac{P}{(\Gamma(\alpha))^2} \int_0^s (s-\sigma)^{\alpha-1} \int_\sigma^L (\sigma'-\sigma)^{\alpha-1} (L-\sigma')^\alpha \cos \theta(\sigma') d\sigma' d\sigma. \end{aligned} \tag{8}$$

Applying the Fubini's theorem, we obtain

$$\begin{aligned} \theta(s) &= \frac{P}{(\Gamma(\alpha))^2} \int_0^s \int_0^\tau (s-x)^{\alpha-1} (\tau-x)^{\alpha-1} (L-\tau)^\alpha \cos \theta(\tau) dx d\tau \\ &\quad + \frac{P}{(\Gamma(\alpha))^2} \int_s^L \int_0^s (s-x)^{\alpha-1} (\tau-x)^{\alpha-1} (L-\tau)^\alpha \cos \theta(\tau) dx d\tau. \end{aligned}$$

Then, we obtain its solution in terms of a Green function. \square

Remark 2. When the load P is small, we may consider the behavior of the solution of the following boundary value problem

$$\begin{cases} {}_s^C \mathcal{D}_L^\alpha ({}_0^C \mathcal{D}_s^\alpha \theta(s)) = \frac{P}{\Gamma(\alpha+1)} (L-s)^\alpha, & 0 < s < L, \\ \theta(0) = {}_0^C \mathcal{D}_s^\alpha \theta(s) \Big|_{s=L} = 0. \end{cases}$$

Here, we approximate $\cos \theta$ by 1, which is a legitimate approximation when P is small. For this problem, we obtain

$$\begin{aligned} \theta(s) &= \frac{P}{(\Gamma(\alpha))^2} \int_0^s (s-\sigma)^{\alpha-1} \int_\sigma^L (\sigma'-\sigma)^{\alpha-1} (L-\sigma')^\alpha d\sigma' d\sigma \\ &= \frac{\alpha P}{\Gamma(2\alpha+1)} \int_0^s (s-\sigma)^{\alpha-1} (L-\sigma)^{2\alpha} d\sigma. \end{aligned}$$

Lemma 3. The function \mathcal{H} is continuous, non-negative and

$$\mathcal{H}(s, \tau) \leq \frac{L^{2\alpha-1}}{(2\alpha-1)(\Gamma(\alpha))^2}, \quad \text{for all } s, \tau \in [0, L].$$

4. Existence and Uniqueness Results

This section is devoted to proving the existence and uniqueness of the solutions for the following problem:

$$\begin{cases} {}^C\mathcal{D}_L^\alpha({}^C\mathcal{D}_s^\alpha\theta(s)) = \frac{P}{\Gamma(\alpha+1)}(L-s)^\alpha \cos\theta(s), & 0 < s < L, \\ \theta(0) = {}^C\mathcal{D}_s^\alpha\theta(s)\Big|_{s=L} = 0 \end{cases} \quad (9)$$

where $\frac{1}{2} < \alpha < 1$. We apply Corollary 1 to define the integral operator \mathcal{K} from $C[0, L]$ to $C[0, L]$ as

$$\begin{aligned} (\mathcal{K}\theta)(s) &= \frac{P}{\Gamma(\alpha+1)(\Gamma(\alpha))^2} \int_0^s (s-x)^{\alpha-1} \int_x^L (\tau-x)^{\alpha-1} (L-\tau)^\alpha \cos\theta(\tau) d\tau dx \\ &= \frac{P}{\Gamma(\alpha+1)} \int_0^L \mathcal{H}(s, \tau) (L-\tau)^\alpha \cos\theta(\tau) d\tau. \end{aligned} \quad (10)$$

Theorem 2. *The initial value problem (9) attains at least one solution θ in $C[0, L]$.*

Proof. Schaefer's fixed-point theorem is used to show that the operator \mathcal{K} given by (10) has a fixed point. We outline the proof in the following steps.

Step 1: \mathcal{K} is a continuous operator.

Let $\{\theta_m\}$ be a convergent sequence with $\theta_m \rightarrow \theta$ in $C[0, L]$. For each $s \in [0, L]$, we have

$$\begin{aligned} |(\mathcal{K}\theta_m)(s) - (\mathcal{K}\theta)(s)| &\leq \frac{P}{\Gamma(\alpha+1)} \int_0^L |\mathcal{H}(s, \tau)| (L-\tau)^\alpha |\cos\theta(\tau) - \cos\theta_m(\tau)| d\tau \\ &\leq \frac{P}{\Gamma(\alpha+1)} \sup_{\tau \in [0, L]} |(L-\tau)^\alpha (\cos\theta(\tau) - \cos\theta_m(\tau))| \int_0^L |\mathcal{H}(s, \tau)| d\tau \\ &\leq \frac{P}{\Gamma(\alpha+1)} L^\alpha \sup_{\tau \in [0, L]} |\theta(\tau) - \theta_m(\tau)| \int_0^L \frac{L^{2\alpha-1}}{(2\alpha-1)(\Gamma(\alpha))^2} d\tau \\ &\leq \frac{P}{(2\alpha-1)\Gamma(\alpha+1)(\Gamma(\alpha))^2} L^{3\alpha} \sup_{\tau \in [0, L]} |\theta(\tau) - \theta_m(\tau)|. \end{aligned}$$

It follows that

$$\|\mathcal{K}\theta_m - \mathcal{K}\theta\|_\infty \leq \frac{P}{(2\alpha-1)\Gamma(\alpha+1)(\Gamma(\alpha))^2} L^{3\alpha} \|\theta - \theta_m\|_\infty$$

which implies that

$$\|\mathcal{K}\theta_m - \mathcal{K}\theta\|_\infty \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Step 2: The image of a bounded set in $C[0, L]$ under \mathcal{K} is also a bounded set.

We show that there is a positive constant $\ell > 0$ such that

$$\forall \theta \in B_{\eta^*} = \{\theta \in C[0, L] : \|\theta\|_\infty \leq \eta^*\},$$

and $\|\mathcal{K}\theta\|_\infty \leq \ell$ for $\eta^* > 0$. Indeed, for any $s \in [0, L]$, by the boundedness of the nonlinear term we have

$$\begin{aligned} |(\mathcal{K}\theta)(s)| &\leq \frac{P}{\Gamma(\alpha+1)} \int_0^L \mathcal{H}(s, \tau) (L-\tau)^\alpha |\cos\theta(\tau)| d\tau \\ &\leq \frac{PL^\alpha}{\Gamma(\alpha+1)} \int_0^L \mathcal{H}(s, \tau) d\tau \end{aligned}$$

$$\leq \frac{PL^{3\alpha}}{(2\alpha - 1)\Gamma(\alpha + 1)(\Gamma(\alpha))^2},$$

thus,

$$\|\mathcal{K}\theta\|_\infty \leq \ell$$

where

$$\ell = \frac{PL^{3\alpha}}{(2\alpha - 1)\Gamma(\alpha + 1)(\Gamma(\alpha))^2}.$$

Step 3: The image of a bounded set in $C[0, L]$ under \mathcal{K} is an equicontinuous set.

Let $s_1, s_2 \in [0, L]$ such that $s_1 < s_2$ and $\theta \in B_{\eta^*}$, which is a bounded set of $C[0, L]$, as above. Then, we see that

$$\begin{aligned} & |(\mathcal{K}\theta)(s_2) - (\mathcal{K}\theta)(s_1)| \\ & \leq \frac{P}{\Gamma(\alpha + 1)} \int_0^L |(\mathcal{H}(s_2, \tau) - \mathcal{H}(s_1, \tau))(L - \tau)^\alpha \cos \theta(\tau)| d\tau \\ & \leq \frac{PL^\alpha}{\Gamma(\alpha + 1)} \int_0^L |\mathcal{H}(s_2, \tau) - \mathcal{H}(s_1, \tau)| d\tau \\ & = \frac{PL^\alpha}{\Gamma(\alpha + 1)} \left(\int_0^{s_1} |\mathcal{H}(s_2, \tau) - \mathcal{H}(s_1, \tau)| d\tau + \int_{s_1}^{s_2} |\mathcal{H}(s_2, \tau) - \mathcal{H}(s_1, \tau)| d\tau \right. \\ & \quad \left. + \int_{s_2}^L |\mathcal{H}(s_2, \tau) - \mathcal{H}(s_1, \tau)| d\tau \right) \\ & = \frac{PL^\alpha}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \left(\int_0^{s_1} \left| \int_0^\tau [(s_2 - t)^{\alpha-1} - (s_1 - t)^{\alpha-1}] (\tau - t)^{\alpha-1} dt \right| d\tau \right. \\ & \quad \left. + \int_{s_1}^{s_2} \left| \int_0^\tau (s_2 - t)^{\alpha-1} (\tau - t)^{\alpha-1} dt - \int_0^{s_1} (s_1 - t)^{\alpha-1} (\tau - t)^{\alpha-1} dt \right| d\tau \right. \\ & \quad \left. + \int_{s_2}^L \left| \int_0^{s_2} (s_2 - \tau)^{\alpha-1} (\tau - t)^{\alpha-1} dt - \int_0^{s_1} (s_1 - \tau)^{\alpha-1} (\tau - t)^{\alpha-1} dt \right| d\tau \right) \\ & \leq \frac{PL^\alpha}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \left(\int_0^{s_1} \int_0^\tau ((s_1 - t)^{\alpha-1} - (s_2 - t)^{\alpha-1}) (\tau - t)^{\alpha-1} dt d\tau \right. \\ & \quad \left. + \int_{s_1}^L \int_0^{s_1} ((s_1 - t)^{\alpha-1} - (s_2 - t)^{\alpha-1}) (\tau - t)^{\alpha-1} dt d\tau \right. \\ & \quad \left. + \int_{s_1}^{s_2} \int_{s_1}^\tau (s_2 - t)^{\alpha-1} (\tau - t)^{\alpha-1} dt d\tau \right. \\ & \quad \left. + \int_{s_2}^L \int_{s_1}^{s_2} (s_2 - t)^{\alpha-1} (\tau - t)^{\alpha-1} dt d\tau \right) \\ & \leq \frac{PL^\alpha}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \left(\frac{(1 - \alpha)(s_2 - s_1)s_1^{2\alpha-1}}{(2\alpha - 2)(2\alpha - 1)} + \frac{(1 - \alpha)(s_2 - s_1)(L - s_1)s_1^{2\alpha-2}}{2\alpha - 2} \right. \\ & \quad \left. + \frac{(s_2 - s_1)^{2\alpha-1}(L - s_2)}{(2\alpha - 1)} + \frac{(s_2 - s_1)^{2\alpha}}{2\alpha(2\alpha - 1)} \right). \end{aligned}$$

As $s_1 \rightarrow s_2$, the right-hand side of the above inequality tends to zero. Following Step 1 to Step 3 and the Arzelá–Ascoli theorem, $\mathcal{K} : C[0, L] \rightarrow C[0, L]$ is completely continuous.

Step 4: A priori bounds. Let $\varepsilon = \{\theta \in C[0, L] : \theta = \lambda\mathcal{K}\theta \text{ for some } 0 < \lambda < 1\}$. We claim that ε is bounded. Let $\theta \in \varepsilon$, then $\theta = \lambda\mathcal{K}\theta$ for some $0 < \lambda < 1$. Hence, $\forall s \in [0, L]$,

$$\theta = \lambda\mathcal{K}\theta = \lambda \left(\frac{P}{\Gamma(\alpha + 1)} \int_0^L \mathcal{H}(s, \tau)(L - \tau)^\alpha \cos \theta(\tau) d\tau \right).$$

By the condition in Step 2, we obtain

$$|\theta(s)| \leq \frac{L^{3\alpha} P}{(2\alpha - 1)\Gamma(\alpha + 1)(\Gamma(\alpha))^2}$$

and hence for every $s \in [0, L]$,

$$\|\theta\|_\infty \leq \frac{L^{3\alpha} P}{(2\alpha - 1)\Gamma(\alpha + 1)(\Gamma(\alpha))^2} := R.$$

This implies the boundedness of the set ε .

Consequently, Schaefer’s fixed-point theorem assures that \mathcal{K} attains a fixed point, which is a solution of the boundary value problem (9). \square

Theorem 3. Problem (9) has a unique solution θ in $C[0, L]$ if

$$\frac{PL^{3\alpha}}{\Gamma(2\alpha + 1)} < 1.$$

Proof. We show that \mathcal{K} is a contraction. For any $\theta, \tilde{\theta} \in C[0, L]$ and $s \in [0, L]$, we have

$$\begin{aligned} & |(\mathcal{K}\theta)(s) - (\mathcal{K}\tilde{\theta})(s)| \\ &= \frac{P}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \left| \int_0^s (s - x)^{\alpha-1} \int_x^L (\tau - x)^{\alpha-1} (L - \tau)^\alpha (\cos \theta(\tau) - \cos \tilde{\theta}(\tau)) d\tau dx \right| \\ &\leq \frac{P}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \int_0^s (s - x)^{\alpha-1} \int_x^L (\tau - x)^{\alpha-1} (L - \tau)^\alpha |\theta(\tau) - \tilde{\theta}(\tau)| d\tau dx \\ &\leq \frac{P}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \|\theta - \tilde{\theta}\|_\infty \int_0^s (s - x)^{\alpha-1} \int_x^L (\tau - x)^{\alpha-1} (L - \tau)^\alpha d\tau dx \\ &= \frac{PB(\alpha, \alpha + 1)}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \|\theta - \tilde{\theta}\|_\infty \int_0^s (s - x)^{\alpha-1} (L - t)^{2\alpha} dt \\ &\leq \frac{\alpha P}{\Gamma(2\alpha + 1)} \|\theta - \tilde{\theta}\|_\infty L^{2\alpha} \int_0^s (s - x)^{\alpha-1} dt \\ &= \frac{\alpha P}{\Gamma(2\alpha + 1)} \|\theta - \tilde{\theta}\|_\infty L^{2\alpha} \left(\frac{s^\alpha}{\alpha}\right) \\ &\leq \frac{PL^{3\alpha}}{\Gamma(2\alpha + 1)} \|\theta - \tilde{\theta}\|_\infty. \end{aligned}$$

It follows that $\|\mathcal{K}\theta - \mathcal{K}\tilde{\theta}\|_\infty \leq \frac{PL^{3\alpha}}{\Gamma(2\alpha + 1)} \|\theta - \tilde{\theta}\|_\infty$. \square

We also establish the existence and uniqueness results with respect to the L^2 norm.

Theorem 4. Problem (9) has a unique solution θ in $L^2[0, L]$ if

$$\frac{PL^{3\alpha}}{2\alpha\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{2(2\alpha - 1)\Gamma(4\alpha)}} < 1. \tag{11}$$

Proof. It is obvious that \mathcal{K} is self-mapping on $L^2[0, L]$. We show that \mathcal{K} is a contraction.

For any $\theta, \tilde{\theta} \in L^2[0, L]$ and $s \in [0, L]$, we use Hölder’s inequality to obtain

$$\begin{aligned} & |(\mathcal{K}\theta)(s) - (\mathcal{K}\tilde{\theta})(s)| \\ &= \frac{P}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \left| \int_0^s (s - x)^{\alpha-1} \int_x^L (\tau - x)^{\alpha-1} (L - \tau)^\alpha (\cos \theta(\tau) - \cos \tilde{\theta}(\tau)) d\tau dx \right| \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{P}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \int_0^s (s - x)^{\alpha-1} \int_x^L (\tau - x)^{\alpha-1} (L - \tau)^\alpha |\theta(\tau) - \tilde{\theta}(\tau)| d\tau dx \\
 &\leq \frac{P}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \int_0^s (s - x)^{\alpha-1} \left(\int_x^L (\tau - x)^{2\alpha-2} (L - \tau)^{2\alpha} d\tau \right)^{\frac{1}{2}} \\
 &\quad \times \left(\int_x^L |\theta(\tau) - \tilde{\theta}(\tau)|^2 d\tau \right)^{\frac{1}{2}} dx \\
 &\leq \frac{P \|\theta - \tilde{\theta}\|_{L^2[0,L]}}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{\Gamma(4\alpha)}} \int_0^s (s - x)^{\alpha-1} (L - t)^{2\alpha-\frac{1}{2}} dx \\
 &\leq \frac{P \|\theta - \tilde{\theta}\|_{L^2[0,L]}}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{\Gamma(4\alpha)}} \left(\int_0^s (s - x)^{2\alpha-2} dx \right)^{\frac{1}{2}} \left(\int_0^s (L - x)^{4\alpha-1} dx \right)^{\frac{1}{2}} \\
 &= \frac{P}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{\Gamma(4\alpha)}} \left(\frac{L^{4\alpha}}{4\alpha} \right)^{\frac{1}{2}} \left(\frac{s^{2\alpha-1}}{2\alpha-1} \right)^{\frac{1}{2}} \|\theta - \tilde{\theta}\|_{L^2[0,L]}.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 &\|\mathcal{K}\theta - \mathcal{K}\tilde{\theta}\|_{L^2[0,L]} \\
 &= \left(\int_0^L |(\mathcal{K}\theta)(s) - (\mathcal{K}\tilde{\theta})(s)|^2 ds \right)^{\frac{1}{2}} \\
 &\leq \left(\int_0^L \left| \frac{P}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{\Gamma(4\alpha)}} \left(\frac{L^{4\alpha}}{4\alpha} \right)^{\frac{1}{2}} \left(\frac{s^{2\alpha-1}}{2\alpha-1} \right)^{\frac{1}{2}} \|\theta - \tilde{\theta}\|_{L^2[0,L]} \right|^2 ds \right)^{\frac{1}{2}} \\
 &= \frac{PL^{2\alpha}}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{4\alpha\Gamma(4\alpha)}} \left(\int_0^L \frac{s^{2\alpha-1}}{2\alpha-1} ds \right)^{\frac{1}{2}} \|\theta - \tilde{\theta}\|_{L^2[0,L]} \\
 &= \frac{PL^{2\alpha}}{\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{4\alpha\Gamma(4\alpha)}} \left(\frac{L^{2\alpha}}{(2\alpha-1)2\alpha} \right)^{\frac{1}{2}} \|\theta - \tilde{\theta}\|_{L^2[0,L]} \\
 &= \frac{PL^{3\alpha}}{2\alpha\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{2(2\alpha-1)\Gamma(4\alpha)}} \|\theta - \tilde{\theta}\|_{L^2[0,L]}.
 \end{aligned}$$

This implies that \mathcal{K} is a contraction satisfying (11). Hence, the uniqueness of a fixed point of the map \mathcal{K} in $L^2[0, L]$ follows from the Banach contraction principle. \square

5. Continuous Dependence of Minimizers on Varying Loads

To study the continuity of minimizers when the load P changes, we begin with a definition in terms of the branch of solutions.

Definition 4. Let us denote by θ_P a solution of (5) with $P = \bar{P}$, $\theta(0) = 0$ and ${}^C_0\mathcal{D}_s^\alpha \theta(s)|_{s=L} = 0$. We say that the set of minimizers θ_P for $P \in [0, B]$, with $B > 0$ is a branch of solutions if the maps from $[0, B]$ to $L^2[0, L]$ given by $P \mapsto \theta_P$ and $P \mapsto {}^C_0\mathcal{D}_s^\alpha \theta_P$ are continuous functions of P .

According to the defined problem of the cantilever beam subjected to the downward uniformly distributed load, the deflection shape of the beam is a concave down. Consequently, the angle on $[0, L]$ will be in the range of 0 to $\frac{\pi}{2}$ according to the deflection shape. Then, we obtain the following results.

Lemma 4. Let θ_P be a minimizer of the energy functional (explicitly depends on P). The function $P \mapsto E_P(\theta_P)$ is a decreasing function.

Proof. Let be $P_2 > P_1$. Then, we have

$$\begin{aligned} E_{P_2}(\theta_{P_2}) - E_{P_1}(\theta_{P_1}) &= E_{P_2}(\theta_{P_2}) - E_{P_2}(\theta_{P_1}) + E_{P_2}(\theta_{P_1}) - E_{P_1}(\theta_{P_1}) \\ &\leq \frac{P_1 - P_2}{\Gamma(\alpha + 1)} \int_0^L (L - s)^\alpha \sin \theta_{P_1}(s) ds. \end{aligned}$$

which implies that $E_{P_2}(\theta_{P_2}) - E_{P_1}(\theta_{P_1})$ is negative because $\sin \theta_{P_1}$ is positive on $[0, L]$ (see Figure 1). \square

Proposition 1. *The set of minimizers for the functional (2) forms a branch of solutions.*

Proof. By Theorem 4, there exist P and P_0 , which satisfy (11) and provide the existence and uniqueness of the minimizers for E_P and E_{P_0} , respectively. We prove that $\{\theta_P\}$ is a minimizing sequence for E_{P_0} if $P \rightarrow P_0$.

For every ψ such that ψ and ${}_0^C D_s^\alpha \psi$ in $L^2[0, L]$, we have

$$E_P(\psi) - E_{P_0}(\psi) = \frac{P - P_0}{\Gamma(\alpha + 1)} \int_0^L (L - s)^\alpha \sin \psi(s) ds \leq \frac{L^{\alpha+1}}{(\alpha + 1)\Gamma(\alpha + 1)} |P - P_0|.$$

Since θ_P is the minimizer of E_P , it follows that

$$\begin{aligned} E_{P_0}(\psi) &= E_P(\psi) + E_{P_0}(\psi) - E_P(\psi) \\ &\geq E_P(\theta_P) - \frac{L^{\alpha+1}}{(\alpha + 1)\Gamma(\alpha + 1)} |P - P_0| \\ &= E_P(\theta_P) - E_{P_0}(\theta_P) + E_{P_0}(\theta_P) - \frac{L^{\alpha+1}}{(\alpha + 1)\Gamma(\alpha + 1)} |P - P_0| \\ &\geq E_{P_0}(\theta_P) - \frac{2L^{\alpha+1}}{(\alpha + 1)\Gamma(\alpha + 1)} |P - P_0|. \end{aligned}$$

As the result holds for every ψ such that ψ and ${}_0^C D_s^\alpha \psi$ in $L^2[0, L]$, this yields that

$$E_{P_0}(\theta_P) \leq E_{P_0}(\theta_{P_0}) + \frac{2L^{\alpha+1}}{(\alpha + 1)\Gamma(\alpha + 1)} |P - P_0|$$

or equivalently,

$$E_{P_0}(\theta_P) - E_{P_0}(\theta_{P_0}) \leq \frac{2L^{\alpha+1}}{(\alpha + 1)\Gamma(\alpha + 1)} |P - P_0|.$$

It follows that $\{\theta_P\}$ is a minimizing sequence for E_{P_0} .

Now, we consider the difference $E_{P_0}(\theta_P) - E_{P_0}(\theta_{P_0})$ and ${}_0^C D_s^\alpha \theta_P(s) - {}_0^C D_s^\alpha \theta_{P_0}(s)$ as

$$\begin{aligned} E_{P_0}(\theta_P) - E_{P_0}(\theta_{P_0}) &= \frac{1}{2} \int_0^L \left(\left| {}_0^C D_s^\alpha \theta_P(s) \right|^2 - \left| {}_0^C D_s^\alpha \theta_{P_0}(s) \right|^2 \right) ds - \frac{P_0}{\Gamma(\alpha + 1)} \int_0^L (L - s)^\alpha (\sin \theta_P(s) - \sin \theta_{P_0}(s)) ds \end{aligned}$$

and

$$\begin{aligned} &{}_0^C D_s^\alpha \theta_P(s) - {}_0^C D_s^\alpha \theta_{P_0}(s) \\ &= \frac{1}{\Gamma(\alpha + 1)} \left({}_s^C I_L^\alpha (L - s)^\alpha (P \cos \theta_P(s) - P_0 \cos \theta_{P_0}(s)) \right) \\ &= \frac{P - P_0}{\Gamma(\alpha + 1)} \left({}_s^C I_L^\alpha (L - s)^\alpha (\cos \theta_P(s)) \right) + \frac{P_0}{\Gamma(\alpha + 1)} \left({}_s^C I_L^\alpha (L - s)^\alpha (\cos \theta_P(s) - \cos \theta_{P_0}(s)) \right) \\ &= \frac{P - P_0}{\Gamma(\alpha + 1)\Gamma(\alpha)} \int_s^L (\tau - s)^{\alpha-1} (L - \tau)^\alpha \cos \theta_P(\tau) d\tau \end{aligned}$$

$$+ \frac{P_0}{\Gamma(\alpha + 1)\Gamma(\alpha)} \int_s^L (\tau - s)^{\alpha-1} (L - \tau)^\alpha (\cos \theta_P(\tau) - \cos \theta_{P_0}(\tau)) d\tau.$$

Then, by direct calculation and using the dominated convergence theorem, we obtain

$$\begin{aligned} & \left\| {}_0^C \mathcal{D}_s^\alpha (\theta_P - \theta_{P_0}) \right\|_{L^2[0,L]}^2 \\ &= 2(E_{P_0}(\theta_P) - E_{P_0}(\theta_{P_0})) - 2 \int_0^L ({}_0^C \mathcal{D}_s^\alpha \theta_{P_0}(s)) ({}_0^C \mathcal{D}_s^\alpha (\theta_P - \theta_{P_0})) ds \\ &+ \frac{2P_0}{\Gamma(\alpha + 1)} \int_0^L (L - s)^\alpha (\sin \theta_P(s) - \sin \theta_{P_0}(s)) ds \\ &\leq \frac{4L^{\alpha+1}}{(\alpha + 1)\Gamma(\alpha + 1)} |P - P_0| \\ &- \frac{2}{\Gamma(\alpha + 1)} \int_0^L ({}_0^C \mathcal{D}_s^\alpha \theta_{P_0}(s)) ({}_s^C \mathcal{I}_L^\alpha (L - s)^\alpha P \cos \theta_P(s) - {}_s^C \mathcal{I}_L^\alpha (L - s)^\alpha P_0 \cos \theta_{P_0}(s)) ds \\ &+ \frac{2P_0}{\Gamma(\alpha + 1)} \int_0^L (L - s)^\alpha (\sin \theta_P(s) - \sin \theta_{P_0}(s)) ds \rightarrow 0 \end{aligned}$$

as $P \rightarrow P_0$, which completes the proof. \square

6. Approximate Solutions and Numerical Examples

The homotopy perturbation method (HPM) was pioneered and developed by He [21]. This technique involves the introduction of an expanding parameter, which serves as an alternative approach. Let A be a differential operator and B be a boundary operator. In general, the HPM can be applied to nonlinear differential equations of the form

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{12}$$

with boundary conditions

$$B\left(u(r), \frac{\partial u(r)}{\partial n}\right) = 0, \quad r \in \Gamma$$

where Γ denotes the boundary of the domain Ω and f is a given analytical function.

The basic idea of the HPM is to split the operator A into linear and nonlinear parts denoted by L and N , respectively, so that (12) can be rewritten as

$$L(u)(r) + N(u)(r) - f(r) = 0, \quad r \in \Omega.$$

Consider a homotopy $v(r, p) : \Omega \times [0, 1] \rightarrow \mathbb{R}$ satisfying

$$H(v, p) = p[A(v) - f(r)] + (1 - p)[L(v) - L(u_0)] = 0 \tag{13}$$

or

$$H(v, p) = p[N(v) - f(r)] + L(v) - L(u_0) + pL(u_0) = 0 \tag{14}$$

where u_0 is the first approximation of (12) in correspondence with the boundary conditions and $p \in [0, 1]$ is an embedding parameter. It can be readily seen from (13) or (14) that

$$\begin{aligned} H(v, 0) &= L(v) - L(u_0) = 0 \\ H(v, 1) &= A(v) - f(r) = 0. \end{aligned}$$

Clearly, when $p = 0$, (13) or equivalently, (14), is a linear equation, whereas when $p = 1$, it is the original nonlinear problem. Hence, changing the embedding parameter p

from 0 to 1 is equivalent to $L(v) - L(u_0) = 0$ with $A(v) - f(r) = 0$. The basic idea of the HPM is thus to continuously deform a simpler problem into the more difficult original one.

We write the solution of (13) or (14) in terms of a power series in p :

$$v = v_0 + pv_1 + p^2v_2 + \dots \tag{15}$$

By choosing $p = 1$, we obtain an approximate solution of Equation (12):

$$u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \dots .$$

The power series (15) converges in most circumstances. Nonetheless, the rate of convergence depends on the nonlinear operator $A(v)$.

To illustrate this method, we solve the boundary value problem (9) with $L = 1$. We set the following homotopy:

$${}_s^C D_1^\alpha ({}_0^C D_s^\alpha \theta(s)) - p \left[\frac{P}{\Gamma(\alpha + 1)} (L - s)^\alpha \cos(\theta(s)) \right] = 0. \tag{16}$$

If we expand the nonlinear term in (16) using the Taylor series, we obtain

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} - \dots$$

and

$$(1 - s)^{\alpha+1} = 1 - (\alpha + 1)s + \frac{(\alpha + 1)\alpha}{2!} s^2 - \frac{(\alpha + 1)\alpha(\alpha - 1)}{3!} s^3 + \frac{(\alpha + 1)\alpha(\alpha - 1)(\alpha - 2)}{4!} s^4 + \dots$$

Hence, we can approximate (16) as follows:

$${}_s^C D_1^\alpha ({}_0^C D_s^\alpha \theta(s)) = p \left[\frac{P}{\Gamma(\alpha + 1)} (1 - s)^\alpha \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \right) \right].$$

Substituting (15) into the homotopy (16) and applying the initial conditions, we obtain a set of linear differential equations from the coefficients of terms with equal powers of p as follows:

$$\begin{aligned} p^0 : {}_s^C D_1^\alpha ({}_0^C D_s^\alpha \theta_0(s)) &= 0, \quad \theta_0(0) = {}_s^C D_1^\alpha \theta_0(1) = 0, \\ p^1 : {}_s^C D_1^\alpha ({}_0^C D_s^\alpha \theta_1(s)) &= \frac{P}{\Gamma(\alpha + 1)} (1 - s)^\alpha \left(1 - \frac{1}{2} \theta_0^2 + \frac{1}{24} \theta_0^4 - \frac{1}{720} \theta_0^6 \right), \quad \theta_1(0) = {}_s^C D_1^\alpha \theta_1(1) = 0, \\ p^2 : {}_s^C D_1^\alpha ({}_0^C D_s^\alpha \theta_2(s)) &= \frac{P}{\Gamma(\alpha + 1)} (1 - s)^\alpha \left(-\theta_0 \theta_1 + \frac{1}{6} \theta_0^3 \theta_1 - \frac{1}{120} \theta_0^5 \theta_1 \right), \\ &\theta_2(0) = {}_s^C D_1^\alpha \theta_2(1) = 0, \\ p^3 : {}_s^C D_1^\alpha ({}_0^C D_s^\alpha \theta_3(s)) &= \frac{P}{\Gamma(\alpha + 1)} (1 - s)^\alpha \left(-\frac{1}{2} \theta_1^2 + \frac{1}{4} \theta_0^2 \theta_1^2 - \frac{1}{48} \theta_0^4 \theta_1 - \theta_0 \theta_2 + \frac{1}{6} \theta_0^3 \theta_2 - \frac{1}{120} \theta_0^5 \theta_2 \right), \\ &\theta_3(0) = {}_s^C D_1^\alpha \theta_3(1) = 0, \\ p^4 : {}_s^C D_1^\alpha ({}_0^C D_s^\alpha \theta_4(s)) &= \frac{P}{\Gamma(\alpha + 1)} (1 - s)^\alpha \left(\frac{1}{6} \theta_0 \theta_1^3 - \frac{1}{36} \theta_0^3 \theta_1^3 - \theta_1 \theta_2 + \frac{1}{2} \theta_0^2 \theta_1 \theta_2 - \frac{1}{24} \theta_0^4 \theta_1 \theta_2 \right. \\ &\quad \left. - \theta_0 \theta_3 + \frac{1}{6} \theta_0^3 \theta_3 - \frac{1}{120} \theta_0^5 \theta_3 \right), \quad \theta_4(0) = {}_s^C D_1^\alpha \theta_4(1) = 0, \\ &\vdots \end{aligned}$$

By applying Corollary 1 to the above equation, we obtain

$$\begin{aligned} \theta_0 &= 0, \\ \theta_1 &= \frac{Ps^\alpha}{\Gamma(2\alpha + 1)} \text{Hypergeometric}_2F_1(1, -2\alpha; \alpha + 1; s) \\ \theta_2 &= 0 \\ \theta_3 &= -\frac{P}{2(\Gamma(\alpha))^2\Gamma(\alpha)} \int_0^s (s - \sigma)^{\alpha-1} \int_\sigma^1 (\sigma' - \sigma)^{\alpha-1} (1 - \sigma')^\alpha (\theta_1(\sigma'))^2 d\sigma' d\sigma \\ \theta_4 &= 0 \dots, \end{aligned}$$

so that the approximate solution of the problem is given by

$$\begin{aligned} \theta(s) &= \lim_{p \rightarrow 1} \sum_{i=0}^{\infty} p^i \theta_i(s) \\ &= \frac{Ps^\alpha}{\Gamma(2\alpha + 1)} \text{Hypergeometric}_2F_1(1, -2\alpha; \alpha + 1; s) \\ &\quad - \frac{P}{2(\Gamma(\alpha))^2} \int_0^s (s - \sigma)^{\alpha-1} \int_\sigma^1 (\sigma' - \sigma)^{\alpha-1} (1 - \sigma')^\alpha \theta_1(\sigma') d\sigma' d\sigma + \dots \end{aligned}$$

From the tangent angle θ of a cantilever beam, we can calculate the deflection shape of a cantilever beam at the equilibrium position in xy -coordinates based on the following equations:

$$\frac{dx}{ds} = \cos \theta \quad \text{and} \quad \frac{dy}{ds} = \sin \theta.$$

Example 1. Consider Problem (9), with $L = 1$, $P = 0.73$, and taking values of $\alpha_1 = 0.6$, $\alpha_2 = 0.75$, and $\alpha_3 = 0.95$.

This example corresponds to a cantilever beam with a uniformly distributed load $P = 0.73$.

Then, we obtain for $\varphi_\alpha = \frac{L^{3\alpha}}{\Gamma(2\alpha + 1)}$ and $\gamma_\alpha = \frac{L^{3\alpha}}{2\alpha\Gamma(\alpha + 1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha + 1)\Gamma(2\alpha - 1)}{2(2\alpha - 1)\Gamma(4\alpha)}}$ as follows:

$$\begin{aligned} P\varphi_{\alpha_1} &\approx 0.6626 < 1, \\ P\varphi_{\alpha_2} &\approx 0.5491 < 1, \\ P\varphi_{\alpha_3} &\approx 0.3995 < 1, \\ P\gamma_{\alpha_3} &\approx 0.9795 < 1, \\ P\gamma_{\alpha_2} &\approx 0.3827 < 1, \\ P\gamma_{\alpha_3} &\approx 0.1772 < 1. \end{aligned}$$

Then, by Theorems 2 and 3, we conclude that there is a unique tangent angle $\theta(s)$ in $C[0, 1]$ that minimizes the energy functional of the beam. Furthermore, in light of Theorem 4, the boundary value problem has a unique solution in $L^2[0, 1]$.

Applying the HPM, the approximate solution for the tangent angle is shown in Figure 2. Furthermore, the deflection shape of the beam under a uniformly distributed load at the equilibrium position can be depicted, as shown in Figure 3. It is highlighted that the curvature of the beam is larger for smaller orders of the fractional derivative.

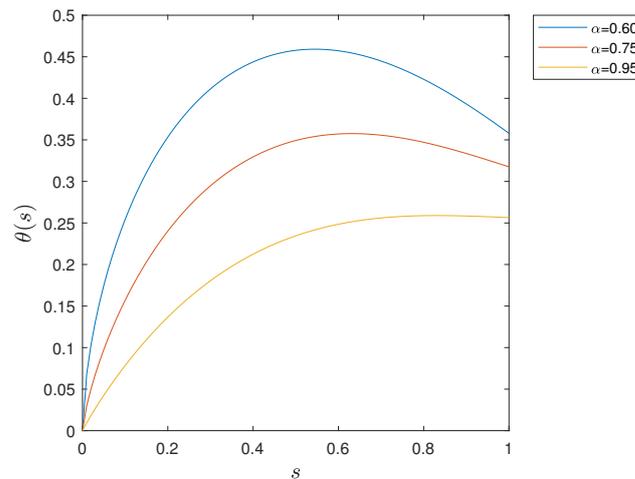


Figure 2. Approximated solutions for θ (tangent angle) of a cantilever beam with a uniformly distributed load $P = 0.73$ for $\alpha = 0.6, 0.75$, and 0.95 .

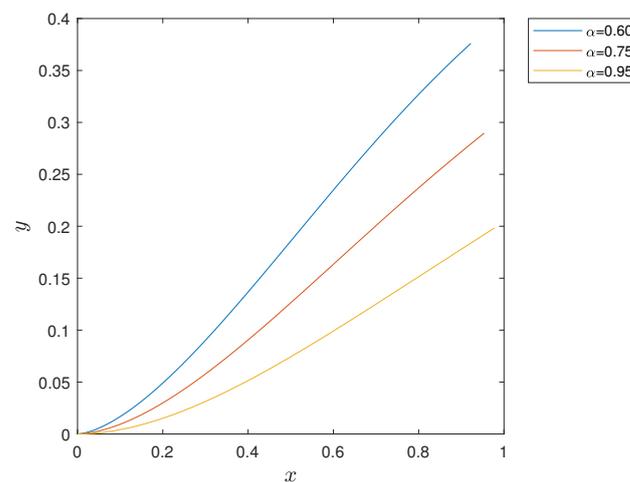


Figure 3. Deflection shapes of a cantilever beam under a uniformly distributed load $P = 0.73$ at the equilibrium position for $\alpha = 0.6, 0.75$, and 0.95 .

Example 2. Consider Problem (9) with $L = 1$, $\alpha = 0.85$, and P taking the values $P_1 = 0.5$, $P_2 = 1.6$, $P_3 = 1.7$, $P_4 = 1.8$, and $P_5 = 2.8$.

This example corresponds to a cantilever beam with a uniformly distributed load with $\alpha = 0.85$.

We obtain $\gamma_\alpha = \frac{L^{3\alpha}}{2\alpha\Gamma(\alpha+1)(\Gamma(\alpha))^2} \sqrt{\frac{\Gamma(2\alpha+1)\Gamma(2\alpha-1)}{2(2\alpha-1)\Gamma(4\alpha)}}$ as follows:

$$P_1\gamma_\alpha \approx 0.1742 < 1,$$

$$P_2\gamma_\alpha \approx 0.5574 < 1,$$

$$P_3\gamma_\alpha \approx 0.5923 < 1,$$

$$P_4\gamma_\alpha \approx 0.6271 < 1,$$

$$P_5\gamma_\alpha \approx 0.9755 < 1.$$

It follows from Theorem 4 that there is a unique tangent angle $\theta(s)$ in $L^2[0, 1]$ that minimizes the energy functional of the beam. Applying the homotopy perturbation method, the approximate solution of the tangent angle can be determined, as shown in Figure 4. Furthermore, the deflection shape of the beam subjected to a uniformly distributed load at the equilibrium position can be depicted, as shown in Figure 5. It can be seen that the minimizers behave continuously when the

loads P_2 and P_4 are close to P_3 , inducing continuous dependence on the loads, which is in agreement with Proposition 1. It is highlighted that the curvature of the beam depends on the load.

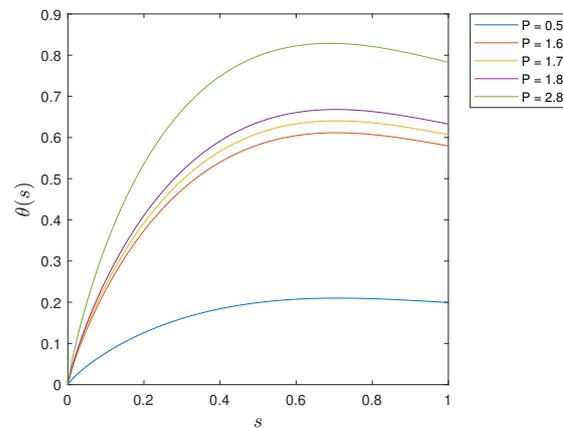


Figure 4. Approximated solutions for θ (tangent angle) of a cantilever beam with a uniformly distributed load $P = 0.5, 1.7,$ and 2.8 when $\alpha = 0.85$.

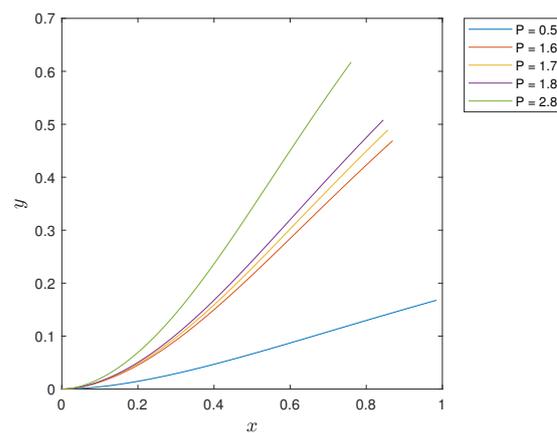


Figure 5. Deflection shapes of a cantilever beam under a uniformly distributed load $P = 0.5, 1.6, 1.7, 1.8,$ and 2.8 at the equilibrium position when $\alpha = 0.85$.

7. Conclusions

We apply the Euler–Lagrange condition for the minimization problem of the energy functional describing the deflection shape of a cantilever beam at the equilibrium position in the fractional calculus framework. By considering boundary value problems, we represent the minimizers in an integral form involving a Green’s function and prove the existence and uniqueness of the minimizers using the Banach contraction principle and Schaefer’s fixed-point theorem. When the load parameters in the energy functional are varied, the set of minimizers forms a branch of solutions with continuous dependence on the load parameters. Finally, the analytical solution is numerically approximated by the homotopy perturbation method to illustrate the deflection shape of cantilever beams at the equilibrium position when various loads and fractional orders are applied. Moreover, the results illustrate that the deformations are larger for smaller orders of the fractional derivative, which is in agreement with [16].

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the final draft of the manuscript. All authors have read and agreed to the published version of the manuscript.

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