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Approximate Controllability of Neutral Differential Systems with Fractional Deformable Derivatives

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Abstract: This article deals with the existence and uniqueness of solutions, as well as the approximate controllability of fractional neutral differential equations (ACFNDEs) with deformable derivatives. The findings are achieved using Banach's, Krasnoselskii's, and Schauder's fixed-point theorems and semigroup theory. Three numerical examples are used to illustrate the application of the theories discussed in the conclusion.

Keywords: fractional differential equations; FDEs; semigroup theory; deformable fractional derivative; Krasnoselskii's fixed point; bounded linear operators; BLOs; approximate controllability

MSC: 26A33; 93B05

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1. Introduction

There are several branches of science and engineering in which the dynamic behaviour of natural processes is swiftly modelled using the concept of integrals, which have non-integer order and are covered in fractional calculus of functions and their derivatives, etc. Since FDEs include non-local relationships for both time and space, the study of fractional calculus has begun to advance more quickly all over the world. Additionally, fractional differential equations are an excellent tool for explaining memory and genetic traits in a variety of processes. The authors of [1,2] employed fractional differential operators for a different use. These operators were used to understand population growth models. Fractional calculus underwent significant growth after the 19th century, primarily as a result of its suitability for a wide range of disciplines and the emergence of numerous definitions for fractional derivatives, each of which had its own unique characteristics. Examples of these definitions include the Wieyl definition, the Riemann–Liouville definition, the Caputo definition, and others. Ironically, the integral form appears in the majority of the formulations of fractional derivatives. Readers are directed to [3–5] for a comprehensive understanding of fractional calculus and [6–16] for an overview of fractional differential equations (FDEs).

The majority of definitions of fractional derivatives utilise the integral form, as previously established. However, in 2014, R. Khalil et al. [17] proposed a limit-based approach, similar to the usual derivative, characterising it as conformable and analogous to a conventional derivative. Subsequently, F. Zulfeqarr et al. [18] introduced the new concept of deformable derivatives, which was notably more straightforward than Khalil's definition. This derivative was inspired by Khalil's work and addressed its limitations while also accommodating a broader range of applications.

It is common knowledge that engineering and mathematical control theory both strongly depend on the idea of controllability. As a result, several researchers have thoroughly examined the controllability of various nonlinear systems in recent years, for example [19–25], and the references therein. Finding an appropriate control function that will allow researchers to move the dynamical system's state towards the targeted final state is the controllability problem. While exact controllability steers towards an exact final state, approximate controllability allows researchers to steer the system towards an arbitrarily small neighbourhood of the final state. Approximate controllability, therefore, applies more to dynamical systems.

Since Hille and Yosida's discovery of generation theory in 1948, the study of semigroups of BLOs has undergone significant developments, making it a substantial area of mathematics that is widely employed in several analytical fields. To solve differential equations, it is necessary to understand the concept of semigroups of BLOs. It has been effective in resolving a considerable class of differential and integro-differential problems in recent years. Using semigroups, Pazy [26] investigated the EaU of classical solutions, strong solutions, and mild solutions to evolution systems.

Further investigation started with a discussion of previously published works [10,27–29]. Specifically, the existence and uniqueness of the mild solutions and approximate controllability of fractional evolution equations with deformable derivatives were investigated in [29,30]

$$\begin{aligned} &D^{\rho}\omega(\beta) = Q\omega(\beta) + f(\beta,\omega(\beta)), \quad \beta \in I = [0,b], \quad 0<\rho<1, \\ &\omega(0) = \omega_0, \end{aligned}$$

and the conclusions were made possible using Banach's and Schauder's fixed-point theorems in semigroup theory.

The authors of [10] examined the existence and uniqueness of solutions to the Cauchy problem for fractional differential equations with non-local conditions

$$\begin{aligned} &D^{\rho}\omega(\beta) = f(\beta,\omega(\beta)), \quad \beta \in I = [0,Z], \quad 0<\rho<1, \\ &\omega(0) + g(\omega) = \omega_0, \end{aligned}$$

in a Banach space.

Also, the authors of [16] further investigated the properties of the deformable derivatives and used the results to study the existence of solutions to the integro-differential equation

$$\begin{aligned} &D^{\rho}\omega(\beta) = k(\omega(\beta)) + f(\beta,\omega(\beta)) + \int_0^{\beta} K(\beta,\sigma,\omega(\sigma))d\sigma, \quad I=[0,Z] \quad and \quad \beta \in I, \quad 0<\rho<1, \\ &\omega(0) = \omega_0, \end{aligned}$$

achieving their results using Weissinger's fixed-point theorem and Krasnoselskii's fixed-point theorem.

Later, M. Etefa and Guerekata et al. [27] studied the results of sufficient conditions for the existence of solutions for a class of initial value problems for impulsive fractional differential equations involving the deformable fractional derivative

$$\begin{aligned} &D^{\rho}\omega(\beta) = f(\beta,\omega(\beta)), \quad \beta \in I = [0,\chi], \ \beta \neq \beta_l, \ l = 1,2,\ldots,p, \\ &\Delta\omega|_{\beta=\beta_l} = I_l(\omega(\beta_l^-)), \\ &\omega(0) = \omega_0. \end{aligned}$$

Further, they obtained their results using the Banach contraction principle and the alternative Leray–Schauder fixed-point theorems.

Drawing inspiration from the above-mentioned works, we study the existence and uniqueness results for FNDE with $^{\mathcal{DD}}$ of the model

$$^{\mathcal{DD}}D^{\rho}[\omega(\beta) - A_1(\beta, \omega(\beta))] = Q[\omega(\beta) - A_1(\beta, \omega(\beta))] + A_2(\beta, \omega(\beta)), \tag{1}$$
$$\omega(0) = \omega_0, \tag{2}$$

and the corresponding controllability model

$$^{\mathcal{DD}}D^{\rho}[\omega(\beta) - A_1(\beta, \omega(\beta))] = Q[\omega(\beta) - A_1(\beta, \omega(\beta))] + A_2(\beta, \omega(\beta)) + Bv(\beta), \tag{3}$$
$$\omega(0) = \omega_0, \tag{4}$$

where $^{\mathcal{DD}}D^{\rho}$ is the deformable fractional derivative of order $\rho \in (0,1)$ *and* $\beta \in [0, \chi]$, $\chi > 0$ is a constant.
$Q : D(Q) \subset \mathbb{X} \to \mathbb{X}$ is an infinitesimal generator of a C_0-semigroup $Z(\beta)(\beta \geq 0)$, and $A_2 : [0, \chi] \times \mathbb{X} \to \mathbb{X}$ are continuous functions. $A_1 : [0, \chi] \times \mathbb{X} \to \mathbb{X}$ is a continuously differentiable function and $\omega_0 \in \mathbb{X}$, where $\mathbb{X}$ is an appropriate space, $v \in L^2([0, \chi], V)$, where V is a Hilbert space, and $B : V \to \mathbb{X}$ is a BLO.

This study's primary findings are as follows:

1. We obtain the solutions to systems (1) and (2) and present them in Theorems 5 and 6. Also, we prove that systems (3) and (4) have approximate controllability.
2. The results of this work improve and generalise other studies that have been reported in the literature [10,27–29].

The rest of this article is organised as follows. In Section 2, we discuss the basic definitions, essential properties, and theorems. Our results were obtained using Krasnoselskii's fixed-point theorem and the Banach contraction principle. In Section 3, we discuss the main results, i.e., the existence and uniqueness of the solution to systems (1) and (2), using appropriate fixed-point theorems. Then, we show that systems (3) and (4) are approximately controllable. Moreover, in Section 4, we present three numerical examples to illustrate our results.

2. Preliminaries

The objective of this section is to present a summary of the key concepts and results associated with deformable derivatives. These ideas and outcomes are instrumental in our efforts to derive our primary conclusions.

Definition 1 (see [18]). *The deformable derivative of order $\rho \in [0,1]$ for a function $\omega : (e_1, e_2) \to \mathbb{R}$ is defined by*

$$\lim_{\epsilon \to 0} \frac{(1 + \epsilon\mu)\omega(\beta + \epsilon\rho) - \omega(\beta)}{\epsilon},$$

where $\rho + \mu = 1$. If this limit exists, we denote it by $D^{\rho}\omega(\beta)$.

Remark 1. *One can note that definition (1) is compatible with $\rho = 0, 1$.*
If $\rho = 0$, $D^0\omega(\beta) = \omega(\beta)$, which is the usual convention, and if $\rho = 1$, $D\omega(\beta) = \omega'(\beta)$.

Definition 2 (see [18]). *Let ω be a continuous function defined on the interval $[e_1, e_2]$. The ρ-fractional integral of ω is as follows:*

$$I^{\rho}_{e_1}\omega(\beta) = \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_{e_1}^{\beta} e^{\frac{\mu}{\rho}\sigma} \omega(\sigma) d\sigma, \quad \textit{where} \quad \rho + \mu = 1, \quad \rho \in (0,1]. \tag{5}$$

Remark 2. *If $e_1 = 0$, Equation (5) becomes*

$$I^{\rho}\omega(\beta) = \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_{0}^{\beta} e^{\frac{\mu}{\rho}\sigma} \omega(\sigma) d\sigma.$$

Theorem 1 (see [18])**.** *Let ω be a differentiable function at a point $\beta \in (e_1, e_2)$ that is differentiable for all ρ at that point. So far, we have*

$$D^{\rho}\omega(\beta) = \mu\omega(\beta) + \rho D\omega(\beta),$$

where $D\omega = \frac{\mathrm{d}}{\mathrm{d}\beta}\omega$.

Theorem 2 (see [18])**.** *By assuming that ω is continuous over the interval $[e_1, e_2]$, it follows that $I^{\rho}_{e_1}\omega$ is differentiable with respect to ρ in the open interval (e_1, e_2), which can also be expressed as*

$$D^{\rho} I^{\rho}_{e_1}\omega(\beta) = \omega(\beta) \quad \text{and} \quad I^{\rho}_{e_1} D^{\rho}\omega(\beta) = \omega(\beta) - e^{\frac{\mu}{\rho}(e_1-\beta)}\omega(e_1).$$

We refer the reader to [10,18,27] for further information on the properties and outcomes of deformable derivatives.

Assume that Q is a linear operator from $D(Q) \subset \mathbb{X}$ into $\mathbb{X}$ and $\omega_0 \in \mathbb{X}$.

Lemma 1 ([31])**.** *The Cauchy problem for the deformable fractional derivative is governed by the parameter $\rho \in (0, 1]$*

$$D^{\rho}[\omega(\beta) - k(\beta)] = Q[\omega(\beta) - k(\beta)] \quad \beta \in [0, \chi], \tag{6}$$

$$\omega(0) = \omega_0, \tag{7}$$

which has the solution

$$\omega(\beta) = e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - k(0)] + k(\beta),$$

where β is a non-negative real number, $\omega_0 \in D(Q)$, and $k : [0, \chi] \to \mathbb{X}$ is an appropriate space.

Let $\rho \in (0, 1]$. The ensuing fractional inhomogeneous deformable Cauchy system is

$$D^{\rho}[\omega(\beta) - k(\beta)] = Q[\omega(\beta) - k(\beta)] + f(\beta) \quad \beta \in [0, \chi], \tag{8}$$

$$\omega(0) = \omega_0, \tag{9}$$

where β is a non-negative real number, $\omega_0 \in D(Q)$, and $f,\ k : [0, \chi] \to \mathbb{X}$ are suitable functions.

Theorem 3 ([31])**.** *Consider ω as the solution to systems (8) and (9) and $f \in L^1([0, \chi], \mathbb{X})$. Then, ω satisfies*

$$\omega(\beta) = e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - k(0)] + k(\beta) + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^{\beta} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta - \sigma}{\rho}\right) f(\sigma) d\sigma.$$

Theorem 4 ([31])**.** *Consider Q as the infinitesimal generator of a C_0 semigroup and $f \in C([0, \chi], \mathbb{X})$. If $f(\sigma) \in D(Q)$ for $\sigma \in [0, \beta]$ & $Qf(\sigma) \in L^1([0, \chi], \mathbb{X})$, for every $\omega_0 - k(0) \in D(Q)$, then $\omega : [0, \chi] \to \mathbb{X}$ described by*

$$\omega(\beta) = e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - k(0)] + k(\beta) + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^{\beta} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta - \sigma}{\rho}\right) f(\sigma) d\sigma. \tag{10}$$

is a mild solution to (8) and (9).

Proof. If $y(\beta) = e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - k(0)] + k(\beta)$ and $z(\beta) = \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^{\beta} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta - \sigma}{\rho}\right) f(\sigma) d\sigma$, then $\omega(Q)$ can be written as $y(\beta) + z(\beta)$. Since $\omega_0 - k(0) \in D(Q)$, $y(\beta)$ is differentiable and according to Lemma 1, it is a known fact that $D^{\rho}[\omega(\beta) - k(\beta)] = Q[\omega(\beta) - k(\beta)]$.

From Theorem 3.5 [29], we have $z'(\beta) = \frac{1}{\rho}(-\mu z(\beta) + Qz(\beta) + f(\beta))$. From Theorem 1, we obtain

$$\begin{aligned} D^{\rho} z(\beta) &= \mu z(\beta) + \rho z'(\beta) \\ &= \mu z(\beta) + \rho \cdot \frac{1}{\rho}(-\mu z(\beta) + Qz(\beta) + f(\beta)) \\ &= Qz(\beta) + f(\beta). \end{aligned}$$

As a result,

$$\begin{aligned} D^{\rho}\omega(\beta) &= D^{\rho} y(\beta) + D^{\rho} z(\beta) \\ &= Q[y(\beta) - k(\beta)] + D^{\rho} k(\beta) + Qz(\beta) + f(\beta). \end{aligned}$$

$\Longrightarrow$

$$\begin{aligned} D^{\rho}[\omega(\beta) - k(\beta)] &= Qy(\beta) - k(\beta)] + zQ(\beta) + f(\beta) \\ &= Qy(\beta) + Qz(\beta) - Qk(\beta) + f(\beta) \\ &= Q\omega(\beta) - Qk(\beta) + f(\beta) \\ &= Q[\omega(\beta) - k(\beta)] + f(\beta). \end{aligned}$$

By substituting the initial conditions for y and z into the equation $\omega(0) = y(0) + z(0) = \omega_0 - k(0) + k(0) = \omega_0$, we can see that $\omega(\beta)$, as given by Equation (10), represents a mild solution to (8) and (9). $\square$

3. Main Results

The following section begins with a discussion of the EaU of a mild solution to systems (1) and (2), after which the researchers describe and demonstrate the requirements for the approximate controllability of systems (3) and (4).

3.1. Existence Results

Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$, and $\mathcal{C}([0,\chi],\mathbb{X})$ be a Banach space of all continuous functions from $[0,\chi]$ into $\mathbb{X}$ endowed with the supremum norm $\|\omega\|_{\mathcal{C}} = \sup\limits_{\beta\in[0,\chi]} \|\omega(\beta)\|$.

By using the information in the preceding Theorem 4, we are able to determine the solution to our addressed systems (1) and (2).

Definition 3. *A function $\omega \in \mathcal{C}$ is considered to be a mild solution to systems (1) and (2)*

$$\begin{aligned} \omega(\beta) &= e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0,\omega_0)] + A_1(\beta,\omega(\beta)) \\ &\quad + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^{\beta} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta-\sigma}{\rho}\right) A_2(\sigma,\omega(\sigma))d\sigma, \quad \beta \in [0,\chi], \end{aligned} \tag{11}$$

provided the integral exists. To study systems (1) and (2), the conditions that follow need to be listed:

(M0) Q is the infinitesimal generator of a C_0 semigroup of BLOs $\{W(\beta)\}_{\beta\geq 0}$ such that $\mathcal{M} = \sup_{\beta\in[0,\chi]} \|W(\beta)\|_{L(\mathbb{X})}$, where the term $L(\mathbb{X})$ is Banach space on $\mathbb{X}$, $N \geq 1$.

(M1) A_2 is the function, A_2: $[0,\chi] \times \mathbb{X} \to \mathbb{X}$ is continuous, and $A_1 : [0,\chi] \times \mathbb{X} \to \mathbb{X}$ is continuously differentiable and we can find the positive constants $\mathcal{I}_{A_2}, \mathcal{I}_{A_1}$ in such a way that:

(i)

$$\|A_2(\beta,u) - A_2(\beta,\bar{u})\| \leq \mathcal{I}_{A_2}\|u - \bar{u}\|, \quad \textit{for every} \quad \beta \in [0,\chi],\ u,\bar{u} \in \mathbb{X}$$

and $\overline{\mathcal{I}}_{A_2} = \sup\limits_{\beta\in[0,\chi]} \|A_2(\beta,0)\|$.

(ii)

$$\|A_1(\beta,u) - A_1(\beta,\bar{u})\| \le \mathcal{I}_{A_1}\|u-\bar{u}\|, \quad \textit{for every} \quad \beta \in [0,\chi],\ u,\bar{u} \in \mathbb{X}$$

and $\overline{\mathcal{I}}_{A_1} = \sup_{\beta\in[0,\chi]} \|A_1(\beta,0)\|.$

(M2) The function $A_2 : [0,\chi]\times\mathbb{X} \to \mathbb{X}$ *is continuous,* $A_1 : [0,\chi]\times\mathbb{X}\to\mathbb{X}$ *is continuously differentiable, and the positive constants* $\widehat{\mathcal{I}}_{A_2}, \widehat{\mathcal{I}}_{A_1}, \widetilde{\mathcal{I}}_{A_2}, \widetilde{\mathcal{I}}_{A_1}$ *can be found in such a way that:*

(i) $\exists$ *positive constants* $\widehat{\mathcal{I}}_{A_2}, \widetilde{\mathcal{I}}_{A_2}$ *in such a way that*

$$\|A_2(\beta,\omega)\| \le \widehat{\mathcal{I}}_{A_2} + \widetilde{\mathcal{I}}_{A_2}\|\omega\|, \quad \beta\in[0,\chi], \quad \omega\in\mathbb{X}.$$

(ii) $\exists$ *positive constants* $\widehat{\mathcal{I}}_{H}, \widetilde{\mathcal{I}}_{H}$ *in such a way that*

$$\|A_1(\beta,\omega)\| \le \widehat{\mathcal{I}}_{A_1} + \widetilde{\mathcal{I}}_{A_1}\|\omega\|, \quad \beta\in[0,\chi], \quad \omega\in\mathbb{X}.$$

Theorem 5. *Assume that* A_1 *and* A_2 *meet the requirements of (M0)-(M1) and that*

$$\Lambda = \left[\mathcal{I}_{A_1} + \frac{\mathcal{M}\mathcal{I}_{A_2}}{\mu}\right] < 1, \tag{12}$$

then, systems (1) and (2) have a unique solution on $[0,\chi]$*.*

Proof. We modify systems (1) and (2) into a fixed-point problem. Define $\Upsilon : \mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned}(\Upsilon\omega)(\beta) &= e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0,\omega_0)] + A_1(\beta,\omega(\beta)) \\ &\quad + \frac{1}{\rho}e^{\frac{-\mu}{\rho}\beta}\int_0^\beta e^{\frac{\mu}{\rho}s} W\left(\frac{\beta-\sigma}{\rho}\right) A_2(\sigma,\omega(\sigma))d\sigma, \quad \beta\in[0,\chi].\end{aligned} \tag{13}$$

We now show that $\Upsilon B_Q \subset B_Q$, where $B_Q = B(0,Q) = \{\omega\in\mathcal{C}([0,\chi],\mathbb{X}) : \|\omega\|_{\mathcal{C}} \le Q\}$, and radius $Q > \dfrac{\|\Omega_1\|}{1-\widetilde{\mu}}$, where $\|\Omega_1\| = \|\Omega\| + \left(\overline{\mathcal{I}}_{A_1} + \frac{\mathcal{M}\overline{\mathcal{I}}_{A_2}}{\mu}\right)$, $\|\Omega\| = \mathcal{M}[\|\omega_0\| + \|A_1(0,\omega_0)\|]$, $\widetilde{\mu} = \mathcal{I}_{A_1} + \frac{\mathcal{M}\mathcal{I}_{A_2}}{\mu}$.

Obviously, let $\omega\in B_Q$,

$$\begin{aligned}\|(\Upsilon\omega)(\beta)\| &= \Bigg\| e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0,\omega_0)] + A_1(\beta,\omega(\beta)) \\ &\quad + \frac{1}{\rho}e^{\frac{-\mu}{\rho}\beta}\int_0^\beta e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta-\sigma}{\rho}\right) A_2(\sigma,\omega(\sigma))d\sigma\Bigg\| \\ &\le \|\Omega\| + \|A_1(\beta,p(\beta)) - A_1(\beta,0)\| + \|A_1(\beta,0)\| \\ &\quad + \frac{\mathcal{M}}{\rho}e^{\frac{-\mu}{\rho}\beta}\int_0^\beta e^{\frac{\mu}{\rho}\sigma}(\|A_2(\sigma,\omega(\sigma)) - A_2(\sigma,0)\| + \|A_2(\sigma,0)\|)d\sigma \\ &\le \|\Omega\| + \mathcal{I}_{A_1}Q + \overline{\mathcal{I}}_{A_1} + (\mathcal{I}_{A_2}Q + \overline{\mathcal{I}}_{A_2})\frac{\mathcal{M}}{\rho}e^{\frac{-\mu}{\rho}\beta}\int_0^\beta e^{\frac{\mu}{\rho}\sigma}d\sigma \\ &\le \|\Omega\| + \mathcal{I}_{A_1}Q + \overline{\mathcal{I}}_{A_1} + (\mathcal{I}_{A_2}Q + \overline{\mathcal{I}}_{A_2})\frac{\mathcal{M}}{\mu}(1 - e^{\frac{-\mu}{\rho}\beta}) \\ &= \|\Omega\| + \mathcal{I}_{A_1}Q + \overline{\mathcal{I}}_{A_1} + \frac{\mathcal{M}}{\mu}[(\mathcal{I}_{A_2}Q + \overline{\mathcal{I}}_{A_2})](1 - e^{\frac{-\mu}{\rho}\beta}) \\ &\le \|\Omega_1\| + \left[\mathcal{I}_{A_1} + \frac{\mathcal{M}\mathcal{I}_{A_2}}{\mu}\right]Q.\end{aligned}$$

As a result, for $\beta \in [0, \chi]$ and $\omega \in B_Q$, we have

$$\|\mathrm{Y}\omega)\|_{\mathcal{C}} \leq \|\Omega_1\| + \left[\mathcal{I}_{A_1} + \frac{\mathcal{M}\mathcal{I}_{A_2}}{\mu}\right]Q < Q.$$

The above results establish that the Y causes the ball B_Q to be transformed into itself. To proceed further with $\omega, \overline{\omega} \in B_Q$, we define

$$\begin{aligned}
&\|(\mathrm{Y}\omega)(\beta) - (\mathrm{Y}\overline{\omega})(\beta)\| \\
&\leq \|A_1(\beta, \omega(\beta)) - A_1(\beta, \overline{\omega}(\beta))\| + \frac{\mathcal{M}}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} \|A_2(\sigma, \omega(\sigma)) - A_2(\sigma, \overline{\omega}(\sigma))\| d\sigma \\
&\leq \mathcal{I}_{A_1} \|\omega - \overline{\omega}\|_{\mathcal{C}} + \frac{\mathcal{M}\mathcal{I}_{A_2}}{\mu} \|\omega - \overline{\omega}\|_{\mathcal{C}} (1 - e^{\frac{-\mu}{\rho}\beta}) \\
&\leq \left[\mathcal{I}_{A_1} + \frac{\mathcal{M}\mathcal{I}_{A_2}}{\mu}\right] \|\omega - \overline{\omega}\|_{\mathcal{C}}.
\end{aligned}$$

As a result, for $\beta \in [0, \chi]$, the researchers arrived at

$$\|\mathrm{Y}(\omega) - \mathrm{Y}(\overline{\omega})\|_{\mathcal{C}} \leq \Lambda \|\omega - \overline{\omega}\|_{\mathcal{C}},$$

where $\Lambda = \Lambda_{\mathcal{M}, \mathcal{I}_{A_2}, \mathcal{I}_{A_1}, \mu}$ are the parameters of the system. According to (12), $\Lambda < 1$, so Y is a contraction. As a result, systems (1) and (2) have a unique solution on $[0, \chi]$ in accordance with Lemma 2.2 [32] of the Banach contraction principle. □

Now, we can prove that there exist solutions to (1) and (2) by applying Krasnoselskii's fixed-point theorem (Lemma 2.3, [32]).

Theorem 6. *Assumptions (M0), (M1)(ii), and (M2) hold with $\mathcal{I}_{A_1} < 1$. In this case, there is at least one solution to systems (1) and (2) on $[0, \chi]$.*

Proof. Let us define two operators using system (11) as follows:

$$(\mathrm{Y}_1\omega)(\beta) = e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0, \omega_0)] + A_1(\beta, \omega(\beta)), \ \beta \in [0, \chi] \tag{14}$$

and

$$(\mathrm{Y}_2\omega)(\beta) = \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta - \sigma}{\rho}\right) A_2(\sigma, \omega(\sigma)) d\sigma, \quad \beta \in [0, \chi]. \tag{15}$$

We now show that $\mathrm{Y}B_Q \subset B_Q$ and the radius $Q > \frac{\|\Omega_1^*\|}{1 - \widetilde{\mu}_1}$, where $\|\Omega_1^*\| = \|\Omega\| + \left(\widehat{\mathcal{I}}_{A_1} + \frac{\mathcal{M}\widehat{\mathcal{I}}_{A_2}}{\mu}\right)$, $\|\Omega\| = \mathcal{M}[\|\omega_0\| + \|A_1(0, \omega_0)\|]$, $\widetilde{\mu}_1 = \widetilde{\mathcal{I}}_{A_1} + \frac{\mathcal{M}\widetilde{\mathcal{I}}_{A_2}}{\mu}$.

For $\omega, \omega_1 \in B_Q$, we find that

$$\begin{aligned}
&\|\mathrm{Y}_1\omega(\beta) + \mathrm{Y}_2\omega_1(\beta)\| \\
&\leq \left\| e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0, \omega_0)] + A_1(\beta, \omega(\beta)) + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta - \sigma}{\rho}\right) F(\sigma, \omega_1(\sigma)) d\sigma \right\| \\
&\leq \|\Omega\| + \widehat{\mathcal{I}}_{A_1} + \widetilde{\mathcal{I}}_{A_1} Q + \frac{\mathcal{M}}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}s} [\widehat{\mathcal{I}}_{A_2} + \widetilde{\mathcal{I}}_{A_2} Q] d\sigma \\
&\leq \|\Omega\| + \widehat{\mathcal{I}}_{A_1} + \widetilde{\mathcal{I}}_{A_1} Q + \frac{\mathcal{M}}{\mu} \widehat{\mathcal{I}}_{A_2} + \frac{\mathcal{M}}{\mu} \widetilde{\mathcal{I}}_{A_2} Q
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\Omega\| + \widehat{\mathcal{I}}_{A_1} + \frac{\mathcal{M}\widehat{\mathcal{I}}_{A_2}}{\mu} + \left[\widetilde{\mathcal{I}}_{A_1} + \frac{\mathcal{M}\widetilde{\mathcal{I}}_{A_2}}{\mu}\right]Q \\
&\leq \|\Omega_1^*\| + \left[\widetilde{\mathcal{I}}_{A_1} + \frac{\mathcal{M}\widetilde{\mathcal{I}}_{A_2}}{\mu}\right]Q.
\end{aligned}
$$

Thus, for $\beta \in [0, \chi]$ and $\omega \in B_Q$, we have

$$
\|\Upsilon_1(\omega) + \Upsilon_2(\omega_1)\|_{\mathcal{C}} \leq \|\Omega_1^*\| + \left[\widetilde{\mathcal{I}}_{A_1} + \frac{\mathcal{M}\widetilde{\mathcal{I}}_{A_2}}{\mu}\right]Q < Q.
$$

Thus, $\Upsilon_1(\omega) + \Upsilon_2(\omega_1) \in B_Q$. As a next step, we prove that Υ_1 is a contraction. As A_1 is continuous, so is Υ_1. By letting $\omega, \overline{\omega} \in B_Q$, from (14) and (M0), (M1)(ii), we obtain

$$
\|(\Upsilon_1\omega)(\beta) - (\Upsilon_1\overline{\omega})(\beta)\| \leq \mathcal{I}_{A_1}\|\omega - \overline{\omega}\|_{\mathcal{C}}.
$$

Therefore, for $\beta \in [0, \chi]$ and $\omega \in B_Q$, the researchers arrived at

$$
\|(\Upsilon_1\omega) - (\Upsilon_1\overline{\omega})\|_{\mathcal{C}} \leq \mathcal{I}_{A_1}\|\omega - \overline{\omega}\|_{\mathcal{C}}.
$$

Hence, Υ_1 is a contraction. The continuous operator Υ_2 has been deduced from the fact that the function A_2 is continuous. Moreover, Υ_2 is uniformly bounded on B_Q as

$$
\begin{aligned}
\|(\Upsilon_2\omega)(\beta)\| &\leq \frac{\mathcal{M}}{\rho}e^{\frac{-\mu}{\rho}\beta}\int_0^\beta e^{\frac{\mu}{\rho}\sigma}\|A_2(\sigma, \omega(\sigma))\|d\sigma \\
&\leq \frac{\mathcal{M}}{\mu}(\widehat{\mathcal{I}}_{A_2} + \widetilde{\mathcal{I}}_{A_2}Q) = \overline{A},
\end{aligned}
$$

which suggests that $\|\Upsilon_2\omega\|_{\mathcal{C}} \leq \overline{A}$. So, the value of Υ_2 is uniformly bounded, but it is still necessary to demonstrate that Υ_2 is equi-continuous to show that the operator is compact. Now, we find that for every κ_1, κ_2 in $[0, \chi]$ with κ_1, κ_2, and ω in B_Q

$$
\begin{aligned}
&\|(\Upsilon_2\omega)(\kappa_2) - (\Upsilon_2\omega)(\kappa_1)\| \\
&= \left\|\frac{1}{\rho}e^{\frac{-\mu}{\rho}\kappa_2}\int_{\kappa_1}^{\kappa_2} e^{\frac{\mu}{\rho}\sigma}W\left(\frac{\kappa_2 - \sigma}{\rho}\right)A_2(\sigma, \omega(\sigma))d\sigma\right\| \\
&\quad + \left\|\frac{1}{\rho}e^{\frac{-\mu}{\rho}\kappa_1}\int_0^{\kappa_1} e^{\frac{\mu}{\rho}\sigma}\left[W\left(\frac{\kappa_2 - \sigma}{\rho}\right) - W\left(\frac{\kappa_1 - \sigma}{\rho}\right)\right]A_2(\sigma, \omega(\sigma))d\sigma\right\| \\
&= I_1 + I_2.
\end{aligned} \tag{16}
$$

By utilising (M0) and (M1)(i), we obtain

$$
I_1 = \frac{\mathcal{M}(\widehat{\mathcal{I}}_{A_2} + \widetilde{\mathcal{I}}_{A_2}Q)}{\mu}[1 - e^{\frac{-\mu}{\rho}(\kappa_2 - \kappa_1)}].
$$

For $\kappa_1 = 0$, it is simple to see that $I_2 = 0$. If $\kappa_1 > 0$ and $\epsilon > 0$ are small enough, we have

$$
\begin{aligned}
I_2 \leq &\left\|\frac{1}{\rho}e^{\frac{-\mu}{\rho}\kappa_1}\int_0^{\kappa_1 - \epsilon} e^{\frac{\mu}{\rho}\sigma}\left[W\left(\frac{\kappa_2 - \sigma}{\rho}\right) - W\left(\frac{\kappa_1 - \sigma}{\rho}\right)\right]A_2(\sigma, \omega(\sigma))d\sigma\right\| \\
&+ \left\|\frac{1}{\rho}e^{\frac{-\mu}{\rho}\kappa_1}\int_{\kappa_1 - \epsilon}^{\kappa_1} e^{\frac{\mu}{\rho}\sigma}\left[W\left(\frac{\kappa_2 - \sigma}{\rho}\right) - W\left(\frac{\kappa_1 - \sigma}{\rho}\right)\right]F(\sigma, \omega(\sigma))d\sigma\right\|
\end{aligned}
$$

$$\leq \frac{\widehat{\mathcal{I}}_{A_2} + \widetilde{\mathcal{I}}_{A_2} Q}{\rho} e^{\frac{-\mu}{\rho}\kappa_1} \int_0^{\kappa_1-\epsilon} e^{\frac{\mu}{\rho}\sigma} d\sigma \sup_{\sigma\in[0,\kappa_1-\epsilon]} \left\| W\left(\frac{\kappa_2-\sigma}{\rho}\right) - W\left(\frac{\kappa_1-\sigma}{\rho}\right) \right\|$$
$$+ \frac{2\mathcal{M}(\widehat{\mathcal{I}}_{A_2} + \widetilde{\mathcal{I}}_{A_2} Q)}{\rho} e^{\frac{-\mu}{\rho}\kappa_1} \int_{\kappa_1-\epsilon}^{\kappa_1} e^{\frac{\mu}{\rho}\sigma} d\sigma.$$

Because of the operator's compactness $W(\beta), \beta > 0$, the fact that $I_i \to 0 (i = 1,2)$ as $\kappa_2 \to \kappa_1, \epsilon \to 0$. Consequently, $\|(\mathrm{Y}_2\omega)(\kappa_2) - (\mathrm{Y}_2\omega)(\kappa_1)\| \to 0$ as $\kappa_2 \to \kappa_1$. Thus, Y_2 is equi-continuous. Because $\mathrm{Y}_2(\mathbb{X}) \subset \mathbb{X}$, Y has at least one fixed point, according to Arzela-Ascoli's theorem, and Y_2 is considered to be compact, then the problem obtained through the associated system has at least one solution. □

3.2. Approximate Controllability

$\mathbb{X}$ is taken to be a Hilbert space for the sake of this subsection. In this subsection, we define and provide the requirements for the ACFNDE in (3) and (4). First, we define the mild solution to systems (3) and (4).

Definition 4. *A function $\omega \in \mathcal{C}$ is termed a mild solution to (3) and (4) if for any $v \in L^2([0,\chi], \mathbb{X})$ and*

$$\begin{aligned}\omega(\beta) = {} & e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0,\omega_0)] + A_1(\beta, \omega(\beta)) \\ & + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta-\sigma}{\rho}\right)[A_2(\sigma,\omega(\sigma)) + Bv(\sigma)]d\sigma, \quad \beta \in [0,\chi],\end{aligned} \tag{17}$$

provided the integral exists.

Let $\omega_\chi(\omega_0, v)$ be the state value of (3) and (4) at terminal time χ, corresponding to the control v and the initial value ω_0. We present the set $\mathcal{R}(\chi,\omega_0) = \{\omega_\chi(\omega_0, v) : v \in L^2([0,\chi], V)\}$, which is called the reachable set of the model in (3) and (4) at terminal time χ, and its closure in $\mathbb{X}$ is described by $\overline{\mathcal{R}(\chi,\omega_0)}$.

Definition 5 (see [20])**.** *Given any $\epsilon > 0$, it is possible to steer from the point ω_0 to within a distance ϵ from all points in the state space $\mathbb{X}$ at time χ, and systems (3) and (4) are considered approximately controllable on $[0,\chi]$ if $\overline{\mathcal{R}(\chi,\omega_0)} = \mathbb{X}$.*

Let us take the subsequent linear fractional differential model corresponding to (3) and (4)

$$\begin{aligned} {}^{\mathcal{DD}}D^\rho \omega(\beta) &= Q\omega(\beta) + Bv(\beta), \quad \beta \in [0,\chi], \\ \omega(0) &= \omega_0. \end{aligned} \tag{18}$$

Definition 6 (see [29])**.**

(a) A controllability map for system (18) on $[0,\chi]$ is a bounded linear map $\mathfrak{B}^\chi$: $L^2([0,\chi], V) \to \mathbb{X}$, which is defined as

$$\mathfrak{B}^\chi v := \frac{1}{\rho} e^{\frac{-\mu}{\rho}\chi} \int_0^\chi e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\chi-\sigma}{\rho}\right) Bv(\sigma)d\sigma.$$

(b) System (18) is called AC on $[0,\chi]$ if $\overline{\mathrm{ran}\,\mathfrak{B}^\chi} = \mathbb{X}$.

(c) The controllability gramian of (18) on $[0,\chi]$ is defined as $\Gamma_0^\chi = \mathfrak{B}^\chi(\mathfrak{B}^\chi)^*$.

Lemma 2 (see [29,33])**.** *The model (18) is AC on $[0,\chi]$ if and only if $\theta R(\theta, \Gamma_0^\chi) \to 0$ as $\theta \to 0^+$ in strong operator topology, where $R(\theta, \Gamma_0^\chi) = (\theta I + \Gamma_0^\chi)^{-1}$.*

We refer the reader to [20–22,29,33] for background information and more details on approximate controllability.

We create the following conditions to evaluate the approximate controllability of (3) and (4):

$(M0)^{*}$ *Q is the infinitesimal generator of a C_0 semigroup of BLOs $\{W(\beta)\}_{\beta \geq 0}$ on $\mathbb{X}$, and $W(\beta)(\beta > 0)$ is compact. Further, denote $\mathcal{M} = \sup_{\beta \in [0,\chi]} \|W(\beta)\|_{L(\mathbb{X})}$, where $L(\mathbb{X})$ represents the Banach space of all linear and bounded operators on $\mathbb{X}$ and $\mathcal{M} \geq 1$.*

(M3) For each $\beta \in [0,\chi]$, the function $A_2(\beta,\cdot) : \mathbb{X} \to \mathbb{X}$ is continuous, and for all $u \in \mathbb{X}$, the function $A_2(\cdot,u) : [0,\chi] \to \mathbb{X}$ is Lebesgue measurable.

(M4) There exists a constant $\rho_1 \in (0,\rho)$ and a function $\widetilde{\mathcal{I}}_{A_2} \in L^{\frac{1}{\rho_1}}([0,\chi],\mathbb{R}^+)$ in such a way that

$$\|A_2(\beta,u)\| \leq \widetilde{\mathcal{I}}_{A_2}(\beta), \quad \forall u \in \mathbb{X}; \beta \in [0,\chi].$$

(M5) The linear control system (18) is AC on $[0,\chi]$.

We notate things as follows for simplicity:
In B_Q, for each finite constant $Q > 0$, we denote

$$\mathcal{I}_B = \|B\|, \mathcal{I}_{A_2} = \|\widetilde{\mathcal{I}}_{A_2}\|_{L^{\frac{1}{\rho_1}}([0,\chi],\mathbb{R}^+)}, \quad t = \frac{1}{1-\rho_1}, \text{ and } \quad \widetilde{N} = \frac{1}{\rho}\left(\frac{\rho}{t\mu}\right)^{\frac{1}{t}} \mathcal{I}_{A_2}.$$

The feedback control function for systems (3) and (4) is now selected as follows:

$$v(\beta) = v_\theta(\beta,\omega) = B^* T^*\left(\frac{\chi-\beta}{\rho}\right) R\big(\theta,\Gamma_0^\chi\big) q(\omega),$$

where

$$\begin{aligned} q(\omega) &= \omega_\chi - e^{\frac{-t}{\rho}\chi} W\left(\frac{\chi}{\rho}\right)[\omega_0 - A_1(0,\omega_0)] - A_1(\chi,\omega(\chi)) \\ &\quad - \frac{1}{\rho} e^{\frac{-\mu}{\rho}\chi} \int_0^\chi e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\chi-\sigma}{\rho}\right) A_2(\sigma,\omega(\sigma)) d\sigma. \end{aligned} \tag{19}$$

For any $\theta > 0$, we describe the operator $\Upsilon_\theta : \mathcal{C}([0,\chi],\mathbb{X}) \to \mathcal{C}([0,\chi],\mathbb{X})$ as follows:

$$\begin{aligned} (\Upsilon_\theta \omega)(\beta) &= e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0,\omega_0)] + A_1(\chi,\omega(\chi)) \\ &\quad + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta-\sigma}{\rho}\right)[A_2(\sigma,\omega(\sigma)) + Bv_\theta(\sigma,\omega)] d\sigma. \end{aligned} \tag{20}$$

Lemma 3. *If assumptions $(M0)^*$-(M4) hold, for any $\beta \in [0,\chi]$, we obtain:*

(i) $\frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} \left\| W\left(\frac{\beta-\sigma}{\rho}\right) A_2(\sigma,\omega(\sigma)) \right\| d\sigma \leq \mathcal{M}\widetilde{N}.$

(ii) $\|v_\theta(\beta,\omega)\| \leq \frac{\mathcal{M}\mathcal{I}_B}{\theta}\Big[\|\omega_\chi\| + \mathcal{M}[\|\omega_0\| + \|A_1(0,\omega_0)\| + \widetilde{N}] + \widehat{\mathcal{I}}_{A_1} + \widetilde{\mathcal{I}}_{A_1} Q\Big].$

Proof. (i) Given Holder's inequality and (Q3), we obtain

$$\begin{aligned} &\frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} \left\| W\left(\frac{\beta-\sigma}{\rho}\right) A_2(\sigma,\omega(\sigma)) \right\| d\sigma \\ &\leq \frac{\mathcal{M}}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} \widetilde{\mathcal{I}}_{A_2}(\sigma) d\sigma \\ &\leq \frac{\mathcal{M}}{\rho} e^{\frac{-\mu}{\rho}\beta} \left(\int_0^\beta \left(e^{\frac{\mu}{\rho}\sigma}\right)^{\frac{1}{1-\rho_1}} d\sigma \right)^{1-\rho_1} \left(\int_0^\beta \left(\widetilde{\mathcal{I}}_{A_2}(\sigma)\right)^{\frac{1}{\rho_1}} d\sigma \right)^{\rho_1} \\ &\leq \frac{\mathcal{M}}{\rho} e^{\frac{-\mu}{\rho}\beta} \left(\int_0^\beta e^{\frac{t\mu}{\rho}\sigma} d\sigma \right)^{\frac{1}{t}} \|\widetilde{\mathcal{I}}_{A_2}\|_{L^{\frac{1}{\rho_1}}([0,\chi],\mathbb{R}^+)} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\rho}\left(\frac{\rho}{t\mu}\right)^{\frac{1}{t}}\mathcal{I}_{A_2}\mathcal{M}e^{\frac{-\mu}{\rho}\beta}(e^{\frac{\mu}{\rho}\beta}-1)\\
&\leq \mathcal{M}\widetilde{N}.
\end{aligned}
$$

(ii) From (19), (20), and (i), we have

$$
\begin{aligned}
\|v_\theta(\beta,\omega)\| &\leq \left\|B^*T^*\left(\frac{\chi-\beta}{\rho}\right)R(\theta,\Gamma_0^\chi)q(\omega)\right\|\\
&\leq \frac{\mathcal{M}\mathcal{I}_B}{\theta}\|q(\omega)\|\\
&\leq \frac{\mathcal{M}\mathcal{I}_B}{\theta}\Big[\|\omega_\chi\|+\mathcal{M}[\|\omega_0\|+\|A_1(0,\omega_0)\|]+\widehat{\mathcal{I}}_{A_1}+\widetilde{\mathcal{I}}_{A_1}Q+\mathcal{M}\widetilde{N}\Big]\\
&\leq \frac{\mathcal{M}\mathcal{I}_B}{\theta}\Big[\|\omega_\chi\|+\mathcal{M}[\|\omega_0\|+\|A_1(0,\omega_0)\|+\widetilde{N}]+\widehat{\mathcal{I}}_{A_1}+\widetilde{\mathcal{I}}_{A_1}Q\Big].
\end{aligned}
$$

□

Theorem 7. *If hypotheses* $(M0)^*$*-(M3) are true, a mild solution exists for the FN control systems (3) and (4).*

Proof. We must demonstrate that Υ_θ has a fixed point to show that the FN control systems ((3) and (4)) have a mild solution. The proof is divided into the following stages for simplicity:

Step 1: For any $\theta>0$, we can find a constant $\Lambda=\Lambda(\theta)>0$ in such a way that $\Upsilon_\theta(B_\Lambda)\subset B_\Lambda$. For any positive integer $Q>0$ and $\omega\in B_Q$, if $\beta\in[0,\chi]$, then from Lemma 3, we obtain

$$
\begin{aligned}
\|(\Upsilon_\theta\omega)(\beta)\| = {}& e^{\frac{-\mu}{\rho}\beta}\|W\left(\frac{\beta}{\rho}\right)[\omega_0-A_1(0,\omega_0)]\|+\|A_1(\beta,\omega(\beta))\|\\
&+\frac{1}{\rho}e^{\frac{-\mu}{\rho}\beta}\int_0^\beta e^{\frac{\mu}{\rho}\sigma}\|W\left(\frac{\beta-\sigma}{\rho}\right)[A_2(\sigma,\omega(\sigma))+Bv_\theta(\sigma,\omega)]\|d\sigma
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathcal{M}[\|\omega_0\|+||A_1(0,\omega_0)\|]+\widehat{\mathcal{I}}_{A_1}+\widetilde{\mathcal{I}}_{A_1}Q+\frac{1}{\rho}e^{\frac{-\mu}{\rho}\beta}\int_0^\beta e^{\frac{\mu}{\rho}\sigma}\left\|W\left(\frac{\beta-\sigma}{\rho}\right)A_2(\sigma,\omega(\sigma))\right\|d\sigma\\
&\quad+\frac{1}{\rho}e^{\frac{-\mu}{\rho}\beta}\int_0^\beta e^{\frac{\mu}{\rho}\sigma}\left\|W\left(\frac{\beta-\sigma}{\rho}\right)Bv_\theta(\sigma,\omega)\right\|d\sigma\\
&\leq \mathcal{M}[\|\omega_0\|+||A_1(0,\omega_0)\|]+\widehat{\mathcal{I}}_{A_1}+\widetilde{\mathcal{I}}_{A_1}Q+\mathcal{M}\widetilde{N}\\
&\quad+\frac{1}{\mu}\mathcal{M}\mathcal{I}_Be^{\frac{-\mu}{\rho}\beta}(e^{\frac{\mu}{\rho}\beta}-1)||v_\theta(\beta,\omega)\|\\
&\leq \mathcal{M}[\|\omega_0\|+||A_1(0,\omega_0)\|+\widetilde{N}]+\widehat{\mathcal{I}}_{A_1}+\widetilde{\mathcal{I}}_{A_1}Q\\
&\quad+\frac{\mathcal{M}\mathcal{I}_B}{\mu}\cdot\frac{\mathcal{M}\mathcal{I}_B}{\theta}\Big[\|\omega_\beta\|+\mathcal{M}[\|\omega_0\|+\|A_1(0,\omega_0)\|+\widetilde{N}]+\widehat{\mathcal{I}}_{A_1}+\widetilde{\mathcal{I}}_{A_1}Q\Big]\\
&\leq \left(1+\frac{\mathcal{M}^2\mathcal{I}_B^2}{\mu\theta}\right)[\mathcal{M}[\|\omega_0\|+||A_1(0,\omega_0)\|+\widetilde{N}]+\widehat{\mathcal{I}}_{A_1}+\widetilde{\mathcal{I}}_{A_1}Q]+\frac{\mathcal{M}^2\mathcal{I}_B^2}{\mu\theta}\|\omega_\beta\|.
\end{aligned}
$$

For this, we conclude that for large $\Lambda>0, \Upsilon_\theta(B_\Lambda)\subset B_\Lambda$ holds.

Step 2: For any $\beta\in[0,\chi]$, the set $\{(\Upsilon_\theta\omega)(\beta):\omega\in B_\Lambda\}$ is compact in $\mathbb{X}$. If $\beta=0$, the set is obviously compact in $\mathbb{X}$. Let $0<\beta\leq\chi$ be fixed and let ϵ fulfil $0<\epsilon<\beta$. For $\omega\in B_\Lambda$, we describe

$$
\begin{aligned}
(\Upsilon_\theta^\epsilon \omega)(\beta) &= e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0,\omega_0)] + A_1(\beta,\omega(\beta)) \\
&\quad + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^{\beta-\epsilon} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta-\sigma}{\rho}\right)[A_2(\sigma,\omega(\sigma)) + Bv_\theta(\sigma,\omega)]d\sigma \\
&= e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\beta}{\rho}\right)[\omega_0 - A_1(0,\omega_0)] + A_1(\beta,\omega(\beta)) \\
&\quad + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} W\left(\frac{\epsilon}{\rho}\right) \int_0^{\beta-\epsilon} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta-\sigma-\epsilon}{\rho}\right)[A_2(\sigma,\omega(\sigma)) + Bv_\theta(\sigma,\omega)]d\sigma.
\end{aligned}
$$

Since $W(\beta)$ is compact for $\beta > 0$, we find that the set $\{(\Upsilon_\theta\omega)(\beta) : \omega \in B_\Lambda\}$ is relatively compact in $\mathbb{X}$. Moreover, from Lemma 3, we obtain

$$
\begin{aligned}
||(\Upsilon_\theta\omega)(\beta) - (\Upsilon_\theta^\epsilon\omega)(\beta)\| &\le \left\| \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_{\beta-\epsilon}^{\beta} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta-\sigma}{\rho}\right)[A_2(\sigma,\omega(\sigma)) + Bv_\theta(\sigma,\omega)]d\sigma \right\| \\
&\to 0 \quad \text{as} \quad \epsilon \to 0.
\end{aligned}
$$

As a result, the set $\{(\Upsilon_\theta\omega)(\beta) : \omega \in B_\Lambda\}$, $\beta \in (0,\chi]$ is relatively compact in $\mathbb{X}$.
Step 3: A family of functions $\{\Upsilon_\theta\omega : \omega \in B_\lambda\}$ is equi-continuous on $[0,\chi]$.
Let $0 \le \kappa_1 < \kappa_2 \le \chi$, and for any $\omega \in B_\lambda$, we have

$$
\begin{aligned}
\|(\Upsilon_\theta\omega)(\kappa_2) - (\Upsilon_\theta p)(\kappa_1)\| &\le \left\| e^{\frac{-\mu}{\rho}\kappa_1}\left[W\left(\frac{\kappa_2}{\rho}\right) - W\left(\frac{\kappa_1}{\rho}\right)\right][\omega_0 - A_1(0,\omega_0)] \right\| \\
&\quad + \|A_1(\kappa_2,\omega(\kappa_2)) - A_1(\kappa_1,\omega(\kappa_1))\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left\| \frac{1}{\rho} e^{\frac{-\mu}{\rho}\kappa_2} \int_{\kappa_1}^{\kappa_2} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\kappa_2-\sigma}{\rho}\right) A_2(\sigma,\omega(\sigma))d\sigma \right\| \\
&+ \left\| \frac{1}{\rho} e^{\frac{-\mu}{\rho}\kappa_2} \int_{\kappa_1}^{\kappa_2} e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\kappa_2-\sigma}{\rho}\right) Bv_\theta(\sigma,\omega)d\sigma \right\| \\
&+ \left\| \frac{1}{\rho} e^{\frac{-\mu}{\rho}\kappa_1} \int_0^{\kappa_1} e^{\frac{\mu}{\rho}\sigma} \left[W\left(\frac{\kappa_2-\sigma}{\rho}\right) - W\left(\frac{\kappa_1-\sigma}{\rho}\right)\right] A_2(\sigma,\omega(\sigma))d\sigma \right\| \\
&+ \left\| \frac{1}{\rho} e^{\frac{-\mu}{\rho}\kappa_1} \int_0^{\kappa_1} e^{\frac{\mu}{\rho}\sigma} \left[W\left(\frac{\kappa_2-\sigma}{\rho}\right) - W\left(\frac{\kappa_1-\sigma}{\rho}\right)\right] Bv_\theta(\sigma,\omega)d\sigma \right\| \\
&\le \sum_{i=1}^{6} I_i.
\end{aligned}
$$

Given Lemma 3 and Theorem 6, we have

$$
\begin{aligned}
I_1 &\le \left\| W\left(\frac{\kappa_2}{\rho}\right) - W\left(\frac{\kappa_1}{\rho}\right) \right\| [||\omega_0|| + \|A_1(0,\omega_0)||] \\
I_1 &\le \|A_1(\kappa_2,\omega(\kappa_2)) - A_1(\kappa_1,\omega(\kappa_1))\| \\
I_3 &\le \frac{\mathcal{M}\mathcal{I}_{A_2}}{\rho}\left(\frac{\rho}{t\mu}\right)^{\frac{1}{t}}\left[e^{\frac{t\mu}{\rho}\kappa_2} - e^{\frac{t\mu}{\rho}\kappa_1}\right]^{\frac{1}{t}} \\
I_4 &\le \frac{\mathcal{M}^2\mathcal{I}_B^2}{\mu\theta}[\mathcal{M}[\|\omega_0\| + ||A_1(0,\omega_0)\| + \widetilde{N}] + \widehat{\mathcal{I}}_{A_1} + \widetilde{\mathcal{I}}_{A_1}Q]\left[e^{\frac{\mu}{\rho}\kappa_2} - e^{\frac{\mu}{\rho}\kappa_1}\right].
\end{aligned}
$$

Given that the operator $W(\beta)$ is compact and $\beta > 0$, we see that $I_i \to 0$ $(i = 1,2,3,4,5,6)$ as $\kappa_2 \to \kappa_1, \epsilon \to 0$. Consequently, $\|(\Upsilon_2\omega)(\kappa_2) - (\Upsilon_2\omega)(\kappa_1)\| \to 0$ as $\kappa_2 \to \kappa_1$. Thus, Υ_2 is equi-continuous. For the same reason that $\Upsilon_2(\mathbb{X}) \subset \mathbb{X}$, Υ_2 is considered to be compact. Hence, according to Schauder's fixed-point theorem (Theorem 2.8, [29]), we conclude that the operator Υ_θ has a fixed point, which is a mild solution of the model in (3) and (4). □

Theorem 8. *Suppose that* $(M0)^*$*-(M4) are true. Furthermore, if the function* A_2 *is uniformly bounded by the positive constant* C*, then the FN model in (3) and (4) is AC on* $[0, \chi]$*.*

Proof. Let ω_θ be a fixed point of Υ_θ in B_Λ. Any fixed point of Υ_θ is a mild solution of the system in (3) and (4) under the control

$$v_\theta(\beta, \omega_\theta) = B^* T^* \left(\frac{\chi - \beta}{\rho}\right) R(\theta, \Gamma_0^\chi) q(\omega_\theta),$$

where

$$\begin{aligned} q(\omega_\theta) = \omega_\chi - e^{\frac{-\mu}{\rho}\chi} W\left(\frac{\chi}{\rho}\right)[\omega_0 - A_1(0, \omega_0)] - A_1(\beta, \omega_\theta(\beta)) \\ - \frac{1}{\rho} e^{\frac{-\mu}{\rho}\chi} \int_0^\chi e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\chi - \sigma}{\rho}\right) A_2(\sigma, \omega_\theta(\sigma)) d\sigma, \end{aligned}$$

and fulfils the subsequent inequality

$$\begin{aligned} \omega_\theta(\chi) &= e^{\frac{-\mu}{\rho}\chi} W\left(\frac{\chi}{\rho}\right)[\omega_0 - A_1(0, \omega_0)] + A_1(\beta, \omega_\theta(\beta)) \\ &\quad + \frac{1}{\rho} e^{\frac{-\mu}{\rho}\chi} \int_0^\chi e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\chi - \sigma}{\rho}\right)[A_2(\sigma, \omega_\theta(\sigma)) + Bv_\theta(\sigma, \omega_\theta)] d\sigma \\ &= \omega_\chi - q(\omega_\theta) + R(\theta, \Gamma_0^\chi) q(\omega_\theta) \\ &= \omega_\chi - \theta R(\theta, \Gamma_0^\chi) q(\omega_\theta). \end{aligned} \tag{21}$$

Since A_2 is uniformly bounded, we have

$$\int_0^\chi \|A_2(\sigma, \omega_\theta(\sigma))\|^2 d\sigma \leq C^2 \chi.$$

Thus, the sequence $A_2(\cdot, \omega_\theta(\cdot))$ is bounded in $L^2([0, \chi], \mathbb{X})$. Then, we can find a subsequence of $\{A_2(\cdot, \omega_\theta(\cdot) : \theta > 0\}$, still signified by it, that converges weakly to some $A_2(\sigma) \in L^2([0, \chi], \mathbb{X})$. Denote

$$v = \omega_\chi - e^{\frac{-\mu}{\rho}\chi} W\left(\frac{\chi}{\rho}\right)[\omega_0 - A_1(0, \omega_0)] - A_1(\beta, \omega_\theta(\beta)) - \frac{1}{\rho} e^{\frac{-\mu}{\rho}\chi} \int_0^\chi e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\chi - \sigma}{\rho}\right) A_2(\sigma) d\sigma.$$

It is proposed that

$$\begin{aligned} \|q(\omega_\theta) - v\| &= \left\| \frac{1}{\rho} e^{\frac{-\mu}{\rho}\chi} \int_0^\chi e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\chi - \sigma}{\rho}\right)[A_2(\sigma, \omega_\theta(\sigma)) - A_2(\sigma)] d\sigma \right\| \\ &\leq \sup_{\beta \in [0, \chi]} \left\| \frac{1}{\rho} e^{\frac{-\mu}{\rho}\beta} \int_0^\beta e^{\frac{\mu}{\rho}\sigma} W\left(\frac{\beta - \sigma}{\rho}\right)[A_2(\sigma, \omega_\theta(\sigma)) - A_2(\sigma)] d\sigma \right\|. \end{aligned}$$

Consequently,

$$\|q(\omega_\theta) - v\| \to 0 \quad \text{as} \quad \theta \to 0^+.$$

Then, from the above discussion, we have

$$\begin{aligned} ||\omega_\theta(\chi) - \omega_\chi\| &\leq \|\theta R(\theta, \Gamma_0^\chi) q(\omega_\theta)\| \\ &\leq \|\theta R(\theta, \Gamma_0^\chi) v\| + \|\theta R(\theta, \Gamma_0^\chi)\| \|q(\omega_\theta) - v\| \\ &\leq \|\theta R(\theta, \Gamma_0^\chi) v\| + \|q(\omega_\theta) - v\| \to 0 \quad \text{as} \quad \theta \to 0^+. \end{aligned}$$

Consequently, this proves the approximate controllability of the model in (3) and (4). □

4. Applications

Example 1. *Consider the following deformable FN partial differential equation of the form*

$$
{}^{\mathcal{DD}}D^{\frac{1}{2}}\left[\omega(\beta,z) - \frac{e^{-\beta}}{25+e^{\beta}}\cdot\frac{|\omega(\beta,z)|}{(1+|\omega(\beta,z)|)}\right] = \frac{\partial^2}{\partial z^2}\omega(\beta,z) + \frac{e^{-\beta}|\omega(\beta,z)|}{(16+e^{\beta})(1+|\omega(\beta,z)|)},
$$

$$\beta\in(0,1),\ z\in(0,1); \tag{22}$$

$$\omega(\beta,0)=\omega(\beta,1)=0,\quad 0\le\beta\le 1; \tag{23}$$

$$\omega(0,z)=\omega_0(z),\quad 0\le z\le 1; \tag{24}$$

where $\mathbb{X}=L^2[0,1], \omega_0(z)\in\mathbb{X}$.
Define $Q\omega=\omega''$ *with* $D(Q)=\{\omega\in\mathbb{X}:\omega,\omega'$ *are absolutely continuous and* $\omega''\in\mathbb{X}$, $\omega(0)=\omega(1)=0\}$.

Then,

$$Q\omega=\sum_{n=1}^{\infty}-n^2\langle\omega,\Theta_n\rangle\Theta_n,\quad \omega\in D(Q),$$

where $\Theta_n(z)=\sqrt{\frac{2}{\pi}}\sin(nz), 0\leqslant z\leqslant 1, n=1,2,\ldots$. *It is well-known that* Q *generates a* C_0 *semigroup* $W(\beta)(\beta\geqslant 0)$ *on* $\mathbb{X}$, *which is given by*

$$W(\beta)\omega=\sum_{n=1}^{\infty}\Theta^{-n^2\beta}\langle\omega,\Theta_n\rangle\Theta_n,\ \omega\in\mathbb{X}, \tag{25}$$

with $\|W(\beta)\|\leqslant 1$, *for any* $\beta\geqslant 0$. *Put* $\omega(\beta)=\omega(\beta,\cdot)$, *that is,* $\omega(\beta)(z)=\omega(\beta,z),\beta$, $z\in[0,1]$, *and*

$$
\begin{aligned}
A_2(\beta,\omega(\beta)) &= \frac{e^{-\beta}}{(16+e^{\beta})}\cdot\frac{|\omega(\beta,\cdot)|}{1+|\omega(\beta,\cdot)|};\\
A_1(\beta,\omega(\beta)) &= \frac{e^{-\beta}}{(25+e^{\beta})}\cdot\frac{|\omega(\beta,\cdot)|}{1+|\omega(\beta,\cdot)|}.
\end{aligned}
$$

Then, let $x,\overline{x}\in[0,\infty)$ *and* $\beta\in[0,1]$. *Then, there is*

$$
\begin{aligned}
\|A_2(\beta,x)-A_2(\beta,\overline{x})\| &\le \frac{e^{-\beta}}{(16+e^{\beta})}\left\|\frac{x}{1+x}-\frac{\overline{y}}{1+\overline{y}}\right\|\\
&\le \frac{1}{17}\|x-\overline{x}\|,
\end{aligned}
$$

$$
\begin{aligned}
\|A_1(\beta,x)-A_1(\beta,\overline{y})\| &\le \frac{e^{-\beta}}{(25+e^{\beta})}\left\|\frac{x}{1+x}-\frac{\overline{y}}{1+\overline{y}}\right\|\\
&\le \frac{1}{26}\|x-\overline{y}\|.
\end{aligned}
$$

Assumption (M1) holds, with $\mathcal{I}_{A_2}=\frac{1}{17}$ *and* $\mathcal{I}_{A_1}=\frac{1}{26}$. *Since* $\rho=\frac{1}{2}$ *and* $\rho+\mu=1$, *we have* $\mu=\frac{1}{2}$.

If $\mathcal{M}=2$, *we have*

$$
\begin{aligned}
\Lambda &= \left[\mathcal{I}_{A_1}+\frac{\mathcal{M}}{\mu}\mathcal{I}_{A_2}\right]\\
&= \frac{1}{26}+\frac{2}{0.5}\cdot\frac{1}{17}\\
&= 0.2737.
\end{aligned}
$$

Therefore, (12) *holds, where* $\Lambda = 0.2737 < 1$. *According to Theorem* 5*, the given problem in* (22)–(24) *has a unique solution in the interval* $[0,1]$.

Example 2. *Consider the following subsequent FN system using deformable derivatives of the form*

$$^{\mathcal{DD}}D^{\frac{1}{2}}\left[\omega(\beta,z) - \left(\frac{\beta^2}{4} + \frac{\sin\beta}{64}|\omega(\beta,z)|\right)\right] = \frac{\partial^2}{\partial z^2}\omega(\beta,z) + \frac{\beta}{4} + \frac{\sin\beta}{36}|\omega(\beta,z)|,$$
$$z \in (0,1) \quad and \quad \beta \in [0,1]. \tag{26}$$

Using the preliminaries in Example 1*, we set*

$$A_2(\beta,\omega(\beta)) = \frac{\beta}{4} + \frac{\sin\beta}{36}|\omega(\beta,\cdot)|;$$
$$A_1(\beta,\omega(\beta)) = \frac{\beta^2}{4} + \frac{\sin\beta}{64}|\omega(\beta,\cdot)|.$$

Then,

let $x,\overline{y} \in [0,\infty)$ *and* $\beta \in [0,1]$. *Then, we have*

$$\begin{aligned}\|A_2(\beta,x) - A_2(\beta,\overline{y})\| &\leq \left\|\frac{\beta}{4} + \frac{\sin\beta}{36}x - \frac{\beta}{4} - \frac{\sin\beta}{36}\overline{y}\right\| \\ &\leq \frac{1}{36}\|x - \overline{y}\|.\end{aligned}$$

and

$$\begin{aligned}\|A_1(\beta,x) - A_1(\beta,\overline{y})\| &\leq \left\|\frac{\beta^2}{4} + \frac{\sin\beta}{64}x - \frac{\beta^2}{4} - \frac{\sin\beta}{64}\overline{y}\right\| \\ &\leq \frac{1}{64}\|x - \overline{y}\|.\end{aligned}$$

For all $\beta \in [0,1]$ *and* $x \in [0,\infty)$*, we have*

$$\begin{aligned}\|A_2(\beta,x)\| &\leq \left\|\frac{\beta}{4} + \frac{\sin\beta}{36}x\right\| \\ &\leq \left\|\frac{\beta}{4}\right\| + \frac{\sin\beta}{36}\|x\| \\ &\leq \frac{1}{4} + \frac{1}{36}\|x\|.\end{aligned}$$

and

$$\begin{aligned}\|A_1(\beta,x)\| &\leq \left\|\frac{\beta^2}{4} + \frac{\sin\beta}{64}x\right\| \\ &\leq \left\|\frac{\beta^2}{4}\right\| + \frac{\sin\beta}{64}\|x\| \\ &\leq \frac{1}{4} + \frac{1}{64}\|x\|.\end{aligned}$$

So, assumptions (M1)(i)(ii) and (M2)(i)(ii) hold with $\mathcal{I}_{A_2} = \frac{1}{36}, \mathcal{I}_{A_1} = \frac{1}{64}, \widehat{\mathcal{I}}_{A_2} = \frac{1}{4}, \widehat{\mathcal{I}}_{A_1} = \frac{1}{4}, \widetilde{\mathcal{I}}_{A_2} = \frac{1}{36}, \widetilde{\mathcal{I}}_{A_1} = \frac{1}{64}$. *Since* $\rho = \frac{1}{2}$ *and* $\rho + \mu = 1$*, we have* $\mu = \frac{1}{2}$. *According to Theorem* 6*, the given FN system* (26) *with conditions* (23) *and* (24) *has at least one solution in* $[0,1]$.

Example 3. *Consider the following FN system using deformable derivatives of the form*

$$
{}^{\mathcal{DD}}D^{\frac{1}{2}}\left[\omega(\beta,z)-\left(\frac{\beta^2}{4}+\frac{\sin\beta}{64}|\omega(\beta,z)|\right)\right]=\frac{\partial^2}{\partial z^2}\omega(\beta,z)+v(\beta,\omega)+\frac{1}{4}\cdot\frac{e^{-\beta}}{1+e^{\beta}}\cdot\frac{|\omega(\beta,z)|}{(1+|\omega(\beta,z)|)},
$$
$$
\beta\in[0,1], z\in(0,1), \tag{27}
$$

where $\mathbb{X}=V=L^2([0,1])$.

From Example 1, we obtain

$$
A_2(\beta,\omega(\beta))=\frac{1}{4}\cdot\frac{e^{-\beta}}{1+e^{\beta}}\cdot\frac{|\omega(\beta,\cdot)|}{1+|\omega(\beta,\cdot)|};
$$
$$
A_1(\beta,\omega(\beta))=\frac{\beta^2}{4}+\frac{\sin\beta}{64}|\omega(\beta,\cdot)|.
$$

Furthermore, the bounded linear operator $B:V\to\mathbb{X}$ *is defined as* $Bv(\beta)=v(\beta,\cdot)$. *Then, the system (27) with (23) and (24) is transformed into the abstract form of the system in (1) and (2).*

Moreover, assumptions (M3)–(M4) hold with $\widetilde{\mathcal{I}}_{A_2}=\dfrac{e^{-\beta}}{1+e^{\beta}}$ *and* $C=\frac{1}{4}$.

According to Theorem 3.12 [29], the linear system corresponding to (27) with conditions (23) and (24) is AC on $[0,\chi]$ *iff*

$$
B^*T^*\left(\frac{\chi-\beta}{\rho}\right)\omega=0,\ \chi\in[0,\chi]\Longrightarrow x=0. \tag{28}
$$

From (25), we notice that

$$
B^*T^*\left(\frac{\chi-\beta}{\rho}\right)\omega=\sum_{n=1}^{\infty}e^{-n^2\left(\frac{\chi-\beta}{\rho}\right)}\langle\omega,\Theta_n\rangle\Theta_n,\ \omega\in\mathbb{X},\beta\in[0,\chi].
$$

Therefore, condition (28) holds and hence assumption (M5) holds. Thus, according to Theorem 8, system (27) with conditions (23) and (24) is AC on $[0,\chi]$.

5. Conclusions

The concept of a deformable derivative was novel when it was first proposed by F. Zulfeqarr, A. Ujlayan, and P. Ahuja [18]. The limit approach is utilised in this new derivative in the same manner as the traditional derivative. It was given the term "deformable" because it had the inherent capacity to continuously deform functions into derivatives. This idea opens up new avenues for study, allowing one to look at both qualitative and quantitative behaviour in a variety of systems. In this work, we used the novel findings from [18] to address our systems ((1), (2), (3), and (4)). By using the Banach contraction principle and Krasnoselskii's and Schauder's fixed-point theorems, we established Theorems 5 and 6, which demonstrated the existence and uniqueness of the solutions for the addressed systems. We then proved that systems (3) and (4) were approximately controllable in Theorems 7 and 8, and we were able to validate all of the assumptions (M0) and $(M0)^*$-(M5). An appropriate fixed-point theorem might be utilised to develop the usefulness of present research to establish existence and approach controllability with non-instantaneous impulses for a number of different models. The use of the proper fixed-point theorem would make this achievable.

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Abbreviations

The following abbreviations are used in this manuscript:

EaU	existence and uniqueness
ACFNDE	approximate controllability of fractional neutral differential equations
$\mathcal{DD}$	deformable derivative
FDE	fractional differential equation
BLOs	bounded linear operators
FN	fractional neutral
FNDEs	fractional neutral differential equations

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