fractal and fractional

*Article*

# Constructing Analytical Solutions of the Fractional Riccati Differential Equations Using Laplace Residual Power Series Method

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**Abstract:** In this article, a hybrid numerical technique combining the Laplace transform and residual power series method is used to construct a series solution of the nonlinear fractional Riccati differential equation in the sense of Caputo fractional derivative. The proposed method is implemented to construct analytical series solutions of the target equation. The method is tested for eminent examples and the obtained results demonstrate the accuracy and efficiency of this technique by comparing it with other numerical methods.

**Keywords:** fractional Riccati differential equation; Laplace transform; approximate solution; residual power series

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## 1. Introduction

The Riccati equation was named after Count Jacopo Francesco Riccati (1676–1754), an Italian nobleman [1]. This sort of equation has a long history of use in random processes, optimal control and other fields [2–9]. In many research studies, the fractional Riccati differential equation has emerged as a more complete form with a different value of fractional derivative order.

Numerous methods, including Laplace transforms, the Chebyshev wavelet operational matrix approach, the homotopy method, the Pade–variational iteration method, the finite difference method, the Adomian decomposition method and others [10–24] are devoted to solving fractional differential equations. The fractional Riccati equation has been successfully solved using the residual power series method (RPSM) [25]. It is a model and simple technique for generating approximate series solutions to differential equations. The Laplace residual power series method (LRPSM), on the other hand, was created in [26] to solve both linear and nonlinear fractional differential equations.

The LRPSM recommends using an approach that replicates the RPSM, but with a novel mechanism that is simpler than the RPSM, in which it uses the concept of the limit in determining the expansion coefficients, which speeds up the work of the MATHEMATICA software in performing symbolic and numerical calculations of the problem.

The motivation of this paper is to apply the LRPSM to solve the quadratic fractional Riccati differential equation. The accuracy and effectiveness of the method is clarified by displaying numerical examples and comparing the solutions with the results of some approved approaches.

The novelty of this work is illustrated in presenting the solution of the Ricatti equation in a form of rapidly convergent series using the LRPSM. The proposed method is simpler and faster than other numerical methods in establishing series solutions without needing linearization, discretization or differentiations. As a result, it could generate many terms of the series solutions with fewer efforts in comparison to other numerical methods and it could also be programed easily using computer software such as Mathematica.

## 2. Basic Concepts and Theorems

The definition of the Caputo fractional derivative and its properties are introduced in this section. In addition, we present some essential theories related to fractional Taylor expansion that will be used to create a series solution of the nonlinear fractional Riccati differential equation.

**Definition 1.** *The Caputo fractional derivative of order $\beta > 0$ is given by*

$$\mathfrak{D}^{\beta}u(t) = \begin{cases} \frac{1}{\Gamma(\beta)}\int_0^t (t-\omega)^{\beta-1}\, u^{(n)}(\omega)\, d\omega, & n-1 < \beta < n\,,\ t > \omega \geq 0 \\ u^{(n)}(t), & \beta = n. \end{cases}$$

In the following, we mention some popular properties of $\mathfrak{D}^{\beta}$ that are useful in our work.

$$\mathfrak{D}^{\beta}c = 0$$

$$\mathfrak{D}^{\beta}t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\beta)}t^{\alpha-\beta}$$

where $\alpha > -1$, $c \in \mathbb{R}$, $n-1 < \beta < n$ and $t \geq 0$. For more properties about the operator $\mathfrak{D}^{\beta}$, see Ref. [27].

**Definition 2.** *Let $u(t)$ be a piecewise continuous function on the interval $[0,\infty]$ of exponential order $\delta$, the Laplace transform of $u(t)$, $U(s)$ is given by*

$$U(s) = \mathcal{L}[u(t)] = \int_0^{\infty} e^{-st}u(t)dt,\ s > \delta,$$

*and the inverse Laplace transform of $U(s)$ is given by*

$$u(t) = \mathcal{L}^{-1}[U(s)] = \int_{z-i\infty}^{z+i\infty} e^{st}U(s)ds, \quad z = Re(s) > z_0.$$

We summarize the necessary properties of the Laplace transform and its inverse in the following lemma.

**Lemma 1 ([26]).** *Let $u(t), v(t)$ be a piecewise continuous function on the interval $[0,\infty]$. If $U(s) = \mathcal{L}[u(t)]$, $V(s) = \mathcal{L}[v(t)]$ and $a, b \in \mathbb{R}$, then*

*i.* $\mathcal{L}[au(t) + bv(t)] = aU(s) + bV(s)$.

*ii.* $\mathcal{L}^{-1}[aU(s) + bV(s)] = au(t) + bv(t)$.

*iii.* $\lim_{s\to\infty} sU(s) = u(0)$

*iv.* $\mathcal{L}\left[t^{\beta}\right] = \frac{\Gamma(\beta+1)}{s^{\beta+1}}, n-1 < \beta < n$.

*v.* $\mathcal{L}\left[\mathfrak{D}^{\beta}u(t)\right] = s^{\beta}U(s) - \sum_{k=0}^{n-1} s^{\beta-k-1}u^{(k)}(0),\ n-1 < \beta < n$.

**Theorem 1 ([26]).** *Assume that $u(t)$ has the following fractional power series representation at about $t = 0$*

$$u(t) = \sum_{n=0}^{\infty} h_n t^{n\beta},\ 0 < \beta \leq 1\,,\ 0 \leq t < R, \tag{1}$$

*where R is the radius of convergence of the series. If $\mathfrak{D}^{n\beta}u(t)$ is continuous on $(0,R)$, $n = 0,1,2,\ldots$, then the coefficients $h_n{}'s$ are given by*

$$h_n = \frac{\mathfrak{D}^{n\beta}y(0)}{\Gamma(n\beta+1)},\ n = 0,1,2,\ldots. \tag{2}$$

Moreover, researchers in [26] have presented the basic theories and results that they used in the LRPSM as follows:

**Theorem 2.** *Assume that the fractional power series representation of the function $U(s) = \mathcal{L}[u(t)]$ is*

$$U(s) = \sum_{n=0}^{\infty} \frac{h_n}{s^{n\beta+1}}, 0< \beta \le 1,\ s > 0, \tag{3}$$

*then $h_n = \mathfrak{D}^{n\beta}u(0)$, where $\mathfrak{D}^{n\beta} = \mathfrak{D}^{\beta}\mathfrak{D}^{\beta}\ldots\mathfrak{D}^{\beta}$ (n -times).*

The next theorem provides the conditions of convergence of the series expansion in the previous equation.

**Theorem 3.** *Assume that $U(s) = \mathcal{L}[u(t)]$ has the fractional Taylor expansion in Equation (4). If $\left| s\,\mathcal{L}\left[\mathfrak{D}^{(n+1)\beta}u(t)\right]\right| \le \mathcal{K}$ on $(\delta_1,\delta_2]$, where $0 < \beta \le 1$, then the remainder $\mathcal{R}_n(s)$ will satisfy the inequality*

$$|\mathcal{R}_n(s)| \le \frac{\mathcal{K}}{s^{(n+1)\beta+1}},\ \ \delta_1 < s \le \delta_2\ . \tag{4}$$

## 3. Constructing the Laplace Residual Power Series Solution for the Nonlinear Fractional Riccati Differential Equation

This section explains how to use the LRPSM to solve the nonlinear fractional Riccati differential equation. The basic idea behind the offered approach is to apply the RPSM to the Laplace transform, which may be achieved by applying the Laplace transform to the stated equation before taking into consideration the suggested fractional Taylor series to describe the resulting equation's solution. We quickly calculate the unknown coefficients using justifications similar to those found in the typical RPSM. In order to obtain the solution in the original space, we then perform the inverse Laplace transform on the series expansion.

Now, we explain the algorithm of the LRPSM to construct a solution of the nonlinear fractional Riccati differential equation:

$$\mathfrak{D}^{\beta}u(t) + au(t) + bu^2(t) = c, \tag{5}$$

with initial condition

$$u(0) = d, \tag{6}$$

where $0 < \beta \le 1,\ t \ge 0$ and $a,b,c,d$ are constants.

Firstly, operate the Laplace transform on both sides of Equation (5) to get

$$\mathcal{L}\left[\mathfrak{D}^{\beta}u(t)]+a\mathcal{L}[u(t)]+b\mathcal{L}[u^2(t)\right] = \mathcal{L}[c] \tag{7}$$

Applying Lemma 1 and using the initial condition in Equation (6), Equation (7) can be written as

$$U(s) + \frac{a}{s^{\beta}}U(s) + \frac{b}{s^{\beta}}\mathcal{L}\left[\left(\mathcal{L}^{-1}[U(s)]\right)^2\right] - \frac{d}{s} - \frac{c}{s^{\beta+1}} = 0,\ \ s > \delta \ge 0. \tag{8}$$

Now, we construct a series solution for the nonlinear ordinary differential Equation (8); let the solution of Equation (8) have the representation form

$$U(s) = \sum_{n=0}^{\infty} \frac{h_n}{s^{n\beta+1}} \ , \ s > \delta \geq 0 \tag{9}$$

and the kth truncated series of $U(s)$ have the form

$$U_k(s) = \sum_{n=0}^{k} \frac{h_n}{s^{n\beta+1}} \ , \ s > \delta \geq 0, \tag{10}$$

where $h_n$ are the to-be-determined constant coefficients.

The initial condition in Equation (6) with Lemma 1 part (iii) yielded that $h_0 = d$. So, the kth truncated series of $U(s)$ can be written as

$$U_k(s) = \frac{d}{s} + \sum_{n=1}^{k} \frac{h_n}{s^{n\beta+1}} \ , \ \ s > \delta \geq 0 \tag{11}$$

In the next step, to obtain the series' coefficients in Equation (11), we define the Laplace–residual function of Equation (8):

$$LRes(s) = U(s) + \frac{a}{s^\beta}U(s) + \frac{b}{s^\beta}\mathcal{L}\left[\left(\mathcal{L}^{-1}[U(s)]\right)^2\right] - \frac{d}{s} - \frac{c}{s^{\beta+1}}, \ \ s > \delta \geq 0 \tag{12}$$

The $k$th Laplace–residual function is as follows:

$$LRes_k(s) = U_k(s) + \frac{a}{s^\beta}U_k(s) + \frac{b}{s^\beta}\mathcal{L}\left[\left(\mathcal{L}^{-1}[U_k(s)]\right)^2\right] - \frac{d}{s} - \frac{c}{s^{\beta+1}}, \ \ s > \delta \geq 0. \tag{13}$$

It is clear that $LRes(s) = 0, \ s > 0$, and thus $s^{k\beta+1}LRes(s) = 0, k = 0, 1, 2, \ldots$. Therefore,

$$\lim_{s\to\infty}\left(s^{k\beta+1}LRes(x,s)\right) = 0, \ k = 0, 1, 2, \ldots \ . \tag{14}$$

With a view to finding the first unknown coefficient $h_1$ in Equation (11), we substitute $U_1(s) = \frac{d}{s} + \frac{h_1}{s^{\beta+1}}$ in the first Laplace–residual function $LRes_1(s)$ to get

$$\begin{aligned} LRes_1(s) &= \tfrac{h_1}{s^{\beta+1}} + \tfrac{a}{s^\beta}\left(\tfrac{d}{s} + \tfrac{h_1}{s^{\beta+1}}\right) + \tfrac{b}{s^\beta}\mathcal{L}\left[\left(\mathcal{L}^{-1}\left[\tfrac{d}{s} + \tfrac{h_1}{s^{\beta+1}}\right]\right)^2\right] - \tfrac{c}{s^{\beta+1}} \\ &= \tfrac{h_1}{s^{\beta+1}} + \tfrac{ad}{s^{\beta+1}} + \tfrac{a\,h_1}{s^{2\beta+1}} + \tfrac{b}{s^\beta}\mathcal{L}\left[\left(d + \tfrac{h_1 t^\beta}{\Gamma(\beta+1)}\right)^2\right] - \tfrac{c}{s^{\beta+1}} \\ &= \tfrac{1}{s^{\beta+1}}\left(h_1 + ad + bd^2 - c\right) + \tfrac{1}{s^{2\beta+1}}\left(a\,h_1 + 2bd\,h_1\right) + \tfrac{1}{s^{3\beta+1}}\left(\tfrac{bh_1^2\ \Gamma(2\beta+1)}{\Gamma^2(\beta+1)}\right). \end{aligned} \tag{15}$$

Now, multiply both sides of the previous equation by $s^{\beta+1}$ to get

$$s^{\beta+1}\ LRes_1(r,s) = \left(h_1 + ad + bd^2 - c\right) + \frac{1}{s^\beta}(ah_1 + 2bd\,h_1) + \frac{1}{s^{2\beta}}\left(\frac{bh_1^2\ \Gamma(2\beta+1)}{\Gamma^2(\beta+1)}\right). \tag{16}$$

Finding the limit of Equation (16) as $s \to \infty$ and using the fact in Equation (14), we can calculate

$$h_1 = c - ad - bd^2. \tag{17}$$

Now, we substitute $U_2(r,s) = \frac{d}{s} + \frac{h_1}{s^{\beta+1}} + \frac{h_2}{s^{2\beta+1}}$ into the second Laplace–residual function $LRes_2(s)$ to find the second unknown coefficient $h_2$ as follows

$$\begin{aligned} LRes_2(s) = {} & \tfrac{1}{s^{\beta+1}}\left(h_1 + ad + bd^2 - \mathrm{c}\right) + \tfrac{1}{s^{2\beta+1}}\left(h_2 + ah_1 + 2bd\,h_1\right) \\ & + \tfrac{1}{s^{3\beta+1}}\left(ah_2 + 2bd\,h_2 + \tfrac{bh_1^2\ \Gamma(2\beta+1)}{\Gamma^2(\beta+1)}\right) + \tfrac{1}{s^{4\beta+1}}\left(\tfrac{2bh_1\ h_2\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)}\right) \\ & + \tfrac{1}{s^{5\beta+1}}\left(\tfrac{bh_2^2\ \Gamma(4\beta+1)}{\Gamma^2(2\beta+1)}\right). \end{aligned} \tag{18}$$

Thus, $h_2$ is obtained by inserting the value of $h_1$ into Equation (18), then multiplying the resulting equation by $s^{2\beta+1}$ and valuating the limit as $s \to \infty$:

$$h_2 = -(a + 2bd)h_1. \tag{19}$$

Again, to find $h_{3,}$ we substitute $U_3(r,s) = \frac{d}{s} + \frac{h_1}{s^{\beta+1}} + \frac{h_2}{s^{2\beta+1}} + \frac{h_3}{s^{3\beta+1}}$ into the third Laplace–residual function $LRes_3(s)$ to get

$$\begin{aligned} LRes_3(s) = {} & \tfrac{1}{s^{\beta+1}}\left(h_1 + ad + bd^2 - c\right) + \tfrac{1}{s^{2\beta+1}}\left(h_2 + ah_1 + 2bdh_1\right) \\ & + \tfrac{1}{s^{3\beta+1}}\left(h_3 + ah_2 + 2bdh_2 + \tfrac{bh_1^2\Gamma(2\beta+1)}{\Gamma^2(\beta+1)}\right) \\ & + \tfrac{1}{s^{4\beta+1}}\left(ah_3 + 2bdh_3 + \tfrac{2bh_1h_2\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)}\right) \\ & + \tfrac{1}{s^{5\beta+1}}\left(\tfrac{bh_2{}^2\Gamma(4\beta+1)}{\Gamma^2(2\beta+1)} + \tfrac{2bh_1h_3\Gamma(4\beta+1)}{\Gamma(\beta+1)\Gamma(3\beta+1)}\right) + \tfrac{1}{s^{6\beta+1}}\left(\tfrac{2bh_2h_3\Gamma(5\beta+1)}{\Gamma(2\beta+1)\Gamma(3\beta+1)}\right) \\ & + \tfrac{1}{s^{7\beta+1}}\left(\tfrac{bh_3{}^2\Gamma(6\beta+1)}{\Gamma^2(3\beta+1)}\right). \end{aligned} \tag{20}$$

Thus, $h_3$ is obtained by inserting the values of $h_1, h_2$ into Equation (20), then multiplying the resulting equation by $s^{3\beta+1}$ and evaluating the limit as $s \to \infty$:

$$h_3 = -\left((a + 2bd)h_2 + \frac{bh_1{}^2\Gamma(2\beta+1)}{\Gamma^2(\beta+1)}\right). \tag{21}$$

To find $h_4$, substitute $U_3(r,s) = \frac{d}{s} + \frac{h_1}{s^{\beta+1}} + \frac{h_2}{s^{2\beta+1}} + \frac{h_3}{s^{3\beta+1}} + \frac{h_4}{s^{4\beta+1}}$ into the fourth Laplace–residual function $LRes_4(s)$ to get

$$\begin{aligned} LRes_4(s) = {} & \tfrac{1}{s^{\beta+1}}\left(h_1 + ad + bd^2 - c\right) + \tfrac{1}{s^{2\beta+1}}\left(h_2 + ah_1 + 2bd\,h_1\right) \\ & + \tfrac{1}{s^{3\beta+1}}\left(h_3 + ah_2 + 2bd\,h_2 + \tfrac{bh_1^2\Gamma(2\beta+1)}{\Gamma^2(\beta+1)}\right) \\ & + \tfrac{1}{s^{4\beta+1}}\left(h_4 + ah_3 + 2bdh_3 + \tfrac{2bh_1h_2\ \Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)}\right) \\ & + \tfrac{1}{s^{5\beta+1}}\left(ah_4 + 2bdh_4 + \tfrac{bh_2{}^2\Gamma(4\beta+1)}{\Gamma^2(2\beta+1)} + \tfrac{2bh_1h_3\Gamma(4\beta+1)}{\Gamma(\beta+1)\Gamma(3\beta+1)}\right) \\ & + \tfrac{1}{s^{6\beta+1}}\left(\tfrac{2bh_2h_3\Gamma(5\beta+1)}{\Gamma(2\beta+1)\Gamma(3\beta+1)} + \tfrac{2bh_1h_4\Gamma(5\beta+1)}{\Gamma(\beta+1)\Gamma(4\beta+1)}\right) \\ & + \tfrac{1}{s^{7\beta+1}}\left(\tfrac{bh_3{}^2\Gamma(6\beta+1)}{\Gamma^2(3\beta+1)} + \tfrac{2bh_2h_4\Gamma(6\beta+1)}{\Gamma(2\beta+1)\Gamma(4\beta+1)}\right) + \tfrac{1}{s^{8\beta+1}}\left(\tfrac{2bh_3h_4\Gamma(7\beta+1)}{\Gamma(3\beta+1)\Gamma(4\beta+1)}\right) \\ & + \tfrac{1}{s^{9\beta+1}}\left(\tfrac{bh_4{}^2\Gamma(8\beta+1)}{\Gamma^2(4\beta+1)}\right). \end{aligned} \tag{22}$$

With steps similar to the above, we have

$$h_4 = -\left((a + 2bd)h_3 + \frac{2bh_1h_2\Gamma(3\beta+1)}{\Gamma(\beta+1)\Gamma(2\beta+1)}\right). \tag{23}$$

If we proceed in the same way by substituting the kth truncated series $U_k(s)$ into $LRes_k(s)$, multiplying the result by $s^{k\beta+1}$ and evaluating the limit as $s \to \infty$, $h_{k+1}$ for $k \geq 2$ can be obtained by the following recurrence relation

$$h_{k+1} = -\left((a+2bd)h_k + \sum_{\substack{i+j=k \\ i,j\in\mathbb{Z}^+}}^{\infty} \frac{rbh_ih_j\Gamma(k\beta+1)}{\Gamma(i\beta+1)\Gamma(j\beta+1)}\right), \tag{24}$$

where $= \begin{cases} 2, & |i-j| \neq 0 \\ 1, & |i-j| = 0 \end{cases}$, for $k = 2,3,4,\ \ldots,\ i+j=k$.

According to what was introduced, the series solution of Equation (8) is

$$U(s) = \frac{d}{s} + \frac{\left(c-ad-bd^2\right)}{s^{\beta+1}} - \frac{(a+2bd)\left(c-ad-bd^2\right)}{s^{2\beta+1}} + \sum_{k=3}^{\infty} \frac{h_k}{s^{k\beta+1}},\ \ s > \delta \geq 0. \tag{25}$$

So, the series solution of the nonlinear fractional Riccati differential Equation (5) can be obtained by applying the inverse Laplace transform in the solution in Equation (25). Therefore, the Laplace residual power series solution of Equation (5) is given by

$$u(t) = d + \frac{\left(c-ad-bd^2\right)t^{\beta}}{\Gamma(\beta+1)} - \frac{(a+2bd)\left(c-ad-bd^2\right)t^{2\beta}}{\Gamma(2\beta+1)} + \sum_{k=3}^{\infty} \frac{h_k\, t^{k\beta}}{\Gamma(k\beta+1)}, t \geq 0. \tag{26}$$

To test the accuracy of the proposed method, we introduce two kinds of error: absolute error and relative error, that are defined as follows:

$$\text{Absolute error} = |Exact\ value - Approximate\ value|,$$

$$\text{Relative error} = \left|\frac{Exact\ value - Approximate\ value}{Exact\ value}\right|$$

## 4. Illustrative Example

**Example 1.** *Consider the nonlinear fractional Riccati differential equation*

$$\mathfrak{D}^{\beta}u(t) - 2u(t) + u^2(t) = 1,\ \ 0 < \beta \leq 1, \tag{27}$$

*with the initial condition*

$$u(0) = 0 \tag{28}$$

.

Comparing Equations (27) and (28) with Equations (5) and (6), we find that $a = -2, b = 1$, $c = 1$ and $d = 0$. Therefore, the Laplace residual power series solution of Equation (27), according to the construction in Section 3, is as follows:

$$\begin{aligned} u(t) = {} & \frac{t^{\beta}}{\Gamma(1+\beta)} + \frac{2t^{2\beta}}{\Gamma(1+2\beta)} + \frac{t^{3\beta}\left(4-\frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)}\right)}{\Gamma(1+3\beta)} + \frac{t^{4\beta}\left(2\left(4-\frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)}\right)-\frac{4\Gamma(1+3\beta)}{\Gamma(1+\beta)\Gamma(1+2\beta)}\right)}{\Gamma(1+4\beta)} \\ & + \frac{t^{5\beta}\left(2\left(2\left(4-\frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)}\right)-\frac{4\Gamma(1+3\beta)}{\Gamma(1+\beta)\Gamma(1+2\beta)}\right)-\frac{4\Gamma(1+4\beta)}{\Gamma^2(1+2\beta)}-\frac{2\left(4-\frac{\Gamma(1+2\beta)}{\Gamma^2(1+\beta)}\right)\Gamma(1+4\beta)}{\Gamma(1+\beta)\Gamma(1+3\beta)}\right)}{\Gamma(1+5\beta)} \\ & + \ldots . \end{aligned} \tag{29}$$

It should be noted here that the series solutions of Equations (27) and (28) obtained by the LRPSM are identical to those obtained by the RPSM [25].

In a particular case, when $\beta = 1$, the Laplace residual power series solution of the problem (27) is

$$u(t) = t + t^2 + \frac{t^3}{3} - \frac{t^4}{3} - \frac{7t^5}{15} - \frac{7t^6}{45} + \frac{53t^7}{315} + \frac{71t^8}{315} + \frac{197t^9}{2835} - \frac{1213t^{10}}{14175} + \dots \tag{30}$$

which matches with the identical terms of the series expansions of the exact solution

$$u(t) = 1 + \sqrt{2}\tanh\left(\sqrt{2}t + \frac{1}{2}Log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right). \tag{31}$$

Figure 1 depicts the behavior of the approximate solutions $u(t),\ t\ \in I = [0,\ 0.5]$ of problem (27), (28) for various values of $0 < \beta \leq 1$.

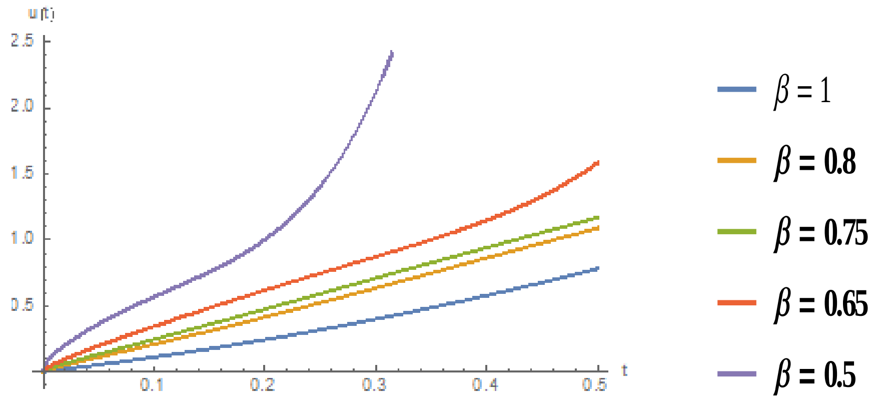


**Figure 1.** The behavior of the approximate solutions $u(t),\ t\ \in I = [0,\ 0.5]$ of problem (27) and (28).

Table 1 presents comparison between our approach of problems (27) and (28) and other existing numerical methods, the Laplace–Adomian–Pade method (LAPM) [28] and the modified homotopy perturbation method (HPM) [29] for $\beta = 0.5$ and $\beta = 0.75$, respectively. As shown in the table, the results produced by the 10-term LRPS approximate solution compare favorably with those of other techniques, especially when the result is close to 1 while having components that are simple to calculate. As a result, by adding more terms, higher precision may be gained for varied values of $\beta$.

**Table 1.** Comparison for the solution of problems (27) and (28) with different methods for $\beta = 0.5, \beta = 0.75$.

| $\beta = 0.5$ | LRPS | LAPM | HPM | $\beta$=0.75 | LRPS | LAPM | HPM |
|---|---|---|---|---|---|---|---|
| t = 0.1 | 0.593178 | 0.356803 | 0.321730 | t = 0.1 | 0.245431 | 0.193401 | 0.216866 |
| t = 0.2 | 0.955952 | 0.922865 | 0.629666 | t = 0.2 | 0.475107 | 0.454602 | 0.428892 |
| t = 0.3 | 1.389321 | 1.634139 | 0.940941 | t = 0.3 | 0.710342 | 0.784032 | 0.654614 |
| t = 0.4 | 2.351819 | 2.204441 | 1.250737 | t = 0.4 | 0.941954 | 1.161985 | 0.891404 |
| t = 0.5 | 4.693042 | 2.400447 | 1.549439 | t = 0.5 | 1.169808 | 1.543881 | 1.132763 |

**Example 2.** *Consider the nonlinear fractional Riccati differential equation*

$$\mathfrak{D}^{\beta}u(t) + u^2(t) = 1, \quad 0 < \beta \leq 1, \quad t \geq 0, \tag{32}$$

*with the initial condition*

$$u(0) = 0 \tag{33}$$

.

Comparing Equations (32) and (33) with Equations (5) and (6), we find that $a = 0, \ b = 1, \ c = 1$ and $d = 0$. Therefore, the Laplace residual power series solution of Equation (27), according to the construction in the previous section, is

$$
\begin{aligned}
u(t) = {} & \frac{t^{\beta}}{\Gamma[1+\beta]} - \frac{t^{3\beta}\Gamma[1+2\beta]}{\Gamma[1+\beta]^2\Gamma[1+3\beta]} + \frac{2t^{5\beta}\Gamma[1+2\beta]\Gamma[1+4\beta]}{\Gamma[1+\beta]^3\Gamma[1+3\beta]\Gamma[1+5\beta]} + \frac{t^{7\beta}\left(-\frac{\Gamma[1+2\beta]^2\Gamma[1+6\beta]}{\Gamma[1+\beta]^4\Gamma[1+3\beta]^2} - \frac{4\Gamma[1+2\beta]\Gamma[1+4\beta]\Gamma[1+6\beta]}{\Gamma[1+\beta]^4\Gamma[1+3\beta]\Gamma[1+5\beta]}\right)}{\Gamma[1+7\beta]} + \\
& t^{9\beta}\left(\frac{4\Gamma[1+2\beta]^2\Gamma[1+4\beta]\Gamma[1+8\beta]}{\Gamma[1+\beta]^5\Gamma[1+3\beta]^2\Gamma[1+5\beta]\Gamma[1+9\beta]} - \frac{2\left(-\frac{\Gamma[1+2\beta]^2\Gamma[1+6\beta]}{\Gamma[1+\beta]^4\Gamma[1+3\beta]^2} - \frac{4\Gamma[1+2\beta]\Gamma[1+4\beta]\Gamma[1+6\beta]}{\Gamma[1+\beta]^4\Gamma[1+3\beta]\Gamma[1+5\beta]}\right)\Gamma[1+8\beta]}{\Gamma[1+\beta]\Gamma[1+7\beta]\Gamma[1+9\beta]}\right) + \dots.
\end{aligned}
\tag{34}
$$

It should be noted here that the series solution of Equations (32) and (33) obtained by the LRPSM is identical to that obtained by the RPSM [25].

In a particular case, when $\beta = 1$, the Laplace residual power series solution of the problem (32) is

$$
u_{11}(t) = t - \frac{t^3}{3} + \frac{2t^5}{15} - \frac{17t^7}{315} + \frac{62t^9}{2835} - \frac{1382t^{11}}{155925} + \dots \tag{35}
$$

which matches with the identical terms of the series expansions of the exact solution

$$
u(t) = \left(e^{2t} - 1\right)\left(e^{2t} + 1\right)^{-1}.
$$

Figure 2 depicts the behavior of the approximate solutions $u(t), \ t \in I = [0, \ 0.5]$ of problems (32) and (33) for various values of $0 < \beta \leq 1$.

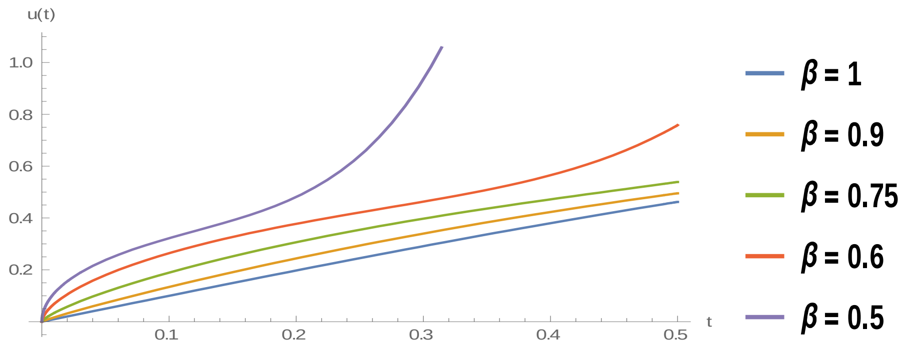


**Figure 2.** The behavior of the approximate solutions $u(t), \ t \in I = [0, \ 0.5]$ of problems (32) and (33).

To test the accuracy of the approximate solution in Equation (35), we calculate in Table 2 two types of error, the absolute error and the relative error, that are defined, respectively, as follows:

**Table 2.** The absolute and relative errors of the 11th approximate LRPS solution of problems (32) and (33) at $\beta = 1$.

| | Exact Solution | Approximate Solution | Absolute Error | Relative Error |
|---|---|---|---|---|
| t = 0.1 | 0.099667 | 0.099667 | $1.68079 \times 10^{-11}$ | $1.686388 \times 10^{-10}$ |
| t = 0.2 | 0.197375 | 0.197375 | $2.11082 \times 10^{-10}$ | $1.069444 \times 10^{-9}$ |
| t = 0.3 | 0.291312 | 0.291312 | $1.51948 \times 10^{-8}$ | $5.215977 \times 10^{-8}$ |
| t = 0.4 | 0.379949 | 0.379949 | $3.4917 \times 10^{-7}$ | $9.189918 \times 10^{-7}$ |
| t = 0.5 | 0.462121 | 0.462121 | $3.92967 \times 10^{-6}$ | $8.503622 \times 10^{-6}$ |

$E_{u,11} = |u - u_{11}|, \ RE_{u,11} = \left|\frac{u - u_{11}}{u}\right|$, where $u$ is the exact value and $u_{11}$ is the 11th approximate value obtained by the LRPSM.

## 5. Conclusions

Despite the fact that there are several numerical and analytical methods for solving a fractional differential equation, certain methods have benefits over others. Some are precise and efficient, but they need mathematical operations that can be time-consuming, complex or even fail. Others are quick and easy, yet they may not provide precise results. It should be noted that one of the most significant advantages of the LRPSM is that the MATHEMATICA program works faster while performing numerical and symbolic computations of the problem, since we do not need to calculate fractional derivatives during the steps of executing mathematical operations to extract results. It should be emphasized at the conclusion of this paper that this approach may be used to generate precise and approximate solutions for various kinds of integral and differential equations of fractional or nonfractional orders that fit the method's requirements.

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## References

1. Reid, T. *Riccati Differential Equations*; Elsevier: Amsterdam, The Netherlands, 1972.
2. Lasiecka, I.; Triggiani, R. (Eds.) *Differential and Algebraic Riccati Equations with Application to Boundary Point Control Problems: Continuous Theory and Approximation Theory*; Springer: Berlin/Heidelberg, Germany, 1991.
3. Khalil, I.S.; Doyle, J.C.; Glover, K. *Robust and Optimal Control*; Prentice Hall: Hoboken, NJ, USA, 1996.
4. Benner, P.; Li, J.-R.; Penzl, T. Numerical solution of large-scale Lyapunov equations, Riccati equations, and linear-quadratic optimal control problems. *Numer. Linear Algebra Appl.* **2008**, *15*, 755–777. [CrossRef]
5. Yong, J. Linear-quadratic optimal control problems for mean-field stochastic differential equations. *SIAM J. Control. Optim.* **2013**, *51*, 2809–2838. [CrossRef]
6. El Karoui, N.; Peng, S.; Quenez, M.C. Backward stochastic differential equations in finance. *Math. Financ.* **1997**, *7*, 1–71. [CrossRef]
7. Bittanti, S.; Laub, A.J.; Willems, J.C. (Eds.) *The Riccati Equation*; Springer Science & Business Media: Berlin/Heidelberg, Germany, 2012.
8. Raja, M.A.Z.; Khan, J.A.; Qureshi, I.M. A new stochastic approach for solution of Riccati differential equation of fractional order. *Ann. Math. Artif. Intell.* **2010**, *60*, 229–250. [CrossRef]
9. Suazo, E.; Sergei, K.S.; José, M.V. The Riccati differential equation and a diffusion-type equation. *N. Y. J. Math* **2011**, *17*, 225–244.
10. Kazem, S. Exact solution of some linear fractional differential equations by Laplace transform. *Int. J. Nonlinear Sci.* **2013**, *16*, 3–11.
11. Yavuz, M.; Ozdemir, N. Numerical inverse Laplace homotopy technique for fractional heat equations. *Therm. Sci.* **2018**, *22*, 185–194. [CrossRef]
12. Jafari, H.; Tajadodi, H. He's variational iteration method for solving fractional Riccati differential equation. *Int. J. Diff. Equ.* **2010**, *2010*, 764738. [CrossRef]
13. El-Ajou, A.; Odibat, Z.; Momani, S.; Alawneh, A. Construction of analytical solutions to fractional differential equations using homotopy analysis method. *Int. J. Appl. Math.* **2010**, *40*, 43–51.
14. Dehghan, M.; Manafian, J.; Saadatmandi, A. Solving nonlinear fractional partial differential equations using the homotopy analysis method. *Numer. Methods Partial. Differ. Equ.* **2009**, *26*, 448–479. [CrossRef]
15. Sweilam, N.H.; Khader, M.M.; S Mahdy, A.M. Numerical studies for solving fractional Riccati differential equation. *Appl. Appl. Math. Int. J. (AAM)* **2012**, *7*, 8.
16. Liu, J.; Li, X.; Wu, L. An operational matrix of fractional differentiation of the second kind of Chebyshev polynomial for solving multiterm variable order fractional differential equation. *Math. Probl. Eng.* **2016**, *2016*, 7126080. [CrossRef]
17. Wang, Q. Numerical solutions for fractional KdV–Burgers equation by Adomian decomposition method. *Appl. Math. Comput.* **2006**, *182*, 1048–1055. [CrossRef]
18. Saadeh, R.; Qazza, A.; Burqan, A. A new integral transform: ARA transform and its properties and applications. *Symmetry* **2020**, *12*, 925. [CrossRef]

19. Neamaty, A.; Agheli, B.; Darzi, R. The shifted Jacobi polynomial integral operational matrix for solving Riccati differential equation of fractional order. *Appl. Appl. Math. Int. J. (AAM)* **2015**, *10*, 16.
20. Qazza, A.; Burqan, A.; Saadeh, R. A new attractive method in solving families of fractional differential equations by a new transform. *Mathematics* **2021**, *9*, 3039. [CrossRef]
21. Burqan, A.; Saadeh, R.; Qazza, A. A novel numerical approach in solving fractional neutral pantograph equations via the ara integral transform. *Symmetry* **2021**, *14*, 50. [CrossRef]
22. Burqan, A.; El-Ajou, A.; Saadeh, R.; Al-Smadi, M. A new efficient technique using Laplace transforms and smooth expansions to construct a series solution to the time-fractional Navier-Stokes equations. *Alex. Eng. J.* **2021**, *61*, 1069–1077. [CrossRef]
23. Khandaqji, M.; Burqan, A. Results on sequential conformable fractional derivatives with applications. *J. Comput. Anal. Appl.* **2021**, *29*, 1115–1125.
24. Qiang, X.; Mahboob, A.; Chu, Y.M. Numerical approximation of fractional-order Volterra integrodifferential equation. *J. Funct. Spaces* **2020**, *2020*, 8875792. [CrossRef]
25. Ali, M.; Jaradat, I.; Alquran, M. New computational method for solving fractional Riccati equation. *J. Math. Comput. Sci.* **2017**, *17*, 106–114. [CrossRef]
26. Eriqat, T.; El-Ajou, A.; Oqielat, M.N.; Al-Zhour, Z.; Momani, S. A new attractive analytic approach for solutions of linear and nonlinear neutral fractional pantograph equations. *Chaos Solitons Fractals* **2020**, *138*, 109957. [CrossRef]
27. Oldham, K.; Spanier, J. *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*; Elsevier: Amsterdam, The Netherlands, 1974.
28. Khan, N.A.; Ara, A.; Khan, N.A. Fractional-order Riccati differential equation: Analytical approximation and numerical results. *Adv. Differ. Equ.* **2013**, *2013*, 185. [CrossRef]
29. Odibat, Z.; Momani, S. Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order. *Chaos Solitons Fractals* **2008**, *36*, 167–174. [CrossRef]