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Article

Switching-Jumps-Dependent Quasi-Synchronization Criteria for Fractional-Order Memrisive Neural Networks

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Abstract: This paper investigates the switching-jumps-dependent quasi-synchronization issue for fractional-order memristive neural networks (FMNNs). First, a simplied linear feedback controller is applied. Then, in terms of several fractional order differential inequalities and two kinds of Lyapunov functions, two quasi-synchronization criteria expressed by linear matrix inequality (LMI)-based form and algebraic form are established, respectively. Meanwhile, the co-designed scheme for error bound and control gain is established. Compared with the previous quasi-synchronization results, a strong assumption that the system states must be bounded is removed. Finally, some simulation examples are carried out to display the feasibility and validity of the proposed analysis methods.

Keywords: quasi-synchronization; fractional-order memristive neural networks; error bound; switching-jumps

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1. Introduction

Neural networks have self-organization ability, fault tolerance, adaptability, fast computing speed, and strong associative ability, and they are widely used in image processing [1], fault diagnosis [2], signal processing [3], pattern recognition [4], fluid mechanics [5] and other fields with broad application prospects and are a research hotspot. A rich body of research has studied FMNNs and developed neural network models such as convolution neural network [6], Hopfield neural network [7], artificial neural network [8], fuzzy rough neural network [9], and spiking neural network [10]. In recent decades, many scholars nationally and internationally have found that fractional calculus operators have memory and non-locality [11], and they are widely used in neural networks because of these characteristics. Fractional neural networks have achieved many excellent results, such as in [12,13]. Chua [14] pioneered the concept of the memristor, which was discovered by Hui Pu Labs [15] as the fourth basic circuit element. Memristors have the characteristics of low energy consumption, high storage, small size and non-volatility. They have very similar functions to biological neuron synapses and can act as synapses of artificial neurons. Therefore, the fractional-order neural network based on a memristor is established, namely the fractional-order memristive neural network. Many excellent works on FMNNs have been studied in [16–18].

Synchronization is an interesting research hotspot. Up to now, some published works have been published for FMNNs. For example, the Mittag–Leffler synchronization issue has been investigated for complex-valued FMNNs by designing complex-valued adaptive controller and using fractional-order Lyapunov theory [19]. The pinging synchronization control issues have been addressed along with delay pulse FMNNs and multi-delays FMNNs in [20,21], respectively. Combining open-loop control with time-delay feedback control, the projection synchronization issue has been considered for a class of multi-delay FMNNs [22]. By designing a chatter-free sliding mode controller, the synchronization of fractional-order chaotic systems has been discussed in combination with a neural network observer [23]. Using the interval matrix method, a linear state feedback controller was constructed, and then, the quasi-synchronization problem of FMNNs was solved [24]. In [25], the error system was processed by the closure algorithm. In addition, the linear state feedback controller has been applied, and the robust synchronization of the fractional Hopfield memristive neural network has been realized. Nevertheless, almost all the mentioned works are focused on the complete synchronization. Only a few works focus on the quasi-synchronization of FMNNs in the presence of switching jumps. In addition, most existing synchronization controllers may be too complex for FMNNs. Therefore, it is necessary to study the potential mechanism of the impact of switching jumps on synchronization. However, as far as we know, this is still an outstanding issue which deserves further investigation. As such, we need to study the switching-jumps-dependent quasi-synchronization issue for FMNNs.

It is worth noting that the previous works associated with the synchronization, the stability, and the stabilization problems of FMNNs are all expressed in terms of algebraic conditions [26–29]. Nevertheless, the defect of this method is that it requires a large amount of calculation because the conditions must be checked n times one by one. To overcome this defect, the LMI-based approach, based on which and with the help of MATLAB LMI toolbox the feasible solutions can be easily obtained, is considered to be a convenient and effective method. Unfortunately, the existing LMI-based approach is not applicable to the FMNNs systems studied in this paper. To this end, two new inequalities in differential inclusion are identified and the criterion of quasi-synchronization in the form of LMIs is obtained. On the other hand, the previous works on quasi-synchronization [30] cannot be derived unless there is a strong assumption that the system states must be bounded. This implies that the trajectory boundary needs to be given beforehand. That is to say, the state estimation issue is a prerequisite for the research of dynamic systems control design. Therefore, such a problem has become particularly meaningful and achieved a lot of excellent results. However, it indeed poses a difficult and challenging problem for obtaining the real-time status information. The main reason lies in that the boundaries of chaotic systems are dependent on the initial value, and then the boundary must be recalculated as long as the initial value changes. To address this issue, we will develop a novel analysis method that is not limited by chaotic trajectories so as to implement some improved quasi-synchronization criteria.

Inspired by the above discussion, this paper mainly discusses the quasi-synchronization of FMNNs based on linear feedback control. The main contributions are listed as follows:

(1) In contrast with the previous results [24,30], a strong assumption that the system states must be bounded is removed.

(2) In view of several fractional order differential inequalities and two kinds of Lyapunov functions, two quasi-synchronous criteria given by LMIs-based form and algebraic form are developed, respectively. In addition, the co-designed scheme for the error bound and control gain is established.

In addition, some useful notations are displayed in Table 1.

Table 1. Notations and Descriptions.

Notation	Description
$\lambda_{\max}(Q)$	the maximum eigenvalue of matrix Q
Q^T(or Q^{-1})	transpose (or inverse) of matrix Q
$*$	the symmetric element
$Q > 0$	Q is a positive
(or $Q \geq 0$)	definite (or semi-definite) matrix
diag$(\dots)$	a block diagonal matrix

2. Preliminaries and Model Description

Consider the following FMNNs as the drive system

$$ {}_0D_t^{\iota}\nu_p(t) = -\delta_p\nu_p(t) + \sum_{q=1}^{n}\alpha_{pq}\big(\nu_q(t)\big)\psi_q\big(\nu_q(t)\big), \quad p = 1,\ldots,n, \tag{1} $$

where order $0 < \iota < 1$; $\nu_p(t)$ denotes the state of the pth neuron; $\delta_p > 0$ denotes the system state; $\alpha_{pq}(\nu_q(t))$ is the memristive connection weights; and

$$ \alpha_{pq}(\nu_q(t)) = \frac{W_{fpq}}{\iota_p} \times \iota_{pq}, $$

where $\iota_{pq} = 1$ if $p \neq q$, otherwise $\iota_{pq} = -1$. W_{fpq} is the memductances of voltage-controlled memristors M_{fpq}, respectively; M_{fpq} is the memristor connecting $\psi_q\big(\nu_q(t)\big)$ and $\nu_p(t)$; $\psi_q(\cdot)$ is the activation functions satisfying $\psi_q(0) = 0$, $|\psi_q(\nu) - \psi_q(\mu)| \leq \psi_q|\nu - \mu|$, where ψ_q is Lipschitz constant, and $\nu, \mu \in R$ and $q = 1,\ldots,n$. The initial condition $\nu_p(t) = \phi_p(t)$, for $p = 1,\ldots,n$. The memristive synaptic weights $\alpha_{pq}\big(\nu_q(t)\big)$ is represented by

$$ \alpha_{pq}\big(\nu_q(t)\big) = \begin{cases} \hat{\alpha}_{pq}, & \big|\nu_q(t)\big| < T_q, \\ \check{\alpha}_{pq}, & \big|\nu_q(t)\big| > T_q, \end{cases} $$

for $p, q = 1,\ldots,n$, where $\hat{\alpha}_{pq}$, $\check{\alpha}_{pq}$, are constants and the switching jumps $T_q > 0$.

Accordingly, consider the response system

$$ {}_0D_t^{\iota}\mu_p(t) = -\delta_p\mu_p(t) + \sum_{q=1}^{n}\alpha_{pq}\big(\mu_q(t)\big)\psi_q\big(\mu_q(t)\big) + u_p(t), \quad p = 1,\ldots,n, \tag{2} $$

where $u_p(\nu(t))$ is the stabilizing control law, $\mu_p(t)$ denotes the state of the pth neuron; and

$$ \alpha_{pq}\big(\mu_q(t)\big) = \begin{cases} \hat{\alpha}_{pq}, & \big|\mu_q(t)\big| < T_q, \\ \check{\alpha}_{pq}, & \big|\mu_q(t)\big| > T_q, \end{cases} $$

where $T_q > 0$ denotes the switching jumps. The initial condition $\mu_p(t) = \vartheta_p(t)$, for $p = 1,\ldots,n$.

Based on [31], define the multivalued maps for system (1) as

$$ K\big[\alpha_{pq}\big(\nu_q(t)\big)\big] = \begin{cases} \hat{\alpha}_{pq}, & \big|\nu_q(t)\big| > T_q, \\ co\big\{\hat{\alpha}_{pq}, \check{\alpha}_{pq}\big\}, & \big|\nu_q(t)\big| = T_q, \\ \check{\alpha}_{pq}, & \big|\nu_q(t)\big| < T_q. \end{cases} $$

Clearly, $co\big\{\hat{\alpha}_{pq}, \check{\alpha}_{pq}\big\} = \Big[\underline{\alpha}_{pq}, \overline{\alpha}_{pq}\Big]$ for $p, q = 1,\ldots,n$, where $\underline{\alpha}_{pq} = min\big\{\hat{\alpha}_{pq}, \check{\alpha}_{pq}\big\}$, $\overline{\alpha}_{pq} = max\big\{\hat{\alpha}_{pq}, \check{\alpha}_{pq}\big\}$.

For $p = 1,\ldots,n$, the set-valued map

$$ \nu_p(t) \mapsto -\delta_p\nu_p(t) + \sum_{q=1}^{n} K\big[\alpha_{pq}\big(\nu_q(t)\big)\big]\psi_q\big(\nu_q(t)\big), $$

is compact, nonempty and convex. In addition, it is upper semi-continuous [32].

From (1), it yields

$$ {}_0D_t^{\iota}\nu_p(t) \in -\delta_p\nu_p(t) + \sum_{q=1}^{n} K\big[\alpha_{pq}\big(\nu_q(t)\big)\big]\psi_q\big(\nu_q(t)\big). \tag{3} $$

Moreover, in view of the measurable selection theorem [33], there exists $\gamma_{pq}\big(\nu_q(t)\big) \in K\big[\alpha_{pq}\big(\nu_q(t)\big)\big]$ such that

$$ {}_0D_t^\iota \nu_p(t) = -\delta_p \nu_p(t) + \sum_{q=1}^{n} \gamma_{pq}\big(\nu_q(t)\big)\psi_q\big(\nu_q(t)\big). \tag{4} $$

Similarly, for response system (2), define

$$ K\big[\alpha_{pq}\big(\mu_q(t)\big)\big] = \begin{cases} \hat{\alpha}_{pq}, & \big|\mu_q(t)\big| > T_q, \\ co\big\{\hat{\alpha}_{pq}, \check{\alpha}_{pq}\big\}, & \big|\mu_q(t)\big| = T_q, \\ \check{\alpha}_{pq}, & \big|\mu_q(t)\big| < T_q. \end{cases} $$

Then, from (2), it yields

$$ {}_0D_t^\iota \mu_p(t) \in -\delta_p \mu_p(t) + \sum_{q=1}^{n} K\big[\alpha_{pq}\big(\mu_q(t)\big)\big]\psi_q\big(\mu_q(t)\big) + u_p(t). \tag{5} $$

Moreover, there exists $\gamma_{pq}\big(\mu_q(t)\big) \in K\big[\alpha_{pq}\big(\mu_q(t)\big)\big]$ such that

$$ {}_0D_t^\iota \mu_p(t) = -\delta_p \mu_p(t) + \sum_{q=1}^{n} \gamma_{pq}\big(\mu_q(t)\big)\psi_q\big(\mu_q(t)\big) + u_p(t). \tag{6} $$

Let $\varepsilon_p(t) = \mu_p(t) - \nu_p(t)$, and the controller

$$ u_p(t) = -k_p \varepsilon_p(t), \tag{7} $$

where $k_p > 0$, $p = 1, 2, \ldots, n$. Then, according to (4) and (6), one has

$$ {}_0D_t^\iota \varepsilon_p(t) = -(\delta_p + k_p)\varepsilon_p(t) + \sum_{q=1}^{n} \big[\gamma_{pq}\big(\mu_q(t)\big)\psi_q\big(\mu_q(t)\big) - \gamma_{pq}\big(\nu_q(t)\big)\psi_q\big(\nu_q(t)\big)\big]. \tag{8} $$

The initial condition is defined by $\varepsilon_p(0) = \Phi_p(0)$, where $\Phi_p(s) = \vartheta\mu_p(s) - \vartheta\nu_p(s)$.

Denote $\nu(t) = [\nu_1(t), \ldots, \nu_n(t)]^T$, $\mu(t) = [\mu_1(t), \ldots, \mu_n(t)]^T$, $\varepsilon(t) = [\varepsilon_1(t), \ldots, \varepsilon_n(t)]^T$, and $\phi(s) = [\phi_1(s), \ldots, \phi_n(s)]^T$.

Remark 1. *During the dynamic evolution of FMNN, asynchronous switching time interval (ASTI) and synchronous switching time interval (SSTI), i.e., $\alpha_{pq}(\nu_q(t)) \neq \alpha_{pq}(\mu_q(t))$ and $\alpha_{pq}(\nu_q(t)) = \alpha_{pq}(\mu_q(t))$, appear alternately. The drive-response systems are sometimes consistent and sometimes different. When the drive-response systems switch with consistent jumps and inconsistent initial values, it is impossible to eliminate errors for ensuring complete synchronization, because fractional nonlinear systems based on continuous controllers do not have finite time stability [34]. Therefore, the error that is not completely eliminated will result in the state of the drive-response systems not reaching the switch jump T_q at the same time. This means that the drive-response systems cannot be switched synchronously, which will result in $\alpha_{pq}(\nu_q(t)) \neq \alpha_{pq}(mu_q(t))$. In other words, ASTI will appear. On the other hand, mismatches between join weights can also lead to new synchronization errors. Therefore, the linear feedback controller cannot achieve complete synchronisation due to the total error of two different subsystems in ASTI. After analysis, ASTI plays an important role in FMNN synchronization. In fact, even if the initial value is the same, if the switch jumps are not equal, i.e., $\Delta T \neq 0$, the linear feedback control systems (1)–(2) may not be able to achieve full synchronization, because ASTI does exist.*

To facilitate the analysis, the following definition and lemmas are proposed.

Definition 1 ([35]). *The Caputo fractional derivative of order $0 < \iota < 1$ of a function $\chi(t)$ is defined as*

$$_{t_0}D_t^{\iota}\chi(t) = \frac{1}{\Gamma(1-\iota)}\int_{t_0}^{t}(t-\tau)^{-\iota}\chi'(\tau)d\tau.$$

Lemma 1 ([36]). *If continuously differentiable function $h(t) \in C^1([0,+\infty), R)$, for any $\iota \in (0,1)$, one has*

$$_0D_t^{\iota}\nu(t) \leq \mathrm{sign}(h(t))_0\delta_t^{\iota}h(t).$$

Lemma 2 ([37]). *If vector $\nu(t) \in R^n$ denotes a differentiable function. Then, for $t \geq t_0$, one has*

$$\frac{1}{2}{}_{t_0}D_t^{\iota}\nu^T(t)P\nu(t) \leq \nu^T(t)P_{t_0}D_t^{\iota}\nu(t) \qquad \forall\iota \in (0,1],$$

where $P > 0 \in R^{n\times n}$.

Lemma 3 ([38]). *The LMI*

$$\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{12}^T & \mathbb{O}_{22} \end{bmatrix} < 0,$$

is equivalent to

- $\mathbb{O}_{11} < 0, \qquad \mathbb{O}_{22} - \mathbb{O}_{12}^T\mathbb{O}_{11}^{-1}\mathbb{O}_{12} < 0,$
- $\mathbb{O}_{22} < 0, \qquad \mathbb{O}_{11} - \mathbb{O}_{12}\mathbb{O}_{22}^{-1}\mathbb{O}_{12}^T < 0,$

where $\mathbb{O}_{11} = \mathbb{O}_{11}^T$, $\mathbb{O}_{22} = \mathbb{O}_{22}^T$.

3. Main Results

Theorem 1. *If the following algebraic condition*

$$\min_{1\leq p\leq n}\left\{\delta_p + k_p - \sum_{q=1}^{n}\alpha_{qp}^{u}L_p\right\} > 0$$

holds, then systems (1)–(2) achieve quasi-synchronization with error bound $\frac{\theta}{\lambda}$ via the controller (7) , where $\lambda = \min_{1\leq p\leq n}\left\{\delta_p + k_p - \sum_{q=1}^{n}\alpha_{qp}^{u}L_p\right\}$, $\theta = \sum_{p=1}^{n}\sum_{q=1}^{n}|\Delta\alpha_{pq}|L_qT_{\max}$.

Proof. Take the following Lyapunov function:

$$V(t) = \sum_{p=1}^{n}|\varepsilon_p(t)|.$$

or any given $p, q = 1, 2, \ldots, n$, and $\nu_q, \mu_q \in R$, one has
If $|\nu_q(t)| < T_q, |\mu_q(t)| < T_q$, then

$$\begin{aligned}\left|K[\alpha_{pq}(\mu_q(t))]\psi_q(\mu_q(t)) - K[\alpha_{pq}(\nu_q(t))]\psi_q(\nu_q(t))\right| &= \left|\hat{\alpha}_{pq}\psi_q(\mu_q(t)) - \hat{\alpha}_{pq}\psi_q(\nu_q(t))\right| \\ &\leq |\hat{\alpha}_{pq}| \ |\psi_q(\mu_q(t)) - \psi_q(\nu_q(t))| \\ &\leq \alpha_{pq}^{u}L_q|\varepsilon_q(t)|,\end{aligned}$$

where $\alpha_{pq}^{u} = \max\{|\hat{\alpha}_{pq}|, |\check{\alpha}_{pq}|\}$.
If $|\nu_q(t)| > T_q, |\mu_q(t)| > T_q$, then

$$\begin{aligned}\left|K[\alpha_{pq}(\mu_q(t))]\psi_q(\mu_q(t)) - K[\alpha_{pq}(\nu_q(t))]\psi_q(\nu_q(t))\right| &= \left|\check{\alpha}_{pq}\psi_q(\mu_q(t)) - \check{\alpha}_{pq}\psi_q(\nu_q(t))\right| \\ &\leq |\check{\alpha}_{pq}||\psi_q(\mu_q(t)) - \psi_q(\nu_q(t))| \\ &\leq \alpha_{pq}^{u}L_q|\varepsilon_q(t)|.\end{aligned}$$

If $|\nu_q(t)| \leq T_q$, $|\mu_q(t)| \geq T_q$, then

$$\begin{aligned}
&|K[\alpha_{pq}(\mu_q(t))]\psi_q(\mu_q(t)) - K[\alpha_{pq}(\nu_q(t))]\psi_q(\nu_q(t))| \\
=&|\check{\alpha}_{pq}\psi_q(\mu_q(t)) - \hat{\alpha}_{pq}\psi_q(\nu_q(t))| \\
=&|\check{\alpha}_{pq}[\psi_q(\mu_q(t)) - \psi_q(\nu_q(t))] + (\check{\alpha}_{pq} - \hat{\alpha}_{pq})\psi_q(\nu_q(t))| \\
\leq&\alpha^u_{pq}L_q|\varepsilon_q(t)| + |\hat{\alpha}_{pq} - \check{\alpha}_{pq}|L_q|\nu_q(t)| \\
\leq&\alpha^u_{pq}L_q|\varepsilon_q(t)| + |\Delta\alpha_{pq}|L_q T_{\max}.
\end{aligned}$$

If $|\nu_q(t)| \geq T_q$, $|\mu_q(t)| \leq T_q$, we also have

$$\begin{aligned}
&|K[\alpha_{pq}(\mu_q(t))]\psi_q(\mu_q(t)) - K|\alpha_{pq}(\nu_q(t))]\psi_q(\nu_q(t))\ | \\
\leq&\alpha^u_{pq}L_q|\varepsilon_q(t)| + |\Delta\alpha_{pq}|L_q T_{\max}.
\end{aligned}$$

Hence, one can obtain

$$|K[\alpha_{pq}(\mu_q(t))]\psi_q(\mu_q(t)) - K[\alpha_{pq}(\nu_q(t))]\psi_q(\nu_q(t))| \leq \alpha^u_{pq}L_q|\varepsilon_q(t)| + |\Delta\alpha_{pq}|L_q T_{\max}.$$

In view of Lemma 1, one has

$$\begin{aligned}
{}_0\delta^\iota_t V(t) =&\delta^\iota_t \sum_{p=1}^{n}|\varepsilon_p(t)| \leq \sum_{p=1}^{n}\operatorname{sign}(\varepsilon_p(t))\delta^\iota_t\varepsilon_p(t) \\
=&\sum_{p=1}^{n}\operatorname{sign}(\varepsilon_p(t))\{-(\delta_p + k_p)\varepsilon_p(t) \\
&+\sum_{q=1}^{n}[\gamma_{pq}(\mu_q(t))\psi_q(\mu_q(t)) - \gamma_{pq}(\nu_q(t))\psi_q(\nu_q(t))]\} \\
\leq&-\sum_{p=1}^{n}(\delta_p + k_p)|\varepsilon_p(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}\alpha^u_{pq}L_q|\varepsilon_q(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}|\Delta\alpha_{pq}|L_q T_{\max} \\
=&-\sum_{p=1}^{n}\left(\delta_p + k_p - \sum_{q=1}^{n}\alpha^u_{qp}L_p\right)|\varepsilon_p(t)| + \sum_{p=1}^{n}\sum_{q=1}^{n}|\Delta\alpha_{pq}|L_q T_{\max}.
\end{aligned}$$

Let $\lambda = \min_{1\leq p\leq n}\left\{\delta_p + k_p - \sum_{q=1}^{n}\alpha^u_{qp}L_p\right\}, \theta = \sum_{p=1}^{n}\sum_{q=1}^{n}|\Delta\alpha_{pq}|L_q\ T_{\max},\ T_{\max} = \max\{T_q\}$. Then, one has

$$\begin{aligned}
\delta^\iota_t V(t) \leq& -\lambda\sum_{p=1}^{n}|\varepsilon_p(t)| + \theta \\
=& -\lambda V(t) + \theta.
\end{aligned}$$

Based on Fractional Halanay inequality [39], it yields

$$\|\varepsilon(t)\|_1 \leq \frac{\theta}{\lambda}, \quad t \to +\infty.$$

Obviously, the synchronization error belongs to region D, where

$$D = \left\{\varepsilon(t) : \|\varepsilon(t)\|_1 \leq \frac{\theta}{\lambda}\right\}, \quad t \to +\infty,$$

which means that the quasi-synchronization of systems (1)–(2) can be achieved with error bound $\frac{\theta}{\lambda}$. $\square$

Theorem 2. *For given matrices $P > 0, Q_1 > 0, Q_2 > 0, G$, a scalar $\lambda > 0$, and*

$$\begin{bmatrix} -2PC - 2G + P\tilde{A}L + L\tilde{A}^T P + \lambda P & 0 & P \\ * & -Q_1 & 0 \\ * & * & -Q_2 \end{bmatrix} < 0,$$

holds, then the quasi-synchronization of systems (1)–(2) can be achieved with error bound $\sqrt{\frac{\theta}{\min(p_p)\lambda}}$, where $C = \mathrm{diag}(\delta_1, \ldots, \delta_n), M = \mathrm{diag}(M_1, \ldots, M_n), L = \mathrm{diag}(L_1, \ldots, L_n)$, $\theta = H^T Q_2 H$, $H = \Delta A L \tilde{T}_{\max}, \tilde{T}_{\max} = (T_{\max}, \ldots, T_{\max})^T, \tilde{A} = \left(\alpha^u_{pq}\right)_{n\times n}, \Delta A = \left(|\Delta\alpha_{pq}|\right)_{n\times n}$. Moreover, feedback gain $K = P^{-1}G$, $P = \mathrm{diag}(p_1, \ldots, p_n)$ and $K = \mathrm{diag}(k_1, \ldots, k_n)$.

Proof. Consider the Lyapunov function

$$V(t) = |\varepsilon(t)|^T P |\varepsilon(t)|.$$

where $|\varepsilon(t)| = (|e_1(t)|, |e_2(t)|, \ldots, |e_n(t)|)^T, P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$

In view of Lemmas 1–3, one has

$$\begin{aligned}
{}_0\delta^\iota_t V(t) \leq& 2|\varepsilon(t)|^T P {}_0\delta^\iota_t |\varepsilon(t)| = 2\sum_{p=1}^n |\varepsilon_p(t)| p_p \delta^\iota_t |\varepsilon_p(t)| \\
\leq& 2\sum_{p=1}^n |\varepsilon_p(t)| p_p \mathrm{sign}(\varepsilon_p(t)) {}_0\delta^\iota_t \varepsilon_p(t) \\
=& 2\sum_{p=1}^n |\varepsilon_p(t)| p_p \mathrm{sign}(\varepsilon_p(t)) \{-(\delta_p + k_p)\varepsilon_p(t) \\
& + \sum_{q=1}^n [\gamma_{pq}(\mu_q(t))\psi_q(\mu_q(t)) - \gamma_{pq}(\nu_q(t))\psi_q(\nu_q(t))]\} \\
\leq& -2\sum_{p=1}^n |\varepsilon_p(t)| p_p (\delta_p + k_p) |\varepsilon_p(t)| \\
& + \sum_{p=1}^n \sum_{q=1}^n |\varepsilon_p(t)| p_p |\gamma_{pq}(\mu_q(t))\psi_q(\mu_q(t)) - \gamma_{pq}(\nu_q(t))\psi_q(\nu_q(t))| \\
\leq& -2\sum_{p=1}^n |\varepsilon_p(t)| p_p (\delta_p + k_p) |\varepsilon_p(t)| + 2\sum_{p=1}^n \sum_{q=1}^n |\varepsilon_p(t)| p_p \Big(\alpha^u_{pq} L_q |\varepsilon_q(t)| \\
& + |\Delta\alpha_{pq}| L_q T_{\max}) \\
=& -2\sum_{p=1}^n |\varepsilon_p(t)| p_p (\delta_p + k_p) |\varepsilon_p(t)| \\
& + 2\sum_{p=1}^n \sum_{q=1}^n |\varepsilon_p(t)| p_p \alpha^u_{pq} L_q |\varepsilon_q(t)| + 2\sum_{p=1}^n \sum_{q=1}^n |\varepsilon_q(t)| p_p |\Delta\alpha_{pq}| L_q T_{\max} \\
=& -2|\varepsilon(t)|^T P(C+K)|\varepsilon(t)| + 2|\varepsilon(t)|^T P\tilde{A}L|\varepsilon(t)| + 2|\varepsilon(t)|^T PH \\
\leq& -2|\varepsilon(t)|^T P(C+K)|\varepsilon(t)| + 2|\varepsilon(t)|^T P\tilde{A}L|\varepsilon(t)| \\
& + |\varepsilon(t)|^T Q_2^{-1} P^2 |\varepsilon(t)| + H^T Q_2 H \\
=& |\varepsilon(t)|^T \Big(-2PC - 2PK + P\tilde{A}L + L\tilde{A}^T P + Q_2^{-1}P^2 + \lambda P\Big)|\varepsilon(t)| \\
& - \lambda |\varepsilon(t)|^T P |\varepsilon(t)| + H^T Q_2 H.
\end{aligned}$$

By Lemma 3, the LMI of Theorem 2 is equivalent to

$$-2PC - 2PK + P\tilde{A}L + L\tilde{A}^T P + Q_2^{-1}P^2 + \lambda P < 0.$$

Hence, one has

$$\begin{aligned} {}_0\delta_t^\iota V(t) &\leq -\lambda|\varepsilon(t)|^T P|\varepsilon(t)| + H^T Q_2 H \\ &\leq -\lambda|\varepsilon(t)|^T P|\varepsilon(t)| + \theta \\ &= -\lambda V(t) + \theta. \end{aligned}$$

Based on Fractional Halanay inequality [39], one has

$$V(t) \leq \frac{\theta}{\lambda}, \quad t \to +\infty. \tag{9}$$

From $V(t) = |\varepsilon(t)|^T P|\varepsilon(t)| = \sum_{p=1}^n p_p |\varepsilon_p(t)|^2$, one has

$$\min(p_p)\|\varepsilon(t)\|_2^2 \leq V(t) \leq \max(p_p)\|\varepsilon(t)\|_2^2. \tag{10}$$

According to (9) and (10), one can obtain

$$\min(p_p)\|\varepsilon(t)\|_2^2 \leq V(t) \leq \frac{\theta}{\lambda}, \quad t \to +\infty$$

Hence, it has

$$\|\varepsilon(t)\|_2 \leq \sqrt{\frac{\theta}{\min(p_p)\lambda}}, \quad t \to +\infty,$$

which means that the quasi-synchronization of systems (1)–(2) can be achieved with error bound $\sqrt{\frac{\theta}{\min(p_p)\lambda}}$. □

Remark 2. *Because of the fractional derivative definition, some traditional methods applied for MNNs, i.e., the Lyapunov Krasovskii functional, cannot be simply used to find conclusions and apply them to FMNNs. By employing the existing LMI-based analysis techniques and by constructing an appropriate fractional Lyapunov function, the LMI-based conditions of FMNNs are given, which are easy to solve. The analysis technology in this paper provides a new idea for the research of FMNNs.*

4. Numerical Examples

Example 1. *Consider a* $2-$*neurons FMNNs (1),* $\iota = 0.78$, $\psi_q(\nu_q) = tanh(\nu_q)$, $q = 1, 2$. *Take* $\delta_1 = 2.6$, $\delta_2 = 2.6$, $T_q = 1$ *and*

$$\alpha_{11}(\nu_1) = \begin{cases} 2.0, & |\nu_1| < 1 \\ 1.2, & |\nu_1| > 1 \end{cases}, \alpha_{12}(\nu_2) = \begin{cases} -2.0, & |\nu_2| < 1 \\ -2.1, & |\nu_2| > 1 \end{cases},$$

$$\alpha_{21}(\nu_1) = \begin{cases} -0.35, & |\nu_1| < 1 \\ -0.30, & |\nu_1| > 1 \end{cases}, \alpha_{22}(\nu_2) = \begin{cases} 2.55, & |\nu_2| < 1 \\ 2.60, & |\nu_2| > 1 \end{cases}.$$

Assume that the response system (2) is different from the drive system only by switching jumps $T_q = 0.1, q = 1, 2$, *and* $\vartheta_\nu(s) = \vartheta_\mu(s) = (0.8, -0.5)^T$.

Figure 1a,b show the state trajectories of drive-response systems, respectively. Taking $k_1 = 12.5958, k_2 = 4.7486$, by simple calculation, one has $T_{\max} = 1$,

$$\lambda = \min_{1 \leq p \leq n} \left\{ \delta_p + k_p - \sum_{q=1}^n \alpha_{qp}^u L_p \right\} = 2.6486,$$

$$\theta = \sum_{p=1}^n \sum_{q=1}^n |\Delta\alpha_{pq}| L_q T_{\max} = 1.$$

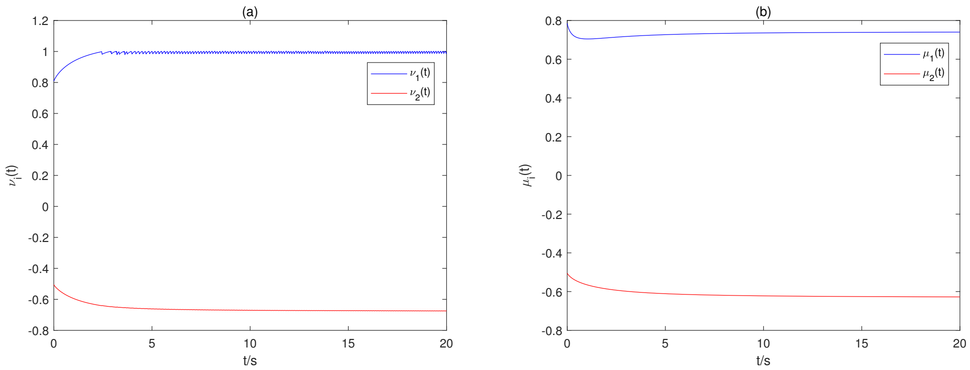

Figure 1. (**a**) Time responses of $\nu_1(t)$, $\nu_2(t)$; (**b**) $\mu_1(t)$, $\mu_2(t)$.

In view of Theorem 1, the quasi-synchronization can be achieved for systems (1)–(2) in the presence of error bound $\|\varepsilon(t)\|_1 \leq \frac{\theta}{\lambda} = 0.3776$, which is verified by Figure 2. From Figure 3, one can conclude that the synchronization error can be influenced by ΔT.

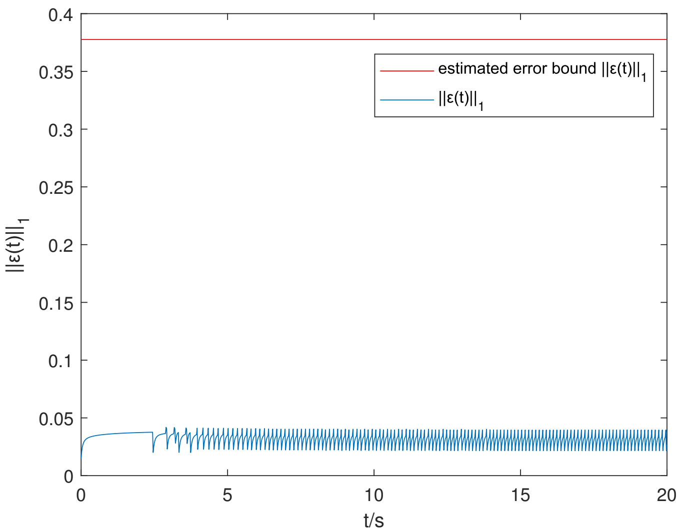

Figure 2. Error bound and $\|\varepsilon(t)\|_1$ of Example 1.

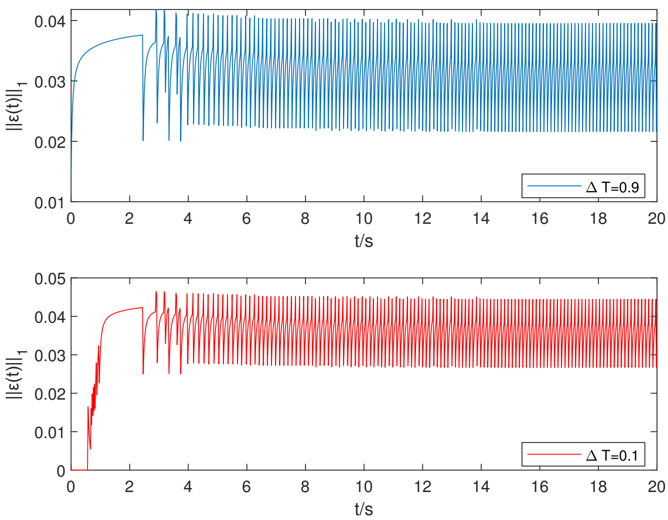

Figure 3. $\|\varepsilon(t)\|_1$ with different ΔT.

Example 2. *Consider a* $3-$*neuron FMNN (1),* $\iota = 0.85$*,* $\psi_q(\nu_q) = tanh(\nu_q)$*. Set* $\delta_1 = 2.2$*,* $\delta_2 = 1.2$*,* $\delta_3 = 1.8$*,* $\vartheta_\nu(s) = (1, -0.5, 0.7)^T$*,* $T_q = 1$*,* $q = 1, 2, 3$*, and*

$$\alpha_{11}(\nu_1) = \begin{cases} 2.2, & |\nu_1| \leq T_q \\ 2, & |\nu_1| > T_q \end{cases}, \alpha_{12}(\nu_2) = \begin{cases} -2, & |\nu_2| \leq T_q \\ -2.1, & |\nu_2| > T_q \end{cases},$$

$$\alpha_{13}(\nu_3) = \begin{cases} 2, & |\nu_3| \leq T_q \\ 1.8, & |\nu_3| > T_q \end{cases}, \alpha_{21}(\nu_1) = \begin{cases} -0.8, & |\nu_1| \leq T_q \\ -0.6, & |\nu_1| > T_q \end{cases},$$

$$\alpha_{22}(\nu_2) = \begin{cases} 5.71, & |\nu_2| \leq T_q \\ 5.68, & |\nu_2| > T_q \end{cases}, \alpha_{23}(\nu_3) = \begin{cases} 1.15, & |\nu_3| \leq T_q \\ 1.1, & |\nu_3| > T_q \end{cases},$$

$$\alpha_{31}(\nu_1) = \begin{cases} -4.75, & |\nu_1| \leq T_q \\ -4.5, & |\nu_1| > T_q \end{cases}, \alpha_{32}(\nu_2) = \begin{cases} -1, & |\nu_2| \leq T_q \\ -0.8, & |\nu_2| > T_q \end{cases},$$

$$\alpha_{33}(\nu_3) = \begin{cases} 1.2, & |\nu_3| \leq T_q \\ 1.25, & |\nu_3| > T_q \end{cases}.$$

Assume that the response system (2) is different from the drive system only by switching jumps $T_q = 0.2$*,* $q = 1, 2, 3$*.*

Figure 4a,b display the state trajectories of systems (1)–(2) with $\vartheta_\nu(s) = (1.5, -0.6, 0.3)^T$ and $\vartheta_\mu(s) = (-1.5, 5.1, -4.6)^T$, respectively.

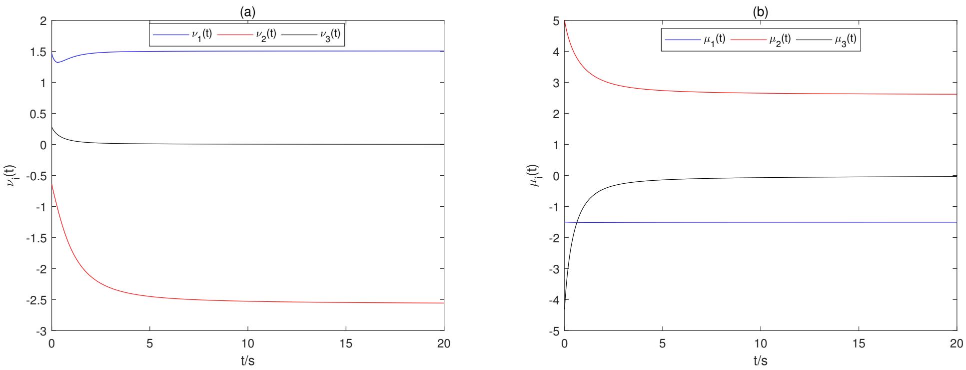

Figure 4. (**a**) Time responses of $\nu_p(t)$; (**b**) $\mu_p(t)$, $p = 1, 2, 3$.

Taking $\lambda = 6$, the values of P, G, Q_1, Q_2, K are given by solving the LMI in Theorem 2.

$$P = \begin{bmatrix} 0.4924 & 0 & 0 \\ 0 & 0.7393 & 0 \\ 0 & 0 & 0.3226 \end{bmatrix},$$

$$G = \begin{bmatrix} 4.1480 & 0 & 0 \\ 0 & 8.2232 & 0 \\ 0 & 0 & 3.4612 \end{bmatrix},$$

$$Q_1 = Q_2 = \begin{bmatrix} 5.3417 & 0 & 0 \\ 0 & 5.3417 & 0 \\ 0 & 0 & 5.3417 \end{bmatrix},$$

$$K = P^{-1}G = \begin{bmatrix} 8.4243 & 0 & 0 \\ 0 & 11.1226 & 0 \\ 0 & 0 & 10.7301 \end{bmatrix}.$$

By simple calculation, one has

$$H = \Delta AL\tilde{T}_{\max} = (0.2, 0.03, 0.05)^T,$$

$$\theta = \left\| H^T Q_2 H \right\|_2 = 0.2137.$$

Based on Theorem 2, the quasi-synchronization can be ensured for systems (1)–(2) with error bound $\|\varepsilon(t)\|_2 \leq \sqrt{\frac{\theta}{\min(p_p)\lambda}} = 0.3322$, which is verified by Figure 5.

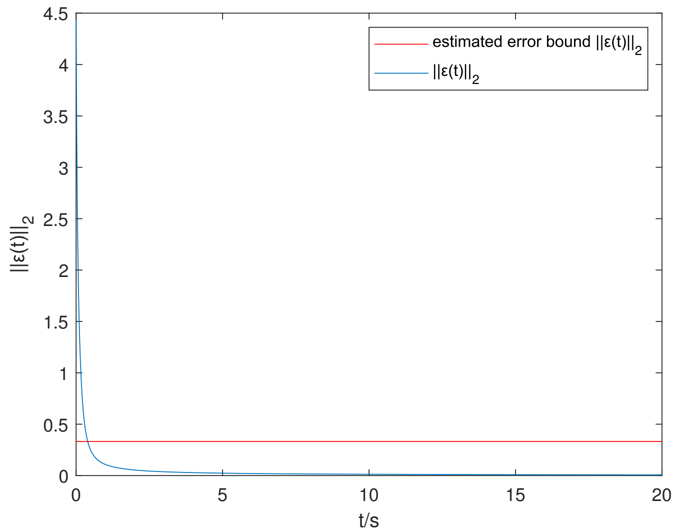

Figure 5. Error bound and $\|\varepsilon(t)\|_1$ of Example 2.

5. Conclusions

This paper investigates the switching-jumps-dependent quasi-synchronization problem for FMNNs. To derive the quasi-synchronization criteria, a simple linear feedback controller is applied. To obtain the improved results, a strong assumption that the system states must be bounded is removed. In combination with several fractional-order differential inequalities and two kinds of Lyapunov functions, two quasi-synchronous criteria expressed by LMIs-based conditions and algebraic conditions are derived, respectively. Finally, the theoretical results are verified by two numerical examples.

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Abbreviations

The following abbreviations are used in this manuscript:

ASTI	asynchronous switching time interval
FMNNs	fractional-order memristive neural networks
SSTI	asynchronous switching time interval
LMI	linear matrix inequality

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