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Application of Riemann–Liouville Derivatives on Second-Order Fractional Differential Equations: The Exact Solution

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Abstract: This paper applies two different types of Riemann–Liouville derivatives to solve fractional differential equations of second order. Basically, the properties of the Riemann–Liouville fractional derivative depend mainly on the lower bound of the integral involved in the Riemann–Liouville fractional definition. The Riemann–Liouville fractional derivative of first type considers the lower bound as a zero while the second type applies negative infinity as a lower bound. Due to the differences in properties of the two operators, two different solutions are obtained for the present two classes of fractional differential equations under appropriate initial conditions. It is shown that the zeroth lower bound implies implicit solutions in terms of the Mittag–Leffler functions while explicit solutions are derived when negative infinity is taken as a lower bound. Such explicit solutions are obtained for the current two classes in terms of trigonometric and hyperbolic functions. Some theoretical results are introduced to facilitate the solutions procedures. Moreover, the characteristics of the obtained solutions are discussed and interpreted.

Keywords: Riemann–Liouville fractional derivative; fractional differential equations; Laplace transform; exact solution

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1. Introduction

The fractional calculus (FC) is a growing field of research due to its numerous applications in several areas of sciences and engineering. The FC is a natural extension of classical calculus (CC) and has been utilized to analyze a considerable number of physical and engineering problems [1–3]. In this context, various models have been studied in the literature such as Narahari et al. [4] who applied the FC concept on the dynamics of the fractional oscillator. Propagation of ultrasonic wave in human cancelous bone was introduced by Sebaa et al. [5] via the FC approach. The physical aspect of the fractional Heisenberg equation has been addressed by Tarasov [6]. Application of the FC on the HIV infectious disease has been discussed by Ding and Yea [7]. In quantum mechanics, Wang et al. [8] investigated the time-fractional diffusion equation while other fractional models in different areas of research can be found in Refs. [9–14]. In addition, the fractional models of the projectile motion were solved by Ebaid [15] and Ebaid et al. [16] utilizing the Caputo fractional derivative (CFD) and by Ahmed et al. [17] by means of the Riemann–Liouville fractional derivative (RLFD).

In Refs. [18,19], the FC was extended to solve an astronomical model using the CFD while El-Zahar et al. [20] derived a closed form solution for the same model via applying the RLFD. Moreover, Aljohani et al. [21] obtained the exact solution of the chlorine transport model in fractional form in terms of the Mittag–Leffler function. Furthermore, the application of the RLFD on a class of engineering oscillatory problems was addressed by Ebaid and Al-Jeaid [22] for a class of first-order fractional initial value problems in which the dual solution was obtained. In addition, Seddek et al. [23] applied the RLFD to solve non-homogeneous fractional differential system containing periodic terms. Very recently, Algehyne et al. [24] presented a promise application of the FC on the concept of time dilation.

The objective of this paper is to extend the application of the RLFD to solve the following two classes:

$$ {}_{c}^{RL}D_t^{2\beta}y(t) + \omega^2 y(t) = a\cos(\Omega t), \ \frac{1}{2} < \beta \leq 1, \tag{1} $$

and

$$ {}_{c}^{RL}D_t^{2\beta}y(t) - \omega^2 y(t) = a\cos(\Omega t), \ \frac{1}{2} < \beta \leq 1, \tag{2} $$

where β is none-integer order of the Riemann–Liouville derivative and a, ω, Ω, and A are constants. The two classes are to be solved under the initial conditions (ICs):

$$ {}_{c}^{RL}D_t^{2\beta-2}y(0) = A, \ {}_{c}^{RL}D_t^{2\beta-1}y(0) = B, \tag{3} $$

at two different cases for c, mainly when $c \to 0$ and $c \to -\infty$. The properties of the Riemann–Liouville derivatives ${}_{0}^{RL}D_t$ and ${}_{-\infty}^{RL}D_t$ are completely different and accordingly the nature of solutions of the present two classes are also different. The exact solution, when available, is the optimal solution for any physical/engineering model. So, the obtained exact solution reflects the importance and the main contribution of this paper. The paper is organized as follows. In Section 2, some preliminaries are introduced. In Section 3, theoretical results are derived for the particular solution of class (1). Section 4 is devoted to obtain the exact solution of class (1) while Section 5 presents the solution of class (2) in addition to the behavior of the obtained solution. The paper is concluded in Section 6.

2. Preliminaries

The Riemann–Liouville fractional integral of order α of function $f : [c, d] \to \mathbb{R}$ $(-\infty < c < d < \infty)$ is defined as [1–3]

$$ {}_{c}I_t^{\alpha}f(t) = \frac{1}{\gamma(\alpha)}\int_c^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau, \quad t > c, \ \alpha > 0. \tag{4} $$

The Riemann–Liouville fractional derivative (RLFD) of order $\alpha \in (1,2)$ is [1–3]

$$ {}_{c}^{RL}D_t^{\alpha}f(t) = \frac{1}{\gamma(2-\alpha)}\frac{d^2}{dt^2}\left(\int_c^t \frac{f(\tau)}{(t-\tau)^{\alpha-1}}d\tau\right), \quad t > c. \tag{5} $$

For $t \in \mathbb{R}$ and $\alpha = 2\beta$ $(\frac{1}{2} < \beta \leq 1)$, we have the following RLFD of the functions $e^{i\omega t}$, $\cos(\omega t)$, and $\sin(\omega t)$ as $c \to -\infty$ [22,23]:

$$ \begin{aligned} {}_{-\infty}^{RL}D_t^{2\beta}e^{i\omega t} &= (i\omega)^{2\beta}e^{i\omega t}, \\ {}_{-\infty}^{RL}D_t^{2\beta}\cos(\omega t) &= \omega^{2\beta}\cos(\omega t + \beta\pi), \\ {}_{-\infty}^{RL}D_t^{2\beta}\sin(\omega t) &= \omega^{2\beta}\sin(\omega t + \beta\pi). \end{aligned} \tag{6} $$

The Laplace transform (LT) of the RLFD (5) as $c \to 0$ is [22]

$$ \mathcal{L}\left[{}_{0}^{RL}D_t^{\alpha}y(t)\right] = s^{\alpha}Y(s) - {}_{0}^{RL}D_t^{\alpha-1}y(0) - s\,{}_{0}^{RL}D_t^{\alpha-2}y(0), \tag{7} $$

which yields

$$ \mathcal{L}\left[{}_{0}^{RL}D_t^{2\beta}y(t)\right] = s^{2\beta}Y(s) - {}_{0}^{RL}D_t^{2\beta-1}y(0) - s\,{}_{0}^{RL}D_t^{2\beta-2}y(0), \tag{8} $$

for $\alpha = 2\beta$. The Mittag–Leffler function of two parameters is defined by [1–3]

$$ E_{\delta,\gamma}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\gamma(\delta n + \gamma)}, \quad (\delta > 0, \ \gamma > 0). \tag{9} $$

In particular, we have the following properties

$$E_{2,1}(-z^2) = \cos(z), \quad E_{2,1}(z^2) = \cosh(z), \quad E_{2,2}(-z^2) = \frac{\sin z}{z}, \quad E_{2,2}(z^2) = \frac{\sinh z}{z}. \tag{10}$$

The inverse LT of some expressions can be given via the Mittag–Leffler function as [2,3]

$$\mathcal{L}^{-1}\Big(\frac{s^{\delta-\gamma}}{s^\delta+\omega^2}\Big) = t^{\gamma-1}E_{\delta,\gamma}(-\omega^2 t^\alpha), \quad Re(s) > |\omega^2|^{\frac{1}{\delta}}, \tag{11}$$

which gives the equalities [16,22,23]:

$$\mathcal{L}^{-1}\Big(\frac{s^{\delta-1}}{s^\delta+1}\Big) = E_\alpha(-t^\delta), \tag{12}$$

$$\mathcal{L}^{-1}\Big(\frac{1}{s^\delta+\omega^2}\Big) = t^{\delta-1}E_{\delta,\delta}(-\omega^2 t^\delta), \qquad Re(s) > |\omega^2|^{\frac{1}{\delta}}, \tag{13}$$

$$\mathcal{L}^{-1}\Big(\frac{s^{-1}}{s^\delta+\omega^2}\Big) = t^{\delta}E_{\delta,\delta+1}(-\omega^2 t^\delta), \qquad Re(s) > |\omega^2|^{\frac{1}{\delta}}. \tag{14}$$

3. Analysis

Theorem 1. *The particular solution $y_p(t)$ of the class (1) as $c \to -\infty$ is given by*

$$y_p(t) = \lambda_1(\beta)\cos(\Omega t) + \lambda_2(\beta)\sin(\Omega t), \tag{15}$$

where $\lambda_1(\beta)$ and $\lambda_2(\beta)$ are given by

$$\lambda_1(\beta) = a\Big(\frac{\omega^2 + \Omega^{2\beta}\cos(\pi\beta)}{\omega^4 + \Omega^{4\beta} + 2\omega^2\Omega^{2\beta}\cos(\pi\beta)}\Big), \ \ \lambda_2(\beta) = a\Big(\frac{\Omega^{2\beta}\sin(\pi\beta)}{\omega^4 + \Omega^{4\beta} + 2\omega^2\Omega^{2\beta}\cos(\pi\beta)}\Big), \tag{16}$$

and hence,

$$y_p(t) = a\Big(\frac{\omega^2\cos(\Omega t) + \Omega^{2\beta}\cos(\Omega t - \pi\beta)}{\omega^4 + \Omega^{4\beta} + 2\omega^2\Omega^{2\beta}\cos(\pi\beta)}\Big). \tag{17}$$

Proof. Suppose that y_p is in the form of Equation (15), then

$$\begin{aligned} {}_{-\infty}^{RL}D_t^{2\beta} y_p &= \lambda_1(\beta)\, {}_{-\infty}^{RL}D_t^{2\beta}\cos(\Omega t) + \lambda_2(\beta)\, {}_{-\infty}^{RL}D_t^{2\beta}\sin(\Omega t), \\ &= \Omega^{2\beta}\cos(\Omega t)(\lambda_1(\beta)\cos(\pi\beta) + \lambda_2(\beta)\sin(\pi\beta)) + \\ &\quad \Omega^{2\pi\beta}\sin(\Omega t)(\lambda_2(\beta)\cos(\pi\beta) - \lambda_1(\beta)\sin(\pi\beta)), \end{aligned} \tag{18}$$

and hence

$$\begin{aligned} {}_{-\infty}^{RL}D_t^{2\beta} y_p + \omega^2 y_p &= \Big[\Big(\Omega^{2\beta}\cos(\pi\beta) + \omega^2\Big)\lambda_1(\beta) + \Omega^{2\beta}\sin(\pi\beta)\lambda_2(\beta)\Big]\cos(\Omega t) + \\ &= \Big[\Big(\Omega^{2\beta}\cos(\pi\beta) + \omega^2\Big)\lambda_2(\beta) - \Omega^{2\beta}\sin(\pi\beta)\lambda_1(\beta)\Big]\sin(\Omega t). \end{aligned} \tag{19}$$

The unknowns $\lambda_1(\beta)$ and $\lambda_2(\beta)$ can be obtained by solving the following coupled algebraic equations:

$$\begin{aligned} &\Big(\Omega^{2\beta}\cos(\pi\beta) + \omega^2\Big)\lambda_1(\beta) + \Omega^{2\beta}\sin(\pi\beta)\lambda_2(\beta) = a, \\ &\Big(\Omega^{2\beta}\cos(\pi\beta) + \omega^2\Big)\lambda_2(\beta) - \Omega^{2\beta}\sin(\pi\beta)\lambda_1(\beta) = 0, \end{aligned} \tag{20}$$

which give

$$\lambda_1(\beta) = a\Big(\frac{\omega^2 + \Omega^{2\beta}\cos(\pi\beta)}{\omega^4 + \Omega^{4\beta} + 2\omega^2\Omega^{2\beta}\cos(\pi\beta)}\Big), \ \ \lambda_2(\beta) = a\Big(\frac{\Omega^{2\beta}\sin(\pi\beta)}{\omega^4 + \Omega^{4\beta} + 2\omega^2\Omega^{2\beta}\cos(\pi\beta)}\Big). \tag{21}$$

Therefore, y_p takes the form:

$$y_p(t) = a\left(\frac{\omega^2\cos(\Omega t) + \Omega^{2\beta}\cos(\Omega t - \pi\beta)}{\omega^4 + \Omega^{4\beta} + 2\omega^2\Omega^{2\beta}\cos(\pi\beta)}\right), \tag{22}$$

which completes the proof. □

Lemma 1. *The particular solution $y_p(t)$ of the class (2) as $c \to -\infty$ is given by*

$$y_p(t) = a\left(\frac{-\varpi^2\cos(\Omega t) + \Omega^{2\beta}\cos(\Omega t - \pi\beta)}{\varpi^4 + \Omega^{4\beta} - 2\varpi^2\Omega^{2\beta}\cos(\pi\beta)}\right). \tag{23}$$

Proof. The proof follows immediately by replacing ω with $-i\varpi$ in Equation (17) of theorem 1, where $i = \sqrt{-1}$. □

4. Solution of the First Class: ${}^{RL}_{c}D^{2\beta}_{t}y(t) + \omega^2 y(t) = a\cos(\Omega t)$

In this section, two types of solutions are to be determined for the class (1) when $c \to 0$ and $c \to -\infty$, respectively. The analysis introduced in Refs. [22,23] is followed here to obtain such types of solutions.

4.1. Solution in Terms of the Mittag–Leffler Function as $c \to 0$

In this case, the first class takes the form:

$${}^{RL}_{0}D^{2\beta}_{t}y(t) + \omega^2 y(t) = a\cos(\Omega t), \quad \frac{1}{2} < \beta \le 1, \tag{24}$$

under the ICs:

$${}^{RL}_{0}D^{2\beta-2}_{t}y(0) = A, \quad {}^{RL}_{0}D^{2\beta-1}_{t}y(0) = B. \tag{25}$$

Applying the LT on Equation (24) yields

$$s^{2\beta}Y(s) - {}^{RL}_{0}D^{2\beta-1}_{t}y(0) - s\,{}^{RL}_{0}D^{2\beta-2}_{t}y(0) + \omega^2 Y(s) = \frac{as}{s^2 + \Omega^2}. \tag{26}$$

Solving (26) for $Y(s)$ gives

$$Y(s) = \frac{As}{s^{2\beta} + \omega^2} + \frac{B}{s^{2\beta} + \omega^2} + \frac{as}{(s^{2\beta} + \omega^2)(s^2 + \Omega^2)}. \tag{27}$$

Applying the inverse LT on $Y(s)$, then $y(t)$ is given by

$$y(t) = At^{2\beta-2}E_{2\beta,2\beta-1}\left(-\omega^2 t^{2\beta}\right) + Bt^{2\beta-1}E_{2\beta,2\beta}\left(-\omega^2 t^{2\beta}\right) + a\mathcal{L}^{-1}\left(\frac{1}{s^{2\beta} + \omega^2}\right) * \mathcal{L}^{-1}\left(\frac{s}{s^2 + \Omega^2}\right), \tag{28}$$

where $(*)$ refers to the convolution operation, hence

$$\begin{aligned} y(t) = {} & At^{2\beta-2}E_{2\beta,2\beta-1}\left(-\omega^2 t^{2\beta}\right) + Bt^{2\beta-1}E_{2\beta,2\beta}\left(-\omega^2 t^{2\beta}\right) + \\ & a\int_0^t \tau^{2\beta-1}E_{2\beta,2\beta}\left(-\omega^2\tau^{2\beta}\right)\cos[\Omega(t-\tau)]d\tau, \end{aligned} \tag{29}$$

which can be written as

$$\begin{aligned} y(t) = {} & At^{2\beta-2}E_{2\beta,2\beta-1}\left(-\omega^2 t^{2\beta}\right) + Bt^{2\beta-1}E_{2\beta,2\beta}\left(-\omega^2 t^{2\beta}\right) + a\cos(\Omega t) \times \\ & \int_0^t \tau^{2\beta-1}E_{2\beta,2\beta}\left(-\omega^2\tau^{2\beta}\right)\cos(\Omega\tau)d\tau + a\sin(\Omega t)\int_0^t \tau^{2\beta-1}E_{2\beta,2\beta}\left(-\omega^2\tau^{2\beta}\right)\sin(\Omega\tau)d\tau. \end{aligned} \tag{30}$$

The involved integrals are difficult to compute explicitly. However, the solution in the integral form (30) reduces to the corresponding solution of the ordinary version of the class (1) as $\beta \to 1$.

Special Case as $\beta \to 1$

The solution in the integral form (30) as $\beta \to 1$ becomes

$$y(t) = AE_{2,1}\left(-\omega^2 t^2\right) + BtE_{2,2}\left(-\omega^2 t^2\right) + a\cos(\Omega t) \times \int_0^t \tau E_{2,2}\left(-\omega^2\tau^2\right)\cos(\Omega\tau)d\tau + a\sin(\Omega t)\int_0^t \tau E_{2,2}\left(-\omega^2\tau^2\right)\sin(\Omega\tau)d\tau, \quad (31)$$

i.e.,

$$y(t) = A\cos(\omega t) + \frac{B}{\omega}\sin(\omega t) + \frac{a}{\omega}\cos(\Omega t)\int_0^t \sin(\omega\tau)\cos(\Omega\tau)d\tau + \frac{a}{\omega}\sin(\Omega t)\int_0^t \sin(\omega\tau)\sin(\Omega\tau)d\tau. \quad (32)$$

Performing the integrals, we obtain

$$y(t) = A\cos(\omega t) + \frac{B}{\omega}\sin(\omega t) + a\left(\frac{\cos(\Omega t) - \cos(\omega t)}{\omega^2 - \Omega^2}\right), \quad (33)$$

which is the corresponding solution of the ordinary version $y''(t) + \omega^2 y(t) = a\cos(\Omega t)$ under the ICs $y(0) = A$ and $y'(0) = B$.

Remark 1. *It is noticed that the solution (30) is not analytic at $t = 0 \ \forall \ \beta \in (1/2, 1)$ for the existence of the term $t^{2\beta-2}$. In the next subsection, we are able to derive the analytic solution in the whole domain $t \geq 0$.*

4.2. Solution in Terms of Trigonometric Functions as $c \to -\infty$

As $c \to -\infty$, the first class is in the form:

$${}^{RL}_{-\infty}D_t^{2\beta} y(t) + \omega^2 y(t) = a\cos(\Omega t), \ \ \frac{1}{2} < \beta \leq 1, \quad (34)$$

and the ICs are

$${}^{RL}_{-\infty}D_t^{2\beta-2} y(0) = A, \quad {}^{RL}_{-\infty}D_t^{2\beta-1} y(0) = B. \quad (35)$$

The solution of Equations (34) and (35) consists of the complementary solution y_c and the particular solution $y_p(t)$. However, the $y_p(t)$ is already given by Equation (17) in Theorem 1 while $y_c(t)$ can be assumed in the form [22,23]:

$$y_c(t) = c(\beta)e^{i\sigma t}, \quad (36)$$

where $c(\beta)$ and σ are unknowns and to be determined. The assumption (36) satisfies the homogeneous part of the fractional Equation (34):

$${}^{RL}_{-\infty}D_t^{2\beta} y_c(t) + \omega^2 y_c(t) = 0, \quad (37)$$

if

$$ce^{i\sigma t}\left[(i\sigma)^{2\beta} + \omega^2\right] = 0, \quad (38)$$

which implies two values of σ as

$$\sigma_1 = i\left(-\omega^2\right)^{\frac{1}{2\beta}}, \quad \sigma_2 = -i\left(-\omega^2\right)^{\frac{1}{2\beta}}, \quad (39)$$

or simply

$$\sigma_1 = \nu, \quad \sigma_2 = -\nu, \quad \nu = i\left(-\omega^2\right)^{\frac{1}{2\beta}}. \quad (40)$$

Accordingly, $y_c(t)$ becomes

$$y_c(t) = c_1(\beta)e^{i\nu t} + c_2(\beta)e^{-i\nu t}, \tag{41}$$

where $c_1(\beta)$ and $c_2(\beta)$ are unknown constants. The general solution is

$$y(t) = c_1(\beta)e^{i\nu t} + c_2(\beta)e^{-i\nu t} + y_p(t), \tag{42}$$

where $y_p(t)$ is given by Equation (17). From (42), we have

$$D_t^{2\beta-1}y(t) = c_1(\beta)(i\nu)^{2\beta-1}e^{i\nu t} + c_2(\beta)(-i\nu)^{2\beta-1}e^{-i\nu t} + D_t^{2\beta-1}y_p(t), \tag{43}$$
$$D_t^{2\beta-2}y(t) = c_1(\beta)(i\nu)^{2\beta-2}e^{i\nu t} + c_2(\beta)(-i\nu)^{2\beta-2}e^{-i\nu t} + D_t^{2\beta-2}y_p(t). \tag{44}$$

At $t = 0$, Equations (43) and (44) become

$$D_t^{2\beta-1}y(0) = (i\nu)^{2\beta-1}[c_1(\beta) - c_2(\beta)] + D_t^{2\beta-1}y_p(0), \tag{45}$$
$$D_t^{2\beta-2}y(0) = (i\nu)^{2\beta-2}[c_1(\beta) + c_2(\beta)] + D_t^{2\beta-2}y_p(0). \tag{46}$$

Applying the ICs (35), we obtain

$$c_1(\beta) = \frac{(i\nu)^{1-2\beta}}{2}\left[(B + i\nu A) - \left(D_t^{2\beta-1}y_p(0) + i\nu D_t^{2\beta-2}y_p(0)\right)\right], \tag{47}$$
$$c_2(\beta) = \frac{(i\nu)^{1-2\beta}}{2}\left[(-B + i\nu A) + \left(D_t^{2\beta-1}y_p(0) - i\nu D_t^{2\beta-2}y_p(0)\right)\right]. \tag{48}$$

To calculate $D_t^{2\beta-1}y_p(0)$ and $D_t^{2\beta-2}y_p(0)$, one can use $y_p(t)$ in Equation (15) in terms of λ_1 and λ_2 to obtain

$$D_t^{2\beta-1}y_p(0) = \Omega^{2\beta-1}[\lambda_1\sin(\pi\beta) - \lambda_2\cos(\pi\beta)], \tag{49}$$
$$D_t^{2\beta-2}y_p(0) = -\Omega^{2\beta-2}[\lambda_1\cos(\pi\beta) + \lambda_2\sin(\pi\beta)], \tag{50}$$

where λ_1 and λ_2 are given by Equation (16). Therefor, the solution takes the final form:

$$y(t) = c_1(\beta)e^{-(-\omega^2)^{\frac{1}{2\beta}}t} + c_2(\beta)e^{(-\omega^2)^{\frac{1}{2\beta}}t} + a\left[\frac{\omega^2\cos(\Omega t) + \Omega^{2\beta}\cos(\Omega t - \pi\beta)}{\omega^4 + \Omega^{4\beta} + 2\omega^2\Omega^{2\beta}\cos(\pi\beta)}\right], \tag{51}$$

where $c_1(\beta)$ and $c_2(\beta)$ are defined by Equations (47) and (48), respectively.

Remark 2. *The solution in the case $c \to -\infty$ is obtained in the explicit form (51) unlike the implicit integral form (30) when $c \to 0$. Moreover, the solution (51) is analytic in the whole domain $t \in \mathbb{R}$. In addition, the explicit form (51) is also equivalent to the corresponding solution of ordinary version of the first class as indicated in the below section.*

Special Case as $\beta \to 1$

To check, we have from (51) as $\beta \to 1$ that

$$y(t) = c_1e^{-i\omega t} + c_2e^{i\omega t} + \frac{a\cos(\Omega t)}{\omega^2 - \Omega^2}. \tag{52}$$

From (49) and (50), we have

$$\left[D_t^{2\beta-1}y_p(0)\right]_{\beta\to1} = \Omega[\lambda_2]_{\beta\to1} = 0, \quad \left[D_t^{2\beta-2}y_p(0)\right]_{\beta\to1} = [\lambda_1]_{\beta\to1} = \frac{a}{\omega^2 - \Omega^2}. \tag{53}$$

The quantities c_1 and c_2 in Equations (47) and (48) become

$$c_1 = \frac{(i\nu)^{-1}}{2}\left[(B + i\nu A) - \frac{ai\nu}{\omega^2 - \Omega^2}\right] = -\frac{B}{2i\omega} + \frac{1}{2}\left(A - \frac{a}{\omega^2 - \Omega^2}\right), \tag{54}$$

$$c_2 = \frac{(i\nu)^{-1}}{2}\left[(-B + i\nu A) - \frac{ai\nu}{\omega^2 - \Omega^2}\right] = \frac{B}{2i\omega} + \frac{1}{2}\left(A - \frac{a}{\omega^2 - \Omega^2}\right). \tag{55}$$

Substituting (54) and (55) into (52), yields

$$y(t) = \frac{B}{2i\omega}\left(e^{i\omega t} - e^{-i\omega t}\right) + \frac{1}{2}\left(A - \frac{a}{\omega^2 - \Omega^2}\right)\left(e^{i\omega t} + e^{-i\omega t}\right) + \frac{a\cos(\Omega t)}{\omega^2 - \Omega^2}, \tag{56}$$

or

$$y(t) = \frac{B}{\omega}\sin(\omega t) + \left(A - \frac{a}{\omega^2 - \Omega^2}\right)\cos(\omega t) + \frac{a\cos(\Omega t)}{\omega^2 - \Omega^2}, \tag{57}$$

which is equivalent to the solution of the ordinary version $y''(t) + \omega^2 y(t) = a\cos(\Omega t)$ under the ICs $y(0) = A$ and $y'(0) = B$.

5. Solution of the Second Class: ${}^{RL}_{c}D^{2\beta}_{t}y(t) - \varpi^2 y(t) = a\cos(\Omega t)$

5.1. Solution in Terms of the Mittag–Leffler Function as $c \to 0$

In this case we consider the fractional differential equation:

$${}^{RL}_{0}D^{2\beta}_{t}y(t) - \varpi^2 y(t) = a\cos(\Omega t), \;\; \frac{1}{2} < \beta \leq 1, \tag{58}$$

under the ICs:

$${}^{RL}_{0}D^{2\beta-2}_{t}y(0) = A, \quad {}^{RL}_{0}D^{2\beta-1}_{t}y(0) = B. \tag{59}$$

Following the same analysis in Section 4.1, one can obtain the solution in the form:

$$\begin{aligned} y(t) = {} & At^{2\beta-2}E_{2\beta,2\beta-1}\left(\varpi^2 t^{2\beta}\right) + Bt^{2\beta-1}E_{2\beta,2\beta}\left(\varpi^2 t^{2\beta}\right) + a\cos(\Omega t) \times \\ & \int_0^t \tau^{2\beta-1}E_{2\beta,2\beta}\left(\varpi^2\tau^{2\beta}\right)\cos(\Omega\tau)d\tau + a\sin(\Omega t)\int_0^t \tau^{2\beta-1}E_{2\beta,2\beta}\left(\varpi^2\tau^{2\beta}\right)\sin(\Omega\tau)d\tau. \end{aligned} \tag{60}$$

As $\beta \to 1$, the solution in the integral form (60) reads

$$\begin{aligned} y(t) = {} & AE_{2,1}\left(\varpi^2 t^2\right) + BtE_{2,2}\left(\varpi^2 t^2\right) + a\cos(\Omega t) \times \\ & \int_0^t \tau E_{2,2}\left(\varpi^2\tau^2\right)\cos(\Omega\tau)d\tau + a\sin(\Omega t)\int_0^t \tau E_{2,2}\left(\varpi^2\tau^2\right)\sin(\Omega\tau)d\tau, \end{aligned} \tag{61}$$

i.e.,

$$y(t) = A\cosh(\varpi t) + \frac{B}{\varpi}\sin(\varpi t) + \frac{a}{\varpi}\cos(\Omega t)\int_0^t \sinh(\varpi\tau)\cos(\Omega\tau)d\tau + \frac{a}{\varpi}\sin(\Omega t)\int_0^t \sinh(\varpi\tau)\sin(\Omega\tau)d\tau. \tag{62}$$

Performing the integrals, we obtain

$$y(t) = A\cosh(\varpi t) + \frac{B}{\varpi}\sin(\varpi t) - a\left(\frac{\cos(\Omega t) - \cosh(\varpi t)}{\varpi^2 + \Omega^2}\right), \tag{63}$$

which is the corresponding solution of the ordinary version $y''(t) - \varpi^2 y(t) = a\cos(\Omega t)$ under the ICs $y(0) = A$ and $y'(0) = B$.

5.2. Solution in Terms of Trigonometric and Hyperbolic Functions as $c \to -\infty$

Here, we consider

$$ {}^{RL}_{-\infty}D^{2\beta}_t y(t) - \varpi^2 y(t) = a\cos(\Omega t), \ \frac{1}{2} < \beta \leq 1, \tag{64}$$

under the ICs:

$$ {}^{RL}_{-\infty}D^{2\beta-2}_t y(0) = A, \quad {}^{RL}_{-\infty}D^{2\beta-1}_t y(0) = B. \tag{65}$$

Following the same procedure of Section 4.2, we can get the solution in the form:

$$ y(t) = c_1(\beta)e^{-(\varpi^2)^{\frac{1}{2\beta}}t} + c_2(\beta)e^{(\varpi^2)^{\frac{1}{2\beta}}t} + a\left[\frac{-\varpi^2\cos(\Omega t) + \Omega^{2\beta}\cos(\Omega t - \pi\beta)}{\varpi^4 + \Omega^{4\beta} - 2\varpi^2\Omega^{2\beta}\cos(\pi\beta)}\right], \tag{66}$$

or

$$ y(t) = c_1(\beta)e^{-\varpi^{\frac{1}{\beta}}t} + c_2(\beta)e^{\varpi^{\frac{1}{\beta}}t} + a\left[\frac{-\varpi^2\cos(\Omega t) + \Omega^{2\beta}\cos(\Omega t - \pi\beta)}{\varpi^4 + \Omega^{4\beta} - 2\varpi^2\Omega^{2\beta}\cos(\pi\beta)}\right], \tag{67}$$

where $c_1(\beta)$ and $c_2(\beta)$ can be determined from Equations (47) and (48) by replacing ω with $-i\varpi$ ($i = \sqrt{-1}$), thus

$$ c_1(\beta) = -\frac{\varpi^{1/\beta-2}}{2}\left[B - {}^{RL}_{-\infty}D^{2\beta-1}_t y_p(0) - \varpi^{1/\beta}\left(A - {}^{RL}_{-\infty}D^{2\beta-2}_t y_p(0)\right)\right], \tag{68}$$

$$ c_2(\beta) = -\frac{\varpi^{1/\beta-2}}{2}\left[-\left(B - {}^{RL}_{-\infty}D^{2\beta-1}_t y_p(0)\right) - \varpi^{1/\beta}\left(A - {}^{RL}_{-\infty}D^{2\beta-2}_t y_p(0)\right)\right]. \tag{69}$$

Suppose that

$$ \rho = B - {}^{RL}_{-\infty}D^{2\beta-1}_t y_p(0), \quad \chi = A - {}^{RL}_{-\infty}D^{2\beta-2}_t y_p(0), \tag{70}$$

then

$$ c_1(\beta) = \frac{\varpi^{1/\beta-2}}{2}\left(-\rho + \varpi^{1/\beta}\chi\right), \tag{71}$$

$$ c_2(\beta) = \frac{\varpi^{1/\beta-2}}{2}\left(\rho + \varpi^{1/\beta}\chi\right). \tag{72}$$

Substituting (71) and (72) into (67), we obtain the solution of the system (64)–(65) in terms of the hyperbolic and trigonometric functions as

$$ y(t) = \varpi^{1/\beta-2}\left[\rho\sinh\left(\varpi^{1/\beta}t\right) + \chi\cosh\left(\varpi^{1/\beta}t\right)\right] + a\left[\frac{-\varpi^2\cos(\Omega t) + \Omega^{2\beta}\cos(\Omega t - \pi\beta)}{\varpi^4 + \Omega^{4\beta} - 2\varpi^2\Omega^{2\beta}\cos(\pi\beta)}\right], \tag{73}$$

where the coefficients ρ and χ are given explicitly in the forms:

$$ \rho = B - \Omega^{2\beta-1}[\lambda_1\sin(\pi\beta) - \lambda_2\cos(\pi\beta)], \tag{74}$$

$$ \chi = A + \Omega^{2\beta-2}[\lambda_1\cos(\pi\beta) + \lambda_2\sin(\pi\beta)], \tag{75}$$

and λ_1 and λ_2 are given by

$$ \lambda_1 = a\left(\frac{-\varpi^2 + \Omega^{2\beta}\cos(\pi\beta)}{\varpi^4 + \Omega^{4\beta} - 2\varpi^2\Omega^{2\beta}\cos(\pi\beta)}\right), \ \lambda_2 = a\left(\frac{\Omega^{2\beta}\sin(\pi\beta)}{\varpi^4 + \Omega^{4\beta} - 2\varpi^2\Omega^{2\beta}\cos(\pi\beta)}\right). \tag{76}$$

It should be noted that the expression (76) also reduces to the solution of the ordinary version given in the previous section by Equation (63) as $\beta \to 1$.

5.3. Behavior of the Solution

It can be easily observed from Equation (73) that the solution is real at any given real values of the parameters ϖ and Ω provided that the denominator in Equation (73) does not vanish, i.e., $\varpi^4 + \Omega^{4\beta} - 2\varpi^2\Omega^{2\beta}\cos(\pi\beta) \neq 0$, $(1/2 < \beta < 1)$. The behavior of the solution (73) is examined at some selected values for the involved parameters. The influence of the fractional-order β on the solution is depicted in Figure 1 when $A = 1$, $B = 1$, $\varpi = \frac{1}{3}$, $\Omega = 3$, and $a = 2$. It is observed that the curves oscillate in the first part of the domain, however, such oscillations reduce as the value of β approaches one.

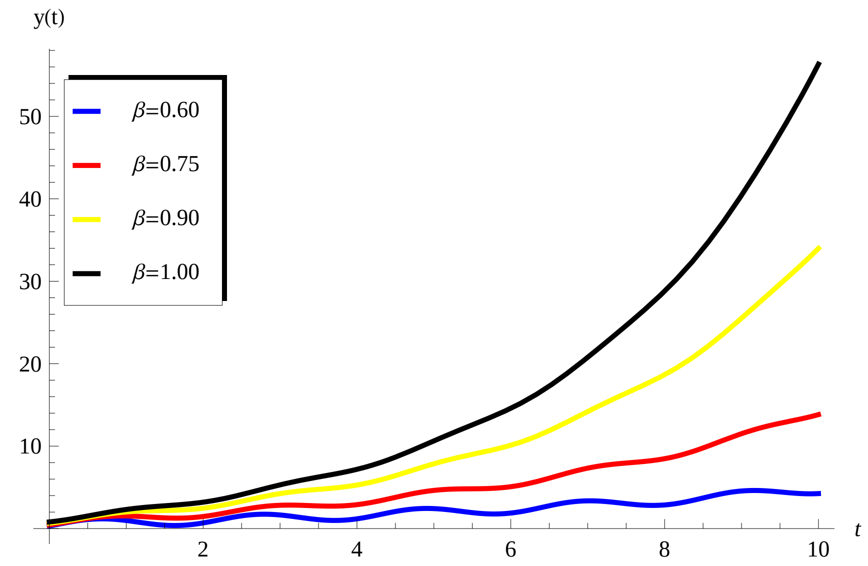

Figure 1. Plots of $y(t)$ in Equation (73) vs t when $A = 1$, $B = 1$, $\varpi = \frac{1}{3}$, $\Omega = 3$, and $a = 2$ at different values of β.

Figure 2 shows the variation of the solution (73) at different values of the coefficient $\varpi > 1$ when $A = 1$, $B = 1$, $\beta = \frac{3}{4}$, $\Omega = 3$, and $a = 2$. It is noticed in Figure 2 that the curves are smooth and have no oscillations. However, the oscillation of the solution $y(t)$ in Equation (73) returns to appear for another set of the ϖ values that are less than unity. This point is declared in Figure 3 which displays behavior for the solution when $A = 1$, $B = 1$, $\beta = \frac{3}{4}$, $\Omega = 3$, and $a = 2$ at different values of $\varpi < 1$.

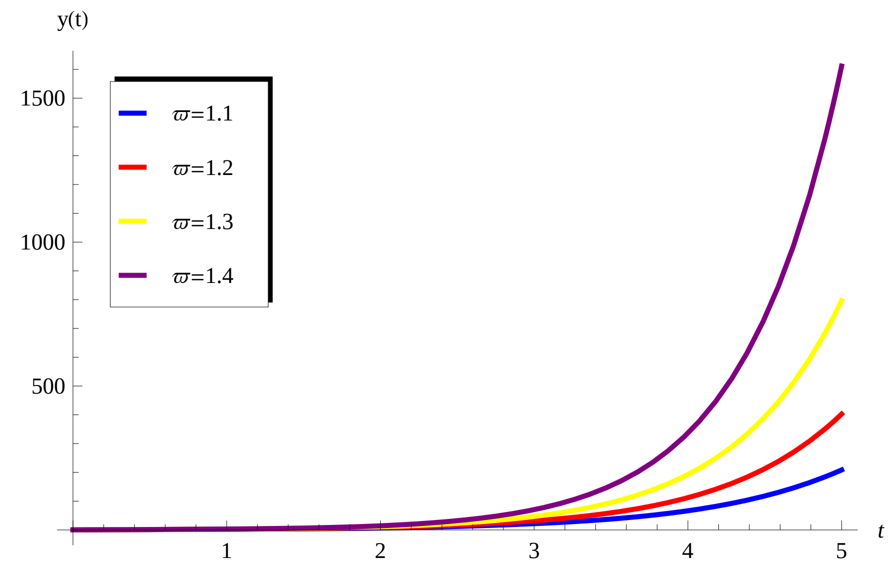

Figure 2. Plots of $y(t)$ in Equation (73) vs. t when $A = 1$, $B = 1$, $\beta = \frac{3}{4}$, $\Omega = 3$, and $a = 2$ at different values of $\varpi > 1$.

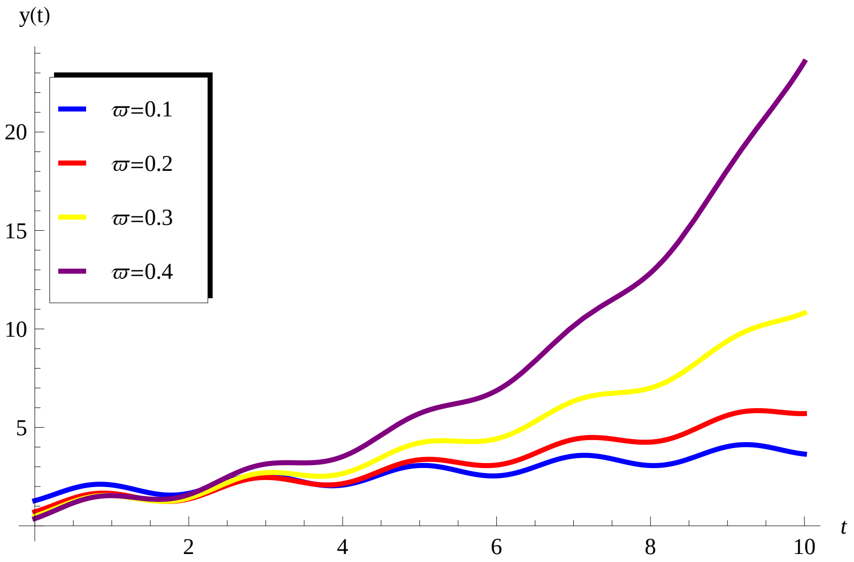

Figure 3. Plots of $y(t)$ in Equation (73) vs. t when $A = 1$, $B = 1$, $\beta = \frac{3}{4}$, $\Omega = 3$, and $a = 2$ at different values of $\varpi < 1$.

6. Conclusions

Two classes of fractional differential equations were solved in this paper by means of two different types of RLFD. The first type considered the lower bound of the integral involved in the RLFD as a zero. The second type treats the lower bound as negative infinity. It was also shown that the solution procedure depends mainly on the implemented type of the RLFD. For the first type of RLFD, the LT method was applied successfully to determine the solutions of the two classes in terms the Mittag–Leffler functions. In addition, a direct analysis was presented to obtain the solutions of the two classes governed by the second type of RLFD, where the solutions were obtained in explicit forms and expressed in terms of trigonometric and hyperbolic functions. Features of the obtained solutions are theoretically discussed and explained. The current analysis may deserve further extension to include other classes of fractional differential equations which describe applications in engineering and physical sciences. In future investigations, other kinds of the fractional derivatives such as Caputo [12,15,16], modified Riemann–Liouville derivative [25], and Atangana–Baleanu derivative [26] will be addressed to solve more complex models such as the nonlinear duffing-oscillator and the nonlinear relativistic oscillator.

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References

1. Miller, K.S.; Ross, B. *An Introduction to the Fractional Calculus and Fractional Differential Equations*; John Wiley & Sons: New York, NY, USA, 1993.
2. Podlubny, I. *Fractional Differential Equations*; Academic Press: San Diego, CA, USA, 1999.
3. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific Publishing Company: Singapore, 2000.
4. Narahari Achar, B.N.; Hanneken, J.W.; Enck, T.; Clarke, T. Dynamics of the fractional oscillator. *Phys. A* **2001**, *297*, 361–367. [CrossRef]
5. Sebaa, N.; Fellah, Z.E.A.; Lauriks, W.; Depollier, C. Application of fractional calculus to ultrasonic wave propagation in human cancellous bone. *Signal Process.* **2006**, *86*, 2668–2677. [CrossRef]
6. Tarasov, V.E. Fractional Heisenberg equation. *Phys. Lett. A* **2008**, *372*, 2984–2988. [CrossRef]
7. Ding, Y.; Yea, H. A fractional-order differential equation model of HIV infection of CD4+T-cells. *Math. Comput. Model.* **2009**, *50*, 386–392. [CrossRef]

8. Wang, S.; Xu, M.; Li, X. Green's function of time fractional diffusion equation and its applications in fractional quantum mechanics. *Nonlinear Anal. Real World Appl.* **2009**, *10*, 1081–1086. [CrossRef]
9. Song, L.; Xu, S.; Yang, J. Dynamical models of happiness with fractional order. *Commun. Nonlinear Sci. Numer. Simul.* **2010**, *15*, 616–628. [CrossRef]
10. Gómez-Aguilara, J.F.; Rosales-García, J.J.; Bernal-Alvarado, J.J. Fractional mechanical oscillators. *Rev. Mex. Física* **2012**, *58*, 348–352.
11. Machado, J.T.; Kiryakova, V.; Mainardi, F. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 1140–1153. [CrossRef]
12. Garcia, J.J.R.; Calderon, M.G.; Ortiz, J.M.; Baleanu, D. Motion of a particle in a resisting medium using fractional calculus approach. *Proc. Rom. Acad. Ser. A* **2013**, *14*, 42–47.
13. Machado, J.T. A fractional approach to the Fermi-Pasta-Ulam problem. *Eur. Phys. J. Spec. Top.* **2013**, *222*, 1795–1803. [CrossRef]
14. Liaqat, M.I.; Akgül, A.; De la Sen, M.; Bayram, M. Approximate and Exact Solutions in the Sense of Conformable Derivatives of Quantum Mechanics Models Using a Novel Algorithm. *Symmetry* **2023**, *15*, 744. [CrossRef]
15. Ebaid, A. Analysis of projectile motion in view of the fractional calculus. *Appl. Math. Model.* **2011**, *35*, 1231–1239. [CrossRef]
16. Ebaid, A.; El-Zahar, E.R.; Aljohani, A.F.; Salah, B.; Krid, M.; Machado, J.T. Analysis of the two-dimensional fractional projectile motion in view of the experimental data. *Nonlinear Dyn.* **2019**, *97*, 1711–1720. [CrossRef]
17. Ahmad, B.; Batarfi, H.; Nieto, J.J.; Oscar, O.-Z.; Shammakh, W. Projectile motion via Riemann-Liouville calculus. *Adv. Differ. Equ.* **2015**, 63. https://doi.org/10.1186/s13662–015-0400–3. [CrossRef]
18. Kumar, D.; Singh, J.; Baleanu, D.; Rathore, S. Analysis of a fractional model of the Ambartsumian equation. *Eur. Phys. J. Plus* **2018**, *133*, 133–259. [CrossRef]
19. Ebaid, A.; Cattani, C.; Al Juhani, A.S.; El-Zahar, E.R. A novel exact solution for the fractional Ambartsumian equation. *Adv. Differ. Equ.* **2021**, *2021*, 88. [CrossRef]
20. El-Zahar, E.R.; Alotaibi, A.M.; Ebaid, A.; Aljohani, A.F.; Gómez Aguilar, J.F. The Riemann-Liouville fractional derivative for Ambartsumian equation. *Results Phys.* **2020**, *19*, 103551. [CrossRef]
21. Aljohani, A.F.; Ebaid, A.; Algehyne, E.A.; Mahrous, Y.M.; Cattani, C.; Al-Jeaid, H.K. The Mittag-Leffler Function for Re-Evaluating the Chlorine Transport Model: Comparative Analysis. *Fractal Fract.* **2022**, *6*, 125. [CrossRef]
22. Ebaid, A.; Al-Jeaid, H.K. The Mittag–Leffler Functions for a Class of First-Order Fractional Initial Value Problems: Dual Solution via Riemann–Liouville Fractional Derivative. *Fractal Fract.* **2022**, *6*, 85. [CrossRef]
23. Seddek, L.F.; Ebaid, A.; El-Zahar, E.R.; Aljoufi, M.D. Exact Solution of Non-Homogeneous Fractional Differential System Containing 2n Periodic Terms under Physical Conditions. *Mathematics* **2023**, *11*, 3308. [CrossRef]
24. Algehyne, E.A.; Aldhabani, M.S.; Areshi, M.; El-Zahar, E.R.; Ebaid, A.; Al-Jeaid, H.K. A Proposed Application of Fractional Calculus on Time Dilation in Special Theory of Relativity. *Mathematics* **2023**, *11*, 3343. [CrossRef]
25. Jumarie, G. Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution. *J. Appl. Math. Comput.* **2007**, *24*, 31–48. [CrossRef]
26. Atangana, A.; Gómez-Aguilar, J.F. Numerical approximation of Riemann-Liouville definition of fractional derivative: From Riemann-Liouville to Atangana-Baleanu. *Numer. Methods Partial. Differ. Eq.* **2018**, *34*, 1502. [CrossRef]