fractal and fractional

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Topological Structure of Solution Sets of Fractional Control Delay Problem

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Abstract: This paper is concerned with the existence of a mild solution for the fractional delay control system. Firstly, we will study the control problem. Then, we will deal with the topological structure of the solution set consisting of the compactness and R_σ property. We will derive a mild solution to the above delay control problem by using the Laplace transform method.

Keywords: mild solution; control problem; topological structure; Caputo fractional derivative; continuous function; Banach spaces; Laplace transform; controllability; continuous semigroup; probability density function

MSC: 34K37; 34B15

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1. Introduction

Fractional calculus is a generalization of classical calculus, related to the operations of the integration and differentiation of a non-integer (fractional) order. The concept of fractional operators was introduced almost simultaneously with the development of classical ones. A fractional analysis is the branch of mathematical analysis that deals with several different possibilities of defining real number powers or the complex power of a differentiation operator D and of integration J. Fractional-order differential equations are generalized and non-integer differential equations that are achieved in time and space with a power-law memory kernel of non-local relationships [1]. Fractional derivatives (which are used as Riemann–Liouville and Caputo derivatives) are used to model viscoelastic damping in certain types of materials, such as polymers. The Caputo derivative is the most appropriate fractional operator to be used in modeling real-world problems. The concept of fractional calculus was initially presented by Leibniz more than 300 years ago. It is worth mentioning that evolution equations including fractional derivatives in time, in some cases, have better effects in applications than traditional evolution equations of an integer order in time. Hilfer gave the implementation of fractional calculus in physics [2,3].

A control system handles, holds, adjusts, or regulates the behavior of other devices or systems using control loops. It can range from a signal home heating controller using a thermostat controlling a domestic boiler to large industrial control systems which are used for controlling processes or machines. A control system is a set of mechanical or electronic devices that regulate other devices or systems by way of control loops. Control systems are a central part of the industry and of automation. A control system provides the desired response by controlling the output. A control problem involves a system that is discussed by state variables; at each time step, the choice of the value of the control variable applied at a time causes a change in state variables of the system at time-step $\nu + 1$.

Fractional delay differential equations have been applied in various neoteric science and engineering fields. The delayed system has an instantaneous control input signal, but that does not affect the signal characteristics. An ideal delay is a delay system that does not affect the signal characteristics at all and that delays the signal for an exact amount of time. The amount of time by which the arrival of a signal is retarded after transmission through physical equipment or the system is known as the delay time. A delay differential equation is known as an equation with delay, compared with those without delay which are more realistic, suitable to discuss many phenomena in nature, and have many applications. Diekmann gave delay equations a functional, complex, and nonlinear analysis. [4]. Sousa, J. V. D. C. et al. [5,6] worked on the mild solutions of fractional differential equations. Niazi et al. [7], Shafqat et al. [8], Alnahdi [9], Khan [10], and Abuasbeh et al. [11] investigated the existence and uniqueness of the fractional evolution equations.

A general form of a time delay equation for $x(\nu)$ is

$$\frac{d}{d\nu}x(\nu) = j(\nu, x(\nu), x_\nu).$$

In [12], the topological structure (compactness and R_σ-property) of the solution set of the following type of fractional delay control problems was studied.

$$\begin{cases} {}^cD_\nu^r x(\nu) = Nx(\nu) + j(\nu, x_\nu) + Bu(\nu),\ \nu \in [-h, a], \\ x(\nu) = \Psi(\nu), \\ 0 < r < 1, \end{cases}$$

where Y and V are Banach spaces. N is a linear closed operator generating a strongly continuous semigroup $T(\nu)$ on Y. Where $r < r < 1$.

Motivated by the above work, we study the topological structure of the solution set of the following fractional control delay problem:

$$\begin{cases} {}^cD_\nu^r Ex(\nu) = Nx(\nu) + j(\nu, x_{(\nu)} + Gu(\nu)\ \nu \in [0, a], \\ x(\nu) = \psi(\nu),\ \nu \in [-h, 0], \\ x'(\nu) = x_1,\ 1 < r < 2, \end{cases} \tag{1}$$

where E, N, and G are linear closed operators generating a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $Y, h \geq 0, {}^c D_\nu^r, 1 < r < 2$, is the Caputo fractional derivative of order q, the state function x takes values in Y, the control function y takes values in V, B is a bounded linear operator from V to X, $\psi \in C([-h, 0]; X), x_t \in C([-h, 0]; X)$ is defined by $x_t(s) = x(t+s)(s \in [-h, 0])$, and $f : [0, b] \times C([-h, 0]; X) \to X$ is, in general, a nonlinear function to be specified later.

As a result of the aforementioned thought, we are compelled to examine the control problem (1) in this work under relatively benign circumstances. The initial goal is to investigate the R_σ-property and compactness of the solution set's topological structure. It is good to mention several controllability issues have been investigated by demonstrating that the reachability set is invariant under a nonlinear perturbation in many works. The authors of these works made the assumption that the solution was unique, which suggests that the solution set is single-pointed and the Lipschitz continuity on the nonlinearity is involved. As a result, there is a limitation to these results. This inspires our work's second goal. More specifically, using the knowledge of the topological structure, we will demonstrate the invariance of its reachability set under nonlinear perturbations under a more general class of nonlinearities f, which does not ensure the uniqueness of a mild solution for the control problem (1). This specifically implies that the control problem (1) is approximately controllable if the corresponding linear problem is approximately controllable. We stress the fact that the lack of uniqueness precludes us from demonstrating the invariance of a reachability set using well-known methods, such as the Banach and Schauder fixed-point theorems.

The characterization of solution sets containing compactness, acyclicity, and an R_σ property are fruitful in the study of corresponding equations. Bader and Kryszewski proved that the set consisting of all mild solutions for a constrained semilinear differential inclusion is a nonempty, compact R_σ set and gave its applications to the periodic problem and the existence of equilibria [13]. Andres and Pavlackon obtained an existence result for the corresponding semilinear systems from information about the structure and a fixed-point index technique in Frechet spaces [14]. Semilinear differential inclusions in Banach spaces are illustrated in [15]. The problem of exact and approximate controllability is to be distinguished. In general infinite-dimensional spaces, the concept of exact controllability is usually too strong [16]. An explanation of multi-valued differential equations is referred to in (Grossmann, 1994). Mahmudo illustrated the controllability of linear stochastic systems referred to in [17]. Controllability is one of the qualitative properties of a control system that has an important place in control theory. Controllable systems have many applications in different branches of science and engineering. Differential inclusions have been used as a model for controlled systems with discontinuities. Fractional evolution inclusions are a type of important differential inclusion mentioning the process behaving in a more complex way with respect to time [18].

Now, let us provide a quick synopsis of this study. We provide some preliminary information in Section 2. For the control problem (1), Section 3 is devoted to the examination of the compactness and R_σ-property of the solution set, followed by the invariance of reachability set under nonlinear perturbations. We provide a sample application in Section 4 to demonstrate the viability of our findings. We include a conclusion in Section 5.

2. Preliminaries

Topology and Topological Structures

Topology is the study of geometrical characteristics and spatial relations unaffected by the continuous changes in the shape or size of figures. In mathematics, topology is concerned with the characteristics of a geometric object that are preserved under continuous deformations, such as stretching, twisting, crumpling, and bending, that is, without closing holes, opening holes, tearing, gluing, or passing through itself [19,20]. Topology deals with characteristics of spaces that are constant under any continuous deformation. It is sometimes called rubber sheet geometry because the objects can be stretched and contracted like rubber but cannot be broken, for example, a square can be deformed into a circle without breaking it.

Some important terms used in topology are stated below:

(i) Connected set: A set that cannot be divided into two nonempty open subsets. Mathematically, it can be written as the union of two subsets A and B is not equal to X. Example: Circle is a connected space.
(ii) Contractible set: A set that can be converted to one of its points by continuous deformation.
(iii) Compact set: A set s; a subset of x is called a compact set if every sequence in S has a subsequence that converges to a point in S.
(iv) Quasi-compact space: A topological space x is called a quasi-compact space if every open covering of Y has a finite subcover.
(v) Convex set: The set of points such that given any two points A, B in that set line AB, joining them lies entirely within that set.
(vi) Frechet space: A space which is equivalently a complete Hausdorff locally convex vector space is metrizable (homeomorphic to metric space).
(vii) Banach space: A complete normed vector space is called a Banach space.

Now, we propose some notations, maintain some conventions, and mention some results which are highly used for deriving the main results. Let $C([a, b]); Y)$ stand for the

Banach space of all continuous functions from $[a,b]$ to Y equipped with super norm $|.|$. Let Y and Z be metric spaces. As usual, 2^Z is the collection of all nonempty subsets of Z.

$$C(Z) = F \in 2^Z \; F \text{ is closed},$$

$\mu : y \to 2^Z$ is a multi-valued map.

Lemma 1 ([21]). *Let $\mu : Y \to 2^Z$ be a closed and quasi-compact multi-valued map. Then, μ is upper semi-continuous (closed, quasi-compact, multi-valued map).*

Definition 1 ([22]). (i) *Y is called an absolute retract (AR space) if for any metric space H and any closed subset $F \subset H$, every continuous function $\zeta : F \to Y$ can be extended to a continuous function $\zeta : H \to Y$.*

(ii) *Y is called absolute neighborhood retracts (ANR space) if for any metric space H, closed subset $F \subset H$, and continuous function $\zeta : F \to Y$ there exists a neighborhood $(F \subset R)$ and continuous extension $\zeta : R \to y \; \zeta$.*

Definition 2. *A nonempty subset F of Y is said to be contractible if there exists a point $y_o \in D$ and a continuous function $J : F \times [0,1] \to F$. Such that $J(x,1) = y_0$ and $J(x,0) = x$ for every $x \in F$.*

Definition 3. *A subset F of metric space is called R_ς set if there exists a decreasing sequence F_n of compact and contractible sets such that $F = \cap_1^\infty F_n$ R_σ set is nonempty, compact, and connected. $compact + convex \subset compact AR - space \subset compact + contractible \subset R_\sigma set$.*

Definition 4. *A multi-valued map $\mu : Y \to 2^Z$ is an R_σ map if μ is upper semi-continuous and $\mu(x)$ is an $R_\sigma$set for each $x \in X$.*

Lemma 2 ([23]). *Let Y be a metric space and M a Banach space. Suppose that $\chi : X \to M$ is a proper map that is χ is continuous. Furthermore, if there is a sequence χ_n of mappings from X into M such as the following:*

(i) *χ_n is proper and χ_n converges to χ uniformly on X.*
(ii) *For a given point $y_0 \in M$ and for every y in a neighborhood of $R(y_0)$ of y_0 in M, there is exactly one solution x_n of the equation $\chi_n = y$. Then, the inverse of $\chi(y_0)$ is an R_σ set.*

Definition 5. *A continuous map $\rho : F \subset W \to W$ is called condensing concerning an $MNC - \beta$ if for all bounded sets $\Omega \subset M$ that is not relatively compact.*

Theorem 1. *Let $\Omega \subset W$ be a bounded open neighborhood of zero and $\rho : \Omega \to W$ a β-condensing map with respect to a monotone non-singular $MNC - \beta$ in W if ρ verifies boundary condition $y \neq \lambda\rho(y)$ where $0 < \lambda \leq 1$ then $Fix(\rho)$ is nonempty and compact.*
Let an abstract operator $\varsigma : \varsigma^1(0,a;Y) \to H([0,a]);Y)$ verify below conditions:

(1) *There exists a constant $H > 0$ such as*

$$||(\varsigma g_1)(\nu) - (\varsigma g_2)(\nu)|| \leq H \int_0^\nu ||g_1(s) - g_2(s)|| ds,$$

for all $g_1, g_2 \in \varsigma^1(0,a;Y)$.

(2) *For each compact set $N \subset Y$ and a progression $g_n \subset \varsigma^1(0,b;X)$ such as that $h_n(\nu) \subset N$ for $\nu \in [0,a]$, the weak convergence $g_n \to g_0$ shows $\varsigma(g_n) \to \varsigma(g_0)$ strongly in $H([0,a];Y)$.*

Remark 1. *A typical example of a ς operator is the Cauchy operator*

$$(Gg)(\nu) = \int_0^\nu g(\mu) d\mu.$$

Lemma 3. *Let ς satisfy both two conditions of the above theorem and a progression $g_n \subset \varsigma^1(0,a;Y)$ be integrable bounded such as that*

$$||g_n(\nu)|| \leq \varrho(\nu),$$

for $\nu \in [0,a]$, $\varrho \in \varsigma^1(0,b)$. Assume that there exists $\varpi(\nu) \in \varsigma^1(0,a)$ such as

$$\begin{aligned} \chi(g_n) &\leq \varpi(\nu) \\ \chi((\varsigma(g_n))(\nu)) &\leq 2C\int_0^\nu \varpi(\mu)d\mu \end{aligned}$$

for $\nu \in [0,a]$ and $T(\nu)$ upon X for $\nu \geq 0$

$$M = \sup ||T(\nu)||$$

for $\nu \geq 0$. Let $1 < r < 2$, and let us give two families of $U_E(\nu)$ and $K_E(\nu)$ of linear operators for $\nu \geq 0$ by

$$U_E(\nu)\omega = \int_0^\infty E\xi_r(\theta)C(\nu^r\theta)d\theta$$

$$K_E(\nu)\omega = \int_0^\infty rE\theta\xi_r(\theta)S(\nu^r\theta)d\theta$$

$$\chi_E(\nu) = \int_0^\nu U_E(\mu)d\mu.$$

We will use probability density function $\xi_r(\theta)$ as

$$\xi_r(\theta) = \frac{1}{r\theta^{1+\frac{1}{r}}}\varpi(\theta^{\frac{-1}{r}}) \geq 0$$

is the function of Wright type defined on $(0,\infty)$ which satisfies

$$\int_0^\infty \xi_r(\alpha)d\alpha = 1$$

$$\varpi(\theta) = \frac{1}{\Pi}\Sigma_{i=1}^\infty(-1)^{i-1}(\theta)^{-ri-1}\frac{\Gamma(ri+1)}{i!}.$$

Lemma 4. *The operators $U_E(\nu)$ and $K_E(\nu)$ hold the following characteristics:*

(i) *For all $\nu \geq 0$, $U_E(\nu)$ and $K_E(\nu)$ are linear and bounded operators on Y. More properly,*

$$||U_E(\nu)(\zeta)|| \leq M||\zeta||, ||K_E(\nu)(\zeta)|| \leq \frac{rM}{\Gamma(1+r)}||\zeta||, \nu \geq 0, \zeta \geq x;$$

(ii) *$U_E(\nu)$ and $K_E(\nu)$, $\nu \geq 0$ are strongly continuous on Y.*

Lemma 5. *If $u(\nu)$ fulfills (1) for $x(\nu) = \Psi(\nu)$ and $x'(\nu) = x_1$, then the solution $x(\nu)$ is given as the fractional delay control problem satisfying the integral equation*

$$Ex(\nu) = E\Psi(0) + Ex_1(\nu) + \frac{1}{\Gamma(r)}\int_0^\nu (\nu-\mu)^{r-1}(Nx(\mu) + j(\mu, x_\mu))d\mu.$$

Moreover, we have:

$$x(\nu) = U_E(\nu)\Psi(0) + \chi_E(\nu)Ex_1 + \int_0^\nu (\nu-\mu)^{r-1}K_E(\nu-\mu)j(\mu, x_\mu)d\mu,$$

$\forall \nu \in [0,b]$, such that

$$U_E(\nu) = \int_0^\infty M_E(\theta)C(\nu^E\theta)d\theta, \chi_E(\nu) = \int_0^\nu U_E s ds, K_E(\nu) = \int_0^\infty E\theta M_E(\theta)C(\nu^E\theta)d\theta,$$

where $U_E(\nu)$ *and* $\chi_E(\nu)$ *are continuous with* $\chi(0) = 1$ *and* $U(0) = 1, \|U_E(\nu)\| \leq c, c > 1, \forall \nu \in [0, T]$.

Proof. We have a delay control problem,

$$\begin{cases} {}^cD_\nu^r Ex(\nu) = Nx(\nu) + j(\nu, x_\nu) + Bu(\nu) \; \nu \in [-h, a], \\ x(\nu) = \Psi(\nu), \\ x'(\nu) = x_1, \; 1 < r < 2. \end{cases}$$

Firstly, consider the following equation of control problem:

$$\begin{aligned} {}^cD_\nu^r Ex(\nu) &= Nx(\nu) + j(\nu, x_\nu) \\ Ex(\nu) - Ex(0) - Ex\prime(o)\nu &= \frac{N}{\Gamma(r)} \int_0^\nu (\nu - \mu)^{r-1} x(\mu) d\mu + \frac{1}{\Gamma(r)} \int_0^\nu (\nu - \mu)^{r-1} j(\mu, x_\mu) d\mu \\ Ex(\nu) - E\Psi(0) - Ex_1(\nu) &= \frac{N}{\Gamma(r)} \nu^{r-1} x(\mu) + \frac{1}{\Gamma(r)} \nu^{r-1} j(\mu, x_\mu) \\ Ex(\nu) &= E\Psi(0) + Ex_1(\nu) + \frac{1}{\Gamma(r)} \int_0^\nu (\nu - \mu)^{r-1} (Nx(\mu) + j(\mu, x_\mu)) d\mu \\ Ex(\nu) &= E\Psi(0) + Ex_1(\nu) + \frac{1}{\Gamma(r)} N\nu^{r-1} x(\mu) + \frac{1}{\Gamma(r)} \nu^{r-1} j(\mu, x_\mu). \end{aligned}$$

Taking Laplace of the above equation,

$$\begin{aligned} ELx(\nu) &= EL\Psi(0) + ELx_1(\nu) + \frac{1}{\Gamma(r)} NL\nu^{r-1} x(\mu) + \frac{1}{\Gamma(r)} L\nu^{r-1} j(\mu, x_\mu) \\ E \int_0^\nu e^{-\lambda\mu} d\mu &= E\Psi(0) \frac{1}{\lambda} + Ex_1 \frac{1}{\lambda^2} + \frac{N}{\Gamma(r)} \frac{\Gamma(r)}{\lambda^r} \int_0^\infty e^{-\lambda\mu} x(\mu) d\mu + \frac{1}{\Gamma(r)} \frac{\Gamma(r)}{\lambda^r} \int_0^\infty e^{-\lambda\mu} j(\mu, x_\mu) d\mu \end{aligned}$$

$$\int_0^\infty e^{-\lambda\mu} x(\mu) d\mu = v(\lambda), \; \int_0^\infty e^{-\lambda\mu} j(\mu, x_\mu) d\mu = \omega(\lambda).$$

The above equation becomes

$$\begin{aligned} Ev(\lambda) &= E\Psi(0) \frac{1}{\lambda} + Ex_1 \frac{1}{\lambda^2} + \frac{N}{\lambda^r} v(\lambda) + \frac{1}{\lambda^r} \omega(\lambda) \\ Ev(\lambda) - \frac{N}{\lambda^r} v(\lambda) &= E\Psi(0) \frac{1}{\lambda} + Ex_1 \frac{1}{\lambda^2} + \frac{1}{\lambda^r} \omega(\lambda) \\ E(\lambda^r I - NE^{-1}) v(\lambda) &= E\Psi(0) \frac{\lambda^r}{\lambda} + Ex_{\frac{\lambda^r}{\lambda}} + \frac{\lambda^r}{\lambda^r} \omega(\lambda) \\ E(\lambda^r I - NE^{-1}) v(\lambda) &= \lambda^{r-1} E\Psi(0) + \lambda^{r-2} Ex_1 + \omega(\lambda) \\ Ev(\lambda) &= \lambda^{r-1} (\lambda^r I - NE^{-1})^{-1} \Psi(0) + \lambda^{r-2} (\lambda^r I - NE^{-1})^{-1} Ex_1 + (\lambda^r I - NE^{-1})^{-1} \omega(\lambda) \\ (\lambda^r I - NE^{-1})^{-1} &= \int_0^\infty E^{-1} e^{-\lambda^r \mu} C(\mu) d\mu \\ Ev(\lambda) &= \lambda^{r-1} \int_0^\infty E^{-1} e^{-\lambda^r \mu} C(\mu) \Psi(0) d\mu + \lambda^{r-2} \int_0^\infty E^{-1} e^{-\lambda^r \mu} C(\mu) Ex_1 d\mu \\ &\quad + \int_0^\infty E^{-1} e^{-\lambda^r \mu} S(\mu) \omega(\lambda) d\mu \\ v(\lambda) &= \lambda^{r-1} \int_0^\infty E^{-1} e^{-\lambda^r \mu} C(\mu) \Psi(0) d\mu + \lambda^{-1} \lambda^{r-1} \int_0^\infty E^{-1} e^{-\lambda^r \mu} Ex_1 d\mu \\ &\quad + \int_0^\infty E^{-1} e^{-\lambda^r \mu} S(\mu) \omega(\lambda) d\mu. \end{aligned}$$

Now, consider one-sided probability density function whose Laplace transform is

$$e^{(-\lambda)^r} = \int_0^\infty e^{-\lambda\theta} \varpi_r(\theta) d(\theta).$$

Firstly, solving first term of the above equation,

$$\begin{aligned}
\lambda^{r-1}\int_0^\infty E^{-1}e^{(-\lambda)^r\mu}\Psi(0)d\mu &= \int_0^\infty \lambda^{r-1}E^{-1}e^{-\lambda^r\nu^r}C(\nu^r)\Psi(0)r\nu^{r-1}d\nu \\
&= \int_0^\infty E^{-1}e^{-(\lambda\nu)^r}C(\nu^r)r\nu^{r-1}\Psi(0)d\nu \\
&= \int_0^\infty r(\lambda\nu)^{r-1}E^{-1}e^{-(\lambda\nu)^r}C\nu^r\Psi(0)d\mu \\
&= \int_0^\infty \frac{-1}{\lambda}\frac{d}{d\nu}(e^{-(\lambda\nu)^r})E^{-1}C(\nu^r)\psi(0)d\nu \\
&= \int_0^\infty\int_0^\infty \frac{-1}{\lambda}\frac{d}{d\nu}(e^{-(\lambda\nu\theta)}\varpi(\theta)E^{-1}C(\nu^r)\psi(0)d\theta d\nu \\
&= \int_0^\infty\int_0^\infty \frac{-1}{\lambda}\frac{d}{d\nu}(e^{-\lambda\frac{\nu\prime}{\theta}(\theta)}\varpi(\theta)E^{-1}C\frac{\nu'^r}{\theta^r}\psi(0)\frac{d\nu'}{\theta}d\theta \\
&= \int_0^\nu\int_0^\nu e^{-\lambda\nu}E^{-1}\varpi(\theta)C(\frac{\nu^r}{\theta^r})\Psi(0)d\nu d\theta \\
&= \int_0^\nu e^{-\lambda\nu}\int_0^\nu E^{-1}\varpi(\theta)C(\frac{\nu^r}{\theta^r})\Psi(0)d\nu d\theta \\
&= \int_0^\nu e^{-\lambda\nu}\int_0^\nu E^{-1}\varpi_r(\theta)^{\frac{-1}{r}}C(\nu^r(\theta')\psi(0)(\frac{-1}{r}(\theta)^{-1-\frac{1}{r}})d\theta' d\nu \\
&= \int_0^\nu e^{-\lambda\nu}\int_0^\nu E^{-1}\varpi_r(\theta)^{\frac{-1}{r}}C(\nu^r(\theta)\Psi(0)\frac{1}{r(\theta)^{1+\frac{1}{r}}}d\theta d\nu \\
&= \int_0^\infty e^{-\lambda\nu}(U_E(\nu)\Psi(0))d\nu \\
&= L'(U_E(\nu)\Psi(0))\lambda.
\end{aligned}$$

Similarly,

$$\lambda^{-1}\lambda^{r-1}\int_0^\infty E^{-1}e^{-(\lambda)^r\mu}C(\mu)Ey_1d\mu = L'[g_1(\nu)]\lambda L'[U_E(\nu)Ey_1])\lambda.$$

Now,

$$\begin{aligned}
&\int_0^\infty E^{-1}e^{-(\lambda)^r\mu}S(\mu)\omega(\lambda)d\mu \\
=& \int_0^\infty E^{-1}e^{-(\lambda)^r(\nu)^r}\omega(\lambda)r\nu^{r-1}d\nu = \int_0^\infty E^{-1}e^{-(\lambda\nu)^r}S(\nu^r)\int_0^\infty e^{-\lambda\mu}f(\mu,y_\mu)d\nu d\mu \\
=& \int_0^\infty\int_0^\infty E^{-1}e^{-(\lambda\nu)^r}S(\nu^r)e^{-\lambda\mu}j(\mu,y_\mu)d\nu d\mu \\
=& \int_0^\infty\int_0^\infty\int_0^\infty E^{-1}e^{-\lambda\nu\theta}\varpi_r(\theta)d\theta S(\nu^r)e^{-\lambda\mu}f(\mu,y_\mu)r\nu^{r-1}d\mu d\nu \\
=& \int_0^\infty\int_0^\infty\int_0^\infty E^{-1}e^{-\lambda\frac{\nu'}{\theta}(\theta)}\varpi_r(\theta)S(\frac{\nu'}{\theta^r}e^{-\lambda\mu}j(\mu,y_\mu)r(\frac{(\nu')^{r-1}}{\theta^{r-1}})\frac{d\nu'}{\theta}d\mu d\theta \\
=& \int_0^\infty\int_0^\infty\int_0^\infty E^{-1}e^{-\lambda(\nu+\mu)}\varpi_r(\theta)S(\frac{\nu^r}{\theta^r})j(\mu,y_\mu)r(\frac{\nu^{r-1}}{\theta^r})d\mu d\nu d\theta \\
=& \int_0^\infty\int_0^\infty\int_0^\infty E^{-1}e^{-\lambda\nu'}\varpi_r(\theta)S(\frac{(\nu'-\mu)^r}{\theta^r})j(\mu,y_\mu)r(\frac{(\nu'-\mu)^{r-1}}{\theta^r})d\mu d\nu d\theta \\
=& \int_0^\infty\int_0^\infty\int_0^\infty E^{-1}e^{-\lambda\nu}\varpi_r(\theta)S(\frac{(\nu-\mu)^r}{\theta^r})j(\mu,y_\mu)r(\frac{(\nu-\mu)^{r-1}}{\theta^r})d\mu d\nu d\theta
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-\lambda\nu}[r\int_0^t\int_0^\infty E^{-1}\varpi_r(\theta)S(\frac{(\nu-\mu)^r}{\theta^r})j(\mu,x_\mu)\frac{(t-\mu)^{r-1}}{\theta^r}d\mu d\theta]d\nu \\
&= L'[r\int_0^\nu(\nu-\mu)^{r-1}\int_0^\infty E^{-1}\varpi(\theta)S(\frac{(\nu-\mu)^r}{\theta^r})j(\mu,x_\mu)\frac{1}{\theta^r}d\theta d\mu]\lambda \\
&= L'[r\int_0^\nu(\nu-\mu)^{r-1}\int_0^\infty E^{-1}\varpi_r(\theta)S(\frac{(\nu-\mu)^r}{\theta^r})j(\mu,x_\mu)\frac{1}{\theta^r}d\theta d\mu]\lambda \\
&= L'[r\int_0^\nu(\nu-\mu)^{r-1}\int_0^\infty E^{-1}\varpi_r(\theta^{\frac{-1}{r}})S(\nu-\mu)^r(\theta')j(\mu,x_\mu)(\theta')(\frac{-1}{r}(\theta')^{-1-\frac{1}{r}}d\theta' d\mu]\lambda \\
&= L'[r\int_0^\nu(\nu-\mu)^{r-1}\int_0^\infty E^{-1}\varpi_r(\theta)^{\frac{-1}{r}})\frac{1}{r\theta^{1+\frac{1}{r}}}S(\nu-\mu)^r(\theta)')j(\mu,x_\mu)(\theta')d\theta' d\mu]\lambda \\
&= L'[r\int_0^\nu(\nu-\mu)^{r-1}\int_0^\infty E^{-1}\varpi_r(\theta)^{\frac{-1}{r}})\frac{1}{r\theta^{1+\frac{1}{r}}}S(\nu-\mu)^r(\theta)j(\mu,x_\mu)(\theta)d\theta d\mu]\lambda \\
&= L'[r\int_0^\nu(\nu-\mu)^{r-1}\int_0^\infty E^{-1}\xi_r(\vartheta)S(\nu-\mu)^r(\theta)j(\mu,x_\mu)(\theta)d\theta d\mu]\lambda \\
&= L'(\int_0^\nu K_E(\nu-\mu)j(\mu,x_\mu)d\mu)\lambda \\
v(\lambda) &= L'(U_E(\nu)\psi(0))\lambda + L^[g_1(\nu))\lambda L'(U_E(\nu)Ex_1)\lambda + L'(\int_0^\nu K_E(\nu-\mu)j(\mu,x_\mu)d\mu.
\end{aligned}
$$

Taking inverse Laplace, we obtain

$$x(\nu) = U_E(\nu)\psi(0) + \chi_E(\nu)x_1 + \int_0^\nu (\nu-\mu)^{r-1}K_E(\nu-\mu)(j(\mu,x_\mu)+Bu(\mu))d\mu$$

which is the required mild solution to our control problem. □

3. Main Results

Our standing assumptions on j are given below:

(i) Let j a function such that $j:[0,a]\times C([-h,0];Y)\to Y$ is continuous.
(ii) There exists $\varsigma \in C[-h,0]$ such that $||j(\nu,v)|| \le \zeta(\nu)(1+|v_0|)$.
(iii) There exists $k \in L^a(0,a)$ such that $\chi(j(\nu,\Omega)) \le k(\nu)$, $\nu \in [0,a]$ and χ is Hausdroff MNC in X.

Theorem 2. *Let $pr > 1$ and assumptions (i),(ii), and (iii) of the above main result be verified and $K_E(\nu)$ be continuous in the uniform operator topology for all ν greater than zero. Given $u \in L^a(0,a;V)$, then $\varrho(u)$ is nonempty and compact. Furthermore, $K_E(\nu)$ is compact for $\nu > 0$, then ϱ is an R_σ set.*
Step 1:
We build the solution map having a mild solution of the control problem in the following:

$$\zeta^u(x(\nu)) = U_E(\nu)\Psi(0) + \chi_E(\nu)Ex_1 + \int_0^\nu (\nu-\mu)^{r-1}K_E(\nu-\mu)(j(\mu,x_\mu)+Bu(\mu)),$$

under the conditions

$$
\begin{aligned}
x(\nu) &= \Psi(\nu), \\
x' &= x_1.
\end{aligned}
$$

It is obvious that $x \in \ominus(u)$ if x is a fixed point of ζ^u. We aim to show that ζ^u accepts at least one fixed point. We claim that ζ^u is β condensing. Consider Ω is a bounded subset of $H([-h,a];X)$ that is not relatively compact such that

$$\beta(\zeta^u(\Omega)) \ge \beta(\Omega)$$

by the definition for β, there exists a progression $y_n \subset \zeta^u(\Omega)$

$$\beta(\zeta^u(\Omega)) = (\gamma y_n), mod_C(y_n)).$$

Assume,

$$\begin{aligned} \sigma(t) &= \frac{2Mr}{\Gamma(1+r)} \int_0^\infty (\nu-\mu)^{r-1} k(\mu) e^{L\mu} d\mu \\ \sup e^{-L\nu}\sigma(\nu) &< 1 \\ \chi(y_n) &= 0. \end{aligned}$$

Now, using point (ii) of the main result, we have

$$\begin{aligned} ||j(\nu, v)|| &\leq \eta(\nu)(1+|v_o|) \\ ||(\nu-\mu)^{r-1} K_E(\nu-\mu) j(\mu, x_n\mu|| &\leq \frac{Mr}{\Gamma(1+r)} (\nu-\mu)^{r-1} \eta(\mu)(1+|x_n\mu(.)|) \\ G_j(\mu) &= (\nu-\mu)^{r-1} K_E(\nu-\mu) j(\mu, x_n\mu). \end{aligned}$$

Moreover, from point (iii) of the main result, it follows

$$\begin{aligned} \chi(j(\nu,\Omega) &\leq k(\nu)\sup\chi(\Omega(\mu)) \\ \chi(G_j(\mu)) &\leq \frac{Mr}{\Gamma(1+r)}(\nu-\mu)^{r-1}k(\mu)\sup\chi(x_n\mu(\mu')) \\ &\leq \frac{Mr}{\Gamma(1+r)}(\nu-\mu)^{r-1}k(\mu)e^{L\mu}\sup e^{(-L)(\mu+\mu')}\chi(x_n(\mu+\mu')) \\ &= \frac{Mr}{\Gamma(1+r)}(\nu-\mu)^{r-1}k(\mu)e^{L\mu}\gamma(x_n) \\ \chi(G_j(\mu)) &\leq \frac{Mr}{\Gamma(1+r)}(\nu-\mu)^{q-1}k(\mu)e^{L\mu}\gamma(\Omega) \\ \chi(\int_0^\nu G_j(\mu)d\mu + (\Phi(Bu))(\nu)) &\leq \sigma(\nu)\gamma(x_n) \\ \gamma(y_n) &\leq \sup e^{-L\nu}\sigma(\nu)\gamma(y_n) \\ \gamma(x_n) &\leq \gamma(y_n) \leq \sup e^{-L\nu}\sigma(\nu)\gamma(x_n) \\ \gamma(y_n) &= 0. \end{aligned}$$

On the other hand, from point (ii) of the main result, it shows that the set $j(.,x_n)$ is bounded in $L^p(0,b;Y)$. We obtain

$$\beta(\Omega) = 0$$

which is a contradiction. This in turn proves that ζ^u is β condensing. Next, take $x \in H([-h,a];Y)$ with $x = \lambda\zeta^u(x)$ for $0 < \lambda \leq 1$. Now, by using point (ii) of the main result, we have

$$\begin{aligned} ||x(\nu)|| &\leq M||\Phi(0)|| + M||Ex_1|| + \frac{rM}{\Gamma(1+r)}\int_0^\infty (\nu-\mu)^{r-1}\eta(\mu)(1+(|x_\mu(.)|)_0)d\mu \\ &+ \frac{rM}{\Gamma(1+r)}\int_0^\infty (\nu-\mu)^{r-1}||Bu(\mu)||d\mu \\ &\leq a_1 + a_2\int_0^\infty (\nu-\mu)^{r-1}(|x_\mu(.)|)|_0 d\mu \\ a_1 &= M||\Phi(0)|| + \frac{rMb^{r-\frac{1}{p}}}{\Gamma(1+r)}(\frac{p-1}{pr-1})^{\frac{p-1}{p}}||Bu||L^p(0,b;X) \\ a_2 &= \frac{rM}{\Gamma(1+r)}\sup\eta(\mu). \end{aligned}$$

Step 2:

Given $\epsilon_n \in (0,1)$ with ϵ_n approaches zero as n approaches infinity. By point (i) of main result, there exists a sequence j_n of locally Lipschitz functions for which $[0,a] \times H([-h,0];Y) \to Y$,

$$||j_n(\nu, v) - j(\nu, v)|| < \epsilon_n.$$

We define the approximation operator by

$$\begin{aligned}
\zeta_n^u &= U_E(\nu)\Phi(0) + \chi_E(\nu)Ex_1 + \int_0^\nu (\nu-\mu)^{r_1} K_E(\nu-\mu)(j_n(\mu, x_\mu) + Bu(\mu))d\mu \\
x(\nu) &= \Psi(\nu), \\
x' &= x_1,\ \nu \in [-h,0] \\
||(I-\zeta_n^u)(x)(\nu) - (I-\zeta^u)(x)(\nu)|| &= 0 \\
||(I-\zeta_n^u)(x)(\nu) - (I-\zeta^u)(x)(\nu)|| &\leq \frac{M}{\Gamma(1+r)} \int_0^\nu (\nu-\mu)^{r-1} ||j_n(\mu, x_\mu) - j(\mu, x_\mu)|| d\mu \leq \frac{Mb^r}{\Gamma(1+r)} \epsilon_n.
\end{aligned}$$

Using below equation

$$||j - n(\nu, v) - j(\nu, v)|| < \epsilon_n.$$

We obtain $I - \zeta_n^u \to I - \zeta^u$ with

$$||j_n(\nu, v)|| \leq 1 + \eta(\nu)(1 + (|v|_0))$$

for any bounded sequence $x_m \subset H([-h,a];Y)$, we have

$$\begin{aligned}
Gj_n(\mu) &= (\nu-\mu)^{r-1} K_E(\nu) j_n(\mu, x_(m\mu)) \\
\chi(\int_0^\nu Gj_n(\mu)d\mu + (\Phi(Bu))(\nu)) &= 0
\end{aligned}$$

for all $y \in C([-h,a];Y)$, the equation

$$(I - \zeta_n^u)(x) = y.$$

For the continuity of $I - \zeta_n^u$ and closeness of K, it is uncomplicated to see that Ω is closed. Let $x_m = y_m$ be progression; we can take a progression $y_m \subset K$ such that

$$\begin{aligned}
x_m - \zeta_n^u &= y_m \\
x_m(\nu) &= U_E(\nu)\Phi(0) + Ex_1 + y_m(\nu) + \int_0^\nu (\nu-\mu)^{r-1} K_E(\nu-\mu)[j_n(\mu, x_(m\mu)) + Bu)\mu)]d\mu \\
& \quad \Phi(\nu) + y_m(\nu)\ \nu \in [-h,0] \\
\chi(\Phi(j_n(\nu, x_(m\nu)) + Bu(\nu)) &= 0 \\
\ominus(u) &= (I - \zeta^u)^{-1}(0)
\end{aligned}$$

for each $t \in [0,a]$. Because for all $\nu \in [-h,a]$, $y_m(\nu)$ is relatively compact in Y, we finalize that

$$\varphi(x_m(\nu)) = 0.$$

This shows that $x_m(\nu)$ is relatively compact for all $\nu \in [-h,b]$.
Observe that the compactness of $K_E(\nu)$ for $\nu > 0$ shows that $K_E(\nu)$ is continuous in the uniform operator topology.

Theorem 3. *Let pr be greater than one. Suppose that assumptions (i),(ii) of the main result are verified and $K_E(\nu)$ is compact for $\nu > 0$. Furthermore, let the hypothesis (iv) there exist $\Psi \in L^p(0,a;V)$ such that*

$$(\Phi(B\Psi))(a) = (\Phi)(a)$$

for each $\Phi \in L^p(0, a; V)$*. Then, there exists* r' *greater than zero such that the reachability set of the control problem is constant under nonlinear perturbations*

$$K_{(a,j)} = K_{(a,0)}.$$

Proof. Step 1:

For every $u \in L^p(0, a; V)$, $\ominus(u)$ is nonempty, compact, and R_σ set. In this step, our purpose is to prove that the multi-valued map ϱ is an R_σ map. Let $u_n \to u$ in $L^p(0, b; V)$ and $x_n \in \varrho(u_n)$, $x_n \to x$ in $HJ[-h, a]; Y)$. It is uncomplicated to observe that x_n holds the following integral equation:

$$\begin{aligned} x_n(\nu) &= U_E(\nu)\Psi(0) + \chi_E(\nu)Ex_1 + \int_0^\nu (\nu-\mu)^{r-1} K_E(\nu-\mu)(j(\mu, x_\mu) + Bu(\mu))d\mu, \\ x(\nu) &= \Psi(\nu), \\ x'(\nu) &= x_1(\nu), \end{aligned}$$

given assumption (i) of the main result

$$j : [0, a] \times H([-h, 0]; X) \to Y.$$

We have $j(\mu, x_(n\mu)) \to j(\mu, x_\mu)$ for all $\mu \in [0, a]$,

$$j(., x_n) + B(u)_n \to j(., x) + B(u(.)).$$

Using limit $n \to \infty$, we see that x verifies the integral equation

$$\begin{aligned} x(\nu) &= U_E(\nu)\Psi(0) + \chi_E(\nu)Ex_1 + \int_0^\nu (\nu-\mu)^{r-1} K_E(\nu-\mu)(j(\mu, x_\mu) + Bu(\mu))d\mu, \\ x(\nu) &= \Psi(\nu), \\ x' &= x_1, \end{aligned}$$

which implies that $x \in \varrho(u)$.

Step 2:

By the hypothesis (iv), there exists a continuous map : $L^a(0, a; Y) \to L^a(0, a; V)$ such that for any $\Phi \in L^a(0, a; y)$,

$$\begin{aligned} (\Phi(BS\Phi))(a) + (\Phi)(a) &= 0 \\ ||S\Phi||_(L^p(0, a; V)) &\leq d||\Phi||_(L^a(0, a; V)). \end{aligned}$$

Now, consider the multi-valued map

$$\begin{aligned} F : L^a(0, a; V) &\to 2^{(L^a(0,a;V))}, \\ F(u) &= S \circ H_j \circ \ominus(u_o + u), \\ H_j : C([-h, a]; Y) &\to L^a(0, a; Y), \\ H_j(x)(\nu) &= j(\nu, x_\nu) \\ ||u||_{L^a(0,a;V)} &\leq d||H_j(x')||_{L^a(0,a;Y)} \\ &\leq d(\int_0^b |\eta(\mu)|^a (1 + |x'(.)|_0)^a d\mu)^{\frac{1}{a}} \\ &\leq da^{\frac{1}{a}} \sup \eta(\mu)(1 + ||x'||_{H([-h,a];Y))} \\ ||u||_{L^a(0,a;V)} &\leq M_1 + M_2||B||_(V \leftrightarrow X)(||u_0||_{L^a(0,a;V)} + ||u||_{L^a(0,a;V))}. \end{aligned}$$

We realize that F accepts a fixed point.

Step 3:

$$
\begin{array}{rcl}
K_(a,j) & = & K(0) \\
x(a,u_0,0) & = & U_E(a)\Phi(0)+(\Phi(Bu_0))(a) \\
u^* & = & SH_j(x) \\
x(a,u_0+u^*,j) & = & U_E(a)\Phi(0)+[\Phi(H_j(x)+B(u_0+u^*))](a) \\
& = & U_E(a)\Phi(0)+(\Phi(Bu_0))(a)+[\Phi(H_j(x)+Bu^*)](a) \\
& = & x(a,u_0,0)+[\Phi(H_j(x)+BSH_j(x))](a), \\
x(a,u,j) & \in & K_(a,j), \\
x(a,u,j) & = & U_E(a)\Phi(0)+[\Phi(H_j(x)+Bu)](a) \\
u' & = & u-SH_j(x) \\
x(a,u',0) & = & U_E(a)\Phi(0)+(\Phi(Bu'))(a)U_E(a)\Phi(0)+[\Phi(H_j(x)+Bu)](a)-[\Phi(H_j(x)+BSH_j(x))](a).
\end{array}
$$

To characterize the approximate controllability of the control problem, let us propose the relevant operator

$$W=\int_0^a (a-\mu)^{r-1}K_E(a-\mu)BB^*K_E^*(a-\mu)d\mu,$$

where B^* and $K_E^*(\nu)$ are adjoints B and $K_E(\nu)$. $\square$

4. Example

Consider the control problem of fractional differential equation with delay in the form given

$$
\begin{cases}
{}^cD_\kappa^r a(\kappa,\varsigma)=\frac{d^2a(\kappa,\varsigma)}{d\varsigma^2}+\sigma u(\kappa,\varsigma)+sin(|a_\kappa(\vartheta,\varsigma)|)\ \kappa\in[0,1],\ \varsigma\in[0,\pi],\ \vartheta\in[-h,0], \\
a(\kappa,0)=x(\kappa,\pi)=0,\ \kappa\in[0,1], \\
a(\kappa,\varsigma)=\phi(t,\varsigma)\ \kappa\in[-h,0],
\end{cases}
$$

here $\frac{1}{2}<r<1$,

$$a_\kappa(\vartheta,\varsigma)=a(\kappa+\vartheta,\varsigma),$$

ϕ is continual and σ is a real number. Suppose $A=V$,

$$v=M^2[0,\pi].$$

Let $N:D(N)\subset A\to A$ be a function given by $N\omega=\frac{d^2}{d\varsigma^2}$ with domain $D(N)=a\in A\omega,\omega'$ which are completely continual and $\omega(0)=\omega(\pi)=0$. It is said that N has a distinct spectrum and eigenvalues are $(-a)^2, a\in A$ with the related eigenvectors. Furthermore, N produces a compact analytic semigroup $T(\kappa)$ on A.

$$
\begin{array}{rcl}
T(\kappa)\omega & = & \Sigma_{a=1}^{\infty}e^{-a^2\kappa}(\omega,\nu_n)\nu_n \\
||T(\kappa)|| & \leq & e^{-\kappa},
\end{array}
$$

for all $\kappa\geq 0$. Denote by $F_{(r,l)}$, the common Mittag-Leffler special function given by

$$F_{(r,l)}(\kappa)=\Sigma_{i=0}^{\infty}\frac{\kappa^i}{\Gamma(ri+l)}.$$

So, we have

$$
\begin{aligned}
Q(\kappa)\omega &= \Sigma_{a=1}^{\infty} F_r(-a^2\kappa^r(\omega, \nu_a)\nu_a \\
P(\kappa)\omega &= \Sigma_{a=1}^{\infty} f_r(-a^2\kappa^r(\omega, \nu_a)\nu_a \\
||Q(\kappa)|| &\leq 1 \\
||P(\kappa)|| &\leq \frac{r}{\Gamma(1+r)}.
\end{aligned}
$$

By compact property of $T(\kappa)$ for $\kappa > 0$, it is clear that $Q(\kappa)$ and $P(\kappa)$ are compact operators for $\kappa > 0$. Furthermore, for $\kappa > 0$ $Q(\kappa)$, $P(\kappa)$ are continual in uniform operator topology. We have

$$
\begin{aligned}
u(\kappa)(\varsigma) &= u(\kappa, \varsigma) \\
a(\kappa)\varsigma &= a(\kappa, \varsigma).
\end{aligned}
$$

It is uncomplicated to check that j is continual from $[0, 1] \times D([-h, 0]; A)$ to A. Furthermore, for all $\kappa \in [0, 1]$ and $\nu \in D([-h, 0]; A)$. We obtain

$$||j(\kappa, \nu)|| \leq \varpi(\kappa)(1 + |\nu_0).$$

Finally, let us propose a definition of bounded linear operator $G : M^2(0, 1; V) \to M^2(0, 1; A)$ by $(Gu)(\kappa) = \sigma u(\kappa)$ for $u(.) \in L^2(0, 1; V)$. G is one-one and onto function.

5. Conclusions

In this paper, we prove the existence of a mild solution for the fractional delay control system. Then, we deal with the topological structure of the solution set consisting of the compactness and R_σ-property. We also derive a mild solution to the above delay control problem by using the Laplace transform method. We intend to learn more about the topological structure of fractional control in our upcoming research. We can identify the uniqueness and existence with uncertainty by using the Caputo derivative. The concept put out in this mission may be expanded upon in future projects, along with the inclusion of the observability and generalization of other activities. There is a lot of research being conducted on this fascinating subject, which may result in a wide range of applications and ideas.

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